Competitive Real Option Risk

Roger Adkins
Alliance Manchester Business School, University of Manchester
Manchester M15 6PB, UK
rogeradkins2020@outlook.com

Alcino Azevedo
Aston Business School, University of Aston
Birmingham B4 7ET, UK
a.azevedo@aston.ac.uk

Dean Paxson[1]
Alliance Manchester Business School, University of Manchester
Manchester M15 6PB, UK
dean.paxson@manchester.ac.uk

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Abstract

We evaluate the risk aspects of a simple portfolio of real options to invest for a duopoly. After summarizing the basic model, covering three sequences, two thresholds, and three strategic and rival options, we look at five risk elements: delta, vega, rho (the conventional option Greeks) along with epsilon (drift) and alpha (market share). The value function of both the leader and follower is most sensitive to revenue (delta), interest rate (rho), drift (epsilon) and market share (alpha) variations, which we view in terms of sensitivities (to percentage changes), partial derivatives (analytical confirmed by numerical) and to a range of each of the input variables. Naturally, delta and rho hedging are plausible and appropriate risk avoidance actions. Maintaining final stage market share is particularly important for the follower.

[1] Corresponding author.

Highlights

- For a duopoly real option investment model, an analytical method is formulated to derive the impact on the value function due to a change in each parameter value.
- The analytical method is validated by comparing the solution with a numerical method based on ever finer differences which converge to the analytical.
- The impact of a change in most parameter values is not constant and varies according to value of the state variable.
- The function representing the impact of a parameter change on the value function is semi-continuous for at least one of the players.

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1 Introduction

What is the appropriate measure of risk for real options in a duopoly? We address this issue through studying (i) the sensitivities of changes in the value functions to 1% changes in the model parameter values, (ii) through calculating the analytical partial derivatives for the thresholds, option coefficients and value functions (reconfirmed with numerical partial derivatives), and (iii) through calculating the changes in value functions across the regimes along a range of changes for each input parameter value[2]. Which is the most appropriate method for observing (and eventually managing) risk?

There is limited literature on most of these approaches. Both pre-emptive and non-pre-emptive duopoly real options usually require a numerical solution for the leader's threshold, and ignore risk exposure, partial derivatives and risk management. Few of the models allow for an operating cost. Few models offer the proofs that the differential equation is solved (or not), and that the value matching and smooth pasting conditions are satisfied. Few authors are concerned with market share derivatives, with risk assessments.

Fudenberg and Tirole (1985) created the foundations of real options in a competitive setting while developing a model of games of timing with a continuous time version of strategy equilibrium. Smets (1993) considered a strategic setting where firms can act under the fear of pre-emption, clearly presented by Dixit and Pindyck (1994) Chapter 9.3. Joaquin and Butler (2000) consider the first mover advantage of lower operating costs. Smit and Trigeorgis (2001) modelled different investment strategies under quantity or reciprocating price competition.

Tsekrekos (2003) studied the sensitivity of the leader and follower value function to market share (with both temporary and pre-emptive permanent market share advantages for the leader), assumed to be constant after the follower enters. Paxson and Pinto (2003) model a leader with an initial market share advantage, which then evolves as new customers arrive (birth) and existing customers depart (death)[3]. Paxson and Pinto (2005) suggest a two-factor model with permanent quantity

[2] Provisionally, these are shown only in Appendix F.

[3] Appendix H reviews the innovations in these two articles regarding some analytical partial derivatives (delta, and alpha), and discussions of the respective leader/follower choices and actions.

advantages accorded to the leader. Paxson and Melmane (2009) provide a two-factor model where the leader starts with a larger but stochastic market share. Bobtcheff and Mariotti (2013) consider a pre-emptive game of two innovative competitors, whose existence may be revealed only by first-mover investment. Azevedo and Paxson (2014) review the literature on developing such real option games. Huberts et al. (2019) show that for a duopoly, entry may be deterred by competitive actions, possibly in a war of attrition or pre-emption, following interesting strategies. Adkins et al. (2022) provide quasi-analytical solutions for switching and divesting opportunities in a duopoly with mutually exclusive options.

We provide six innovations for basic once-off investment opportunities in a duopoly (non-pre-emptive) with variable operating costs: analytical solutions for the thresholds and option coefficients; analytical solutions for the partial derivatives for all of the inputs (except K); confirming all of these solutions with numerical solutions; confirming that these solutions with solve the conventional different equations, and the value matching and smooth pasting conditions (except for the special case of crossing the follower's threshold); confirming that the sensitivities, partial derivatives and simulation of value functions across the basic revenue range are consistent; and finally showing how the delta partial derivative can be used for delta hedging to sharply reduce risk.

The rest of the paper is organized as follows. Section 2 derives the investment real options model for a duopoly with variable operating costs. Section 3 shows sensitivities of the value functions for each of the parameter inputs. Section 4 derives analytical results for each of the partial derivatives, and discusses some of the option coefficient characteristics. Section 5 reviews the additional insights available from considering the evolution of value function across a range of values for each input parameter, useful in hedging risk exposure. Section 6 summarizes and concludes and provides some suggestions for further research and applications.

2 Real Option Model for a Duopoly with Variable Operating Costs

We demonstrate the analytical procedure based on partial derivatives for determining the impact of input parameter variations on the value function of a leader and follower in a duopoly investment opportunity. We find that for one of the rivals in a duopoly model, the derivative of their value function with respect to market share, volatility, interest rate and revenue drift are semi-continuous functions with a jump, which can be both positive and negative, and varies according to the value of the state variable. The partial derivatives have similar characteristics, regarding thresholds and option coefficients, as opposed to the revenue (delta) derivative (where the thresholds and option coefficients remain the same).

Our basic model builds on a monopoly context, where the firm, with no current cash-flow, has a perpetual opportunity to invest in an operating asset that it intends to exercise and operate forever as soon as the asset's prevailing cash-inflow, denoted by v, is sufficiently high. The optimal policy is to retain the investment option for $0 < v < v_1$, where v_1 denotes the threshold cash-inflow, and to exercise the option for $v_1 \leq v < \infty$. While the cash-inflow remains within the inaction region, $v \in (0, v_1)$, the firm does nothing. Whenever v departs from the inaction region, where $v \notin (0, v_1)$, the firm makes the investment. We assume the state variable, the cash-inflow, follows a geometric Brownian motion process:

$$\mathrm{d}v = \delta v \,\mathrm{d}t + \sigma v \,\mathrm{d}W,$$

where δ, σ denote the instantaneous drift and volatility, respectively, and $\mathrm{d}W$ an increment of the standardized Wiener process.

As an extension, we assume there is a simple duopoly where a first mover leader, and a follower share the final market. The leader's initial market share on entering the market is denoted by $m_L = 1$ from capturing the entire market. When the follower subsequently enters the market, its market share is denoted by $0 < m_{FL} < 1$ and simultaneously the leader's final market share reduces to $0 < m_{LF} < 1$ with $m_{LF} + m_{FL} = 1$ and $m_{FL} < m_{LF}$ due to the leader's first mover advantage. Then:

$$0 < m_{FL} < m_{LF} < m_L = 1, m_{LF} + m_{FL} = 1.$$

The nature of the duopoly game is that the leader always commits to a policy change ahead of the follower. By backwardation, we first examine the follower's value function. The value $G_F(v)$ of the follower's perpetual opportunity is:

$$G_F = \left\{ \begin{aligned} & g_{F1} = A_{F1} v^{\beta_1}, \quad v \in (0, v_{F1}), \\ & g_{F2} = \frac{m_{FL}\, v}{r-\delta} - \frac{m_{FL}\, f}{r} - K, \quad v \notin (0, v_{F1}). \end{aligned} \right\} \text{F1,F2} \qquad (1)$$

where K denotes the investment cost, f the operating cash-outflow, and r the risk-free rate with the net adjusted return shortfall ε=r-δ, with an unknown threshold, v_{F1}, an investment option coefficient, A_{F1}, and the option power parameter, β_1,. We assume that, where v=p*q, where q is a constant market volume quantity, p is stochastic, and f is equivalent to a variable operating cost. In (1), the term $A_{F1} v^{\beta_1}$ represents the real option value for the follower of eventually entering the market.

The value $G_L(v)$ of the leader's opportunity is:

$$G_L = \left\{ \begin{aligned} & g_{L1} = A_{L1} v^{\beta_1}, \quad v \in (0, v_{L1}), \\ & g_{L2} = A_{L11} v^{\beta_1} + \frac{m_L\, v}{r-\delta} - \frac{m_L\, f}{r} - K, \quad v \in [v_{L1}, v_{F1}), \\ & g_{L3} = \frac{m_{LF}\, v}{r-\delta} - \frac{m_{LF}\, f}{r} - K, \quad v \notin (0, v_{F1}). \end{aligned} \right\} \text{L1, L2, L3} \qquad (2)$$

In (2), the term $A_{L11} v^{\beta_1}$ represents the value for the leader of the **rival option** (negative value for the leader, when the follower enters the market). The coefficient A_{L11} is obtained from the value conserving condition $g_{L2}(v_{F1}) = g_{L3}(v_{F1})$.

The solutions for the follower's entry threshold, v_{F1}, and coefficient, A_{F1}, the leader's entry threshold, v_{L1}, and coefficients, A_{L1}, A_{L11}, and β_1 are derived as follows.

From (1), the value-matching relationship and smooth-pasting condition[4] for the follower's value function are, respectively:

$$\left.\begin{aligned} g_{F1}\left(v_{F1}\right)-g_{F2}\left(v_{F1}\right) &= A_{F1}v_{F1}^{\beta_1}-\frac{m_{FL}\,v_{F1}}{r-\delta}+\frac{m_{FL}\,f}{r}+K=0,\\ \frac{\partial\left(g_{F1}\left(v_{F1}\right)-g_{F2}\left(v_{F1}\right)\right)}{\partial v} &= \beta_1 A_{F1}v_{F1}^{\beta_1-1}-\frac{m_{FL}}{r-\delta}=0.\end{aligned}\right\} \tag{3a, 3b}$$

Solving for v_{F1} and A_{F1} yields:

$$\left.\begin{aligned} v_{F1} &= \frac{\left(m_{FL}\,f+rK\right)\beta_1\left(r-\delta\right)}{m_{FL}\left(\beta_1-1\right)r},\\ A_{F1} &= \frac{m_{FL}}{\beta_1\left(r-\delta\right)}\left[\frac{\left(m_{FL}\,f+rK\right)\beta_1\left(r-\delta\right)}{m_{FL}\left(\beta_1-1\right)r}\right]^{1-\beta_1}.\end{aligned}\right\} \tag{4a, 4b}$$

From (2), the value-conserving condition for the leader when the follower exercises is:

$$g_{L2}\left(v_{F1}\right)-g_{L3}\left(v_{F1}\right)=A_{L11}v_{F1}^{\beta_1}+\frac{m_L\,v_{F1}}{r-\delta}-\frac{m_L\,f}{r}-\frac{m_{LF}\,v_{F1}}{r-\delta}+\frac{m_{LF}\,f}{r}=0. \tag{5}$$

Solving for A_{L11} yields:

$$A_{L11}=-\left(m_L-m_{LF}\right)\left(\frac{v_{F1}}{r-\delta}-\frac{f}{r}\right)v_{F1}^{-\beta_1}. \tag{6}$$

From (2), the value-matching relationship and smooth-pasting condition for the leader's value function are, respectively:

$$\left.\begin{aligned} g_{L1}\left(v_{L1}\right)-g_{L2}\left(v_{L1}\right) &= A_{L1}v_{L1}^{\beta_1}-A_{L11}v_{L1}^{\beta_1}-\frac{m_L\,v_{L1}}{r-\delta}+\frac{m_L\,f}{r}+K=0,\\ \frac{\partial\left(g_{L1}\left(v_{L1}\right)-g_{L2}\left(v_{L1}\right)\right)}{\partial v} &= \beta_1 A_{L1}v_{L1}^{\beta_1-1}-\beta_1 A_{L11}v_{L1}^{\beta_1-1}-\frac{m_L}{r-\delta}=0.\end{aligned}\right\} \tag{7a, 7b}$$

Solving for v_{L1} and A_{L1} yields:

[4] The conventional approach to such an optimal stopping problem is that if v follows a geometric Brownian motion process, the solution G(v) must satisfy an ordinary differential equation, $\frac{1}{2}\sigma^2v^2G''(v)+(r-\delta)vG'(v)-rG(v)=0$, along with the value matching and smooth pasting boundary (v*) conditions, 3a,b, 5, 7a,b, see Dixit and Pindyck (1994), page 141.

$$\left.\begin{aligned} v_{L1} &= \frac{(m_L f + rK)\beta_1 (r-\delta)}{m_L r(\beta_1 - 1) r}, \\ A_{L1} &= A_{L11} + \frac{m_L}{\beta_1 (r-\delta)} \left[\frac{(m_L f + rK)\beta_1 (r-\delta)}{m_L r(\beta_1 - 1) r} \right]^{1-\beta_1}. \end{aligned}\right\} \tag{8a, 8b}$$

The power parameter β_1 is the positive root of the characteristic Q function:

$$Q(\beta_1) = \tfrac{1}{2}\sigma^2 \beta_1 (\beta_1 - 1) + \beta_1 (r-\delta) - r = 0, \quad \beta_1 > 0. \tag{9}$$

In summary, all option coefficients and thresholds have an analytical solution, with the option coefficients simplified using the threshold expressions:

$$\left.\begin{aligned} v_{F1} &= \frac{\beta_1 (r-\delta)(m_{FL}\ f + rK)}{m_{FL}(\beta_1 - 1) r}, \\ A_{F1} &= \frac{m_{FL}}{\beta_1 (r-\delta)} [v_{F1}]^{1-\beta_1}, \\ v_{L1} &= \frac{\beta_1 (r-\delta)(m_L\ f + rK)}{m_L(\beta_1 - 1) r}, \\ A_{L1} &= A_{L11} + \frac{m_L}{\beta_1 (r-\delta)} [v_{L1}]^{1-\beta_1}, \\ A_{L11} &= -(m_L - m_{LF}) \left(\frac{v_{F1}}{r-\delta} - \frac{f}{r} \right) v_{F1}^{-\beta_1} < 0, \\ \beta_1 &= \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2} \right) + \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2}}. \end{aligned}\right\} \tag{10}$$

Table 1A

Mathematica Thresholds & Option Coefficients for Duopoly Model[5]

v_{L1}	v_{F1}	A_{L1}	A_{F1}	A_{L11}	β_1
8.63437	18.70780	3.88276	1.43595	-2.35747	1.71508

[5] Based on the parameter values (INPUT) in Table 1B.

Table 1B ODE, VM & SP Conditions[6]

	A	B	C	D
1	**CROR MODEL**		ODE	
2	INPUT			
3	v	14.00		
4	K	140.00		
5	σ	0.16		
6	r	0.05		
7	δ	0.03		
8	f	2.00		
9	mLF	0.60		
10	mFL	0.40		
11	OUTPUT			
12	F1(v)	132.6927	IF(B3<B14,B16*(B3^B19),B13)	1a
13	F2(v)	124.0000	B10*(B3/(B6-B7)-B8/B6)-B4	1b
14	vF1	18.7078	(B19/(B19-1))*((B6*B4+B10*B8)*(B6-B7))/(B6*B10)	4a
15	vL1	8.6344	(B19/(B19-1))*(B6*B4+B8)*(B6-B7)/B6	8a
16	AF1	1.4360	(B10/(B19*(B6-B7)))*(B14^(1-B19))	4b
17	AL1	3.8828	B18+(1/(B19*(B6-B7)))*(B15^(1-B19))	8b
18	AL11	-2.3575	(-(B10)*(B14/(B6-B7)-B8/B6)*(B14^-B19))	6
19	β_1	1.7151	0.5-(B6-B7)/(B5^2)+SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2))	9
20	L(v)	302.1527	IF(B3<B15,B21,IF(AND(B3>B15,B3<B14),B22,B23))	2
21	L1(v)	358.7955	B17*(B3^B19)	2a
22	L2(v)	302.1527	B18*(B3^B19)+(B3/(B6-B7)-B8/B6)-B4	2b
23	L3(v)	396.0000	B9*(B3/(B6-B7)-B8/B6)	2c
24	Leader	Pre-Invest v=5		
25	ODE	0.0000	0.5*(B5^2)*(B3^2)*B27+(B6-B7)*B3*B26-B6***B21**	
26	F'(v)	43.9546	IF(B3<B14,B19*B17*(B3^(B19-1)),1)	
27	F''(v)	2.2451	IF(B3<B14,B19*(B19-1)*B17*(B3^(B19-2)),0)	
28	F(vL1)	156.6229	B17*(B15^B19)	VM1
29	V*-K	156.6229	B18*(B15^B19)+(B15/(B6-B7)-B8/B6)-B4	VM1
30	SP1	0.0000	B19*B17*(B15^(B19-1))-(B19*B18*(B15^(B19-1))+1/(B6-B7))	SP1
31	Leader	Post-Invest L, Pre-Invest F v=14		
32	ODE	0.0000	0.5*(B5^2)*(B3^2)*B34+(B6-B7)*B3*B33-B6*B22+(B3-B8)	
33	F'(v)	23.3124	B19*B18*(B3^(B19-1))+1/(B6-B7)	
34	F''(v)	-1.3631	B19*(B19-1)*B18*(B3^(B19-2))	
35	F(vL1)	537.2340	B18*(B14^B19)+(B14/(B6-B7)-B8/B6)	VM2
36	V*	537.2340	B9*((B14/(B6-B7)-B8/B6))	VM2
37	SP2	**-12.8349**	**B19*B18*(B14^(B19-1))+1/(B6-B7)-B9/(B6-B7)**	**SP2**
38	Follower	Post-Invest L, Pre-Invest F v=14		
39	ODE	0.0000	0.5*(B5^2)*(B3^2)*B41+(B6-B7)*B3*B40-B6*B12	
40	FF'(v)	16.2557	B19*B16*(B3^(B19-1))	
41	FF''(v)	0.8303	B19*(B19-1)*B16*(B3^(B19-2))	
42	FF(vF)	218.1560	B16*(B14^B19)	VM3
43	V*-K	218.1560	B10*((B14/(B6-B7)-B8/B6))-B4	VM3
44	SP3	0.0000	B19*B16*(B14^(B19-1))-B10/(B6-B7)	SP3
45	Follower	Pre-Invest L&F v=5		
46	ODE	0.0000	0.5*(B5^2)*(B3^2)*B41+(B6-B7)*B3*B40-B6*B12	
47	FF'(v)	16.2557	B19*B16*(B3^(B19-1))	
48	FF''(v)	0.8303	B19*(B19-1)*B16*(B3^(B19-2))	
49	FF(vF)	218.1560	B16*(B14^B19)	VM4
50	V*-K	218.1560	B10*((B14/(B6-B7)-B8/B6))-B4	VM4
51	SP4	0.0000	B19*B16*(B14^(B19-1))-B10/(B6-B7)	SP4

[6] This spreadsheet shows that the analytical equations solve the ODE, and that the VM and SP conditions are satisfied, except for the special case of the Leader SP post-investment with a jump if the Follower invests.

Figure 1A

F(v)	0.00	4.71	15.48	31.03	50.82	74.51	101.87	132.69	166.84	204.19	244.00	284.00	324.00	364.00
L(v)	0.00	12.75	41.85	83.89	137.41	197.67	252.76	302.15	346.09	384.77	436.00	496.00	556.00	616.00

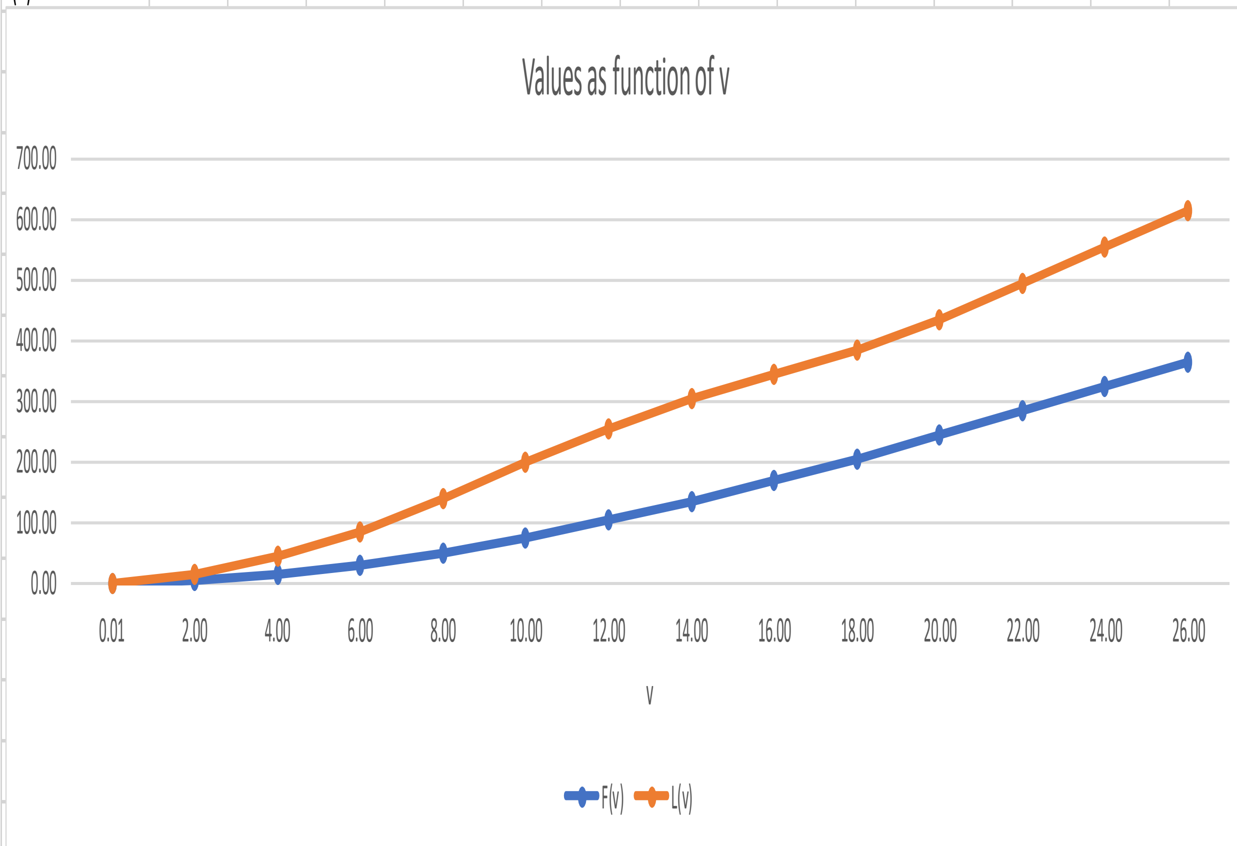
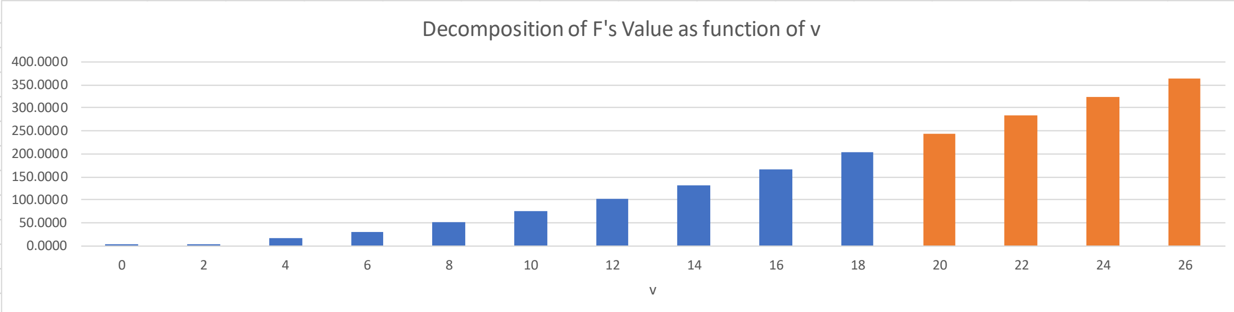
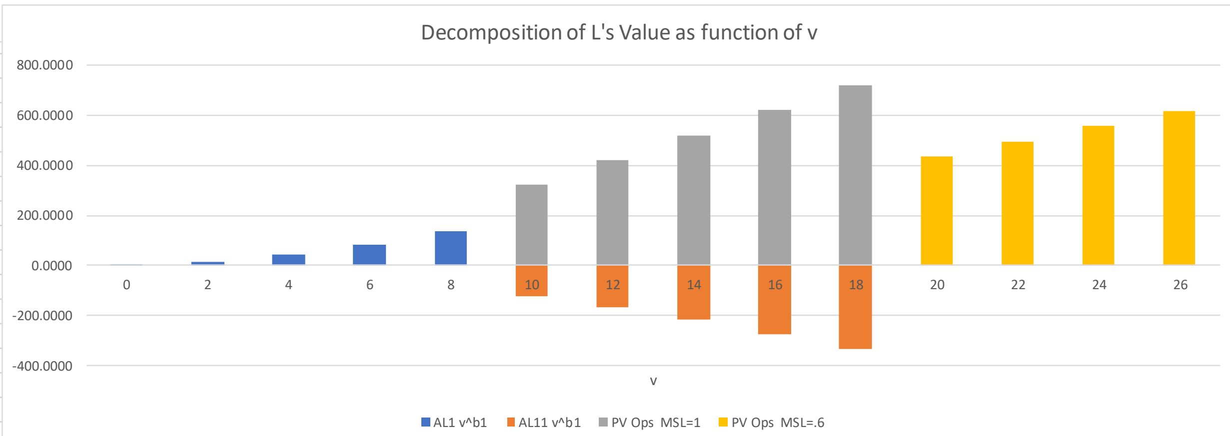


Figure 1A shows that the value functions is an almost linear function of increasing v, despite leader market share falling to 60% after the follower invests. A decomposition of the value function in Figure 1B shows that the follower's value function strategic option value (blue) to invest steadily increases with v, the leader's value is split into the PV of operations after investing (grey) less the negative value of the rival option (orange); when the follower invests, the leader's value is entirely the PV of operations (60% of the market).

Figure 1B

AF1 v^b1	0.0005	4.7145	15.4783	31.0267	50.8179	74.5118	101.8656	132.6927	166.8431	204.1922	0.0000	0.0000	0.0000	0.0000
PV Ops MSF=.4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	244.0000	284.0000	324.0000	364.0000
VF L	0.0014	12.7477	41.8528	83.8949	137.4093	197.6708	252.7628	302.1527	346.0866	384.7690	436.0000	496.0000	556.0000	616.0000
AL1 v^b1	0.0014	12.7477	41.8528	83.8949	137.4093	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
AL11 v^b1	0.0000	0.0000	0.0000	0.0000	0.0000	-122.3292	-167.2372	-217.8473	-273.9134	-335.2310	0.0000	0.0000	0.0000	0.0000
PV Ops MSL=1	0.0000	0.0000	0.0000	0.0000	0.0000	320.0000	420.0000	520.0000	620.0000	720.0000	0.0000	0.0000	0.0000	0.0000
PV Ops MSL=.6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	436.0000	496.0000	556.0000	616.0000





3 Sensitivities

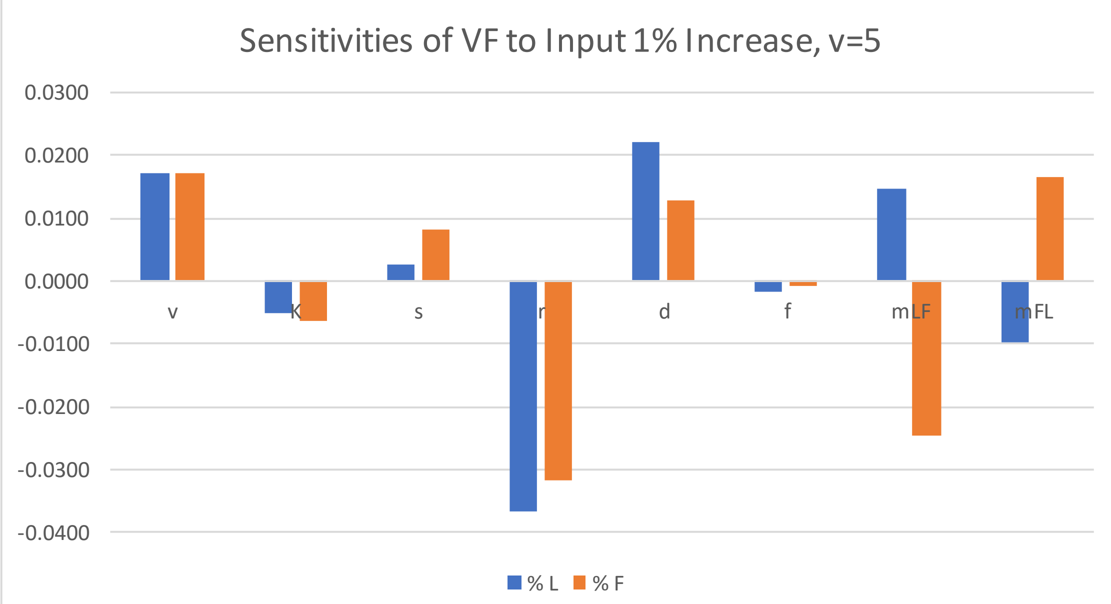
Figure 2 (v=5, regime L1 before either has invested) shows a quick and easy way to assess the sensitivity of the leader and follower value functions to changes in each of the eight parameter values separately. Significance (more than 2%) is indicated in bold. Thresholds are highly sensitive to changes in interest rates and revenue drift, but not generally to changes in the other parameter values (except for the follower's threshold to the leader's final market share).

Figure 2

Percentage Change in Thresholds, Option Coefficients & Value Functions for a 1% Increase in the Parameter Value

	K	σ	r	δ	f	mLF	mFL
vF1	0.0090	0.0050	**0.0298**	**-0.0230**	0.0010	0.0137	-0.0089
vL1	0.0078	0.0050	**0.0286**	**-0.0230**	0.0022	0.0000	0.0000
AF1	-0.0064	0.0181	**-0.0209**	-0.0035	-0.0007	**-0.0245**	0.0165
AL1	-0.0053	0.0124	**-0.0260**	0.0058	-0.0019	0.0145	-0.0098
AL11	0.0060	-0.0147	**0.0242**	-0.0020	0.0011	0.0239	-0.0161
β_1	0.0000	-0.0036	-0.0040	0.0059	0.0000	0.0000	0.0000

		v	K	σ	r	δ	f	mLF	mFL
v=5	% L	0.0172	-0.0053	0.0025	**-0.0368**	**0.0222**	-0.0019	0.0145	-0.0098
	% F	0.0172	-0.0064	0.0081	**-0.0318**	0.0129	-0.0007	**-0.0245**	0.0165
v=10	% L	0.0146	-0.0034	-0.0003	**-0.0351**	**0.0227**	-0.0013	0.0148	-0.0099
	% F	0.0172	-0.0064	0.0038	**-0.0364**	**0.0200**	-0.0007	**-0.0245**	0.0165
v=22	% L	0.0104	0.0000	0.0000	**-0.0249**	0.0158	-0.0004	0.0100	-0.0067
	% F	0.0155	-0.0049	0.0000	**-0.0372**	**0.0236**	-0.0006	**-0.0224**	0.0149



All sensitivities are logical, with the value functions of both the leader and follower with the same sign, except for the increase in m_{LF}=1-m_{FL}, that is the leader's final market share. Naturally, each value function increases with an increase in v, and decreases with an increase in K. Consistent with expected sensitivity for a call option (investment opportunity), each value function increases with increases in volatility and in the net drift rate, but decreases with increases in interest rates, which is the most significant in percentage terms. Changes in the operating costs does not seem to make much of a difference. Also shown are the sensitivities when v=10 (L2 middle regime after the leader has invested) and v=22 (L3 final regime, after both have invested). Observations are that the sensitivities to increases in v are much the same over the regimes. Naturally, increases in K are not relevant for the leader in L3, who spends K in the transition from L1 to L2. Change in the interest rate and drift are important in L1 (affecting the thresholds and option coefficients), and continue to be important in L2 and L3 affecting the present value of operations. Note that there is no effect of changes in volatility in L3, since there are no options in that regime. Maintaining market share is critical for the follower in all regimes, less important for the leader

4 Partial Derivatives

Market Share Partials

From (1), the impact of the market share m_{LF} change on the follower's opportunity value is:

$$G_{F,m_{LF}}{}' = \begin{Bmatrix} \dfrac{\partial g_{F1}}{\partial m_{LF}} = \dfrac{\partial A_{F1}}{\partial m_{LF}} v^{\beta_1} + \dfrac{\partial \beta_1}{\partial m_{LF}} A_{F1} v^{\beta_1} \, log(v), \quad v \in (0, v_{F1}), \\ \dfrac{\partial g_{F2}}{\partial m_{LF}} = -\dfrac{v}{r-\delta} + \dfrac{f}{r}, \quad v \notin (0, v_{F1}). \end{Bmatrix} \qquad (11a, 11b)$$

From (2), the impact of the market share m_{LF} change on the leader's opportunity value is:

$$G_{L,m_{LF}}{}' = \begin{Bmatrix} \dfrac{\partial g_{L1}}{\partial m_{LF}} = \dfrac{\partial A_{L1}}{\partial m_{LF}} v^{\beta_1} + \dfrac{\partial \beta_1}{\partial m_{LF}} A_{L1} v^{\beta_1} \, log(v), \quad v \in (0, v_{L1}), \\ \dfrac{\partial g_{L2}}{\partial m_{LF}} = \dfrac{\partial A_{L11}}{\partial m_{LF}} v^{\beta_1} + \dfrac{\partial \beta_1}{\partial m_{LF}} A_{L11} v^{\beta_1} \, log(v), \quad v \in [v_{L1}, v_{F1}), \\ \dfrac{\partial g_{L3}}{\partial m_{LF}} = \dfrac{v}{r-\delta} - \dfrac{f}{r}, \quad v \notin (0, v_{F1}). \end{Bmatrix} \qquad (12a, 12b, 12c)$$

The derivation of the partial derivatives for each variable with respect to the leader's market share after the follower's entry, m_{LF}, follows the procedure described in Appendix A & B.

$$\begin{aligned} &\frac{\partial v_{L1}}{\partial m_{LF}} = 0, \\ &\frac{\partial v_{F1}}{\partial m_{LF}} = \frac{v_{F1}^{1-\beta_1}}{(\beta_1 - 1)\beta_1 A_{F1}} \left(\frac{(\beta_1 - 1) v_{F1}}{r-\delta} - \frac{\beta_1 f}{r} \right), \end{aligned} \qquad (13)$$

$$\frac{\partial A_{L1}}{\partial m_{LF}} = -\frac{v_{F1}\left(m_L - m_{LF}\right)\left(rv_{F1}\left(\beta_1 - 1\right) - f\left(r-\delta\right)\beta_1\right)}{r\left(r-\delta\right)^2 A_{F1}\left(\beta_1 - 1\right)\beta_1} v_{F1}^{-2\beta_1}$$
$$+\frac{\left(rv_{F1} - f\left(r-\delta\right)\right)}{r\left(r-\delta\right)} v_{F1}^{-\beta_1} - \frac{A_{L11}\left(rv_{F1}\left(\beta_1 - 1\right) - f\left(r-\delta\right)\beta_1\right)}{A_{F1}\left(\beta_1 - 1\right) r\left(r-\delta\right)} v_{F1}^{-\beta_1} \tag{14}$$

$$\frac{\partial A_{F1}}{\partial m_{LF}} = -v_{F1}^{-\beta_1}\left(\frac{rv_{F1} - f(r-\delta)}{r(r-\delta)}\right), \tag{15}$$

$$\frac{\partial A_{L11}}{\partial m_{LF}} = \frac{\left(rv_{F1} - f\left(r-\delta\right)\right)}{r\left(r-\delta\right)} v_{F1}^{-\beta_1}$$
$$-\frac{\left(m_L - m_{LF}\right)v_{F1}\left(rv_{F1}\left(\beta_1 - 1\right) - f\left(r-\delta\right)\beta_1\right)}{r\left(r-\delta\right)^2 A_{F1}\left(\beta_1 - 1\right)\beta_1} v_{F1}^{-2\beta_1} \tag{16}$$
$$-\frac{A_{L11}\left(rv_{F1}\left(\beta_1 - 1\right) - f\left(r-\delta\right)\beta_1\right)}{r\left(r-\delta\right) A_{F1}\left(\beta_1 - 1\right)} v_{F1}^{-\beta_1}$$

$$\frac{\partial \beta_1}{\partial m_{LF}} = 0. \tag{17}$$

Using the base case parameter values, the results calculated in Mathematica are in Table 2A.

Table 2A
Leader's Market Share Partial Derivative Values

$\frac{\partial v_{L1}}{\partial m_{LF}}$	$\frac{\partial v_{F1}}{\partial m_{LF}}$	$\frac{\partial A_{L1}}{\partial m_{LF}}$	$\frac{\partial A_{F1}}{\partial m_{LF}}$	$\frac{\partial A_{L11}}{\partial m_{LF}}$	$\frac{\partial \beta_1}{\partial m_{LF}}$
0.0	41.97263	9.43960	-5.89367	9.43960	0.0

Table 2B

Market Share Partial Derivative Equation Solutions

	A	B	C	D
1	**CROR**		MARKET SHARE Partials	
2	INPUT			
3	v	5.00		
4	K	140.00		
5	σ	0.16		
6	r	0.05		
7	δ	0.03		
8	f	2.00		
9	mLF	0.60		
10	mFL	0.40		
11	OUTPUT			
12	F1(v)	22.6952	IF(B3<B14,B16*(B3^B19),B13)	1a
13	F2(v)	-56.0000	B10*(B3/(B6-B7)-B8/B6)-B4	1b
14	vF1	18.7078	(B19/(B19-1))*((B6*B4+B10*B8)*(B6-B7))/(B6*B10)	4a
15	vL1	8.6344	(B19/(B19-1))*(B6*B4+B8)*(B6-B7)/B6	8a
16	AF1	1.4360	(B10/(B19*(B6-B7)))*(B14^(1-B19))	4b
17	AL1	3.8828	B18+(1/(B19*(B6-B7)))*(B15^(1-B19))	8b
18	AL11	-2.3575	(-(B10)*(B14/(B6-B7)-B8/B6)*(B14^-B19))	6
19	β_1	1.7151	0.5-(B6-B7)/(B5^2)+SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2))	9
20	L(v)	61.3668	IF(B3<B15,B21,IF(AND(B3>B15,B3<B14),B22,B23))	2
21	L1(v)	61.3668	B17*(B3^B19)	2a
22	L2(v)	32.7404	B18*(B3^B19)+(B3/(B6-B7)-B8/B6)-B4	2b
23	L3(v)	-14.0000	B9*(B3/(B6-B7)-B8/B6)-B4	2c
24	(mLF) PARTIALS			
25	δvF1/δLF	41.9726	((B14^(1-B19))/((B19-1)*B19*B16))*B35	13
26	δAF1/δLF	-5.8937	-(B14^-B19)*((B6*B14-B8*(B6-B7))/(B6*(B6-B7)))	15
27	δAL1/δLF	9.4396	(-B14*(1-B9)*(B14^(-2*B19)))*B39/B37+B38-B40	14
28	δAL11/δLF	9.4396	B42-(1-B9)*(B14^(1-2*B19))*B43/B45-B18*(B14^-B19)*B43/B44	16
29	δF(v)/δLF	-93.1490	IF(B3<B14,B26*B3^B19,B30)	*11a, 18a*
30		-210.0000	-B3/(B6-B7)+B8/B6	11b
31	δL(v)/δLF	149.1923	IF(B3<B15,B32,IF(AND(B3>B15,B3<B14),B33,B34))	12
32		149.1923	B27*(B3^B19)	12a
33		149.1923	B28*(B3^B19)	12b
34		210.0000	(B3/(B6-B7)-B8/B6)	12c
35	A	600.2796	(((B19-1)*B14/(B6-B7))-B8*B19/B6)	
36	δAL1/δLF	Parts		
37	B	0.0000	(B6*((B6-B7)^2)*B16*(B19-1)*B19)	
38	C	5.8937	(B14^-B19)*((B6*B14-B8*(B6-B7))/(B6*(B6-B7)))	
39	D	0.6003	(B6*B14*(B19-1)-B8*(B6-B7)*B19)	
40	E	-9.0714	(B18*B39*(B14^-B19))/(B16*(B19-1)*B6*(B6-B7))	
41	δAL11/δLF	Parts		
42	F	5.8937	(B14^-B19)*((B6*B14-B8*(B6-B7))/(B6*(B6-B7)))	
43	G	0.6003	(B6*B14*(B19-1)-B8*(B6-B7)*B19)	
44	H	0.0010	B6*(B6-B7)*B16*(B19-1)	
45	I	0.0000	B6*((B6-B7)^2)*B16*B19*(B19-1)	

Note that the leader's threshold is not affected by the final market share, but the follower's threshold increases significantly with increases in that final market share, naturally. The follower's strategic option to invest declines as the final market share increases; but the leader's strategic and rival option coefficient partial derivatives are the same, (14)=(16). Note that the equations for (14) and (16) have been simplified by using parts (B37:B40) and (B42:B45).

Inserting Table 2 values into (11) and (12) yields respectively:

$$G_{F,m_{LF}}{}' = \begin{Bmatrix} -5.89367\,v^{1.71508}, & v \in (0,18.70780), \\ 40.0-50.0\,v, & v \notin (0,18.70780), \end{Bmatrix} \qquad \text{(18a, 18b)}$$

and:

$$G_{L,m_{LF}}{}' = \begin{Bmatrix} 9.43960\,v^{1.71508}, & v \in (0,8.63437), \\ 9.43960\,v^{1.71508}, & v \in [8.63437,18.70780), \\ 50.0\,v-40.0, & v \notin (0,18.70780). \end{Bmatrix} \qquad \text{(19a, 19b,19c)}$$

Figure 3A v=10

F(v)	107.29	103.82	100.40	97.01	93.67	90.36	87.11	83.89	80.72	77.59	74.51
L(v)	145.57	151.05	156.47	161.83	167.13	172.38	177.56	182.68	187.74	192.74	197.67
mLF	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60

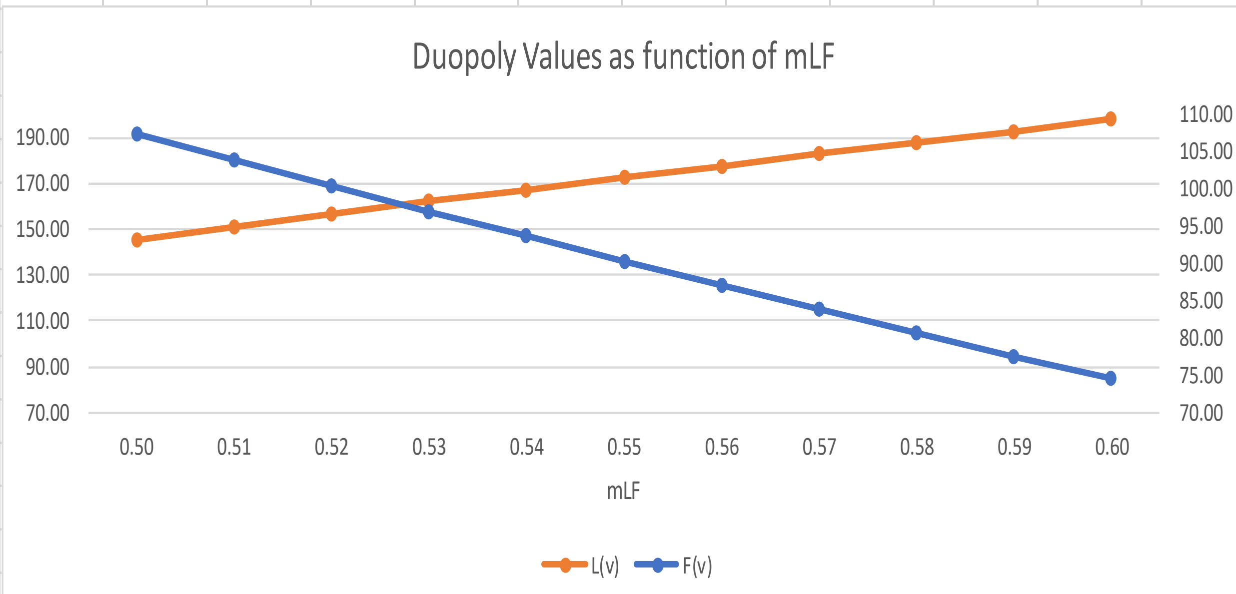
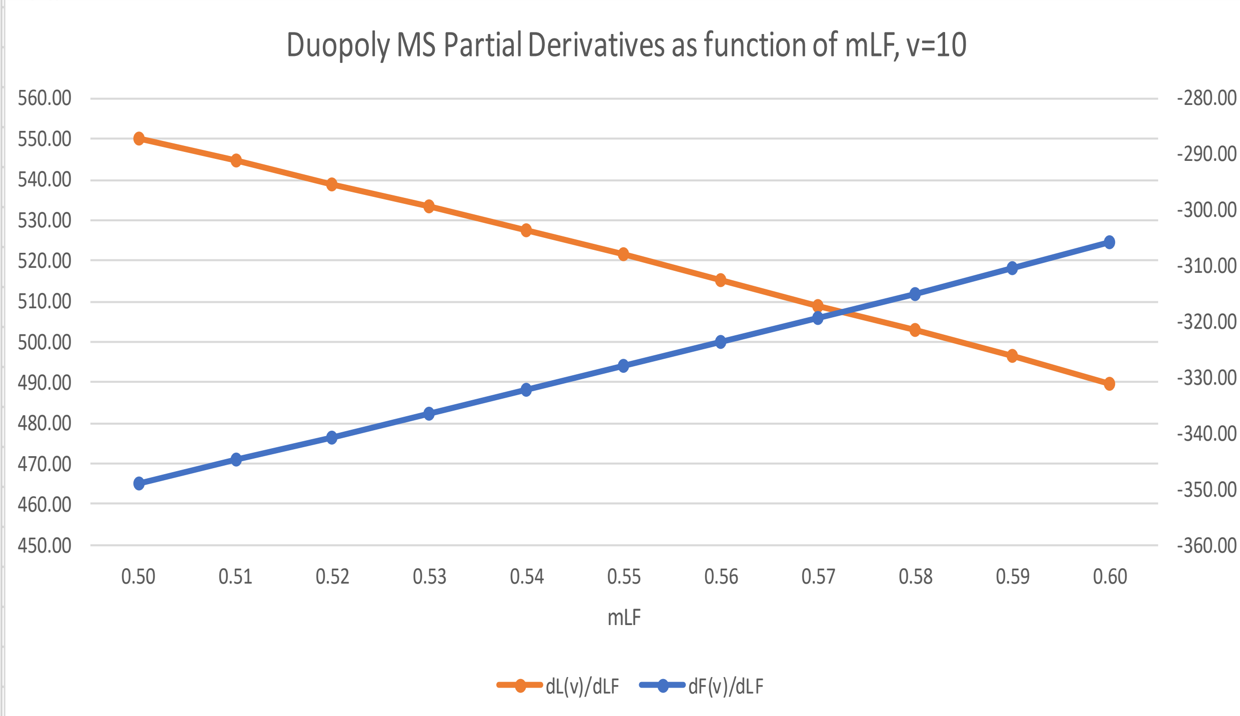


Figure 3A shows that both leader and follower values when vL<v<vF are linear functions of increasing market share (follower negative, naturally) The partial derivatives are also linear, with the follower's increasing as leader's final market share increases, in Figure 3B.

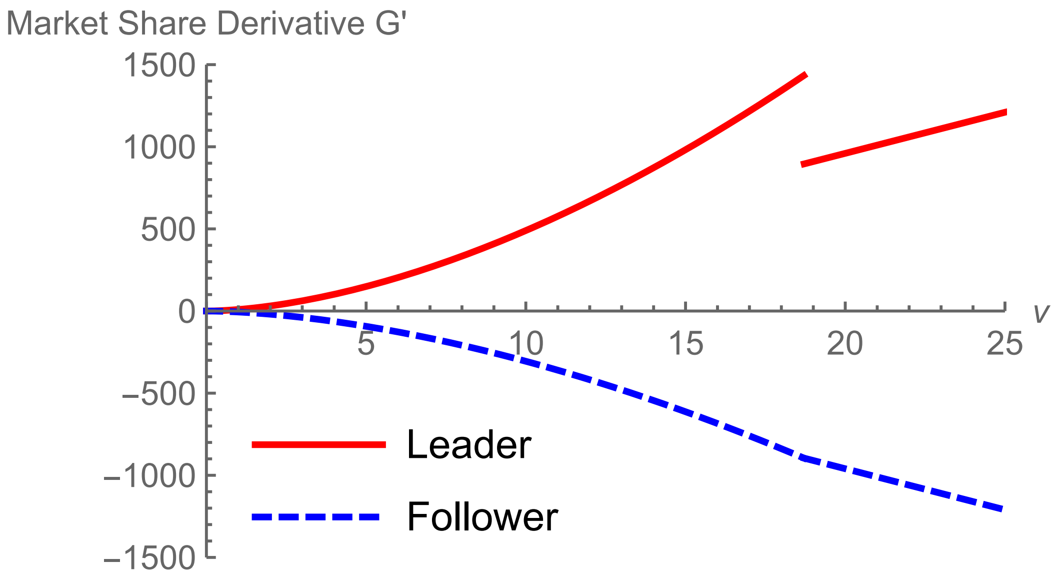
Figure 3B

δF(v)/δLF	-348.85	-344.80	-340.69	-336.53	-332.32	-328.05	-323.72	-319.34	-314.90	-310.39	-305.82
δL(v)/δLF	550.35	544.75	539.07	533.28	527.40	521.41	515.32	509.11	502.80	496.37	489.82



The plots of $G_{L,m_{LF}}'$ and $G_{F,m_{LF}}'$ are presented in Figure 4, as a function of v. $G_{F,m_{LF}}'$ is a decreasing continuous function of the prevailing cash-inflow v, since a positive change in m_{LF} implies a negative change in m_{FL} and adverse consequences for the follower. Although the function is continuous, it is not smooth at the follower's market entry at $v = v_{F1}$. In contrast, $G_{L,m_{LF}}'$ is a semi-continuous function with a downward jump at $v = v_{F1}$. The function is increasing for $0 \le v < v_{F1}$ and for $v > v_{F1}$ since a positive change in m_{LF} benefits the leader.

Figure 4
Impact of Market Share Change on the Leader's and Follower's Value Functions



So, at all levels of v, in this case the leader will benefit from an increase of m_{LF}, except at the follower's threshold v_{F1}, when there is a sudden drop in the leader's value function, which, however, is still positive as a function of v thereafter.

Volatility Partials

The impact of volatility changes on the follower's opportunity value is found from:

$$G_F' = \left\{ \begin{array}{l} \dfrac{\partial g_{F1}}{\partial \sigma} = \dfrac{\partial A_{F1}}{\partial \sigma} v^{\beta_1} + \dfrac{\partial \beta_1}{\partial \sigma} A_{F1} v^{\beta_1} \log(v), \quad v \in (0, v_{F1}), \\ \dfrac{\partial g_{F2}}{\partial \sigma} = 0, \quad v \notin (0, v_{F1}). \end{array} \right\} \qquad (20a, 20b)$$

$$G_L' = \left\{ \begin{array}{l} \dfrac{\partial g_{L1}}{\partial \sigma} = \dfrac{\partial A_{L1}}{\partial \sigma} v^{\beta_1} + \dfrac{\partial \beta_1}{\partial \sigma} A_{L1} v^{\beta_1} \log(v), \quad v \in (0, v_{L1}), \\ \dfrac{\partial g_{L2}}{\partial \sigma} = \dfrac{\partial A_{L11}}{\partial \sigma} v^{\beta_1} + \dfrac{\partial \beta_1}{\partial \sigma} A_{L11} v^{\beta_1} \log(v), \quad v \in [v_{L1}, v_{F1}), \\ \dfrac{\partial g_{L3}}{\partial \sigma} = 0, \quad v \notin (0, v_{F1}). \end{array} \right\} \qquad (21a, 21b, 21c)$$

The derivatives expressed in (20) and (21) are determined in Appendix C, and their solutions are presented below:

$$\frac{\partial v_{L1}}{\partial \sigma}=\frac{v_{L1}\sigma\beta_1}{\left(r+\frac{1}{2}\beta_1^2\sigma^2\right)}, \tag{22}$$

$$\frac{\partial v_{F1}}{\partial \sigma}=\frac{v_{F1}\sigma\beta_1}{\left(r+\frac{1}{2}\beta_1^2\sigma^2\right)}, \tag{23}$$

$$\left.\begin{aligned}\frac{\partial A_{L1}}{\partial \sigma}=&-\frac{\left(m_L-m_{LF}\right)v_{F1}\beta_1\sigma}{\left(r+\frac{1}{2}\beta_1^2\sigma^2\right)\left(r-\delta\right)v_{F1}^{\beta_1}}+\frac{A_{L11}\sigma\beta_1^2\left(\beta_1-1\right)log\left[v_{F1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2}\\&+\frac{\left(A_{L1}-A_{L11}\right)\sigma\beta_1^2\left(\beta_1-1\right)log\left[v_{L1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2}-\frac{A_{L11}\sigma\beta_1^2}{r+\frac{1}{2}\beta_1^2\sigma^2},\end{aligned}\right\} \tag{24}$$

$$\frac{\partial A_{F1}}{\partial \sigma}=\frac{A_{F1}\beta_1^2\left(\beta_1-1\right)\sigma\, log\left[v_{F1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2}, \tag{25}$$

$$\left.\begin{aligned}\frac{\partial A_{L11}}{\partial \sigma}=&-\frac{\left(m_L-m_{LF}\right)v_{F1}\beta_1\sigma}{\left(r+\frac{1}{2}\beta_1^2\sigma^2\right)\left(r-\delta\right)v_{F1}^{\beta_1}}+\frac{A_{L11}\sigma\beta_1^2\left(\beta_1-1\right)log\left[v_{F1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2}\\&-\frac{A_{L11}\sigma\beta_1^2}{r+\frac{1}{2}\beta_1^2\sigma^2},\end{aligned}\right\} \tag{26}$$

$$\frac{\partial \beta_1}{\partial \sigma}=-\frac{\sigma\beta_1^2\left(\beta_1-1\right)}{\left(r+\frac{1}{2}\beta_1^2\sigma^2\right)}. \tag{27}$$

Consequently, the partial derivatives of the value functions with respect to volatility are:

$$\frac{\partial g_{F1}}{\partial \sigma}=\frac{A_{F1}v^{\beta_1}\sigma\beta_1^2\left(\beta_1-1\right)\left(log\left[v_{F1}\right]-log\left[v\right]\right)}{r+\frac{1}{2}\beta_1^2\sigma^2},\quad v\in\left(0,v_{F1}\right), \tag{28}$$

and:

$$\left.\begin{aligned}\frac{\partial g_{L1}}{\partial \sigma}=&-\frac{\left(m_L-m_{LF}\right)v_{F1}\beta_1\sigma}{\left(r+\frac{1}{2}\beta_1^2\sigma^2\right)\left(r-\delta\right)v_{F1}^{\beta_1}}v^{\beta_1}+\frac{A_{L11}\sigma\beta_1^2\left(\beta_1-1\right)log\left[v_{F1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2}v^{\beta_1}\\&+\frac{\left(A_{L1}-A_{L11}\right)\sigma\beta_1^2\left(\beta_1-1\right)log\left[v_{L1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2}v^{\beta_1}-\frac{A_{L11}\sigma\beta_1^2}{r+\frac{1}{2}\beta_1^2\sigma^2}v^{\beta_1}\\&-\frac{A_{L1}\sigma\beta_1^2\left(\beta_1-1\right)}{r+\frac{1}{2}\beta_1^2\sigma^2}v^{\beta_1}\, log\left[v\right],\quad v\in\left(0,v_{L1}\right),\\\frac{\partial g_{L2}}{\partial \sigma}=&-\frac{\left(m_L-m_{LF}\right)v_{F1}\beta_1\sigma}{\left(r+\frac{1}{2}\beta_1^2\sigma^2\right)\left(r-\delta\right)v_{F1}^{\beta_1}}v^{\beta_1}+\frac{A_{L11}\sigma\beta_1^2\left(\beta_1-1\right)log\left[v_{F1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2}v^{\beta_1}\\&-\frac{A_{L11}\sigma\beta_1^2}{r+\frac{1}{2}\beta_1^2\sigma^2}v^{\beta_1}-\frac{A_{L11}\sigma\beta_1^2\left(\beta_1-1\right)}{r+\frac{1}{2}\beta_1^2\sigma^2}v^{\beta_1}\, log\left[v\right],\quad v\in\left[v_{L1},v_{F1}\right).\end{aligned}\right\} \text{(29)(30)}$$

$G_F{}'$ is a continuous but not a smooth function for $v \in \mathrm{R}^+$. The value of the function $\partial g_{F1}/\partial\sigma$ is non-negative for $v \in (0, v_{4F1})$, equals zero at its two end-points, $v = 0, v \geq v_{F1}$, and exhibits a point of maximum at $v = v_{F1}\mathrm{Exp}[-1/\beta_1]$.

Figure 5
Impact of Volatility Changes on the Leader's and Follower's Value Functions

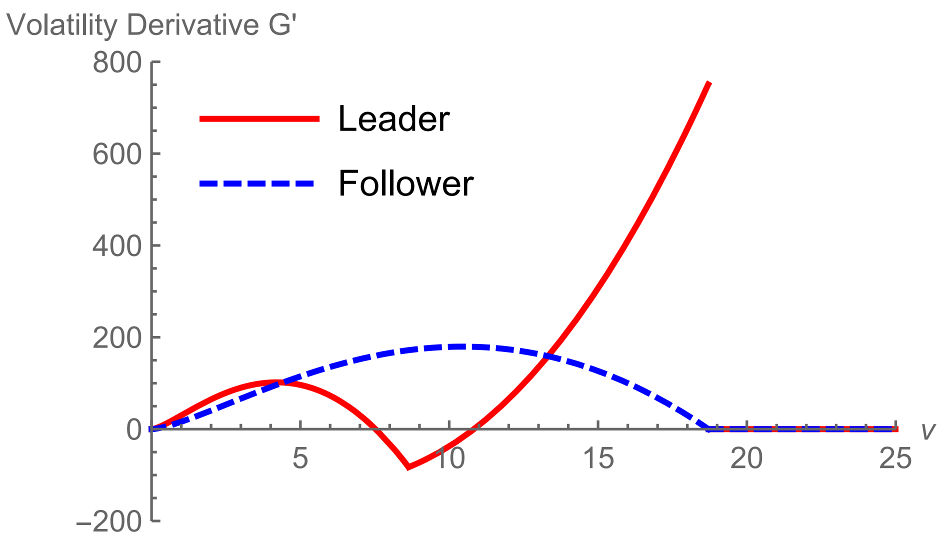


Figure 5 corroborates the predicted properties for the follower's and the leader's value functions. The effect of volatility changes on the follower's value $G_F{}'$ behaves as a continuous function but not continuously differentiable at $v = v_{F1}$, attaining a maximum at $v_{F,MAX} = 10.25626$. The effect of volatility changes on the leader's value $G_L{}'$ behaves as **a** semi-continuous function, continuous for $v < v_{F1}$ and for $v > v_{F1}$ but discontinuously differentiable at $v = v_{L1}$, while having a down-jump discontinuity at $v = v_{F1}$. For $v_{L1} \leq v < v_{F1}$, $G_L{}'$ is an increasing function.

In contrast[7], the shape and behaviour of $G_L{}'$ is less straightforward by being most likely a semi-continuous function and displaying both positive and negative values. First, we note that $\frac{\partial g_{L1}(v_{L1})}{\partial \sigma} < 0$, so $G_L{}'$ displays both positive and negative values and is concave for $v < v_{L1}$. Second, since

$$\frac{\partial g_{L1}(v_{L1})}{\partial \sigma} = \frac{\partial g_{L2}(v_{L1})}{\partial \sigma},$$

$G_L{}'$ is a continuous function for $v < v_{F1}$, for all revenue values prior to the follower's market entry. For $v \geq v_{L1}$, $\frac{\partial g_{L2}}{\partial \sigma}$ is an increasing function, which intersects the abscissa only if the expression:

$$(m_L - m_{LF})\frac{v_{F1}}{r-\delta} + A_{L11}\beta_1 v_{F1}^{\beta_1} \tag{31}$$

is positive, in which case $G_L{}'$ experiences a discontinuous downward jump at $v = v_{F1}$. If (31) is negative, then $\frac{\partial g_{L2}}{\partial \sigma} < 0$ for $v \in [v_{L1}, v_{F1}]$, in which case $G_L{}'$ experiences a discontinuous upward jump at $v = v_{F1}$. If in the unlikely event (31) is zero, then $G_L{}'$ becomes a continuous function for all $v \geq 0$. Clearly, since A_{L11} is negative, the magnitude of $m_L - m_{LF}$ is critical in deciding the sign of (31) and determining whether the jump is upwards or downwards.

These properties of $G_L{}'$ and $G_F{}'$ can be illustrated numerically. Table 3A presents the solution values obtained using Mathematica and their derivative values with respect to volatility. Substituting these values into (20) and (21) yields the impact of volatility changes on the followers' and the leader's opportunity value, respectively:

$$G_F{}' = \begin{cases} \frac{\partial g_{F1}}{\partial \sigma} = 16.14882 v^{1.71508} - 5.51354 v^{1.71508} \log[v], \quad v \in (0, 18.70780), \\ \frac{\partial g_{F2}}{\partial \sigma} = 0, \quad v \notin (0, 18.70780), \end{cases} \tag{32a, 32b}$$

[7] The background for some of the statements made in this paragraph is in Appendix C.

$$G_L' = \begin{cases} \dfrac{\partial g_{L1}}{\partial \sigma} = 30.08801 v^{1.71508} - 14.90837 v^{1.71508} \, log\left[v\right], \quad v \in \left(0, 8.63437\right), \\ \dfrac{\partial g_{L2}}{\partial \sigma} = -21.56416 v^{1.71508} + 9.05181 v^{1.71508} \, log\left[v\right], \quad v \in \left[8.63437, 18.70780\right), \\ \dfrac{\partial g_{L3}}{\partial \sigma} = 0, \quad v \notin \left(0, 18.70780\right). \end{cases}$$

(33a,33b, 33c)

As expected, all thresholds and strategic option coefficients have a positive “vega” (sensitivity to changes in volatility), but the rival option for the leader has a negative vega. This means that the negative value of this rival option becomes more negative as volatility increases, which does not benefit the leader when v is between the leader and follower thresholds.

Table 3A

Mathematica Solution and Volatility Partial Derivative Values for Competitive Model

$\frac{\partial v_{L1}}{\partial \sigma}$	$\frac{\partial v_{F1}}{\partial \sigma}$	$\frac{\partial A_{L1}}{\partial \sigma}$	$\frac{\partial A_{F1}}{\partial \sigma}$	$\frac{\partial A_{L11}}{\partial \sigma}$	$\frac{\partial \beta_1}{\partial \sigma}$
27.03195	58.56922	30.08801	16.14882	-21.56416	-3.83963

Table 3B

	A	B	C	D	E
1	**CROR**		VOLATILITY Partials		
2	INPUT				CROR
3	v	5.00			
4	K	140.00			
5	σ	0.16			
6	r	0.05			
7	δ	0.03			
8	f	2.00			
9	mLF	0.60			
10	mFL	0.40			
11	OUTPUT				
12	F1(v)	22.6952	IF(B3<B14,B16*(B3^B19),B13)		
13	F2(v)	-56.0000	B10*(B3/(B6-B7))-(B10*B8/B6)-B4		
14	vF1	18.70780	(B19/(B19-1))*((B6*B4+B10*B8)*(B6-B7))/(B6*B10)		
15	vL1	8.63437	(B19/(B19-1))*(B6*B4+B8)*(B6-B7)/B6		
16	AF1	1.43595	(B10/(B19*(B6-B7)))*(B14^(1-B19))		
17	AL1	3.88276	B18+(1/(B19*(B6-B7)))*(B15^(1-B19))		
18	AL11	-2.3575	(-(B10)*(B14/(B6-B7)-B8/B6)*(B14^-B19))		
19	β_1	1.7151	0.5-(B6-B7)/(B5^2)+SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2))		
20	L(v)	61.3668	IF(B3<B15,B21,IF(AND(B3>B15,B3<B14),B22,B23))		
21	L1(v)	61.3668	B17*(B3^B19)		
22	L2(v)	32.7404	B18*(B3^B19)+B3/(B6-B7)-B8/B6-B4		
23	L3(v)	-14.0000	B9*B3/(B6-B7)-B9*B8/B6-B4		
24	(σ) PARTIALS				Table 4A
25	δvL1/δσ	27.0319	(B15*B5*B19)/B38	22	27.0320
26	δvF1/δσ	58.5692	(B14*B5*B19)/B38	23	58.5692
27	δAF1/δσ	16.1488	B39/B38	25	16.1488
28	δAL1/δσ	30.0880	B40/((B38*(B6-B7)*(B14^B19)))+B41*(B19-1)*LN(B14)/B38+B42-B41/B38	24	30.0880
29	δAL11/δσ	-21.5642	(B40/((B38*(B6-B7)*(B14^B19))))+(B41*(B19-1)*LN(B14)/B38)-B41/B38	26	-21.5616
30	δF(v)/δσ	114.9829	IF(B3<B14,B27*(B3^B19)+B36*B16*(B3^B19)*LN(B3),B31)	28	
31					
32	δL(v)/δσ	96.3139	IF(B3<B15,B33,IF(AND(B3>B15,B3<B14),B34,B35))		
33		96.3139	B28*(B3^B19)+B36*B17*(B3^B19)*LN(B3)	29	
34		-110.5688	B29*(B3^B19)+B36*B18*(B3^B19)*LN(B3)	30	
35		0.0000		21c	
36	δβ1/δσ	-3.8396	(-B5*(B19^2)*(B19-1))/B38	27	-3.8363
37		Parts			
38	A	0.0877	B6+0.5*(B19^2)*(B5^2)		
39	B	1.4155	B16*(B19^2)*(B19-1)*B5*LN(B14)		
40	C	-2.0535	(-(1-B9)*B14*B19*B5)		
41	D	-1.1095	B18*B5*(B19^2)		
42	E	51.6522	B40/((B38*(B6-B7)*(B14^B19)))+B41*(B19-1)*LN(B15)/B38		
43	ELASTICITY				
44	(σ/F)(δF/δσ)	0.8106	(B5/B12)*B30		
45	(σ/L)(δL/δσ)	0.2511	(B5/B20)*B32		

Which of the two competitors benefits most from a volatility change, and when? Figure 5 can be decomposed into four segments. (I) While $v \in (0, v_{L,MAX}]$, the leader gains more from positive volatility changes due to first mover advantage. (II) For v increasingly greater than $v_{L,MAX}$, the follower gains from volatility increases as the leader loses, approaching ($\partial g_{L2} / \partial \sigma$) a minimum at $v = 6.00676 < v_{L1}$.) (III) The follower vega becomes negative, and $\frac{\partial g_{L2}}{\partial \sigma}$ and $\frac{\partial g_{F1}}{\partial \sigma}$ intersect at $v_{L2,F1} = 12.96882$. Finally, the leader increasingly benefits more from positive volatility changes while $v \in \left(v_{L2,F1}, v_{F1}\right)$, because those changes defer the market entry for the follower since $\frac{\partial v_{F1}}{\partial \sigma} > 0$ and thereby prolong the monopoly position for the leader. Note the equations (24) and (25) have been simplified using parts (B38:B42).

Table 3C

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
43	v	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
44	L SO	**33.22**	**92.62**	**168.73**	**258.23**	**359.22**	**470.42**	**590.90**	**719.95**	856.97	1001.50	1153.13	1311.53	1476.38	1647.43	1824.44	2007.20	2195.51	2389.22	2588.15	2792.16
45	L SO Part	**0.00**	**-27.76**	**-80.14**	**-154.77**	**-249.95**	**-364.41**	**-497.13**	**-647.25**	-814.08	-997.00	-1195.46	-1409.01	-1637.21	-1879.67	-2136.05	-2406.03	-2689.32	-2985.63	-3294.72	-3616.35
46	L RO	-23.94	-66.74	-121.58	-186.07	-258.84	-338.97	-425.78	-518.77	**-617.50**	**-721.65**	**-830.91**	**-945.04**	**-1063.83**	**-1187.08**	**-1314.63**	**-1446.31**	**-1582.01**	**-1721.59**	-1864.93	-2011.94
47	L RO Part	0.00	18.12	52.31	101.01	163.13	237.83	324.45	422.43	**531.31**	**650.69**	**780.22**	**919.59**	**1068.52**	**1226.76**	**1394.09**	**1570.29**	**1755.18**	**1948.56**	2150.29	2360.20
48	F SO	**20.85**	**58.14**	**105.91**	**162.09**	**225.48**	**295.28**	**370.90**	**451.90**	**537.91**	**628.63**	**723.81**	**823.23**	**926.71**	**1034.07**	**1145.18**	**1259.89**	**1378.10**	**1499.68**	1624.55	1752.61
49	F SO Part	**0.00**	**-12.67**	**-36.58**	**-70.63**	**-114.07**	**-166.31**	**-226.88**	**-295.39**	**-371.53**	**-455.00**	**-545.58**	**-643.04**	**-747.18**	**-857.83**	**-974.84**	**-1098.05**	**-1227.34**	**-1362.57**	-1503.63	-1650.41
50	δF(v)/δσ	20.85	45.47	69.34	91.45	111.41	128.97	144.03	156.51	166.38	173.63	178.23	180.19	179.53	176.24	170.34	161.84	150.76	137.12	120.92	102.20
51	δL(v)/δσ	33.22	64.87	88.59	103.46	109.27	106.01	93.78	72.69	42.89	4.50	-42.33	-25.45	4.69	39.68	79.47	123.98	173.17	226.98	285.36	348.27
52		I	I	II	II	II	II	II	II	III	III	III	III	III	IV	IV	IV	IV	IV		
53	Vega F	+	+	+	+	+	+	+	+		-	-	-	-	-	-	-	-	-		
54	Vega L	+	+	-	-	-	-	-	-	+	+	+	+	+	+	+	+	+	+		

Table 3C shows in detail what parts of the leader strategic option (before investing), and rival option (after investing, before follower invests) contribute to the positive and negative vegas. In the first phase I both vegas are positive. In the second phase II the leader's vega is negative, follower's vega is positive, a contrasting risk exposure. In the third phase III both vegas are positive, a similar risk exposure. In the fourth phase IV the leader's vega is positive, the follower's vega is negative, a contrasting risk exposure.

Delta Partials

The leader delta (34) does not involve any change in the thresholds or option coefficients, while the other partial derivatives do.

$$\frac{\partial V_L(v)}{\partial v} = \begin{cases} \frac{\partial V_{L3}(v)}{\partial v} = m_{LF}\frac{1}{r-\delta} & \text{for } v \geq v_{F1} \\ \frac{\partial V_{L2}(v)}{\partial v} = \frac{1}{r-\delta} + \beta_1 A_{L11} v^{\beta_1 - 1} & \text{for } v_{L1} \leq v < v_{F1}, \\ \frac{\partial V_{L1}(v)}{\partial v} = \beta_1 A_{L1} v^{\beta_1 - 1} & \text{for } v < v_{L1}. \end{cases} \tag{34}$$

Differentiate the follower's value function with respect to v yields:

$$\frac{\partial V_F(v)}{\partial v} = \begin{cases} \dfrac{\partial V_{F2}(v)}{\partial v} = m_{FL}\dfrac{1}{r-\delta} & \text{for } v \geq v_{F1} \\ \dfrac{\partial V_{F1}(v)}{\partial v} = \beta_1 A_{F1} v^{\beta_1 - 1} & \text{for } v < v_{F1}, \end{cases} \tag{35}$$

In line with conventional option pricing theory, it could be argued that for L1 and L2

$$\begin{aligned} \frac{\partial V_{L1}(v)}{\partial v} &= \beta_1 A_{L1} v^{\beta_1 - 1} = 21.0498 \quad v = 5 \\ \frac{\partial V_{L2}(v)}{\partial v} &= \frac{1}{r-\delta} + \beta_1 A_{L11} v^{\beta_1 - 1} = 23.3124 \quad v = 14 \end{aligned} \tag{36}$$

a short position 21.0498 when v =5, and VF_L=61.3668 should be used to delta hedge the L value function which includes the strategic investment option $A_{L1} v^{\beta_1 - 1}$ in the initial L1 regime. A short position 23.3124 when v =14, and VF_L=302.1527 should be used to delta hedge the L value function which includes the negative value of the rival investment option $A_{L11} v^{\beta_1}$. These hedging guidelines are not well presented in the literature.

Interest Rate and Drift Partial Derivatives

See Appendix D and E.

5 Value Functions Across Ranges of Input Parameter Values & Hedging

Appendix F shows the effect on the value functions of changes of each of the eight inputs across a range of parameter values and regimes, assuming changes are independent. The summaries for each parameter v, K, σ, r, δ, f, m_{LF} and m_{FL} are given below.

Changing **v** does not affect the thresholds but may move the L/F across the regimes, as shown in Figure 1A. The follower's value function consists solely of the investment option value, until past v=18.7 the value function is the present value (PV) of operations. Over L1, the leader's value function consists solely of the investment option value; over L3, the PV of operations with the effective market share reduced from 100% to 60%. Over L2 the value function L2(v) consists of the PV of operations less the value of the rival option (reduction of market share when the follower invests).

Naturally, both thresholds increase if **K** increases, and both value functions decrease. But over the range of K increasing by 5 over each interval from 115 to 165 in Appendix Figure F4, the absolute decrease of the VF when v=5 is half of that even for the leader, since there is still the option of making the investment. But when v=22 over the L3 regime, the leader's value function decreases by 184, and the follower's by 70 if K increases from 115 to 165 as the follower invests, and the leader's market share is reduced. Such an illustration shows that the value of investment tax credits or subsidies increases as v increases, more for the leader than for the follower.

As expected, both thresholds increase (about the same) as $\boldsymbol{\sigma}$ increases. At L1 when v=5, the value functions consisting solely of the investment option values increase as shown in Figure 5, also at L2 for the follower. At L3 volatility changes do not affect the PVs of either the leader or follower.

As noted in the sensitivities analysis, increases in **r** significantly affect all thresholds and option coefficients, increasing the thresholds, and reducing the investment option coefficients (and reducing the negative value of the rival option). As also noted in sensitivities, increasing r reduces both value functions at all v levels.

In contrast to r, increases in $\boldsymbol{\delta}$ significantly reduce the thresholds, and increase the investment option coefficients (and increase the negative value of the rival option) over this range. Curiously, the investment coefficients first decrease and then increase as δ goes from 1.75% to 4.25%. As noted in sensitivities, increasing δ increases both value functions at all v levels.

Changing **f** does not change much of anything, perhaps because this such an in-the-money investment option after v=7.5 for the leader. The signs for the effect on the thresholds, option coefficients and value functions are the same as indicated in the sensitivities.

As indicated in sensitivities, changes in the leader's final market share $\mathbf{m_{LF}}$ do not affect the leader's threshold but negatively affect the strategic option coefficient, and reduce the negative rival option coefficient, thus increasing the leader's value function over all regimes, reducing the follower's value function significantly, as shown in Figure 4. As indicated in sensitivities, increases in the follower's final market share $\mathbf{m_{FL}}$ do not affect the leader's threshold but reduce the follower's threshold, and increase the investment option coefficient. The leader's investment coefficient is slightly reduced, and the rival option coefficient becomes more negative. As expected, the value functions move in opposite directions, but the effect seems to be constant, as the spread between the value function decreases with the narrowing of the difference in market shares.

Risk hedging may be the most useful activity using these partial derivatives, especially over one regime such as L2 where there are no jumps. Table 4 is an illustration of delta hedging based on equations 34 and 35 for the middle regime L2. Suppose the leader is satisfied with maintaining the value function of 384 after investing (cost 140), with a PV of operations 720 and a rival option value of -355 when v=18. The leader seeks to maintain this value function value (in case v declines) by shorting v for each price interval (adjusting the delta at each interval), and marking-to-market (or model) at each interval, as shown in Table 4 down to v=9. The leader's experiences an unhedged loss for each integer if v declines, which increases with the v decline because the rival option becomes less negative. The deltas are all positive since increasing v benefits both, Δ F2<ΔL2 until just before the follower's investment threshold of 18.7. When v=18, Δv= $\frac{\partial V_{L2}(v)}{\partial v}=\frac{1}{r-\delta}+\beta_1 A_{1LSS}\, v^{\beta_1 -1}=50.0000-31.9416=18.0584$, so a short position Δ/18=1.003 in v should be used to delta hedge the L value function, minimizing the combined unhedged loss and hedge gain[8].

Table 4 shows the leader and follower gross loss (unhedged) for the value function VF as v falls from 18 to 9 in the L2 (after the leader invests). The largest component of the loss for the leader is in the PV of operations, which is constant at 50 for each interval. There is a small loss for the F investment option value at lower v. The mean hedged loss (combining the unhedged and hedging gain/loss) is sharply reduced for both the leader and follower.

[8] This ignores transaction costs and other practical considerations like whether there is an active market in p or p futures, roll-over costs for finite futures, margins, and credit risks.

Table 4

Delta Hedging over v=18 to 9, L2

v	9	10	11	12	13	14	15	16	17	18				
HEDGED LOSS=+	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	**MEAN**	**STDEV**	MAX	MIN
VF F		-0.46	-0.45	-0.44	-0.43	-0.42	-0.41	-0.40	-0.40	-0.39	**-0.42**	**0.02**	-0.39	-0.46
VF L		0.76	0.74	0.72	0.70	0.69	0.67	0.66	0.65	0.64	**0.69**	**0.04**	0.76	0.64
UNHEDGED LOSS=+		LOSS 10->9							LOSS 17->16	LOSS 18->17	**MEAN**	**STDEV**	MAX	MIN
VF F		12.32	13.23	14.12	14.99	15.84	16.67	17.48	18.28	19.07	**15.78**	**2.31**	19.07	12.32
VF L		29.78	28.28	26.82	25.39	24.00	22.64	21.30	19.99	18.70	**24.10**	**3.79**	29.78	18.70
DELTA HEDGE		GAIN 10->9							GAIN 17->16	GAIN 18->17				
VF F GAIN=+		12.78	13.68	14.56	15.42	16.26	17.08	17.88	18.68	19.46	16.20	2.28	19.46	12.78
VF L		29.02	27.54	26.10	24.69	23.31	21.96	20.64	19.34	18.06	23.41	3.75	29.02	18.06
dVF/dv		12.78	13.68	14.56	15.42	16.26	17.08	17.88	18.68	19.46				
dVL/dv		29.02	27.54	26.10	24.69	23.31	21.96	20.64	19.34	18.06				

For the leader, the mean loss (mostly due to the PV operations) and variability is significant unhedged, but sharply reduced with this academic hedging based on the delta partial derivatives, and choice of hedging intervals over these limited intervals. By hedging, the standard deviation of the leader's unhedged losses of 3.79 is reduced to .04. However, trying to delta hedge over the investment thresholds is likely to be problematic.

6 Summary and Conclusions

We provide several possibly unique contributions for the real option solutions and derivatives for basic once-off investment opportunities in a duopoly with variable operating costs: analytical solutions for the thresholds and option coefficients, and for the partial derivatives for all of the inputs; confirming all of these solutions with numerical solutions, and that all of the conventional conditions are satisfied; based on simulations of the solutions and partial derivatives over a range of input parameter values, we show how the delta partial derivative can be used for delta hedging to sharply reduce risk of this portfolio of real options.

We proposed three measures of the risk exposure of the real option portfolio of duopoly investment opportunities: sensitivities, partials, and value functions across a range of input parameter values[9].

(i) Sensitivities show the change in each threshold, option coefficient, value function for a 1% change in the input parameter value for a single v, easy to calculate but not shown (yet) across regimes.

(ii) Partials show the change in continuous time, which are also compared to proportionate change over an almost infinitesimal interval (.0000000001).

(iii) VF Vary Integers enables on a single chart viewing the analytical results over a wide range of integer input parameter values, including across regimes, illustrating L jumps at the F threshold.

An advantage of the analytical solutions for the thresholds and option coefficients (rather than a numerical solution for the leader's threshold as in other papers, and all thresholds as in Adkins et al., 2022) is that all of these calculations can be done immediately, changing other variables as well.

Future research is to use these analytical solutions to introduce stochastic K. What is the relationship between Delta, Vega, Rho, and Alpha in this basic duopoly investment model? How should one use volatility swaps to hedge Vega, interest-rate futures to hedge Rho, and arrangements with third parties and marketing experts (or collusion through industry associations) to hedge Alpha risk? Is there a simple measure like VaR which can be constructed out of these analytical formulae to assess risk for this basic model? Practical examples are warranted.

[9] Of course, each of these formats can be replicated for volatility changes for instance using the Appendix Table C1 (CROR Num PD Vol), using in C2=1.0000000001 for (ii), C2=1.01 for (i), and C2=.17 for the σ interval .16->.17 (iii).

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APPENDICIES

A: Determining the Partial Derivatives

B: Model PD with respect to Market Share

C: Model PD with respect to Volatility

D: Model PD with respect to Risk-free Interest Rate

E: Model PD with respect to Epsilon

F: Values as a Function of a Range of Parameter Values

G: Delta Hedging

H: Literature Review of Some Competitive Real Option Partial Derivatives