Irreversibility, Uncertainty and the Optimal Finance of Public Goods

Yishay D. Maoz

The Open University of Israel

Abstract

This article analyzes the case where a government finances the supply of a public good via a tax on a private good which is produced and traded in a market where demand is stochastic, production requires an initial irreversible investment, and firms can choose optimally when to invest. The government problem is to find the tax rate that maximizes overall welfare, which is a composite of the welfare generated by the production of each of the two goods. The analysis reveals this optimal tax rate and enables characterizing it, as well as characterizing the equilibrium dynamics under the optimal tax rate. Of particular interest is the result that the higher the uncertainty regarding the demand for the private good the higher the optimal tax rate.

Keywords: Investment, Uncertainty, Taxes, Competition, Public Goods, Welfare.

JEL codes: C61, D25, D40, D60, H21, H41

Corresponding author: Yishay D. Maoz, Department of Management and Economics, The Open University of Israel. yishayma@openu.ac.il.

1. Introduction

The study of commodity taxation and its usage for the finance of the supply of public goods has long roots in economic research, going back almost a century ago to the works of Ramsey (1928) and Pigou (1947). The wide research since then covers a variety of different aspects of this issue, and yet, so far it has not been analyzed within the economic environment typically portrayed by the modern literature on investment under uncertainty. It is important to fill this void because this latter literature analyzes the investment choices of firms under several plausible assumptions which strongly affect the resulting market dynamics.[1]

Thus, to fill this void, I present here a model of a private good which is taxed by the government in order to finance the supply of a public good. The private good is assumed to be produced and traded within the economic environment portrayed by the typical models of the literature of investment under uncertainty. In particular, it is assumed that production of this good requires some initial irreversible investment, the profit process firms face is stochastic, and firms can optimally choose the timing of investment. The government is selecting the tax rate so as to maximize a composite of the net welfare from each good. To do so, it has to optimally balance between the welfare loss that the tax inflicts on the market for the private good and the welfare gains from supplying the public good. The objectives of the analysis are to find the optimal tax rate and its main properties, to characterize the resulting equilibrium dynamics, and in particular to find how the profitability uncertainty affects the optimal tax.

This latter question about the effect of the level of uncertainty on the optimal tax collected for the finance of a public good is a novel one. Previous studies about optimal commodity taxation under uncertainty have dealt with a variety of issues such as the provision

[1] See Dixit and Pindyck (1994), Trigeorgis (1995) and Schwartsz and Trigeorgis (2004) for books presenting the fundamental insights, results and methodologies od this literature.

of social security [as in Varian (1980) or Cremer and Gahvari (1995)], making investment occur earlier [as in Barbosa, Carvalho and Pereira (2016) or Di Corato (2016)], or tackling harmful externalities [as in Di Corato and Maoz (1999, 2023)]. Yet, none of them have dealt with the provision of public goods. A single exception is Aronsson and Blomquist (2003), but the uncertainty there is not about profitability, but pertains to the welfare loss caused by the externality created by the production of a certain private good.

As already mentioned, the private good in the model presented here is produced and traded in a market where profitability is a stochastic process, and firms can enter the market and start producing only if they incur an irreversible investment. Due to that – firms enter the market only at times when the profit flow is sufficiently high. Specifically, it is assumed that the source of the profitability uncertainty is stochastic demand dynamics. Due to that, firms enter the market at times when the market price is sufficiently high, exceeding the long-run average cost by a mark-up based on the demand uncertainty.

Under this dynamic and stochastic modelling, the market quantity is not a single level but a stochastic flow, and the tax proceeds (which are based on taxing the market quantity) is also a stochastic flow. Despite this deviation from the simpler concepts of quantity and tax proceeds, a "Laffer curve" pattern emerges here too, as a higher level of the tax rate (i.e., the tax per unit of the private good) has the following two contradicting effects: On the one hand, a higher amount of tax is collected from any sale of a single unit of the good; On the other hand, the flow of the number of units of the good is expected to be lower. The reason for the second effect is that the tax per unit is added to the production cost and thus raises the threshold price at which firms invest and enter the market, and therefore lowers the expected flow of market quantity over time.

These two elements of the "Laffer curve" make the expected value of the discounted flow of tax proceeds an inverse u-shape function of the level of the tax that the government sets. Moreover, as the tax proceeds are used to finance the supply of a welfare-yielding public good, the u-shape form is also translated into the welfare function which is a composite of the welfare of both the private and the public goods. Thus, a unique welfare-maximizing level of the tax rate exists, and the analysis characterizes it. In particular, it is found that the higher the volatility of the demand swings the greater the optimal tax rate levied on the private good. The reason for that is based on the nature of the optimization which is based on the balance between the following three effects of a *marginal* increase of the tax rate:

- More proceeds are collected from each unit traded
- Less proceeds are collected because less units are produced
- Less welfare is gathered from the market of the private good because less units are produced

The first two elements are those that create the "Laffer curve" pattern of the tax proceeds. In equilibrium, the optimal tax rate must be in the range where the curve is increasing because the government does not care merely about maximizing tax proceeds but also cares about the welfare in the taxed market. Thus, in equilibrium, the sum of these two responses to a marginal increase in the tax rate is an increase in tax proceeds which leads to greater production of the public good and more welfare from its consumption. The optimum is characterized with a perfect equality between the welfare added by these two elements and the welfare loss captured by the third element. In the analysis of the model it is proven that an increase in the volatility of the demand for the private good raises the first element, which raises welfare, by more then it raises the second and third elements, which are lowering welfare, and therefore raises the optimal tax rate.

The article is organized as follows. Section 2 presents the basic setting of the model and the optimization of the single firm in the market for the private good, as well as the resulting dynamics in the market of this good. In this section the tax rate is exogenous, as this is how the firms view it. Section 3 presents the considerations of the government in imposing the welfare maximizing tax, and the market equilibrium dynamics under the optimal tax. Section 4 shows how the demand uncertainty regarding the private good affects the optimal tax. Section 5 offers some concluding remark. Some of the more technical parts of the analysis were relegated to two appendices.

2. Basic settings and the optimization of firms

There are two goods in the economy. The first one is a private good produced and traded in a perfectly competitive market, with the exception that the government taxes this markets. The reason for the taxation is the need to finance the activities of the government with regard to the other good, which is a public good and the government produces and offers it to the public.

The market for the private good is modeled as a specific case of the standard model of investment under uncertainty introduced by Dixit (1989) for the case of a single firm, and its extension to the case of perfect competition as presented in Dixit and Pindyck (1994, pages 252-260). The model in the current study follows the perfect competition version as presented by Dixit and Pindyck (1994). The rest of this section describes their results about the optimal choices of firms and the resulting market dynamics.

The market comprises a large number of identical and infinitesimally small, price-taking firms. At each time point $t \geq 0$, the demand for this good is given by:

(1) $$P_t = \frac{X_t}{Q_t^{\alpha}},$$

where Q_t is the quantity offered and consumed in the market at time *t*, and P_t is the price of the good at time *t*. The term X_t, is a demand shift factor that evolves stochastically over time according to the following Geometric Brownian Motion:

(2) $$dX_t = \mu \cdot X_t \cdot dt + \sigma \cdot X_t \cdot dZ_t,$$

where μ is the drift parameter, σ is the instantaneous volatility, and dZ_t is the increment of a standard Wiener process satisfying $E(dZ_t) = 0, \quad E(dZ_t)^2 = dt$ at each time *t*.

Firms are risk-neutral and maximize their expected value. Each firm rationally forecasts the future evolution of the whole market. An active firm produces a flow of one unit of output, at the production cost of $w > 0$. An idle firm can enter the market at any time. The basic model by Dixit (1989) introduces irreversibility of the investment via a fixed entry cost, denoted *k*. He also allows the firm to exit the market if demand goes sufficiently law, where exit bears the fixed cost *l*, which can be negative if by exit the firm sells its capital. Irreversibility in this case springs from the assumption $k + l > 0$, implying that part of the entry cost, *k*, is necessarily lost by entry. Such modelling prevents an analytical solution, and necessitates numerical analysis, and therefore it is often helpful to examine the extreme case where *l* is infinite, implying that the firm cannot exit at all. Dixit and Pindyck (1994) too adopt this assumption in the model which is followed here. It simplifies the analysis greatly, enables analytical solution, and also enables dropping the entry cost k from the model, as irreversibility springs from the commitment to stay in the market and produce at the flow of production cost w.[2]

To prevent the value of the firms from going to infinity it is also assumed that the interest rate r satisfies $r > \mu$. In addition to the production cost w, each unit also costs the firm with a tax at the size of τ.

The decision to enter the market is driven by expected profitability, and therefore takes place only when X_t is sufficiently large. In particular, given the current market quantity, Q_t, a firm enters the market only if X_t hits an entry threshold which denoted by $X^*(Q_t)$. To find this threshold, let $V(X_t, Q_t)$ denote the value of an active firm. An analysis based on a Bellman equation (see Appendix A) shows that $V(X_t, Q_t)$ is of the following form:

(3) $$V(X_t, Q_t) = Y(Q_t) \cdot X^{\beta} + \frac{P_t}{r-\mu} - \frac{w+\tau}{r},$$

where and β is the upper root of the quadratic:

(4) $$\tfrac{1}{2} \cdot \sigma^2 \cdot x^2 + \left(\mu - \tfrac{1}{2} \cdot \sigma^2\right) \cdot x - r = 0,$$

and satisfies $\beta > 1$, while $Y(Q_t)$ is to be found together with the entry threshold $X^*(Q_t)$ via the following two boundary conditions which holds at time instances in which X_t hits the threshold and a new firm enters the market:

[2] See McDonald and Siegel (1986) and Pindyck (2000) as additional examples for influential model assuming no exit in order to obtain a closed form solution.

(5) $$V\left[X^{*}(Q_t), Q_t\right]=0$$

and:

(6) $$V_X\left[X^{*}(Q_t), Q_t\right]=0\,.$$

The condition captured by (5) is known as the *Value Matching* condition. It states that the net value of entry (i.e., of becoming an active firm) is zero, as follows from the perfect competition in this market. It holds for any investment threshold, and not merely for the optimal threshold. (6) on the other hand, presents a condition for an optimal threshold, known as the *Smooth Pasting Condition*. Applying (4) in (5) and (6), yields the optimal entry threshold:

(7) $$X^{*}(Q_t)=\hat{\beta}\cdot\frac{r-\mu}{r}\cdot(w+\tau)\cdot Q_t^{\alpha},$$

where $\hat{\beta}\equiv\frac{\beta}{\beta-1}$. Note that $\hat{\beta}>1$, and thus scales up the investment threshold, compared with the net present value rule, to account for the presence of uncertainty and irreversibility (see Dixit and Pindyck, 1994, Ch. 5, Section 2). Note from (7) that $\frac{\partial X^{*}(Q_t)}{\partial\beta}<0$ and note from implicit differentiation of (4) that $\frac{\partial\beta}{\partial\sigma^2}<0$. This implies that uncertainty delays investment in the sense that the higher the uncertainty the higher the entry threshold.

Figure 1 presents the resulting entry dynamics in the market. At a point like **A** that lies inside the region below $X^{*}(Q_A)$, small movements of the continuous demand shock X_t shift the industry's position vertically but do not provoke changes in industry capacity. As soon as X_t hits $X^{*}(Q_A)$ however, investment occurs, increasing industry capacity. This raises the investment threshold, so the industry lies again below the $X^*(Q_t)$ function and further investment is postponed until the next time in which X_t hits the threshold function.

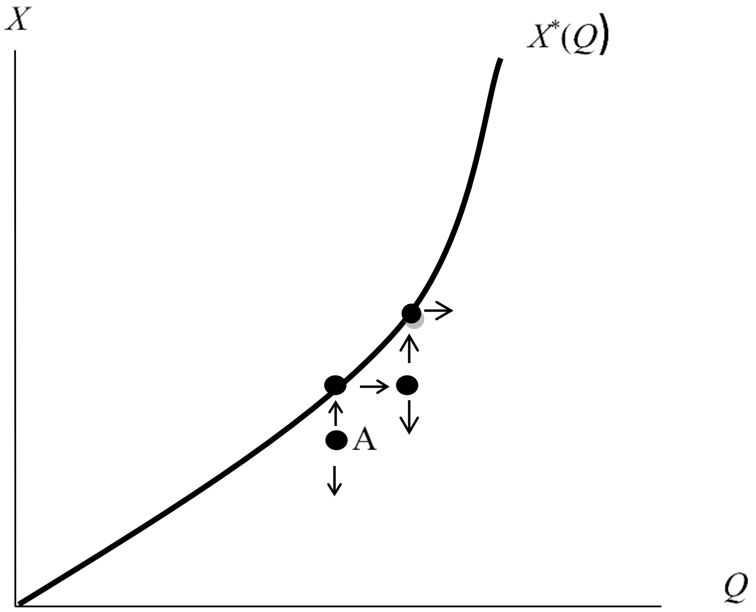

Figure 1: Demand swings and entry dynamics in a competitive industry. When the market is at a point like **A**, below the entry threshold, the swings in the demand shit factor, X_t, do not change market quantity. When X_t hits the threshold function $X^*(Q_t)$, firm entry leads to an incremental increase in Q making X_t once again below the threshold line.

This pattern of investment in a competitive industry results in an endogenous cap on the price process, such that at any time $P_t \le P^*$, where from (1) and (7) it follows that:

(8) $$P^* = \frac{X^*(Q_t)}{Q_t^{\alpha}} = \hat{\beta} \cdot \frac{r-\mu}{r} \cdot (w+\tau).$$

3. The Optimal Tax

At time $t = 0$ the government decides on the optimal tax rate τ. Following this decision, there are two possibilities to the immediate market reaction, given Q_0 and X_0 and based on the formula for the entry threshold, as given in (7).

First, if the government chooses a sufficiently high level of τ, then $X_0 < X^*(Q_0)$, implying that no entry occurs at that time. The market quantity remains at Q_0 for a while and changes only later, when the process X_t will reach the threshold level $X^*(Q_0)$. Specifically, based on (7), this occurs if the government chooses a tax level within the range $\tau \geq \tau^*(Q_0, X_0)$ where:

(9) $$\tau^*(Q_0, X_0) = \frac{r}{r-\mu} \cdot \frac{X_0}{\hat{\beta} \cdot Q_0^{\alpha}} - w.$$

Otherwise, if the government choses τ in the range $\tau < \tau^*(Q_0, X_0)$, then $X_0 \geq X^*(Q_0)$ and there is an immediate entry raising quantity immediately from Q_0 to the level $Q^*(X_0, \tau)$ for which X_0 is no longer above the entry threshold, but equals it. Based on that, $Q^*(X_0, \tau)$ is found, via (7) and the equality $X_0 = X^*(Q_0)$, and therefore given by:

(10) $$Q^*(X_0, \tau) = \left[\frac{r \cdot X_0}{\hat{\beta} \cdot (r-\mu) \cdot (w+\tau)} \right]^{\frac{1}{\alpha}}.$$

3.1 The value of tax collection

In this sub-section the tax collection from time t onwards is calculated, given the time t values of X and Q, and for the case where $X_t < X^*(Q_t)$ so that not entry occurs right away at time t.

For the simplicity of the analysis it shall be assumed that when, at time $t = 0$, the government chooses the size of the tax per unit of output, τ, then it is relevant only for firms entering from that point onwards, and not for the firms that were already active before this decision, and producing the quantity Q_0.

This assumption simplifies the analysis as it helps maintaining the simplifying assumption of extreme irreversibility taken before. Otherwise, the extreme irreversibility may lead to the undesirable result that the optimal tax rate is infinite, as the firms active in the market at time 0 cannot exit, so the quantity Q_0 may be supplied even at an infinite tax rate. As mentioned earlier, allowing exit prevents analytical solutions and leaves only the option of conducting a numerical analysis. The same applies for relaxing the assumption that each firm produces a flow of one unit of output and, instead, allowing temporary production in a variable scale. This, too, will which will enable only numerical analysis rather than an analytical one.

The assumption that the tax is levied only on firms entering after the time in which the tax is set is merely a simplifying one and does not affect the qualitative results. The is so because although the tax levied at time $t = 0$ does not lower market quantity at that point in time, it nonetheless does lower the expected flow of market quantity from then on as the tax is an additional cost for the firms which may enter the market later. Thus, the two main effects of the tax on welfare are preserved: First, welfare in the taxed market is harmed as the tax serves as an additional cost for the firms and makes them lower their production; Second, the tax collection (which finances the supply of the public good) is subject to the classic tradeoff where raising the *tax rate* lowers the *tax base* and thus yields more from every unit of output, but lowers the number of such units.

The function $T(Q_t, X_t, \tau)$ denotes the value of the tax collection from time *t* onwards, given the time *t* values of *X* and *Q* and given the tax rate τ. Based on the assumptions so far, it is defined as:

(11) $$T(Q_t, X_t, \tau) \equiv E\int_t^{\infty}[\tau \cdot (Q_s - Q_0)] \cdot e^{-r\cdot(s-t)} \cdot ds = \tau \cdot H(Q_t, X_t) - \tau \cdot \frac{Q_0}{r}$$

where

(12) $$H(Q_t, X_t) \equiv E\int_t^{\infty} Q_s \cdot e^{-r\cdot(s-t)} \cdot ds \,,$$

and serves therefore as the present value of the flow of taxed market quantity from time *t* onwards. An analysis similar to the one conducted in Appendix A shows that $H(Q_t, X_t)$ is of the form:

(13) $$H(Q_t, X_t) = C(Q_t) \cdot X_t^{\beta} + \frac{Q_t}{r} \,,$$

where $C(Q_t)$ is to be determine by the following boundary condition:

(14) $$H_Q\left[Q_t,\, X^*(Q_t)\right] = 0 .$$

Applying (13) in (14), simplifying, and then applying (7) and (8), yields that:

(15) $$C'(Q_t) = -\frac{1}{r \cdot X^*(Q_t)^{\beta}} = -\frac{1}{r \cdot P^{*\beta} \cdot Q_t^{\alpha\cdot\beta}} .$$

Recall that $H(Q_t, X_t)$ is the present value of the flow of taxed market quantity from time t onwards, and note that the second term in (13), namely $\frac{Q_t}{r}$, is the present value of the flow of quantity if no firms enter after time t so that the quantity remains Q_t. Thus, the first term in (13), namely $C(Q_t)\cdot X_t^{\beta}$, shows the contribution to $H(Q_t, X_t)$ of future entries. This leads to the following boundary condition:

(16) $$\lim_{Q_t \to \infty} C(Q_t) = 0,$$

which holds because when Q_t is infinitely large, then, by (7), the threshold $X^*(Q_t)$ is infinitely large too, and the probability that X_t will ever reach it is 0, so no future entries will take place.

Based on (16), there are two possibilities for the integration of (15) in order to obtain the function $C(Q_t)$. First, if $\alpha\cdot\beta < 1$ then integrating yields that $C(Q_t) \to \infty$. The logic under this result is that the smaller α the more elastic the demand and therefore the smaller the decline in price each time the entry threshold is hit, implying a shorter expected time until the next time the threshold is hit. Thus, the smaller α the faster the process of quantity growth and below a certain level of α this process is so fast that $H(Q_t, X_t)$, the present value of the flow of quantity, is infinite. This is highly unrealistic and therefore this case is not in the focus of the current study.

We therefore focus on the second case and assume henceforth that $\alpha \cdot \beta > 1$. Integrating (15) under this assumption, bearing condition (16) in mind, yields:

(17) $$C(Q_t) = \frac{Q_t}{r \cdot (\alpha \cdot \beta - 1) \cdot X^*(Q_t)^\beta} .$$

From (11), (12), (13) and (17) it follows therefore that:

(18) $$T(Q_t, X_t, \tau) = \frac{\tau \cdot (Q_t - Q_0)}{r} + K(Q_t) \cdot X_t{}^\beta \cdot \frac{\tau}{(w + \tau)^\beta} ,$$

where:

(19) $$K(Q_t) \equiv \frac{r^{\beta - 1}}{(\alpha \cdot \beta - 1) \cdot Q_t^{\alpha \cdot \beta - 1} \cdot \left[\hat{\beta} \cdot (r - \mu)\right]^\beta} .$$

If, at setting the tax at time $t = 0$ the government was maximizing its tax collection, rather then maximizing welfare, then the optimal tax would be achieved via:

(20) $$T_\tau(Q_0, X_0, \tau) = K(Q_0) \cdot X_0{}^\beta \cdot \frac{w - (\beta - 1) \cdot \tau}{(w + \tau)^{\beta + 1}} = 0,$$

which implies that the tax level that maximizes the tax collection is:

(21) $$\tau^{Max} = \frac{w}{\beta - 1}.$$

3.2 The welfare function

From (1) it follows that the present value of the surplus created from time t onwards by production of the private good is given by:

(22) $$S(Q_t, X_t, \tau) = E\int_t^\infty \left(X_s \cdot \frac{Q_s^{1-\alpha}}{1-\alpha} - w \cdot Q \right) \cdot e^{-r\cdot(s-t)} \cdot ds.$$

Based on (22), a procedure similar to the one presented in Appendix A, leads to:

(23) $$S(Q_t, X_t, \tau) = F(Q_t) \cdot X_t{}^\beta + \frac{X_t}{r - \mu} \cdot \frac{Q_t^{1-\alpha}}{1-\alpha} - \frac{w \cdot Q_t}{r},$$

where $F(Q_t)$ is to be determine by the following boundary condition:

(24) $$S_Q\left[Q_t,\, X_t^{\,*}(Q_t),\, \tau\right] = 0\,.$$

Applying (23) in (24), simplifying, and then applying (7) and (8), yields that:

(25) $$F'(Q_t) = -\frac{w+\beta\cdot\tau}{(\beta-1)\cdot r\cdot P^{*\beta}\cdot Q^{\alpha\cdot\beta}}$$

Note that the second and third terms in the RHS of (23) represent the present value of the surplus created by production of the private good from time *t* onwards if quantity does not change and remains Q_t over time. This implies that the first term of (23), namely $F(Q_t)\cdot X_t^{\,\beta}$, represent the contribution of future entries to the present value of the surplus. This leads to a boundary condition:

(26) $$\lim_{Q_t\to\infty} C(Q_t) = 0\,,$$

which holds for similar reasons for those underlying condition (16). Integrating (25), leads, via (26) and by applying (19), to:

(27) $$F(Q_t) = \frac{K(Q_t)\cdot(w+\beta\cdot\tau)}{(\beta-1)\cdot(w+\tau)^{\beta}}.$$

The welfare in the market for the private good is defined as the surplus created by production of that good, less the amount taken from the market via taxation, i.e.:

(28) $$W^{PR}(Q_t, X_t, \tau) \equiv S(Q_t, X_t, \tau) - T(Q_t, X_t, \tau)$$

The welfare that the government generates by supplying the public good is assumed to be positively connected to its expenditures on it, which in turn, is based on the proceeds from taxing the market for the private good. To apply the simplest form for this assumption, it is assumed that the welfare from supplying the public good equals the expenditure on it, i.e:

(29) $$W^{PU}(Q_t, X_t, \tau) = T(Q_t, X_t, \tau).$$

As shall be seen later, this simplifying linearity assumption does not lead to a corner solution. The reason for that is the curvature in the welfare at the market for the private good, which follows from the curvature in the demand for this good as capture by (1).

The overall welfare in this economy at time t, denoted $W(Q_t, X_t, \tau)$, is assumed to be the following composite of the surplus created in the market for the private good and the welfare from supplying the public good:

(30) $$W(Q_t, X_t, \tau) \equiv W^{PR}(Q_t, X_t, \tau) + \delta \cdot W^{PU}(Q_t, X_t, \tau)$$

$$= S(Q_t, X_t, \tau) + (\delta - 1) \cdot T(Q_t, X_t, \tau),$$

where the coefficient δ represents the weight that welfare from the public good has in the overall welfare, and the second equality follows from (28) and (29).

From (30) it follows that taxation contributes to welfare only if the $\delta > 1$, which will be assumed from now on.

Applying (20), (23) and (27) in (30) leads to:

(31) $$W(Q_t, X_t, \tau) = \frac{K(Q_t) \cdot X_t^{\beta}}{\beta - 1} \cdot \frac{w + [1 + (\beta - 1) \cdot \delta] \cdot \tau}{(w + \tau)^{\beta}} + \frac{X_t}{r - \mu} \cdot \frac{Q_t^{1-\alpha}}{1 - \alpha} - \frac{w \cdot Q_t}{r}$$

$$+ (\delta - 1) \cdot \frac{\tau \cdot (Q_t - Q_0)}{r}.$$

3.3 The optimal tax

The optimal tax is found by looking at the welfare at time $t = 0$ in which the government sets the tax. Given the values of X and Q at this time, there are two possibilities. In the first one, the government chooses a value of τ in the range $\tau \geq \tau^*(Q_0, X_0)$ and therefore $X_0 \leq X^*(Q_0)$ so that no entry occurs immediately. In the second case, the government chooses a value of τ in the range $\tau < \tau^*(Q_0, X_0)$ and therefore $X_0 > X^*(Q_0)$ so that entry occurs immediately making quantity immediately rise. In this section both cases will be analyzed, revealing the optimal tax level under each one, and finding the condition that show to which of the two ranges it is better for the government to turn.

3.3.1 The optimal tax in the range $\tau \geq \tau^*(Q_0, X_0)$

In this case, there is no immediate entry after the tax is set, so the quantity remains Q_0 for a while. Recall also that, by construction, (31) represents welfare at a time instant in which there is no entry. Thus, in this case the welfare is captured by evaluating the (31) at (Q_0, X_0), and the optimal tax is found by differentiating it with respect to τ. This yields:

(32) $$W_\tau(Q_0, X_0, \tau) = K(Q_0) \cdot X_0^{\beta} \cdot \frac{(\delta-1)\cdot w - [1+(\beta-1)\cdot\delta]\cdot\tau}{(w+\tau)^{\beta+1}}$$

From (19) it follows that $K(Q_0) > 0$, and this implies that the sign of the derivative is the sign of the numerator on the RHS of (32). As τ goes from 0 to infinity, this numerator continuously falls from $(\delta-1)\cdot w > 0$ to -∞, implying that $W(Q_0, X_0, \tau)$ is an inverse u-shape function of τ, maximized at:

(33) $$\tau^{opt} = \frac{\delta - 1}{\delta\cdot(\beta-1)+1}\cdot w.$$

Note that $\delta > 1$ and $\beta > 1$ assert that τ^{opt} is positive. Another property of the optimal tax is that it is smaller than the level that maximizes tax collection, τ^{Max}, as the government balances between collecting taxes and the welfare in the taxed market. This result is established by:

(34) $$\tau^{opt} = \frac{\delta - 1}{\delta \cdot (\beta - 1) + 1} \cdot w < \frac{\delta - 0}{\delta \cdot (\beta - 1) + 0} \cdot w = \frac{w}{\beta - 1} = \tau^{Max},$$

where τ^{Max} is captured by (21).

Note that the entire analysis in this subs-section, leading to equation (33) for the optimal tax, was done under the assumption that the level of τ that the government chooses is in the range $\tau \geq \tau^*(Q_0, X_0)$. Thus, it is important to verify that the optimal tax level captured by (33) is indeed within that range. This occurs if the right end of the relevant range, namely $\tau^*(Q_0, X_0)$, is within the downward-sloping part of $W(Q_0, X_0, \tau)$, so that the peak of this inverse u-shaped function of τ is indeed within the relevant range. This condition takes the form of the inequality $W_\tau [Q_0, X_0, \tau^*(Q_0, X_0)] < 0$, and applying (9) and (32) in it yields the following necessary and sufficient condition for τ^{opt} to be within the relevant range:

(35) $$P_0 = \frac{X_0}{Q_0{}^\alpha} \leq \frac{(r - \mu) \cdot \delta \cdot \beta \cdot \hat{\beta}}{r \cdot [\delta \cdot (\beta - 1) + 1]} \cdot w \equiv P^{**}.$$

Thus, the initial level of demand for the private good has to be sufficiently low, given the initial supply of that good, to make the government choose the tax level captured by (33) for which no immediate entry occurs in that market.

If (35) does not hold, then, within the range this sub-section focuses on, namely, $\tau > \tau^*(Q_0, X_0)$, the welfare function is a downward-sloping function of τ, which implies that the optimal tax level is in the complementary range $\tau \leq \tau^*(Q_0, X_0)$. The next sub-section analyzes the properties of the welfare function and finds the optimal tax within that range.

3.3.2 The optimal tax in the range $\tau < \tau^*(Q_0, X_0)$

In this subsection the welfare-maximizing tax is found for the case in which the tax that government chooses satisfies $\tau < \tau^*(Q_0, X_0)$ so that $X_0 > X^*(Q_0)$ and therefore there is immediate entry taking quantity at once from Q_0 to $Q^*(X_0, \tau)$, which was introduced earlier and captured by (10). Based on that, in this case the welfare function at time 0 will be denoted by $W^-(Q_0, X_0, \tau)$ and defined by:

(36) $$W^-(Q_0, X_0, \tau) \equiv W\left[Q^*(X_0, \tau), X_0, \tau\right].$$

The optimal tax is found by differentiating the welfare function (36) with respect to τ. The derivative will be denoted $G(Q_t, X_t, \tau)$ and it is characterized by:

(37) $$G(Q_t, X_t, \tau) \equiv \frac{dW^-(Q_0, X_0, \tau)}{d\tau} = \frac{dW\left[Q^*(X_0, \tau), X_0, \tau\right]}{d\tau}$$

$$= W_Q\left[Q^*(X_0, \tau), X_0, \tau\right] \cdot \frac{dQ^*(X_0, \tau)}{d\tau} + W_\tau\left[Q^*(X_0, \tau), X_0, \tau\right]$$

$$= W_\tau\left[Q^*(X_0, \tau), X_0, \tau\right],$$

where the last equality follows from $W_Q\left[Q^*(X_0, \tau), X_0, \tau\right]=0$ which follows from $Q^*(X_0, \tau)$ being the quantity in which entry stops as the current level of X equals the entry threshold, so that the case from the previous sub-section is met and (12), (23) and (27) are satisfied.

Thus, applying $Q^*(X_0, \tau)$, as captured by(10), in (31), taking the partial derivative with respect to τ, and simplifying, yields:

(38)
$$G(Q_0, X_0, \tau)=\frac{K\left[Q^*(X_0, \tau)\right]\cdot X_0{}^{\beta}}{\beta-1}\cdot\frac{(\beta-1)\cdot(\delta-1)\cdot w-\left[1+(\beta-1)\cdot\delta\right]\cdot(\beta-1)\cdot\tau}{(w+\tau)^{\beta+1}}$$
$$+(\delta-1)\cdot\frac{Q^*(X_0, \tau)-Q_0}{r}.$$

Applying $Q^*(X_0, \tau)$ and $K\left[Q^*(X_0, \tau)\right]$, as captured by(10) and (18), and simplifying yields:

(39)
$$G(Q_0, X_0, \tau)=\beta\cdot\frac{(\delta-1)\cdot\alpha\cdot w-(\delta-\delta\cdot\alpha+\alpha)\cdot\tau}{(\alpha\cdot\beta-1)\cdot(w+\tau)}\cdot\frac{Q^*(X_0, \tau)}{r}-(\delta-1)\cdot\frac{Q_0}{r}.$$

The optimal tax rate can be found and characterized via (39), as the following *Proposition 1* establishes:

Proposition 1: For any X_0 and Q_0:

(a) $W^{-}(Q_0, X_0, \tau)$ is an inverse u-shaped function of τ, peaking at the level of τ for which $G(Q_0, X_0, \tau) = 0$,

(b) The level of τ that maximizes $W^{-}(Q_0, X_0, \tau)$ is in the relevant range of this case, namely, $\tau < \tau^{*}(Q_0, X_0)$, if and only if X_0 and Q_0 satisfy $P_0 > P^{**}$.

Proof: In Appendix B. □

3.3.3 The optimal tax – a unified analysis

This sub-section unifies, via *Proposition 2*, the results of subsection 3.3.2, which has focused on the range $\tau < \tau^{*}(Q_0, X_0)$, and the results of subsection 3.3.1, which has focused on the range where $\tau \geq \tau^{*}(Q_0, X_0)$, and presents thus a unified look on the optimal tax rate.

Proposition 2: If Q_0 and X_0 are such that $P_0 < P^{**}$ then the optimal tax rate is located within the range $\tau \geq \tau^{*}(Q_0, X_0)$ and captured by (33).

Otherwise, the optimal tax rate is located within the range $\tau < \tau^{*}(Q_0, X_0)$ and can be found as the single root of $G(Q_0, X_0, \tau) = 0$, where $G(Q_0, X_0, \tau)$ is captured by (39).

Proof: By construction, $W^{-}(Q_0, X_0, \tau)$ and $W(Q_0, X_0, \tau)$ meet one another when $\tau = \tau^{*}(Q_0, X_0)$, as can be verified by (9), (10) and (36).

In the case where $P_0 < P^{**}$ the meeting point is characterized by both $W^{-}{}_{\tau}(Q_0, X_0, \tau) < 0$ and $W_{\tau}(Q_0, X_0, \tau) < 0$ implying that the optimal tax rate is to the left of this point, i.e., within the range $\tau < \tau^{*}(Q_0, X_0)$ and therefore, by *Proposition 1*, it is the root of $G(Q_0, X_0, \tau) = 0$.

The opposite happens in the case where $P_0 \geq P^{**}$. In this case the meeting point is characterized by both $W^{-}{}_{\tau}(Q_0, X_0, \tau) > 0$ and $W_{\tau}(Q_0, X_0, \tau) > 0$, implying that the optimal tax rate is to the right of this point, i.e., within the range $\tau \geq \tau^{*}(Q_0, X_0)$ and therefore, by the analysis in sub-section 3.3.1, it is captured by (33). □

4. The effect of uncertainty on the optimal tax

The purpose of this section is to show that the uncertainty in the economic environment, as captured by the parameter σ^2, raises the optimal tax rate τ^{opt}. We start the analysis with case where $P_0 < P^{**}$ and the optimal tax is given by (33). In this case σ^2 affects τ^{opt} only via β and it is immediate to notice that $\frac{d\tau^{opt}}{d\beta} < 0$. Applying β for x in (4) leads via implicit differentiation conducted in Appendix A, to $\frac{d\beta}{d\sigma^2} < 0$ and therefore asserts that in this case τ^{opt} is an increasing function of σ^2.

We now turn to the complementary case where $P_0 \geq P^{**}$. To analyze this case it is useful to apply (39) in the equation $G(Q_0, X_0, \tau) = 0$ from which the optimal tax, τ^{opt}, is found, and to present resulting equation as follows:

(40) $$G^*\left(\tau^{opt}, \beta\right) = M\left(\tau^{opt}\right) \cdot N(\beta) - (\delta - 1) \cdot \frac{Q_0}{r} = 0\,,$$

where:

(41) $$N(\beta) \equiv \frac{\beta}{\alpha \cdot \beta - 1} \cdot \frac{Q^*(X_t, \tau)}{r}\,,$$

(42) $$M\left(\tau^{opt}\right) \equiv \frac{(\delta - 1) \cdot \alpha \cdot w - (\delta - \delta \cdot \alpha + \alpha) \cdot \tau^{opt}}{w + \tau^{opt}}\,.$$

From the assumption that $\alpha \cdot \beta - 1 > 0$, together with (1), it follows that $N(\beta)$, as captured by (41), is always positive. This, together with (40), asserts that $M\left(\tau^{opt}\right)$ is always positive too. Also note from the definition of $\hat{\beta}$ that:

(43) $$\frac{d\hat{\beta}}{d\beta} = \frac{-1}{(\beta - 1)^2}\,.$$

Differentiating (10), applying (43), and simplifying yields:

(44) $$\frac{dQ^*(X_t, \tau)}{d\beta} = \frac{Q^*(X_t, \tau)}{\alpha \cdot \beta \cdot (\beta - 1)}$$

Differentiating $N(\beta)$, as captured by (41), and using (43) and (44), yields:

(45) $$N'(\beta) = \frac{\alpha - 1}{\alpha \cdot (\beta - 1)} \cdot \frac{Q^*(X_t, \tau)}{r \cdot (\alpha \cdot \beta - 1)} < 0$$

Based on that:

(46) $$G^*{}_{\beta}(\tau^{opt}, \beta) = M(\tau^{opt}) \cdot N'(\beta) < 0 .$$

Appendix B establishes that $G^*{}_{\tau^{opt}}(\tau^{opt}, \beta)$.[3] This leads to:

(47) $$\frac{d\tau^{opt}}{d\beta} = -\frac{G^*{}_{\beta}(\tau^{opt}, \beta)}{G^*{}_{\tau^{opt}}(\tau^{opt}, \beta)} < 0 .$$

Then, from $\frac{d\beta}{d\sigma^2} < 0$ it follows that τ^{opt} is an increasing function of σ^2 also in this case where initial values of X and Q lead to $P_0 \geq P^{**}$.

[3]Specifically, it is established there via equation (B.4).

5. Conclusion

In this study I have presented the problem of a government that searches for the welfare maximizing tax rate, where the tax is levied on a private good in order to finance the supply of a public good. The novelty in the analysis is that the private good is modelled with the standard features of the literature about investment under uncertainty, namely that the profitability firms face is a stochastic process, that production requires an initial irreversible investment, and that firms can chose the timing for when to invest and start producing. The analysis leads to a solution for the optimal tax rate and to a characterization of the market dynamics under the optimal choice of the government. A result of particular importance is that the higher the uncertainty in the demand swings over time the higher the optimal tax rate.

Another comparative static result is that the optimal tax rate is increasing in the weight that the public good captures in the overall welfare function. This is a rather intuitive result, almost trivial, and therefore was not highlighted in the analysis. On the other hand, the results regarding the role of demand elasticity plays in determining the optimal tax rate, are rather ambiguous, in contrast to results from static modelling. On the one hand, demand elasticity should be sufficiently large, at leas at very small quantities, to ensure that welfare is finite (under the standard assumption that the demand curve does not cross the axis in the price-quantity plane). On the other hand, the analysis reveals that demand elasticity also cannot be too elastic, because if it will then the process in which firms enter when demand is sufficiently high will not delay subsequent investment enough, leading to the non-plausible result of an infinitesimally large tax collection. These different effects of the demand elasticity translate to an ambiguous effect on the optimal tax rate.

While the issue of the optimal finance of public goods has been extensively studied over the years, it is important to study it further as in recent years, prices, and in particular housing prices, have soared and public cries for the government to lower taxes are often heard. Thus, if changes in tax rates are to be considered, then, as the results here show, the role of uncertainty should be considered as well.

References

Aronsson, T. and Blomquist, S., 2003. Optimal Taxation, Global Externalities and Labor Mobility. *Journal of Public Economics*, 87: 2749-2764.

Barbosa, D., Carvalho, V.M. and Pereira, P.J., 2016. Public Stimulus for Private Investment: An Extended Real Options Model. *Economic Modelling*, 52: 742-748.

Cremer, H. and Gahvari, F., 1995. Uncertainty, Optimal Taxation and the Direct Versus Indirect Tax Controversy. *The Economic Journal*, 105: 1165-1179.

Di Corato, L., 2016. Investment Stimuli under Government Present-Biased Time Preferences. *Journal of Economics*, 119: 101-111.

Di Corato, L. and Maoz, Y.D., 2023. Externality Control and Endogenous Market Structure under Uncertainty: The price vs. Quantity Dilemma. *Journal of Economic Dynamics and Control*, 150, Article 104640.

Dixit, A., 1989. Entry and Exit Decisions under Uncertainty. *Journal of political Economy*, 97: 620-638.

Dixit, Avinash K., and Robert S. Pindyck. 1994. *Investment under uncertainty*. Princeton university press, 1994.

McDonald, R. and Siegel, D., 1986. The Value of Waiting to Invest. *The Quarterly Journal of Economics*, 101: 707-727.

Pigou, Arthur Cecil. 1947. *A Study in Public Finance*, 3rd edition. London: Macmillan.

Pindyck, R.S., 2000. Irreversibilities and the Timing of Environmental Policy. *Resource and Energy Economics*, 22: 233-259.

Ramsey, Frank P., 1928. "A Contribution to the Theory of Taxation." *The Economic Journal*, 37: 47-61.

Schwartz, E.S. and Trigeorgis, L. eds., 2004. *Real Options and Investment under Uncertainty: Classical Readings and Recent Contributions*. MIT press.

Trigeorgis, L., 1995. *Real Options in Capital Investment: Models, Strategies, and Applications*. Bloomsbury Publishing USA.

Varian, H.R., 1980. Redistributive Taxation as Social Insurance. *Journal of Public Economics*, 14: 49-68.

Appendix A – The value of an active firm

This appendix presents the derivation of the value function $V(X,Q)$, as captured by (3). It does so by following the analysis in Dixit (1989). The starting point of the analysis is noting that by its definition, $V(Q,X)$ satisfies:

(A.1) $$V(Q_0,X_0) = E_{X_0}\left[\int_0^\infty [X_t \cdot f(Q_t) - w] \cdot e^{-rt} \cdot dt\right].$$

(A.1) leads to the following Bellman equation for time instants in which the entry threshold is not reached and Q is unchanged:

$$V(Q_t, X_t) = [X_t \cdot f(Q_t) - w] \cdot dt + \frac{1}{1 + r \cdot dt} \cdot E[V(Q_t, X_{t+dt})]. \tag{A.2}$$

Multiplying by $1 + r \cdot dt$, and rearranging, yield:

$$r \cdot dt \cdot V(Q, X) = [X \cdot f(Q) - w] \cdot dt \cdot (1 + r \cdot dt) + E[dV(Q, X)], \tag{A.3}$$

where $dV(Q_t, X_t) = V(Q_t, X_{t+dt}) - V(Q_t, X_t)$, as we look at a time in which Q is not changed. By Itô's lemma,

$$E[dV(Q, X)] = \left[\frac{1}{2} \cdot \sigma^2 \cdot X^2 \cdot V_{XX}(Q, X) + \mu \cdot X \cdot V_X(Q, X)\right] \cdot dt, \tag{A.4}$$

where time indexes are omitted from here on for notational convenience. Substituting (A.4) into (A.3), dividing by dt, taking the limit $dt \to 0$, and rearranging, yields:

$$\frac{1}{2} \cdot \sigma^2 \cdot X^2 \cdot V_{XX}(Q, X) + \mu \cdot X \cdot V_X(Q, X) - r \cdot V(Q, X) + X \cdot f(Q) - w = 0. \tag{A.5}$$

Trying a solution of the type X^b for the homogenous part of (A.5) and a linear form as a particular solution to the entire equation yields

(A.6) $$V(Q,X) = Z(Q) \cdot X^{\alpha} + Y(Q) \cdot X^{\beta} + \frac{X \cdot f(Q)}{r-\mu} - \frac{w}{r} = 0,$$

where $Z(Q)$ and $Y(Q)$ are to be found later via additional conditions, and $\alpha < 0$ and $\beta > 1$ solve the quadratic

(A.7) $$\frac{1}{2} \cdot \sigma^2 \cdot x \cdot (x-1) + \mu \cdot x - r = 0.$$

Applying $x = 0$ and then $x = 1$, and bearing in mind that $r > \mu$ asserts that (A.7) has two roots, one of them negative and the other exceeds 1.

By the standard properties of a geometric Brownian Motions, it follows that[4]:

(A.8) $$E_{X_0=X}\left[\int_0^{\infty} X_t \cdot e^{-r \cdot t} dt\right] = \frac{X}{r-\mu}.$$

Eq. (A.8) implies that the term $\frac{X \cdot f(Q)}{r-\mu} - \frac{w}{r}$ in (A.6) represents the expected value of the flow of profits if Q remains at its current level forever. The two other terms in (A.6) therefore represent how expected future changes in Q affect the value of the firm.

As a geometric Brownian motion, when X goes to 0 the probability of its ever hitting $X^*(Q) > 0$, and thus of an increase in Q, tends to zero. Therefore

[4] Dixit and Pindyck (1994, page 72) prove (A.6).

(A.9) $$\lim_{X\to 0}\left(Z(Q)\cdot X^{\alpha}+Y(Q)\cdot X^{\beta}\right)=0.$$

Because $\alpha<0$, (A.9) implies $Z(Q)=0$. Substituting into (A.6) then gives Eq. (6) in the text.

Finally, from implicit differentiation of (A.7), evaluated at $x=\beta$, it follows that:

(A.10) $$\frac{d\beta}{d\sigma^2}=-\frac{\frac{1}{2}\cdot\beta\cdot(\beta-1)}{\frac{1}{2}\cdot\sigma^2\cdot(2\cdot\beta-1)+\mu}=-\frac{\frac{1}{2}\cdot\beta\cdot(\beta-1)}{\frac{1}{2}\cdot\sigma^2\cdot\beta+\frac{1}{2}\cdot\sigma^2\cdot(\beta-1)+\mu}$$

$$=-\frac{\frac{1}{2}\cdot\beta\cdot(\beta-1)}{\frac{1}{2}\cdot\sigma^2\cdot\beta+\frac{r}{\beta}}<0,$$

where the last equality follows from (A.7), and the inequality follows from $\beta>1$.

Appendix B – the proof of *Proposition 1*

This appendix presents the proof of *Proposition 1*. The first part of the proposition states that $W^{-}(Q_0,X_0,\tau)$ is an inverse u-shaped function of τ. To establish that, note from the analysis in sub-section 3.3.2 that $G(Q_0,X_0,\tau)$ is the derivative of $W^{-}(Q_0,X_0,\tau)$, and that, based on (39), it satisfies:

(B.1) $$G(Q_0,X_0,0)=\frac{\delta-1}{r}\cdot\left(\frac{1}{\alpha\cdot\beta-1}+1\right)\cdot\left[Q^{*}(X_0,\tau)-Q_0\right]>0,$$

where the inequality follows from $\delta > 1$, $\alpha \cdot \beta > 1$, and also from $Q^*(X_0, \tau) > Q_0$ which follows from $Q^*(X_0, \tau)$ being the outcome of immediate entry of firms that occurs in this case. Another property of $G(Q_0, X_0, \tau)$ that follows from (39) is:

(B.2) $$\lim_{\tau \to \infty} G(Q_0, X_0, \tau) = -(\delta - 1) \cdot \frac{Q_t}{r} < 0.$$

(B.1) and (B.2) prove existence, via continuity. To prove uniqueness, first notice from (10), that:

(B.3) $$\frac{dQ^*(X_0, \tau)}{d\tau} = -\frac{Q^*(X_t, \tau)}{r \cdot \alpha \cdot (w + \tau)}$$

Differentiating (39) with respect to τ, applying (B.3) and simplifying, yields:

(B.4) $$G_\tau(Q_0, X_0, \tau) = -\frac{\beta \cdot w \cdot \delta}{(\alpha \cdot \beta - 1) \cdot (w + \tau)^2} \cdot \frac{Q^*(X_0, \tau)}{r}$$

$$- \beta \cdot \frac{(\delta - 1) \cdot \alpha \cdot w - (\delta - \delta \cdot \alpha + \alpha) \cdot \tau}{(\alpha \cdot \beta - 1) \cdot (w + \tau)} \cdot \frac{Q^*(X_t, \tau)}{r \cdot \alpha \cdot (w + \tau)}$$

$$= -\frac{\beta \cdot w \cdot \delta}{(\alpha \cdot \beta - 1) \cdot (w + \tau)^2} \cdot \frac{Q^*(X_0, \tau)}{r} - \frac{\beta \cdot Q_0}{r \cdot \alpha \cdot (w + \tau)} < 0,$$

where the 2nd equality follows from (39). This proves uniqueness and completes the proof of part (a) of the proposition. To prove part (b) of the proposition note that by definitions of Q^* and τ^*, and therefore by (9) and (10):

(B.5) $$Q^*\left[X_0,\ \tau^*(Q_0, X_0)\right] = Q_0$$

Applying (9) and (B.5) in (39), and simplifying, yields:

(B.6) $$G\left[Q_0, X_0, \tau^*(Q_0, X_0)\right] = \left(\frac{\beta \cdot \delta \cdot w}{\frac{r}{r-\mu} \cdot \frac{X_0}{\hat{\beta} \cdot Q_0^{\alpha}}} - \beta \cdot \delta + \delta - 1 \right) \cdot \frac{Q_0}{(\alpha \cdot \beta - 1) \cdot r}$$

$$= \frac{(\beta - 1) \cdot \delta + \delta + 1}{(\alpha \cdot \beta - 1) \cdot r} \cdot \left(\frac{P^{**}}{P_0} - 1 \right) \cdot Q_0,$$

where the second equality follows from (1) and (35). From (B.6) it follows that if and only if X_0 and Q_0 satisfy $P_0 > P^{**}$ then $G\left[Q_0, X_0, \tau^*(Q_0, X_0)\right] < 0$, implying that the level of τ that maximizes $W^-(Q_0, X_0, \tau)$ satisfies $\tau < \tau^*(Q_0, X_0)$. This establishes part (b) of the proposition.