Contents

Submitted to *Operations Research*
manuscript (Please, provide the manuscript number!)

Real options under ambiguity with rare events

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We consider a real-options problem in which the underlying project value follows a geometric or exponential Lévy process, capturing rare events besides continuous fluctuations. Such rare events lead to ambiguity because of inconclusive empirical data or market incompleteness. We use ambiguity theory —leveraging the notion of variational preferences and g-expectations— to pin down for the general case a pricing kernel under which to value real options and derive the firm's optimal real-option exercise strategy under this pricing kernel. We also provide sufficient conditions for the optimality of a threshold policy in the general case. For the specialized case with multi-priors preferences, we obtain explicit expressions for the optimal investment threshold, expected investment time, and value function and prove comparative statics to assess analytically the effect of small jumps on these. Closed-form expressions are not readily available for multipliers preferences, but we provide approximate solutions for the cases with negligible and deep ambiguity. Rare events, which are priced under ambiguity aversion, generally lead to a higher investment threshold, delayed investment and higher option value.

Key words: Real options, ambiguity, optimal stopping, variational inequalities.

1. Introduction

Recent political and socioeconomic developments (e.g., the unexpected vote for Brexit, the Covid-19 pandemic, war in Ukraine) underscore the impact rare ~~or unexpected~~ events can have on firm decisions. Following Merton's (1976) pioneering work, ~~introduced pure jump processes to model rare events in option pricing theory. A~~ a large class of jump-diffusion models (also known as Lévy processes) (e.g., Duffie et al., 2000; Kou, 2002) have ~~since~~ been used in the literature on asset pricing. These models assume that an asset's price dynamics is driven by two fundamental processes, a Brownian motion implying continuous shocks and a pure jump process implying rare events. Empirical evidence shows that such models fit well with financial time series, with the discontinuous rare events explaining more of the underlying risk in the economy (Ornthanalai, 2014).

A key issue in finance is to assess the fair price of future cashflows by determining an appropriate pricing kernel (e.g., via a constant discount rate). This issue is also fundamental for an operational perspective because it influences the firm's investment decisions under uncertainty, e.g., to launch a new project (Dixit and Pindyck, 1994; Trigeorgis, 1996). If the decision maker can readily identify the factors that drive a project's value (e.g., commodity prices, demand patterns), the time series of these factors' fluctuations ("empirical data") can be quite useful ~~for valuation purposes~~ (see Ross, 1976). In many cases, the decision maker is able to identify traded financial securities or market indexes that are closely correlated with the project's value. The notion that real investment projects can be valued using principles rooted in (financial) asset pricing theory is well accepted under the assumption that the project's prospects are perfectly spanned by traded financial instruments (see, e.g., Constantinides, 1978). However, if a project's value is subject to both continuous fluctuations and rare events, then "ambiguity" ("Knightian uncertainty") may arise for several reasons:

a. Rare events introduce discontinuous shocks, e.g., due to paradigm shifts in technologies, market regulation, or industry structure. Because such events are rare, statistical models for their distribution and dynamics are prone to error, such that several models may well be consistent with the observed empirical data. As a result, decision makers cannot confidently identify one distribution ~~model~~ among such multiple models (e.g., Hansen and Sargent, 2001).

b. When benchmarking against traded financial instruments, inferring the market price of risk or the equivalent martingale measure (EMM) from financial time series is key. If the market is complete, then there is a unique market price of risk (or EMM) which can be used to price an American option allowing the discounting of cashflows at the riskfree rate (Bensoussan, 1984; Karatzas, 1988). However, under market incompleteness (which is typical in markets exhibiting rare events besides continuous fluctuations), there are multiple market prices of risk (or EMMs) consistent with the financial time series data (Riedel, 2009; Thijssen, 2011). Under such ambiguity, there again is a range of models that are consistent with the empirical data: investors may fail to agree on a common prior (see Burzoni et al., 2021). Further, the optimal exercise for real options as well as the price of these options depend on the manager's subjective ambiguity preferences for a particular statistical model.

In this paper, we consider an investment problem in the spirit of McDonald and Siegel (1986) in which the firm decides about the launch of a new project whose value follows a geometric or exponential Lévy process. Samuelson (1965), Merton (1976), and Kou (2002) examine different ~~special cases and~~ variants of such processes. Because of ambiguity, the investment problem embeds two subproblems, one of optimal stopping (for a given choice of a statistical model) and another related to the multiplicity of statistical models that fit the empirical data. We reduce the set of such models by using notions from ambiguity theory, focusing first on a general theory assuming the decision maker has variational preferences (see Maccheroni et al., 2006b,a) and then constructing the optimal exercise policy assuming she has multiple-priors (see Gilboa and Schmeidler, 1989) or multipliers preferences (see Hansen and Sargent, 2001). This approach essentially consists of formulating the alternative models in terms of an ambiguity-adjusted drift, a well-established methodology aligned with the standard treatment of 'outside uncertainty,' model mis-specification, and model ambiguity (see Hansen, 2014). Within this reduced set of models, we show that the firm's optimal investment strategy is unique and characterized by an investment threshold. The value function, which prices in the effect of rare events onto the firm's cashflows and onto its business decisions, is also unique. We express the continuation set, the value function, and the expected stopping time for the geometric Lévy process explicitly in case of multiple-priors preferences. Under assumptions leading to nice financial interpretations (comparable to those in Dixit and Pindyck (1994) and McDonald and Siegel (1986)), we establish that a firm facing rare events besides continuous fluctuations generally faces a higher investment threshold and is thus less prone to investing, its deferral option being more valuable. These results are robust to the choice of arrival rates, jump size distributions and their (positive or negative) contribution to the value process dynamics. These results are also robust to alternative ambiguity preferences such as multipliers preferences. To the best of our knowledge, ours is the first study to prove such results analytically.

2. Literature review

This paper is at the interface of the literatures on Lévy processes and ambiguity theory.

Lévy processes

The geometric Brownian motion—used by, e.g., Samuelson (1965), Black and Scholes (1973), Merton (1973), and McDonald and Siegel (1986)—does not allow for rare events. Merton (1976) introduces Poisson-type rare events in the asset's log-return dynamics assuming that jumps occur at a constant arrival rate and have normally distributed sizes. Bates (1996) extends this geometric jump model to allow for a square-root variance process, while Kou (2002) assumes a double exponential (or Laplace) distribution for the jump size. Duffie, Pan, and Singleton (2000) synthesize the earlier literature via a larger class of geometric Lévy processes in which the drift, diffusion and jump intensity are all affine in the state variable. Empirical evidence suggests such models with jumps exhibit higher skewness and leptokurtosis consistent with observed financial times series and can better explain empirical phenomena such as the "volatility smile" (Duffie et al., 2000; Kou, 2002; Ornthanalai, 2014). For tractability, we focus on one-dimensional processes—setting aside stochastic volatility—and benchmark against the models of Samuelson (1965), Merton (1976), and Kou (2002) which are well-established in the literature.

In the presence of discontinuous rare events, market incompleteness becomes an issue as it leads to multiple EMMs. Merton (1976) assumed that rare events are not systematic (i.e., are firm-specific) and hence command no risk premium, which allows selecting one specific EMM among a larger set. However, if rare events embed systematic risk as the recent experiences with Brexit, Covid 19, and the War in Ukraine suggest, then the key assumption made by Merton (1976, pp. 132-134) in order to price a (European) option written on an underlying asset following (his simple variant of) a geometric Lévy process does not hold. Duffie et al. (2000, Section 3.1) and Kou (2002, Section 6) make less restrictive choices that allow the selection of an arbitrary EMM. Instead of considering a large set of EMMs (within which one makes an arbitrary choice), we narrow down the selection within a subset of EMMs that are aligned with the agent's ambiguity preferences (considering various types of ambiguity preferences in turn).

Ambiguity theory

The literature introduced various types of ambiguity preferences: multiple priors (Gilboa and Schmeidler, 1989), κ-ignorance (Chen and Epstein, 2002), multiplier preferences (Hansen and Sargent, 2001), variational preferences (Maccheroni et al., 2006a,b; Petracou et al., 2022), and smooth preferences (Klibanoff et al., 2005). A key insight from this literature is that the fair pricing of financial claims in the presence of ambiguity involves a minimization over a given set of probability measures, so as to identify a "worse-case scenario" under which to price these claims. This procedure leads one to consider a nonlinear generalization of the concept of conditional expectation (Burzoni et al., 2021)—typically a g-expectation (Pardoux and Peng, 1992). Our paper focus on variational preferences, which encompass both Gilboa and Schmeidler's (1989) multiple-priors and Hansen and Sargent's (2001) multipliers preferences for which we construct the value function.

Our goal is to study a real options problem à la McDonald and Siegel (1986) for which the underlying project value is subject to a geometric Lévy process. Several authors have contributed in this direction already. For instance, Nishimura and Ozaki (2007) study a problem in which an agent has multiple-priors preferences à la Gilboa and Schmeidler and decides when to launch a project whose value follows a geometric Brownian motion. Trojanowska and Kort (2010) and Thijssen (2011) study a similar problem, but under κ-ignorance. Cheng and Riedel (2013) consider a more general problem in which the underlying process has continuous sample paths and the agent has variational preferences. Specifically, Cheng and Riedel (2013) characterize the value process as the smallest g-supermartingale that dominates the payoff process. For cases in which the underlying asset follows a diffusion, Cheng and Riedel (2013) identify a variational inequality (hereafter VI) and proves that, if a function solves this VI (in the ~~weak or~~ distributional sense), then it coincides with the value function of optimal stopping. The authors also provide examples involving κ-ignorance. Chen et al. (2013) pursue similar objectives than Cheng and Riedel (2013) and leverage the techniques for solving optimal stopping problems in case of reflected backward stochastic differential equations (BSDE) introduced by El Karoui et al. (1997). Compared to Cheng and Riedel (2013) and Chen et al. (2013), our focus is on the impact of rare events alongside continuous fluctuations (by considering geometric Lévy processes). In that respect, our paper is closer to Quenez and Sulem (2014) and Dumitrescu et al. (2015) who study optimal stopping problems involving variational preferences and Lévy processes. Quenez and Sulem (2014) characterize the value process as the solution to a reflected BSDE, while Dumitrescu et al. (2015) prove that, in the Markovian case, the value function of optimal stopping solves a VI (in the viscosity sense) and establishes the uniqueness of such a (viscosity) solution. These papers do not fully study the effects of ambiguity on the optimal policy or on the value function (as they focus more on the characterization of the value function rather than its construction). The study of such effects, which have far-reaching consequences from the operations (research) viewpoint, is the main objective of this paper. We state key results in Quenez and Sulem (2014) and Dumitrescu et al. (2015) for the sake of completeness. We further provide closed-form expressions for the investment thresholds, value functions, and expected exercise time in case of Gilboa and Schmeidler's multiple-priors. We further derive useful comparative statics, which yield novel and interesting managerial insights: Rare events are show to lead to a higher investment threshold, a delayed investment and a higher real option value, but a concurrent high degree of ambiguity may reduce the investment threshold and the real option value of waiting. Under Hansen and Sargent's multiplier preferences, we provide approximate solutions for the case with negligible and deep ambiguity.

3. Project valuation under model uncertainty

3.1. Ambiguity about multiple uncertainty sources

Consider a firm contemplating launching a project whose value is subject to continuous fluctuations as well as rare events. On a measurable space $(\Omega, \mathcal{F})$, consider that continuous random fluctuations are modeled by a continuous process W, while rare events are modeled by a suitable random measure N. Let $\mathbb{F} := \{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ be the natural filtration associated with these two fundamental processes. The characterization of the project value dynamics requires the adoption of laws of probability for W and N, which is equivalent to transforming the measurable space $(\Omega, \mathcal{F})$ into a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and selecting the stochastic processes W and N such that they correspond, under the probability measure $\mathbb{P}$, to a Wiener process and a Poisson random measure, respectively. In particular, under the probability measure $\mathbb{P}$, the random measure N counts the number of jumps of size $\mathrm{d}Z_s \in U \subset \mathbb{R}$ occurring in the time interval $[0, t]$:

$$N(t, U) := \sum_{s:0 \le s \le t} \mathbf{1}_U(\mathrm{d}Z_s), \quad \text{where } \mathrm{d}Z_s = Z_s - Z_{s^-} \text{ and } \mathbf{1}_{\cdot} \text{ denotes the indicator function.}$$

The mapping $U \mapsto \mathbb{E}^{\mathbb{P}}[N(1, U)] =: \nu(U)$ defines the *Lévy measure*. This defines a norm given by $\|\nu\| = \int_{\mathbb{R}} \mathrm{d}\nu(z)$.

For the sake of generality, consider first the dynamics in Quenez and Sulem (2014) and Dumitrescu et al. (2015), with the underlying state process solving

$$X_t = x, \quad \text{and } \mathrm{d}X_s = \tilde{\mu}(X_s)\mathrm{d}s + \tilde{\sigma}(X_s)\mathrm{d}W_s + \int_{\mathbb{R}} \tilde{\gamma}(X_s, z)\bar{N}(\mathrm{d}s, \mathrm{d}z) \text{ for } s > t, \tag{1}$$

where $\tilde{\gamma}(X, z)$ captures the impact of the random jumps on the project value of size in the interval $[z, z + \mathrm{d}z]$ at time t, and $\bar{N}$ is the compensated Poisson measure. In our work, we focus (unless specified otherwise) on a standard model of a geometric Lévy process, with

$$\tilde{\mu}(X) = \mu X, \quad \tilde{\sigma}(X) = \sigma X \quad \text{and} \quad \tilde{\gamma}(X, z) = \gamma(z)X, \tag{2}$$

where μ and $\sigma > 0$ are constant parameters and $\gamma(\cdot)$ is a function such that $\gamma(z) \geq -1$ ν-almost everywhere ("ν-a.e."). This process subsumes Samuelson's (1965) geometric Brownian motion (obtained for $\gamma \equiv 0$ or $\nu \equiv 0$) and the jump-diffusion models introduced by Merton (1976), Kou (2002) and Øksendal and Sulem (2007). The assumption $\gamma(\cdot) \geq -1$ ensures, roughly speaking, that the worse impact of jumps on the project value dynamics is for this project value to vanish.

Given a common prior $\mathbb{P}$, it is standard to pin down a single EMM, say $\mathbb{Q}$, for the purpose of fairly pricing a contingent claim paying off an amount modeled as a suitable random variable ξ (see Harrison and Kreps, 1979). In the context of option pricing, the random variable ξ could be interpreted as the discounted value as of time t of receiving an amount $F(X_T)$ at time $T \geq t$ where X is the underlying security modeled in eq. (1) and $F : \mathbb{R}_+ \to \mathbb{R}$ is a known payoff function, i.e., $\xi = e^{-r(T-t)} F(X_T)$. Given this (and denoting by $\mathbb{E}^{\mathbb{Q}}[\cdot \mid \mathcal{F}_t]$ the conditional expectation under the measure $\mathbb{Q}$ with respect to $\mathcal{F}_t$), one then uses a formula given by

$$\xi \mapsto \mathcal{J}(\xi) := \mathbb{E}^{\mathbb{Q}}[\xi \mid \mathcal{F}_t] \text{ to price that contingent claim as at time } t \in [0, T]. \tag{3}$$

In the presence of rare events leading to ambiguity (i.e., multiple probability models for some underlying process), one cannot find a unique EMM, so that for non–replicable assets there are multiple pricing formulas of the form in eq. (3) depending on the choice of ~~prior~~ measure.

3.2. Characterizing variational preferences via backward stochastic differential equations

The dynamics of eq. (1) are based on the modeling assumption that the stochastic value process can be split into a mean behavior and random fluctuations (ideally of zero mean) around the mean behavior. These characteristics depend on the two fundamental processes W and N, and whether they can be modeled as mean-zero fluctuations around a mean behavior under a prior $\mathbb{P}$. However, there is no consensus among economic agents about a common prior $\mathbb{P}$, leading to ambiguity about the dynamics of eq. (1). Typically, several priors are compatible with past observed empirical data, which differ in the drift, diffusion term, jump arrival rate or jump magnitude. As a result, adopting an alternative probability measure $\mathbb{Q}$ for these fluctuations leads to variations in the mean behavior of the dynamics of eq. (1) under the corresponding measure.

Assume there is a reference prior $\mathbb{P}$ such that the relevant set consists only of the probability measures $\mathbb{Q}^\lambda$ that are absolutely continuous with respect to $\mathbb{P}$. One can associate to any of these probability measures a Radon-Nikodym derivative, namely

$$\frac{\mathrm{d}\mathbb{Q}^\lambda}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_T} = Z_T^\lambda, \text{ where } \mathrm{d}Z_t^\lambda = Z_t^\lambda\Big[\lambda_{W,t}\mathrm{d}W(t) + \int_{\mathbb{R}} \lambda_{J,t}(z)N(\mathrm{d}t,\mathrm{d}z)\Big] \text{ and } Z_0^\lambda = 1, \tag{4}$$

which is expressed in terms of a stochastic process $\lambda := (\lambda_W, \lambda_J) \in \mathbb{H}^2 \times \mathbb{H}_\nu^2$. (All relevant functional spaces are defined in Appendix A.) **It is possible that it is more common in finance to write a $-$ sign in the bracket in eq. (4)? For instance on page 156 of Dumas and Luciano (2017). (Sorry, I have limited access to my books, so I cannot find a more common finance references.) In finance, it is common for the market price of risk to be positive, which leads to a negative adjustment to the drift when we have a $-$ sign. If we have a $+$, it may mean that the market price of risk takes negative value? Again, I am not challenging the math, but rather the economic interpretation of the market price of risk. See also comment below.** The process λ is interpreted, in the context of finance or real options, as the market price of risk: the real-valued process λ_W captures the market price of risk from the continuous shocks, while the function-valued process λ_J captures the market price of risk from the rare events. Let $\mathbb{L}_{[t,T]}$ denote the set of market prices of risk λ characterizing the alternative probability measures $\mathbb{Q}^\lambda$ for ξ at time t. Applying Girsanov's Theorem, with use of the Radon-Nikodym derivative in eq. (4), leads us to consider a drift given by

$$\tilde{\mu}^\lambda(X_t) := \tilde{\mu}(X_t) + \Big[\tilde{\sigma}(X_t)\lambda_{W,t} + \int_{\mathbb{R}} \tilde{\gamma}(X_t,z)\lambda_{J,t}(z)\nu(\mathrm{d}z)\Big] \tag{5}$$

under the new measure $\mathbb{Q}^\lambda$.

If ambiguity results from inconclusive empirical data, as in the motivation (a) on page 5, then more than one choices of λ are compatible with the empirical data. If, on the other hand, the decision maker aims to benchmark the project's excess return (over the riskfree rate $r > 0$) to the excess return of a stock market index (following a state process of the same form as eq. (1)), spanning is imperfect as the Poisson-type jumps lead to market incompleteness (motivation (b) on page 5). Market incompleteness introduced by the jumps allows for an infinity of suitable choices for λ within the set $\mathbb{L}_{[t,T]}$. In particular, assuming that the underlying project value in eq. (1) with the choice of eq. (2) is spanned by a traded security, S, whose price dynamics takes a similar form (with μ_S, σ_S, and γ_S in lieu of μ, σ, and γ), market incompleteness on account of the jump process results in an infinity of EMMs, which by Girsanov transformation are related to the reference prior $\mathbb{P}$ through an exponential martingale transform of the form eq. (4), with possible choices of the drift process λ constrained by the following condition:

$$\sigma_S \lambda_W + \int_{\mathbb{R}} \gamma_S(z) \lambda_J(z) \nu(\mathrm{d}z) = \mu_S - r. \tag{6}$$

If jumps are absent (i.e., $\nu \equiv 0$, where the symbol "$\equiv$" signifies "is identically equal to") or inconsequential (i.e., $\gamma_S \equiv 0$), then eq. (6) admits a unique solution, $\lambda_W = [\mu_S - r]/\sigma_S$, the classic Sharpe ratio. **Again, I am not confident with the sign. If $\lambda_W = [\mu_S - r]/\sigma_S$, then the drift of the risky traded security becomes**

$$\tilde{\mu}^{\lambda}(S_t) = \mu_S S_t + \sigma_S S_t \frac{\mu_S - r}{\sigma_S} = \big(2\mu_S - r\big) S_t,$$

while I was expecting r as risk-neutral drift. If the + sign in eq. (5) is mathematically correct, but the market price of risk in the case with no or inconsequential jumps $\lambda_W = -[\mu_S - r]/\sigma_S$, we should correct the text, not necessarily the math. Further, could it be that eq. (6) should read

$$r = \mu_S + \sigma_S \lambda_W + \int_{\mathbb{R}} \gamma_S(z) \lambda_J(z) \nu(\mathrm{d}z) \iff -\sigma_S \lambda_W - \int_{\mathbb{R}} \gamma_S(z) \lambda_J(z) \nu(\mathrm{d}z) = \mu_S - r?$$

In the case of firm-specific jumps (and hence diversifiable rare-event risk), Merton (1976) argues that the European option is replicable at all times except the random times at which the jumps occur. Because the rare events are assumed diversifiable and not priced (i.e., $\lambda_J \equiv 0$), eq. (6) again yields the Sharpe ratio. In all other cases (i.e., if $\nu \not\equiv 0$, $\gamma_S \not\equiv 0$, and $\lambda_J \not\equiv 0$), eq. (6) admits an infinity of solutions, leaving the choice of market prices of risk $\lambda = \big(\lambda_W, \lambda_J(\cdot)\big)$ at the decision maker's discretion.

Ambiguity theory helps one identify a probability measure $\mathbb{Q}^\lambda$ that is consistent with the decision maker's ambiguity preferences by essentially introducing a minimization problem over the space $\mathbb{L}_{[t,T]}$ of equivalent probability measures. This calls for the use of the notion of *g–expectations*, a nonlinear generalization of the concept of conditional expectation that applies under ambiguity.

A widely–used family of g–expectations are the so–called *variational preferences* (see Maccheroni et al., 2006a,b). In this framework, an ambiguity–averse agent determines the fair price $\mathcal{E}_t(\xi,T)$ at time t of receiving an amount ξ payable at $T \geq t$ by solving the problem:

$$\mathcal{E}_t(\xi,T) = \inf_{\lambda\in\mathbb{L}_{[t,T]}} \Big\{\mathbb{E}^{\mathbb{Q}_\lambda}\left[\xi\middle|\mathcal{F}_t\right] + \zeta(t,\lambda,T)\Big\}, \quad \text{with} \quad \zeta(t,\lambda,T) = \mathbb{E}^{\mathbb{Q}_\lambda}\Big[\int_0^T G\big(s,\lambda_s\big)\mathrm{d}s\Big|\mathcal{F}_t\Big]. \tag{7}$$

The fair price $\mathcal{E}_t(\xi,T)$ in eq. (7) results from a worst–case valuation ("inf") among the relevant market prices of risk $\lambda \in \mathbb{L}_{[t,T]}$ for the payoff ξ. Not all models $\lambda \in \mathbb{L}_{[t,T]}$ are given equal consideration: certain models λ are more penalized by the penalty functional or "ambiguity index" ζ of eq. (7). The choice of ζ in eq. (7) is a characteristic of the agent's ambiguity preferences (see Maccheroni et al., 2006a,b). Given the penalty function G in eq. (7), one may define a function $g : [0,T]\times\mathbb{R}\times L^2_\nu \to \mathbb{R}$ by a Fenchel-Legendre transform

$$g\big(t,\Lambda_W,\Lambda_J\big) := \inf_{(\lambda_W,\lambda_J)\in\mathbb{R}\times L^2_\nu} \Big\{G\big(t,\lambda_W,\lambda_J\big) + \lambda_W\Lambda_W + \int_{\mathbb{R}} \lambda_J(z)\Lambda_J(z)\mathrm{d}\nu(z)\Big\}. \tag{8}$$

Following standard arguments in convex analysis, this transformation can be inverted to provide

$$G\big(t,\lambda_W,\lambda_J\big) := \sup_{(\Lambda_W,\Lambda_J)\in\mathbb{R}\times L^2_\nu} \Big\{g\big(t,\lambda_W,\lambda_J\big) - \lambda_W\Lambda_W - \int_{\mathbb{R}} \lambda_J(z)\Lambda_J(z)\mathrm{d}\nu(z)\Big\}. \tag{9}$$

(The problems (8) and (9) are static optimization problems taken over the space $\mathbb{R}\times L^2_\nu$ and not over the space of stochastic processes that take values on the latter space. When we need to emphasize that λ is considered as a stochastic process rather than as a static element, we will use the notation $\lambda_\cdot$.) We can be a little more concrete as to the definition of the set $\mathbb{L}_{[t,T]}$: it is the set of predictable processes λ such that $G(\cdot,\lambda)\in\mathbb{H}^2$. Because the fair price $\mathcal{E}_t(\xi,T)$ in eq. (7)

is itself a random variable depending on the information set $\mathcal{F}_t$ as of time t, one can consider a stochastic process $\{Y_t:\ t\in\mathbb{R}_+\}:=\{\mathcal{E}_t(\xi,T):\ t\in\mathbb{R}_+\}$ that is $\mathbb{F}$-adapted. This stochastic process can be expressed in terms of a backward stochastic differential equation (BSDE, see Pardoux and Peng, 1992; Dumitrescu et al., 2015):

Theorem 1 **(Theorem 5.2 in Quenez and Sulem, 2013)**. *If g is Lipschitz continuous and concave with respect to (Λ_W,Λ_J), and satisfies a monotonicity property for Λ_J, then the process (Y,Λ_W,Λ_J) solves a BSDE of the form*

$$\begin{aligned}-\mathrm{d}Y_t &= g\big(t,\Lambda_{W,t},\Lambda_{J,t}\big)\mathrm{d}t-\Lambda_{W,t}\mathrm{d}W_t-\int_{\mathbb{R}}\Lambda_{J,t}(z)N(\mathrm{d}t,\mathrm{d}z)\ \textit{for}\ t\in[0,T]\\ Y_T&=\xi.\end{aligned}\tag{10}$$

Moreover, there exists a market price of risk $\hat\lambda_\cdot=\big(\hat\lambda_{W,\cdot},\hat\lambda_{J,\cdot}\big)\in\mathbb{L}_{[t,T]}$ such that

$$G\big(t,\hat\lambda_{W,t},\hat\lambda_{J,t}\big)=g\big(t,\Lambda_{W,t},\Lambda_{J,t}\big)-\hat\lambda_{W,t}\Lambda_{W,t}-\int_{\mathbb{R}}\hat\lambda_{J,t}(z)\Lambda_{J,t}(z)\mathrm{d}\nu(z)\ \textit{a.s. for}\ t\in[0,T],$$

with the process $\hat\lambda_\cdot$ being optimal for (7).

The choice of G in eq. (7) uniquely characterizes the penalty functional ζ in eq. (7) and consequently the fair price $\mathcal{E}_t(\xi,T)$ (see Maccheroni et al., 2006a,b). Classical models include *multiple-priors preferences* (Gilboa and Schmeidler, 1989) and Hansen and Sargent's (2001) *multiplier preferences*. Following Theorem 1, one can compute a function g as the outcome of a static problem in eq. (8) and then solve the BSDE in eq. (10). By doing so, one obtains the fair prices Y of the contingent claim paying ξ at time T under ambiguity and determining *the* market price of risk λ consistent with the payoff ξ.

3.3. Real options problem under variational preferences – General results

Assume that the decision–maker has variational preferences of the form introduced in eq. (7) and that the project value process $\{\xi_s\}_s$ is given by

$$\xi_s=f(s,X_s)\ \text{for}\ t<s<T\ \text{and}\ \xi_T=f_T(X_T).\tag{11}$$

Then, the optimal investment rule obtains as the solution of the optimal stopping problem:

$$\sup_{\tau\in\mathbb{T}_{[t,T]}} \mathcal{E}_t(\xi_\tau,\tau) \quad \text{where} \quad \mathcal{E}_t(\xi_\tau,\tau):=\inf_{\lambda\in\mathbb{L}_{[t,T]}} \Big\{\mathbb{E}^{\mathbb{Q}_\lambda}\left[\xi_\tau\middle|\mathcal{F}_t\right]+\zeta(t,\lambda,\tau)\Big\}. \tag{12}$$

For the problem under consideration, a classical choice for the function f is

$$f(s,X_s)=e^{-rs}(X_s-K), \tag{13}$$

the net discounted project value, where the amount $K>0$ is interpreted as a fixed investment cost.

The fair price $\mathcal{E}_t(\xi,T)$ in eq. (12) results from solving two subproblems: one of optimal stopping explicit through the sup operator in eq. (12) and another subproblem embedded in the definition of the conditional g-expectations via the worse-case scenario (inf operator) in eq. (12). In eq. (12), the "sup" is interpreted as the essential supremum and $\mathbb{T}_{[t,T]}$ is the set of all stopping times with values in $[t,T]$. A stopping time $\hat{\tau}$ is optimal for the agent who wants to maximize the fair price $\mathcal{E}_t(\xi_\tau,\tau)$ by choosing the probability measure that corresponds to the worst-case scenario (see Corollary 6.2 in Quenez and Sulem, 2014). (The stated result holds for $\mathbb{L}_{[t,T]}$ compact, or for a bounded convex and closed subset of a separable Hilbert space.) (Note that the ~~optimal stopping~~ problem in eq. (12) has a minimax structure and can be interpreted as a saddle-point equilibrium. In fact, Theorem 6.1 in Quenez and Sulem (2014) guarantees that, for Lipschitz continuous g, a saddle point exists for problem (12), i.e., one may interchange the order of the (essential) sup and inf in problem (12), and there is a pair $(\hat{\tau},\hat{\lambda})$ for which the sup and the inf are respectively attained.)

In the general case allowing for non-Markov processes (see Cheng and Riedel, 2013 and Chen et al. (2013) for processes with continuous sample paths and Quenez and Sulem, 2014 for Lévy processes), the optimal stopping problem in eq. (12) can be solved by constraining the solution of the BSDE in (10) to dominate the payoff process $\{\xi_s; s\in[t,T]\}$ in eq. (11). From this Snell-envelope representation of the value process, Chen et al. (2013) (Theorem 10) and Quenez and Sulem (2014) (Theorem 4.1) deduce that the value process is decreasing in the ambiguity driver g and that the optimal stopping time is increasing pathwise (Chen et al., 2013, Theorem 10).

The specific problem considered in eq. (12) has a Markovian structure, so its solution can be obtained via a value function

$$u(t,x) = \sup_{\tau \in T_{[t,T]}} \mathcal{E}_t(\xi_\tau, \tau), \tag{14}$$

where (t,x) denote the initial conditions of the state equation (1). The principle of optimality also holds, so one can determine a dynamic programming equation that the value function satisfies. This equation is called a variational inequality (VI) in the context of optimal stopping (see Bensoussan and Lions, 1982). Dumitrescu et al. (2015) establish that the value function u is continuous and can be obtained as the viscosity solution of an appropriate VI. To state this result, we introduce the operators $\mathcal{A}, \mathcal{T}, \mathcal{B}$, and $\mathcal{G}$, defined on a suitable function space by:

$$\begin{aligned}
&\mathcal{A}u(t,x) := \frac{1}{2}\tilde{\sigma}^2(x)\frac{\partial^2 u}{\partial x^2}(t,x) + \tilde{\mu}(x)\frac{\partial u}{\partial x}(t,x)\\
&\mathcal{T}u(t,x) := \int_{\mathbb{R}} \Big\{ u\big(t, x+\tilde{\gamma}(x,z)\big) - u(t,x) - \frac{\partial u}{\partial x}(t,x)\tilde{\gamma}(x,z) \Big\} \nu(\mathrm{d}z),\\
&\mathcal{B}u(t,x)(\cdot) := u(t, x+\tilde{\gamma}(x,\cdot)) - u(t,x)\\
&\mathcal{G}u(t,x) := g\Big(t, \tilde{\sigma}(x)\tfrac{\partial u}{\partial x}(t,x), \mathcal{B}u(t,x)(\cdot)\Big),
\end{aligned} \tag{15}$$

where (see eq. (8))

$$g\big(t, \Lambda_W, \Lambda_J\big) := \inf_{(\lambda_W,\lambda_J)\in\mathbb{R}\times L^2_\nu} \Big\{ G(t,\lambda_W,\lambda_J) + \lambda_W \Lambda_W + \int_{\mathbb{R}} \lambda_J(z)\Lambda_J(z)\mathrm{d}\nu(z) \Big\}.$$

Theorem 2 **(see Theorem 3.1 in Dumitrescu et al., 2015)**. *The value function u in eq.* (14) *is a viscosity solution of the variational inequality*

$$\min\Big\{ u(t,x) - f(t,x); -\big(\tfrac{\partial}{\partial t} + \mathcal{A} + \mathcal{T} + \mathcal{G}\big)u(t,x) \Big\} = 0 \tag{16a}$$

$$u(T,x) = f_T(x), \tag{16b}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$.

(Dumitrescu et al. (2015) discuss a dynamic risk measure minimization problem: Their results are stated in a modified form for suitability with our variational utility maximization problem.) Several adjustments are noteworthy. If there are jumps (i.e., $\mathcal{T} \not\equiv 0$) but no ambiguity (i.e., $\mathcal{G} \equiv 0$), then the

VI (16a) simplifies to the evolutionary VI corresponding to an optimal stopping problem of a Lévy process without ambiguity adjustment (see, e.g., Øksendal and Sulem, 2007, Edition 3, Chapter 3). If there is ambiguity (i.e., $\mathcal{G} \not\equiv 0$), but no jumps (i.e., $\mathcal{T} \equiv 0$), then the VI (16a) simplifies to the one in Cheng and Riedel (2013, eq. (8)) with an operator $\mathcal{G}$ given by $\cancel{\mathcal{G}u(t,x) = g\big(t, \sigma(x)\frac{\partial u}{\partial x}(t,x)\big)}$ $\mathcal{G}u(t,x) = g\big(t, \tilde{\sigma}(x)\frac{\partial u}{\partial x}(t,x)\big)$. If there is no ambiguity (i.e., $\mathcal{G} \equiv 0$) and no jumps (i.e., $\mathcal{T} \equiv 0$), then eq. (16a) generates to the evolutionary VI corresponding to the optimal stopping problem of a diffusion (see, e.g., Bensoussan and Lions, 1982, Section 3.2).

In the spirit of McDonald and Siegel (1986) and Dixit and Pindyck (1994), we consider a perpetual American call option: the payoff is of the form in eq. (13), while the infinite horizon $T = \infty$ leads us to impose restrictions on the discount rate $r > 0$ and the dynamics of X. By time–homogeneity of the state equation, and assuming that the penalty function G is time–independent, it follows that the starting–time dependence on t becomes irrelevant in the sense that the value function in eq. (14) can be written as $u(t,x) = e^{-rt}v(x)$, with v given by

$$x \mapsto v(x) := \sup_{\tau \in \mathbb{T}} \mathcal{E}_0\Big(e^{-r\tau}(X_\tau - K), \tau\Big), \tag{17}$$

where $\mathbb{T}$ denotes the set of stopping times with values in $\mathbb{R}_+$. Letting τ go arbitrary to ∞ leads to a lower bound for $v(\cdot)$, so $v(\cdot) \geq 0$. Besides, it can be shown using standard arguments that $v(\cdot)$ is convex on its domain. **Is there an easy argument to show it? For instance, g-expectation are sublinear so**

$$\mathcal{E}_0\Big(e^{-r\tau}\big((\alpha x_1 + (1-\alpha)x_2 X^1_\tau - K), \tau\Big) \leq \alpha\mathcal{E}_0\Big(e^{-r\tau}\big(x_1 X^1_\tau - K), \tau\Big) + (1-\alpha)\mathcal{E}_0\Big(e^{-r\tau}\big(x_2 X^1_\tau - K), \tau\Big), \quad \forall \alpha \in [0,1].$$

Because $\sup(a+b) \leq \sup(a) + \sup(b)$**, we have**

$$v\left(\alpha x_1 + (1-\alpha)x_2\right) \leq \alpha v\big(x_1\big) + (1-\alpha)v\big(x_2\big).$$

It might be good to prove rigorously convexity as it would "motivate" the assumption of convexity we do later.

We look for solutions of eq. (16) in the form $u(t,x) = e^{-rt}v(x)$ using the specific model in eq. (2) (under the integrability condition $\lim_{t\to\infty} e^{-rt}v\left(X_t\right) = 0$). Doing so leads us to the following corollary of Theorem 2:

COROLLARY 1. *The value function in eq.* (17) *is a viscosity solution of the variational inequality*

$$\min\Big\{v(x)-(x-K);-\big(\mathcal{A}+\mathcal{T}-r\mathbb{I}+\mathcal{G}\big)v(x)\Big\}=0,\quad x>0,$$
$$\lim_{x\to 0^+}v(x)=0 \text{ and } \lim_{x\to\infty}v(x)/[x-K]=1 \tag{18}$$

where

$$\begin{aligned}
\mathcal{A}v(x)&:=\frac{1}{2}\sigma^2x^2\frac{\mathrm{d}^2v}{\mathrm{d}x^2}(x)+\mu x\frac{dv}{dx}(x)\\
\mathcal{T}v(x)&:=\int_{\mathbb{R}}\Big\{v\big(x+\gamma(z)x\big)-v(x)-\frac{\mathrm{d}v}{\mathrm{d}x}(x)\gamma(z)x\Big\}\nu(\mathrm{d}z),\\
\mathcal{B}v(x)(\cdot)&:=v(x+\gamma(\cdot)x)-v(x)\\
\mathcal{G}v(x)&:=g\Big(\sigma x\tfrac{\mathrm{d}v}{\mathrm{d}x}(x),\mathcal{B}v(x)(\cdot)\Big).
\end{aligned}\tag{19}$$

If there are neither jumps (i.e., $\mathcal{T}\equiv 0$) or ambiguity (i.e., $\mathcal{G}\equiv 0$), then the VI in eq. (18) degenerates to the one for McDonald and Siegel's (1986) problem.

One can infer from Corollary 1 that the optimal stopping time is $\inf\big\{t\geq 0\big|v(X_t)=X_t-K\big\}$. Yet, at this stage, one cannot readily establish whether the optimal strategy is of the threshold type or whether the stopping region is the union of disconnected sets. We now state sufficient conditions for the case with variational preferences under which the optimal stopping time for the problem in (17) is a **threshold policy**.

PROPOSITION 1 **(Threshold policy)**. *Assume that the PDE*

$$\begin{cases}(\mathcal{A}+\mathcal{T}-r\mathbb{I}+\mathcal{G})\,w(x)=0, & \forall x\in(0,\hat{x}),\\ \qquad\qquad w(x)=x-K, & \forall x\geq\hat{x}\end{cases}\quad \text{with } \lim_{x\to 0+}w(x)=0 \text{ and } w'(\hat{x})=1. \tag{20}$$

has a sufficiently smooth solution $w(\cdot)$. **Do we actually need to add $w'(\hat{x})=1$ as we said that $w\in C^1(\mathbb{R}_+)$?** *We make the parameter restrictions $g\leq 0$ and $-1\leq\gamma(\cdot)\leq 0$ ν-a.e. and assume that the solution $w(\cdot)$ of eq.* (20) *is positive and convex and satisfies the condition*

$$w\left(\frac{r+||v||}{r-\mu}K\right)>K\frac{\mu+||\nu||}{r-\mu}.$$

Then, the function $w(\cdot)$ is strictly increase in $(0,\infty)$ from 0 to ∞ with $0<w'(\cdot)\leq 1$ and coincides with the value function $v(\cdot)$ in eq. (17)*. Further, the optimal strategy is the first-hitting time given by* $\inf\big\{t\geq 0\big|X_t^x\geq\hat{x}\big\}$, *with $\hat{t}$ defined implicitly by eq.* (20)*.*

The sufficient conditions in Proposition 1 hold independently of the specific modeling of ambiguity preferences within the class of variational preferences. Besides, the standard smooth-fit conditions, the Proposition specifies other sufficient conditions. Clearly, if the solution $w(\cdot)$ of eq. (20) fails to be positive or convex, then $w(\cdot)$ cannot be a candidate for the value function in eq. (17) as one can establish the value function satisfies these properties.

4. Real options problem under multiple-priors preferences

We now consider the specialized case of *multiple–priors* preferences à la Gilboa and Schmeidler (1989) for which the penalty function is

$$G(\lambda_W, \lambda_J) = \begin{cases} 0, & \text{if } |\lambda_W| \leq \theta k_1 \text{ and } |\lambda_J(z)| \leq \theta k_2 \text{ for all } z \in \mathbb{R} \\ \infty, & \text{otherwise.} \end{cases} \tag{21}$$

The parameter $\theta \geq 0$ in eq. (21) models the degree of the agent's ambiguity aversion. In this case, the ~~solution of the~~ optimization problem in eq. (8) simplifies to

$$g(\Lambda_W, \Lambda_J) = -\theta\left(k_1|\Lambda_W| + k_2\int_{\mathbb{R}} |\Lambda_J(z)|\mathrm{d}\nu(z)\right), \quad \text{for} \quad \theta, k_1, k_2 > 0, \tag{22}$$

~~which translates to choosing~~ with the infimum attained at the boundaries, namely at $\lambda_W = -\theta k_1$ and $\lambda_J(\cdot) \equiv -\theta k_2$. In this case, the function g in eq. (22) turns out to be negative, a feature consistent with our assumption in Proposition 1.

4.1. Value function and optimal stopping policy

~~To ensure a nice financial interpretation and on account of convexity arguments,~~ We look for a convex solution of eq. (20), so we consider functions of the form $x \mapsto x^\beta$ for a suitable $\beta > 1$ in the continuation region. Given the expressions for the operators in eq. (19) for the case of multiple-priors, upon substituting the above ansatz in the equation, we are led to define β as a root of the function given by

$$\beta \mapsto h(\beta) := r - \frac{1}{2}\sigma^2\beta(\beta-1) - \mu\beta - \phi(\beta) + \theta\Big[k_1\sigma\beta + k_2\psi(\beta)\Big], \tag{23a}$$

where

$$\phi(\beta) := \int_{\mathbb{R}} \Big[(1+\gamma(z))^{\beta} - 1 - \beta\gamma(z)\Big] \mathrm{d}\nu(z) \quad \text{and} \quad \psi(\beta) := \int_{\mathbb{R}} \Big|(1+\gamma(z))^{\beta} - 1\Big| \mathrm{d}\nu(z). \tag{23b}$$

In the absence of jumps (as $\nu \equiv 0$ leads to $\phi \equiv \psi \equiv 0$), the function $h(\cdot)$ in eq. (23) becomes quadratic in β: it has a root $\beta > 1$ already given in eq. (37) in Nishimura and Ozaki (2007). If, further, ambiguity is irrelevant (i.e., $\theta \to 0$), $h(\cdot)$ simplifies to Dixit and Pindyck's (1994) "fundamental quadratic" function; provided $r > \mu$, this function $\beta \mapsto r - \frac{1}{2}\sigma^2\beta(\beta-1) - \mu\beta$ has a root denoted $\beta_0 > 1$ herein. We also consider the quantity

$$\beta_{cr} := \frac{r + \|\nu\|}{\mu + \|\nu\|} \tag{24}$$

and make the

Assumption 1 **(Assumption for multiple-priors optimal stopping problem)**. *Assume that* $-1 \le \gamma(\cdot) \le 0$ ν*-a.e.,* $\mu < r$ *and*

$$0 \le \theta < \theta_{cr} := \frac{\frac{1}{2}\sigma^2\beta_{cr}(\beta_{cr}-1) + \mu\beta_{cr} + \varphi(\beta_{cr}) - r}{k_1\sigma\beta_{cr} + k_2\psi(\beta_{cr})}.$$

If we agree with the new conditions for Proposition 1, Benoit suggests to put the following paragraph just below Proposition 1 instead of here and to slightly rephrase Assumption 1, so we don't repeat our assumptions too often. Because of the condition $\gamma(\cdot) \le 0$ in Assumption 1, the underlying process cannot jump from the continuation set to the interior of the stopping set. Under the stated assumption, the process has a tendency to revert back to the interior of the continuation set; the process may only leave the continuation set because of continuous fluctuations. The assumption of negative jumps is not unduly restrictive, as it captures fairly well most market-wide jumps (e.g., due to stock market crashes, economic recessions, or physical events like hurricanes or the pandemic) and industry or firm-specific shocks from introduction of new technologies or substitute products by competition.

Why do we state the lemma before Assumption 1? I believe, to the contrary, the assumption comes from the Lemma. Can we state the lemma just before "We also consider the quantity $\beta_{\mathrm{cr}} \ldots$"? In order to prove the main result of this paper, we need to establish the following lemma first:

LEMMA 1 **(Root of $h(\cdot)$ in eq. (23)).** *Let*

$$\bar{\theta}(M) := -\frac{r - \frac{1}{2}\sigma^2 M(M-1) - \mu M - \varphi(M)}{k_1 \sigma M + k_2 \psi(M)}. \tag{25}$$

$$\bar{\theta}(M) := \frac{\frac{1}{2}\sigma^2 M(M-1) + \mu M + \varphi(M) - r}{k_1 \sigma M + k_2 \psi(M)}. \tag{26}$$

I suggest re-writing eq. (26) as in the second equation, so the parallel with the inequality in Assumption 1 appears more clearly. *Then, if $\theta < \bar{\theta}(M)$, $h(\cdot)$ has a root β in $(1, M)$~~; otherwise, i.e., if $\theta \geq \bar{\theta}(M)$, there is no root in $(1, M)$~~, but not otherwise.*

~~The proof of the Lemma is provided in Appendix C.~~ We now state ~~the main~~ a key result of ~~the~~ our paper:

THEOREM 3 **(Optimal stopping problem for multiple-priors preferences in eq. (21).).** *Under Assumption 1, the VI in* (18) *Shall we talk about the VI's solution or the value function? The value function may sound less technical to non-VI-fans admits a solution of the form*

$$w(x) = \begin{cases} \frac{K}{\beta-1}\left(\frac{x}{\hat{x}}\right)^{\beta}, & 0 < x < \hat{x}, \\ x - K, & x \geq \hat{x}, \end{cases} \qquad \textit{where} \quad \hat{x} = \frac{\beta}{\beta-1}K, \tag{27}$$

with β defined as the unique root of the function $h(\cdot)$ in eq. (23) *in the interval* $(1, \beta_{cr})$. *The optimal stopping time is $\hat{\tau}(x) := \inf\left\{t \geq 0 \middle| X_t \geq \hat{x}\right\}$.*

~~The proof of the Theorem is provided in Appendix D.~~

The functional form of the value function in eq. (27) is reminiscent of the solution of McDonald and Siegel's (1986) problem. The difference, however, lies in the value of β, which obtains in our case by solving the equation $h(\beta) = 0$, while it obtains from solving a quadratic equation for the former. Provided the degree of ambiguity θ is within the range $[0, \theta_{\mathrm{cr}})$, the optimal stopping rule is a threshold policy whereby the optimal stopping time is the first time that the project value exceeds the project value exceeds the cut-off value in eq. (27).

REMARK 1. Here we assume that $\mu < r$, which is a standard condition for the existence/finiteness of the value function in the non–ambiguous case. Note that in the presence of ambiguity, one may still admit solutions for $\mu > r$, since the effect of ambiguity is to lower the effective drift to $\mu^{\hat{\lambda}} := \mu - \theta\,(k_1 + k_2\|\nu\|)$, hence allowing the above condition for the effective drift as long as θ is sufficiently large.

The explicit nature of this solution allows us to study the effects of ambiguity on the value function and optimal stopping rule:

COROLLARY 2 **(Monotonicity in θ.)**. *Under Assumption 1, the value function w and the investment threshold $\hat{x}$ decrease with the ~~ambiguity parameter~~ degree of ambiguity aversion θ.*

~~See Appendix E for the proof.~~ The result in Corollary 2 is consistent with Theorem 4.1 in Quenez and Sulem (2014) and Theorem 10 in Chen et al. (2013) obtained for the general case based on the Snell envelope representation of the value process. It is intuitive that a larger degree of ambiguity aversion θ leads the firm to discount future cashflows more (or to reduce the ambiguity-adjusted drift more while keeping the discount rate constant). The effect on the optimal investment threshold, which may seem unexpected at first, is consistent with known results in real options analysis (see, e.g., Dixit and Pindyck, 1994, Figure 5.6, p. 157). Because a higher degree of ambiguity aversion leads the expected appreciation of the project value (in the sense of g-expectations) to fall, the opportunity cost of killing the real option is reduced. So, the firm sets a low investment threshold to benefit earlier from a flow of income generated by the project.

4.2. Expected investment time

We next turn our attention to the expected investment time. Under ambiguity, which involves adopting different probability measures for the process $\{X_t; t \geq 0\}$, ~~we need to clarify~~ the notion of 'expected value' is somewhat unclear. ~~To this end, we could~~ We may either consider the expectation under the physical measure $\mathbb{P}$, or under any alternative measure $\mathbb{Q}^{\lambda}$ adopted by the decision–maker and compatible with the measure change rule in eq. (4). Clearly, ~~even though~~ while the stopping time $\hat{\tau}(x) := \inf\left\{t \geq 0 \middle| X_0 = x \text{ and } X_t \geq \hat{x}\right\}$ ~~takes the same values in all cases~~ remains unchanged, the ~~probability by which these values are assigned depends on~~ choice of any probability measure

$\mathbb{Q}^\lambda$ ~~(note that $\mathbb{Q}^0 \equiv \mathbb{P}$)~~ affects ~~, and the same applies for the~~ its expected ~~stopping time~~ value. For example, setting $\lambda = 0$ in eq. (4) leads us to compute its expectation under the physical measure (as $\mathbb{Q}^0 \equiv \mathbb{P}$), while setting $\lambda = \hat{\lambda}$ as in Theorem 1 ~~corresponds to calculating~~ yields its g–expectation ~~of the stopping time~~. ~~Going from one calculation to the~~ Switching from one measure to another essentially corresponds to changing the drift of the process $\{X_t; t \geq 0\}$ to $\tilde{\mu}^\lambda$ in eq. (5), which ~~for the model under consideration~~ under multiple-priors preferences reduces to $\tilde{\mu}^\lambda(X_t) = \mu^\lambda X_t$, where $\mu^\lambda := \mu + \sigma\lambda_W + \int_{\mathbb{R}} \gamma(z)\lambda_J(z)\nu(\mathrm{d}z)$. For a similar discussion for the non–ambiguity case, see Shackleton and Wojakowski (2002). **It indeed seems the market price of risk is negative?**

Proposition 2 derives the expected investment time under various probability measures explicitly:

PROPOSITION 2 **(Expected investment time)**. *We make Assumption 1. ~~and~~ For a given choice of market price of risk λ, suppose that*

$$\delta^\lambda := \mu^\lambda - \frac{1}{2}\sigma^2 - \frac{1}{2}\int_{\mathbb{R}} \gamma^2(z)\nu(\mathrm{d}z) \geq 0. \tag{28}$$

Further, assume that the solution ρ of

$$\frac{1}{2}\sigma^2\rho(\rho - 1) + \mu^\lambda\rho + \psi(\rho) = 0 \tag{29}$$

for $\psi(\cdot)$ given in eq. (23b) satisfies $\rho < 0$. Then, the expectation of the investment time $\hat{\tau}(x) := \inf\left\{t \geq 0 \middle| X_0 = x \text{ and } X_t \geq \hat{x}\right\}$, under the probability measure $\mathbb{Q}^\lambda$, is given by

$$T^\lambda(x) = \frac{1}{\delta^\lambda} \ln\left(\frac{\hat{x}}{\min\{x, \hat{x}\}}\right) \geq 0. \tag{30}$$

For the geometric Brownian motion (GBM), it is known (see, e.g., Chevalier-Roignant and Trigeorgis, 2011, p. 448) that the expected time $T_0(x) := \mathbb{E}^{\mathbb{P}}\left[\inf\{t \geq 0 \mid X_t \geq \hat{x}_0\}\right]$ to reach the investment threshold $\hat{x}_0 := \frac{\beta_0}{\beta_0 - 1}K$ (from below) is finite if

$$\delta_0 := \mu - \frac{1}{2}\sigma^2 > 0, \quad \text{and is given by } T_0(x) = \frac{1}{\delta_0} \ln\left(\frac{\hat{x}_0}{\min\{x, \hat{x}_0\}}\right). \tag{31}$$

We can recover this result from Proposition 2 by taking the $\gamma \equiv 0$ and $\theta = 0$ ($\lambda = 0$ **Do you mean $\lambda_W = 0$ and $\lambda_J \equiv 0$ ν-a.e.?**), that corresponds to the absence of jumps and (nonpriced) ambiguity, respectively. Proposition 2 generalizes the expression for the expected investment time of eq. (31) for the Lévy process of eq. (1) for $\tilde{\mu}$, $\tilde{\sigma}$, and $\tilde{\gamma}$ given in eq. (2), including as well the presence of ambiguity. Although an investment threshold $\hat{x}$ may exist according to Theorem 3, this does not necessarily imply that the first-hitting time $\inf\{t \geq 0 \mid X_t \geq \hat{x}, X_0 = x\}$ has a finite ~~g-expectation~~ **expectation under the chosen measure $\mathbb{Q}^\lambda$?**. For the $\mathbb{Q}^\lambda$-expectation to be finite, it must be that the ambiguity-adjusted project value's growth is sufficiently strong to compensate for the inherent uncertainty arising from both continuous shocks (via σ) and negative Poisson-type jumps (via the Lévy measure ν and the function $\gamma(\cdot)$).

4.3. Impact of rare events on the firm's decisions

~~We make the assumption that~~ Again, multiple ~~risk neutral~~ probability measures arise ~~only as an effect of the jump term~~ from market incompleteness due the combination of continuous and jump events. **Benoit did not like your sentence because incompleteness comes from the combination. If there is only jumps, the market is complete, no?** This assumption implies that μ^λ is affected only by λ_J, while λ_W is set to zero under $\mathbb{P}$ or $(r-\mu)/\sigma$ under $\mathbb{Q}^\lambda$. **Benoit is confused with the previous sentence. Are we arguing that "To focus on the effect of jumps, we set an arbitrary value for λ_J, for instance 0 under $\mathbb{P}$ and $-\frac{\mu-r}{\sigma}$ under the arbitrary choice of $\mathbb{Q}^\lambda$ in Merton (1976)."** Under this assumption we will provide a small jump–size expansion for the solution of the optimal stopping problem, which sheds light on the qualitative features of the solution in Theorem 3 and the effect of jumps on the firm's decisions.

PROPOSITION 3. *Let Assumption 1 and condition* (28) *hold. Express* $\gamma(z)$ *as* $\gamma(z) = \gamma\bar{\gamma}(z)$ *where* $\gamma > 0$ *is a small constant and let* $\bar{\gamma}^2 := \int_{\mathbb{R}} \bar{\gamma}^2(z)\nu(\mathrm{d}z)$. *Denote by the subscript* 0 *the relevant quantities for the GBM, in the absence of rare events. Then, the following results hold:*

(i) The root β *of* $h(\cdot)$ *in eq.* (23) *admits an expansion in* γ *of the form*

$$\beta = \beta_0 + \gamma^2\beta_1 + O(\gamma^3), \quad \textit{where} \quad \beta_1 = -\frac{\frac{1}{2}\bar{\gamma}^2\beta_0(\beta_0 - 1)}{\frac{1}{2}\sigma^2(\beta_0 - 1) + \frac{1}{2}\sigma^2\bar{\gamma}^2\beta_0 + \mu^\lambda} < 0.$$

(ii) The investment threshold $\hat{x}$ in Theorem 3 admits an expansion in γ of the form

$$\hat{x} = \hat{x}_0 + \gamma^2 \hat{x}_1 + O(\gamma^3), \quad \textit{where} \quad \hat{x}_1 = \frac{-\beta_1 \hat{x}_0}{\beta_0(\beta_0 - 1)} > 0.$$

(iii) The expectation of the investment time under the measure $\mathbb{Q}^\lambda$, denoted by $T^\lambda(v)$ in Proposition 2 admits an expansion in γ of the form

$T(x) = T_0(x) + \gamma^2 T_1(x)$ *where δ_0 and $T_0(x)$ are given in eq.* (31) *and* $T_1(x) = \dfrac{1}{\delta_0}\Big(T_0(x) + \dfrac{\hat{x}_1}{\hat{x}_0}\Big) > 0.$

(iv) The value function $v(\cdot)$ in Theorem 3 admits an expansion in γ of the form

$$v(x) = v_0(x) - \beta_1 \gamma^2 v_0(x) \left[\frac{2}{\beta_0} + \ln \frac{\hat{x}_0}{x}\right] + O(\gamma^3).$$

Proposition 3 establishes that a firm facing rare events besides the usual continuous fluctuations of the GBM is less prone to investing. In particular, compared to the GBM benchmarks, the presence of rare events lead to a *decrease* in β (Proposition 3i), an *increase* in the investment threshold $\hat{x}$ (Proposition 3ii), and to *more caution* in the sense that the investment is delayed on average (Proposition 3iii). It also *increases* the value function w in comparison to the GBM benchmark (Proposition 3iv). The option to defer one's investment is more valuable in the presence of rare events, as it offers a further insurance against adverse rare-event-type developments. Provided the jumps have a small magnitude, these results are robust to the modeling choices concerning the jump arrival rates and size distributions (embedded in the Poisson measure ν), as well as concerning the effect of the contributions of the rare events on the project value dynamics (via the function $\cancel{\gamma(\cdot)}$ $\bar{\gamma}(\cdot)$?). To the best of our knowledge, this is the first study to obtain such analytic results. Intuitively, rare events lead to more total risk and consequently more caution by the firm, an effect analogous to when the volatility σ of the continuous fluctuations is increased.

4.4. Illustrative examples

Table 1 ~~summarizes the~~ specifies parameter values and the jump-size distribution functions used ~~for providing benchmark examples~~ as benchmarks of geometric Lévy processes: Samuelson's (1965)

Table 1 Geometric Lévy processes used as benchmark examples.

Project dynamics	Common features	Model-specific features	
		Jump size density $p(\cdot)$	Parameter values
Samuelson (1965)	Discount rate $r=0.05$, drift $\mu=0.035$, volatility $\sigma=0.1$, arrival rate $\Lambda=3$, $\gamma(z)=-\gamma\lvert z\rvert/2$, $\gamma=0.5$ $\theta=0.05$, $k_1=k_2=1$ investment cost $K=1$	$p\equiv 0$	Not applicable
Merton (1976)		$\frac{1}{\sqrt{2\pi s^2}}\exp\left(\frac{1}{2}\left[\frac{z-m}{s}\right]^2\right)$	$m=-0.05$, $s=0.086$
Kou (2002)		$\mathrm{p}\eta_1 e^{-\eta_1 z}\mathbf{1}_{\{z\geq 0\}} + \mathrm{q}\eta_2 e^{\eta_2 z}\mathbf{1}_{\{z<0\}}$, where $\mathrm{p}+\mathrm{q}=1$, $\eta_2>1$ and $\eta_1>0$	$\mathrm{p}=0.30$, $\eta_1=40, \eta_2=12$

Note: Parameter values adapted from Feng and Linetsky (2008)

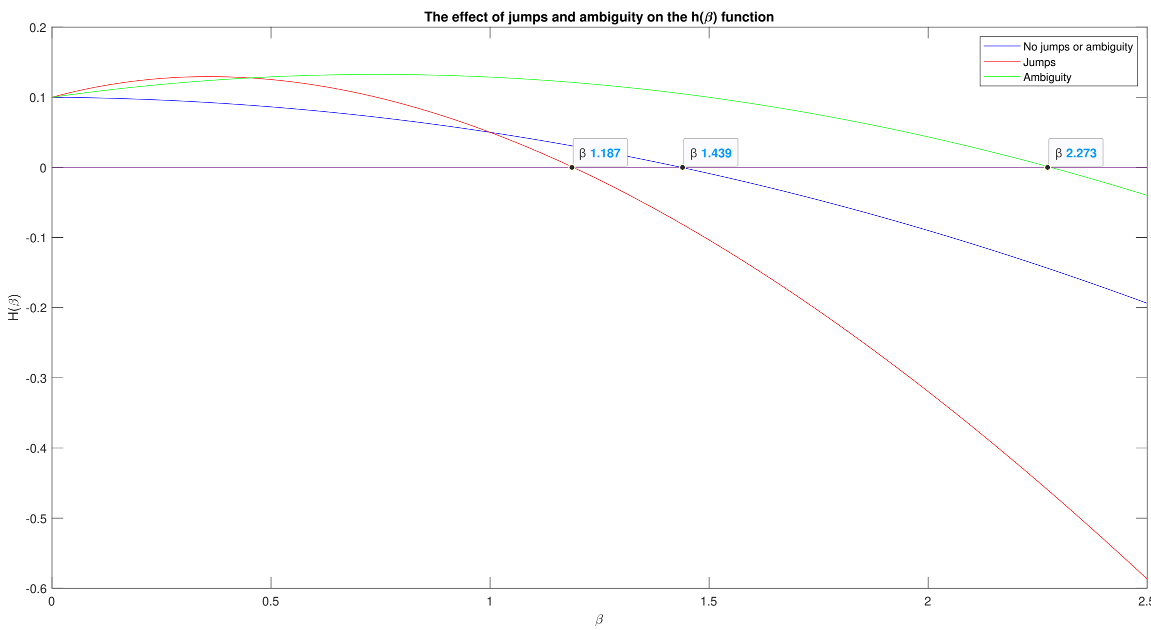

Figure 1: **The effect of jumps and ambiguity on the function $h(\cdot)$ in** (23). The values of all parameters values are provided in Table 1.

no jumps or ambiguity case , Merton's (1976) normally distributed jumps and Kou's (2002) double-exponential jumps. Parameter values are adapted from Feng and Linetsky (2008).

Figure 1 plots the function $\cancel{h(\beta)}$ $h(\cdot)$ in (23) ~~for different values of β. The graph~~ This function $h(\cdot)$ ~~meets~~ crosses the horizontal axis at a root $\beta>1$. The blue line ("No jumps or ambiguity") refers to the Samuelson (1965) model, while the red line ("Jumps") refers to the model in Merton (1976). The green line ("Ambiguity") refers to a model *with jumps and ambiguity* as in Kou (2002). As ~~is apparent from the Figure~~ observed, ~~the introduction of~~ introducing jumps (from the blue to the redlines) decreases the root of $h(\cdot)$ as rare events raise the overall risk profile of the project, while the combined effect of jumps and ambiguity ~~is to increase the root~~ (from the blue to the greenline) leads to a larger root **for our choice of parameter values?**.

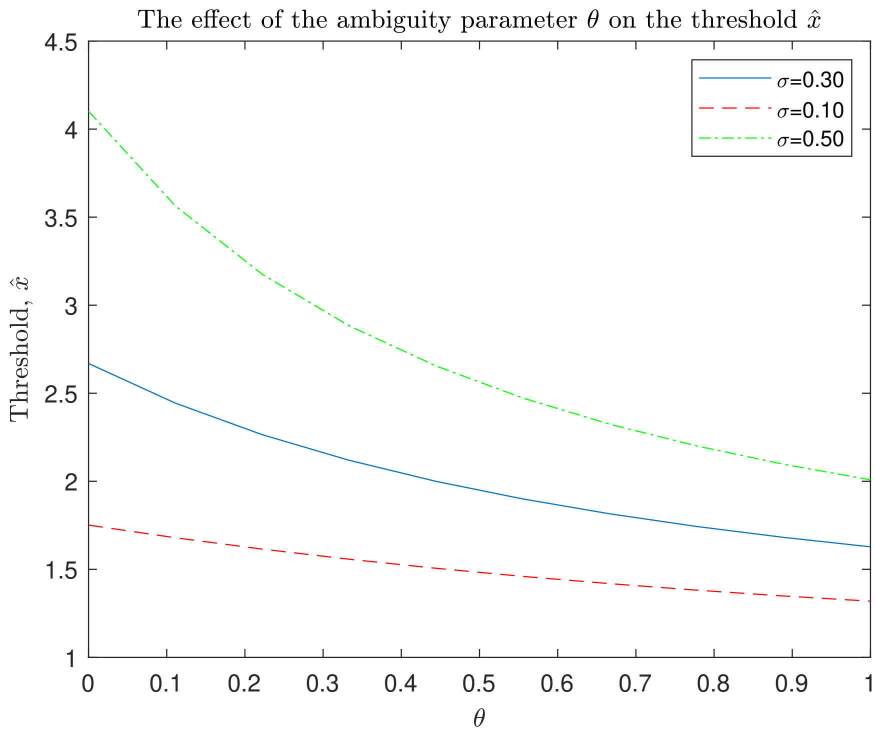

Figure 2: **The effect of the ambiguity parameter θ on the investment threshold $\hat{x}$** for different values of σ. The values of all parameters values are provided in Table 1.

Figure 2 plots the threshold $\hat{x}$ as a function of the ambiguity parameter θ, for different values of the volatility parameter σ. It illustrates a key result in Corollary 2 pertaining to the monotonicity of $\hat{x}$ in θ. A larger degree of ambiguity aversion implies a lower expected future value of the project, reduces the opportunity cost of killing one's option, and so decreases the optimal threshold $\hat{x}$ as the firm prefers accumulating cashflows from the project earlier. Because the downward adjustment to the ambiguity-adjusted drift is increasing in the volatility σ, the effect of ambiguity highlighted above is more pronounced for higher values of σ.

Finally, Figure 3 depicts the value function $w(\cdot)$, highlighting the points at which the value function "touches" the net present value from investing ($x \mapsto x - K$) for the first time (corresponding to the investment thresholds $\hat{x}$). Again, compared to the Samuelson (1965) case (blue line), jumps

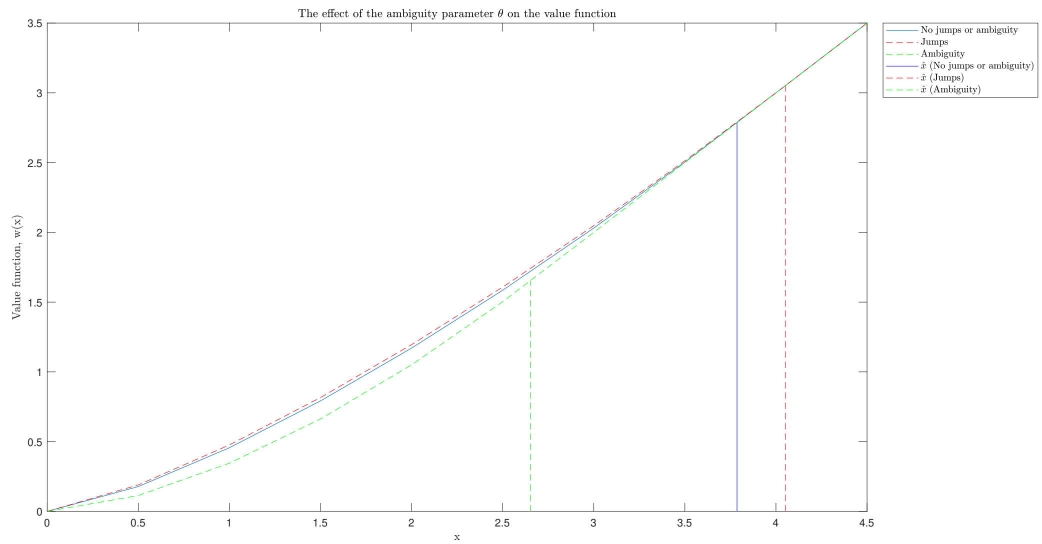

Figure 3: **The effect of the ambiguity parameter θ on the value function $w(\cdot)$ and the threshold $\hat{x}$.** The values of all parameters values are provided in Table 1.

(red line) increase firm value in the continuation region, but jumps and ambiguity (green line) contribute jointly to a reduction in firm value function (and of the threshold) as established in Corollary 2. **If we switch Figure 2 and Figure 3, I suggest adding the sentence: "Building on these first insights, Figure 2 isolates the effect of ambiguity on the threshold."**

5. Real options problem under multipliers preferences

We now consider *multipliers preferences* à la Hansen and Sargent (2001) where the penalty function G is given by

$$G(\lambda_W, \lambda_J) = \frac{1}{2\theta}\left(k_1^{-1}\lambda_W^2 + k_2^{-1}\int_{\mathbb{R}} \lambda_J(z)^2 d\nu(z)\right) \tag{32}$$

for some $k_1, k_2 > 0$. In this case as well, the parameter $\theta \geq 0$ drives the agent's degree of ambiguity aversion. In contrast to the case with Gilboa-Schmeidler preferences, the optimal choice for $(\lambda_W, \lambda_J(\cdot))$ is not obtained on the boundaries of the allowed region~~for the information drifts~~, with the function g in eq. (8) now being given by

$$g(\Lambda_W, \Lambda_J) = -\frac{\theta}{2}\left\{k_1\Lambda_W^2 + k_2\int_{\mathbb{R}} \lambda_J^2(z) d\nu(z)\right\}. \tag{33}$$

Here, again, g is negative, consistently with our assumption in Proposition 1.

For this new case, we cannot obtain the value function in eq. (17) in closed form. We may, however, construct an approximate solution by using a perturbation approach. We consider two ~~extreme~~ cases ~~of interest~~: (a) the limit as $\theta \to 0$, corresponding to probability models ~~cases in which where we restrict ourselves to models which are very~~ close to the common prior $\mathbb{P}$ and (b) the limit as $\theta \to \infty$ which is known as to the deep (Knightian) uncertainty in the literature.

PROPOSITION 4. *Let $\hat{x}_0 = \frac{\beta_0}{\beta_0 - 1}K$ be the free boundary corresponding to the optimal stopping problem with jumps in the absence of ambiguity (i.e., for which the VI is $\min\left\{w(x)-(x-K); -\left(\mathcal{A}+\mathcal{T}-r\mathbb{I}\right)w(x)\right\} = 0$ almost every $x>0$). In the limit as $\theta \to 0$ the VI in Corollary 1* ***Again, shall we call it VI's solution or value function?*** *with g given by eq. (33) admits a solution of the form*

$$v(x) = \begin{cases} \left[1 - \frac{\theta}{2}\frac{\hat{x}_0}{\beta_0}\frac{C_0}{C_1}\left(1-\left(\frac{x}{\hat{x}_0}\right)^{\beta_0}\right)\right]\left(\frac{x}{\hat{x}_0}\right)^{\beta_0}, & x \le \hat{x} := \left(1 - \frac{\theta}{2}\frac{1}{\beta_0(\beta_0-1)}\frac{C_0}{C_1}\right)\hat{x}_0, \\ x-k, & \textit{otherwise}, \end{cases}$$

where C_0 and C_1 are positive constants given in Appendix I and corrections of order $O(\frac{\theta}{2}^2)$ are omitted.

The above solution satisfies the smooth pasting condition up to order $O(\epsilon^2)$. Moreover, the effects of ambiguity in this limit are to decrease the threshold $\hat{x}$ as well as the value function. We established comparable results in Corollary 2 for the case with multiple-priors ambiguity preferences. Again, a higher drift adjustment due to ambiguity depreciates the future expected project value, reduces the opportunity cost of killing one's option, and leads to an earlier exercise.

As far as the deep uncertainty limit $\theta \to \infty$ is concerned, one sees that $x \mapsto g(\sigma x v'(x), \mathcal{B}v(x))$ with g given in eq. (33) diverges to $-\infty$, unless $v(\cdot)$ is a constant **? Benoit does not see $v(\cdot)$ in eq. (33)**. By observing the boundary conditions in eq. (18) with g in eq. (33), we see that $v(\cdot)$ cannot be a constant. Hence, (18) implies that $v(x) = x - K$, i.e., the value function collapses onto the obstacle, **Benoit does not understand the arguments with the boundary conditions. If $g \to -\infty$, then $\mathcal{G} \to -\infty$ and so $-(\mathcal{A}+\mathcal{T}-r1+\mathcal{G}) \to +\infty$, so the** min **in eq. (18) is**

$v(x) - (x - K) = 0$**?** which is not an acceptable solution. **Benoit remembers a discussion about why the value function cannot be the obstacle, but does not remember the key argument. We may want to recall it here. For instance, the superharmonicity does not rule out the possibility for the value function to be linear (unless I am mistaken).**

6. Conclusion

This paper considers an investment problem in which the underlying project value follows a geometric Lévy process. In such circumstances involving rare events, empirical data may be inconclusive to infer the underlying probability model and capital markets may be incomplete, so that the determination of a unique market price of risk is infeasible. In these cases, there are multiple models that may be consistent with the observed empirical data. We narrow down the set of such models to those that are aligned with the decision-maker's ~~multiple-priors ambiguity~~ variational preferences ~~as per Gilboa and Schmeidler (1989).~~ The main contribution to the extant real options literature is the incorporation of the nondiversifiable rare events into the firm's optimal investment problem.

~~We show that, in the presence of rare events, the value function is unique, with the optimal strategy characterized by an investment threshold~~. Under (general) variational preferences, we specify a dynamic programming equation that is satisfied by the value function and embeds adjustments for rare events and ambiguity. We further specify a set of sufficient conditions that ensure the optimality of a threshold policy. For the case with multiple-priors preferences, we obtain analytic expressions for the optimal investment threshold, the expected investment time and the value function , and study (analytically) the impact of rare events on them. ~~Under reasonable assumptions that ensure a nice financial interpretation and comparability with the extant real options literature,~~ Rare events—whatever the arrival rates and distribution of the jumps or their contributions to the project value dynamics—generally lead to a higher investment threshold, a delayed investment and a higher option value. The case with multipliers preferences is less amenable to explicit expressions, but we were able to construct an approximate solution for the limit case of negligible ambiguity by using a perturbative approach. Under both multiple-priors and multipliers preferences (close to the reference prior), a higher degree of ambiguity aversion is proven to lead to an earlier investment and reduced firm value. This is because the project value grows at a lower expected rate under the appropriate ambiguity-adjusted measure, so the incentive to delay the investment is reduced.

Our model has certain limitations which present opportunities for future work in this area. First, we model a relatively simple real options problem in the spirit of McDonald and Siegel (1986); it would be interesting to further explore whether model refinements, such as involving switching options (see, e.g., Pham, Vath, and Zhou, 2009), compound options (see, e.g., Bensoussan and Chevalier-Roignant, 2018) or strategic interactions (see, e.g., Chevalier-Roignant et al., 2011; Hellmann and Thijssen, 2018), ~~will confirm the main results on the impact of rare events on the firm's investment decisions and lead to EMMs consistent with multiple-priors ambiguity preferences~~ would confirm the key insights from our paper or not. Second, if multiple priors are due to market incompleteness, even though our proposed solution identifies a unique ambiguity-adjusted drift at the upper bound of the uncertainty interval, this unique ambiguity-adjusted drift still allows for multiple market prices of risk. This concern is not specific to our setting but is endemic to adopting Gilboa and Schmeidler's (1989) ambiguity preferences in capital market applications. Further work may help resolve this by adopting the minimum entropy approach minimizing over a set of priors subject to penalization or following more general variational preferences as in Maccheroni et al. (2006a). This issue is less relevant if the source of ambiguity relates to multiple priors due to inconclusive empirical data. A promising alternative approach to ambiguity aversion may rely on the Fréchet mean approach to ambiguity, using a convex combination of priors (see, e.g., Petracou, Xepapadeas, and Yannacopoulos, 2022). These alternative ambiguity approaches may lead to solutions that are not at the boundary of the ambiguity set.

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Appendices

Appendix A: Functional spaces

We now define the functional spaces used in the paper:

- The space L_ν^2 comprises all ν-measurable functions f such that $||f||_\nu^2 := \int_{\mathbb{R}} |f(z)|^2 \nu(\mathrm{d}z) < \infty$.
- The space $\mathbb{H}^2$ comprises all $\mathcal{F}$-adapted processes X such that $||X||_{\mathbb{H}^2}^2 := \mathbb{E}\int_0^T X_t^2 \mathrm{d}t < \infty$
- The space $\mathbb{H}_\nu^2$ comprises all $\mathcal{F}$-adapted processes X such that $||X||_{\mathbb{H}_\nu^2}^2 := \mathbb{E}\int_0^T \int_{\mathbb{R}} X_t^2 \mathrm{d}\nu(z)\mathrm{d}t < \infty$.
- $\mathbb{L}_{[t,T]}$ is the set of market prices of risk λ characterizing the alternative probability measures $\mathbb{Q}^\lambda$ for ξ at time t.

Domains of functions and processes used in this paper: The process λ_W is assumed to belong to the Hilbert space $\mathbb{H}^2$, while λ_J belongs to $\mathbb{H}_\nu^2$.

Appendix B: Proof of Proposition 1

Let w be a function with integrable second-order derivatives that solves the free-boundary problem of eq. (20). We look for conditions under which this function $w(\cdot)$ solves the VI in Corollary 1, i.e., under which

$$w(x) \geq x - K, \qquad \forall x \in (0, \hat{x}), \tag{34a}$$

$$(\mathcal{A} + \mathcal{T} - r\mathbb{I} + \mathcal{G})\, w(x) \leq 0, \qquad \forall x > \hat{x}. \tag{34b}$$

We consider the inequality in eq. (34a) first. The nonlocality of the operator $\mathcal{T}$, $\mathcal{B}$ and $\mathcal{G}$ makes the problem somewhat unusual. It follows from eqs. (19) and (20) that

$$(\mathcal{A} + \mathcal{T} - r\mathbb{I})\, w(x) = -g\left(\sigma x w'(x), w(x + \gamma(\cdot)x) - w(x)\right), \quad \forall x \in (0, \hat{x}).$$

If we assume that $g \leq 0$, then

$$(-\mathcal{A} + r\mathbb{I} - \mathcal{T})\, w(x) \leq 0 \quad \forall x \in (0, \hat{x}). \tag{35}$$

We claim that $w'(\cdot)$ does not change sign on $(0, \hat{x})$. Indeed, assume the contrary, i.e., that there exists a $x_0 \in (0, \hat{x})$ at which $w'(\cdot)$ changes sign and such that

- $\underline{w(x_0) > \hat{x} - K}$. Then, there necessarily is a point $x_1 \geq x_0$ at which $w(\cdot)$ attains a local maximum. This is obvious if x_0 is itself a local maximum. If x_0 is a local minimum, then $w(\cdot)$ increases to the right of x_0. But, because $w(\hat{x}) = \hat{x} - K$, there must be a x_1 at which $w(\cdot)$ attains a local maximum. In both cases, we have

$$\left(-\mathcal{A} + r\mathbb{I}\right) w(x_1) = -\frac{1}{2}\sigma^2 x_1^2 \underbrace{w''(x_1)}_{<0} - \mu x_1 \underbrace{w'(x_1)}_{=0} + r \underbrace{w(x_1)}_{>0} > 0.$$

Further, it follows from the definition of $\mathcal{T}$ in eq. (19) and Taylor's theorem that

$$\mathcal{T}w(x_1) = \int_{\mathbb{R}} \int_{x_1}^{x_1+\gamma(z)x_1} w''(\xi)\big(x_1 + \gamma(z)x_1 - \xi\big)\mathrm{d}\xi\nu(\mathrm{d}z)$$

If we assume that $-1 \leq \gamma(\cdot) \leq 0$ and that $w(\cdot)$ is convex, then

$$\mathcal{T}w(x_1) = \int_{\mathbb{R}} \int_{x_1}^{x_1+\gamma(z)x_1} \underbrace{w''(\xi)}_{>0}\underbrace{\big(x_1 + \gamma(z)x_1 - \xi\big)}_{<0}\mathrm{d}\xi\nu(\mathrm{d}z) < 0.$$

But these two inequalities contradict ineq. (35), which rule out the possibility $w'(\cdot)$ changes sign at an $x_0 \in (0,\hat{x})$ such that $w(x_0) > \hat{x} - K$. The above confirms Theorem 3.1.4ii in Garroni and Menaldi (2002) that the function $w(\cdot)$ cannot attain a **global?** maximum at an interior point of the interval $(0,\hat{x})$.

- $\underline{w(x_0) < \hat{x} - K}$. Here, there are four possibilities: either x_0 is a (i) global maximum, (ii) a local local maximum, (iii) a local minimum or (iv) a global minimum. The possibility (i) that x_0 is a global maximum is clearly ruled out as $w(\hat{x}) = \hat{x} - K > w(x_0)$. If (ii) x_0 is a local maximum, then $w(\cdot)$ decreases on the right of x_0, $\cdots$, reaches a local minimum $x_1 \in (x_0,\hat{x})$ and then increases until it reaches $w(\hat{x}) = \hat{x} - K > w(x_0)$. (If there are multiple local maxima and minima, then the previous statement holds provided we interpret x_2 as the largest local minimum in $(0,\hat{x})$.) This behavior implies that $w(\cdot)$ has convex and concave segments, which goes against our **assumption that $w(\cdot)$ is globally convex**. If (iii) x_0 is a local minimum, then there must be a $x_1 \in (0,x_0)$ at which $w(\cdot)$ attains a local maximum. This implies convex and concave segments and contradicts again our **assumption of global convexity** of $w(\cdot)$. If (iv) x_0 is global minimum, then $w(x_0) < w(0) = 0$. As $w(\hat{x}) = \hat{x} - K > 0$, this implies by continuity that there is a $x_1 \in (x_0,\hat{x})$ such that $w < 0$ on $(0,x_1)$ and > 0 on $(x_1,\hat{x})$. **This contradicts our assumption that $w(\cdot) \geq 0$ on $(0,\hat{x})$.**

Because w' does not change sign and we have $w'(\hat{x}) = 1$ according to eq. (20), it follows that $0 < w'(x) < 1$ for $x \in (0,\hat{x})$. By differentiating $x \mapsto W(x) := w(x) - x + K$, it is immediate that

$$W'(\cdot) < 0 \text{ on } (0,\hat{x}), \text{ with } K \geq W(x) \geq 0, \tag{36}$$

where the values of $W(\cdot)$ at 0 and $\hat{x}$) obtain from the boundary conditions in eq. (20). So the inequality in eq. (34a) is satisfied under the stated conditions.

We now consider the ineq. (34b). Define

$$B(x) := \{z \in \mathbb{R} \,:\, x(1+\gamma(z)) \leq \hat{x}\} \subset \mathbb{R}.$$

It follows from the definition of the nonlocal operator $\mathcal{T}$ in eq. (19) and the definition of $w(\cdot)$ as the classical solution of eq. (20) that, for any $x > \hat{x}$,

$$\mathcal{T}w(x) = \int_{B(x)} \Big\{w(x+\gamma(z)x) - (x-K) - \gamma(z)x\Big\}\nu(\mathrm{d}z)$$

$$
\begin{aligned}
&+\int_{B(x)^c}\{\underbrace{x+\gamma(z)x-K-(x-K)-\gamma(z)x}_{=0}\}\nu(\mathrm{d}z)\\
&=\int_{B(x)}\Big\{W(x+\gamma(z)x)+x-K+\gamma(z)x-(x-K)-\gamma(z)x\Big\}\nu(\mathrm{d}z)\\
&=\int_{B(x)}W(x+\gamma(z)x)\nu(\mathrm{d}z),
\end{aligned}
$$

where we recall the definition of $W(\cdot)$. It now follows from the estimates in eq. (36) that

$$0\le \mathcal{T}w(x)\le K||\nu||,\quad \forall x>\hat{x}.$$

It now follows from the definition of $\mathcal{A}$ in eq. (19) that

$$(\mu-r)x+rK\le(\mathcal{A}+\mathcal{T}-r\mathbb{I})\,w(x)\le(\mu-r)x+rK+K||\nu||,\quad \forall x>\hat{x}.$$

If we assume that $\mu<r$ and $g\le 0$—hence, $\mathcal{G}w\le 0$ on $(\hat{x},\infty)$—it thus suffices for ineq. (34b) to be satisfied that

$$\hat{x}>\frac{r+||\nu||}{r-\mu}K.$$

We omit the verification theorem, which follows from standard arguments, and conclude Proposition 1.

Appendix C: Proof of Lemma 1

Re-write the function $h(\cdot)$ in eq. (23) as

$$h(\beta)=h_0(\beta)+\theta h_1(\beta),\ \text{where}\ h_0(\beta):=r-\frac{1}{2}\sigma^2\beta(\beta-1)-\mu\beta-\phi(\beta)\ \text{and}\ h_1(\beta):=k_1\sigma\beta+k_2\psi(\beta).$$

We first study $h_0(\cdot)$. It follows the definition of ϕ in eq. (23) that

$$h_0'(\beta)=-\sigma^2\beta-\frac{\sigma^2}{2}-\mu-\int_{\mathbb{R}}[\ln(1+\gamma(z))e^{\beta\ln(1+\gamma(z))}-\gamma(z)]\mathrm{d}\nu(z). \tag{37}$$

We claim that $h_0'(\beta)\le 0$ for all $\beta>0$ so that h_0 is decreasing. To see this, we differentiate the function $z\mapsto\rho(z):=\ln(1+z)e^{\beta ln(1+z)}-z$ for $z>-1$:

$$\rho'(z)=\frac{1}{1+z}e^{\beta\ln(1+z)}+\beta(\ln(1+z))^2e^{\beta\ln(1+z)}-1$$

so that $\rho'(0)=0$. This corresponds to a local minimum for ρ. In fact, a graph of this function immediately reveals that 0 is a global minimum; hence, $\rho(z)\ge 0$ for all $z\in(-1,\infty)$. Then, the positivity of the Lévy measure implies that $\int_{\mathbb{R}}[\ln(1+\gamma(z))e^{\beta\ln(1+\gamma(z))}-\gamma(z)]d\nu(z)\ge 0$ therefore (37) yields that $h_0'(\beta)\le 0$. It follows from differentiating $h_0(\cdot)$ twice that

$$h_0''(\beta)=-\sigma^2-\int_{\mathbb{R}}\Big(\ln(1+\gamma(z))\Big)^2e^{\beta\ln(1+\gamma(z))}\nu(\mathrm{d}z)\le 0,$$

so $h_0(\cdot)$ is decreasing and concave. The function h_0 diverges to $-\infty$ as $\beta \to \infty$.

We now study $h_1(\cdot)$. We re-write ψ in (23) as

$$\psi(\beta) = \int_{\mathbb{R}} \psi_0(\beta, z)\nu(\mathrm{d}z), \quad \text{where} \quad \psi_0(\beta, z) := \Big|\big[1+\gamma(z)\big]^{\beta} - 1\Big|.$$

Because (as can be easily checked) $\beta \mapsto \psi_0(\beta, z)$ is increasing—for both $\gamma(z) > 0$ and $\gamma(z) < 0$—and ν is a positive measure, it follows that ψ increases and so does h_1. Further, it obtains from the definitions that $\psi(1) = \int_{\mathbb{R}} |\gamma(z)|\nu(\mathrm{d}z) > 0$, so h_1 is positive valued on $(1, \infty)$. It diverges to ∞ as $\beta \to \infty$.

We now study $h(\cdot)$. The function h is the sum of a decreasing concave function (h_0) and of an increasing, positive valued function (h_1). Note that h is continuous due to Lebesgue dominated convergence. We have $\phi(0) = \phi(1) = \psi(0) = 0$, whence

$$h(0) = r > 0 \quad \text{and} \quad h(1) = \underbrace{r - \mu}_{>0} + \theta k_1 \sigma + \theta k_2 \int_{\mathbb{R}} |\gamma(z)|\nu(dz) > 0.$$

Because h_0 is continuous and monotone decreasing on $(1, \infty)$ from $h_0(1) > 0$ to $-\infty$, this function necessarily admits a unique root $\beta > 1$. Fix an arbitrary $M \geq 1$. By the continuity of h, for h to have a root β in $(1, M)$, it suffices that $h(M) < 0$. Because $h(M) = h_0(M) + \theta h_1(M)$ and $h_1 \geq 0$, this implies that

$$\theta < \bar{\theta}(M) := -\frac{h_0(M)}{h_1(M)}.$$

Note that if $h(M) > 0$, i.e., $\theta > \bar{\theta}(M)$, by the properties of $h(\cdot)$ (a sum of a concave function and an increasing function) we do not have a solution of $h(\beta) = 0$ in $(1, M)$. We thus proved Lemma 1.

Appendix D: Proof of Theorem 3

The proof proceeds in several steps.

Step 1. Given preferences à la Gilboa-Schmeidler in (21), it follows from eqs. (8) and (21) that

$$g(\Lambda_W, \Lambda_J) = \inf_{|\lambda_W| \leq \theta k_1,\ |\lambda_J(z)| \leq \theta k_2} \left\{ \lambda_W \Lambda_W + \int_{\mathbb{R}} \lambda_J(z)\Lambda_J(z)\mathrm{d}\nu(z) \right\}$$

By the Cauchy-Schwarz inequality (applied pointwise in the second case), we obtain

$$\inf_{|\lambda_W| \leq \theta k_1} \lambda_W \Lambda_W = -\theta k_1 |\Lambda_W| \quad \text{and} \quad \inf_{|\lambda_J(z)| \leq \theta k_2} \int_{\mathbb{R}} \lambda_J(z)\Lambda_J(z)\mathrm{d}\nu(z) = -\theta k_2 \int_{\mathbb{R}} |\Lambda_J(z)|\mathrm{d}\nu(z)$$

which leads to eq. (22). In the limit as $\theta \to 0$, $g \equiv 0$ and $\mathcal{G} \equiv 0$. So the VI in Corollary 1 reduces to the VI corresponding to the optimal stopping problem of a jump-diffusion without ambiguity (Øksendal and Sulem, 2007, Chapter 2).

Step 2. We anticipate that the solution in the continuation region will be of the form $w(x)=Cx^\beta$ for a suitable $\beta>1$ and that the continuation region will be of the form of an interval $[0,\hat{v})$, again for a suitable $\hat{v}$. The verification will come shortly. The choice of $\beta>1$ follows by the convexity of the value function. Note that the proposed ansatz satisfies the required boundary condition at $v=0$.

Assume that $w(x)=x^\beta$ for $\beta\in(1,\infty)$. Then, it follows from the definitions of the operators, the condition $-1\le\gamma(\cdot)\le 0$ ν-a.e. and the expression for g in eq. (22) that

$$\mathcal{T}w(x)=x^\beta\int_{\mathbb{R}}\Big\{(1+\gamma(z))^\beta-1-\beta\gamma(z)\Big\}\nu(\mathrm{d}z)=:\phi(\beta)x^\beta,$$

$$\mathcal{G}w(x)=g\Big(\sigma x w'(x),\mathcal{B}w(x)\Big)=-\theta\Big(k_1\sigma\beta x^\beta+k_2x^\beta\int_{\mathbb{R}}\Big|\big(1+\gamma(z)\big)^\beta-1\Big|\mathrm{d}\nu(z)\Big)=:-\theta\Big(k_1\sigma\beta+k_2\psi(\beta)\Big)x^\beta,$$

where $\phi(\cdot)$ and $\psi(\cdot)$ are defined in eq. (23). Then,

$$\begin{aligned}\big(\mathcal{A}-r\mathbb{I}+\mathcal{T}+\mathcal{G}\big)w(x)&=\frac{1}{2}\sigma^2x^2F''(x)+\mu xF'(x)+\mathcal{T}F(x)+g\Big(\sigma xF'(x),\mathcal{B}F(x)\Big)-rF(x)\\&=-\underbrace{\left[r-\frac{1}{2}\sigma^2\beta(\beta-1)-\mu\beta-\phi(\beta)+\theta\Big(k_1\sigma\beta+k_2\psi(\beta)\Big)\right]}_{=:h(\beta)\text{ in eq. (23)}}x^\beta.\end{aligned}$$

Step 3. Here, we need the proof of Lemma 1, which is provided in Appendix C.

Step 4. We conjecture a VI's solution of the form

$$w(x)=\begin{cases}Cx^\beta & \text{in the continuation region } D:=(0,\hat{x}),\\ x-K & \text{in the stopping region } [\hat{x},\infty),\end{cases}\tag{38}$$

where β solves $h(\beta)=0$. **Benoit suggests to put the following sentence below Proposition 1:** The validity of the smooth pasting condition in the presence of jumps is justified by the presence of the Brownian motion component in the value process (see Theorem 6 and Proposition 7 in Alili and Kyprianou, 2005; or Theorem 2.2 and Example 2.5 in Øksendal and Sulem, 2007). Using the smooth-fit principle, we can specify $\hat{x}=\beta K/[\beta-1]$ and $C=\hat{x}^{1-\beta}/\beta$.

To verify that this conjecture holds requires us to investigate a set of inequalities. Before we proceed, we establish some auxiliary estimates. First, define $\varphi(x):=x-K-\frac{K}{\beta-1}\big(\frac{x}{\hat{x}}\big)^\beta$ on D. We have

$$\varphi'(x)=1-\frac{K\beta}{\beta-1}\frac{x^{\beta-1}}{(\hat{x})^\beta}=1-\Big(\frac{x}{\hat{x}}\Big)^{\beta-1}\ge 0,\quad\text{for } 0<x<\hat{x};\tag{39}$$

so, $\varphi(\cdot)$ is increasing on D. Because $\varphi(\hat{x})=0$, we just checked that $w(x)\ge x-K$ in D. Second, we also check that

$$L(x,z):=\frac{K}{\beta-1}\left(\frac{x+\gamma(z)x}{\hat{x}}\right)^\beta-(x+\gamma(z)x)\tag{40}$$

satisfies

$$-K \le L(x,z) \le 0, \;\; \forall\, x > \hat{x}, \;\; \forall\, z \; : x + \gamma(z)x \le \hat{x} \tag{41}$$

Indeed, fix $x > \hat{x}$, let $Z = 1 + \gamma(z)$ and define the function

$$Z \mapsto S(Z) := \frac{K}{\beta - 1}\left(\frac{x}{\hat{x}}\right)^{\beta} Z^{\beta} - xZ$$

Note that for the values of z as in (41) it holds that $Z \le \frac{\hat{x}}{x}$, so that

$$\frac{dS}{dZ}(Z) = \frac{\beta K}{\beta - 1}\left(\frac{x}{\hat{x}}\right)^{\beta} Z^{\beta-1} - x \le \frac{\beta K}{\beta - 1}\left(\frac{x}{\hat{x}}\right)^{\beta}\left(\frac{\hat{x}}{x}\right)^{\beta-1} - x = \left(\frac{\beta K}{\beta - 1}\frac{1}{\hat{x}} - 1\right)x = 0.$$

Hence, S is a decreasing function of Z and since $S(0) = 0$, and $S(\frac{\hat{x}}{x}) = -K$ inequality (41) follows. Note, that the left hand side of inequality (40) could follow directly from eq. (39).

We now verify inequalities to establish whether the solution to the free-boundary problem solves the VI. In particular, we will now check whether in D^c $(x \ge x^*)$,

$$\big(\mathcal{A} - r\mathbb{I} + \mathcal{T} + \mathcal{G}\big)w(x) \le 0, \quad \forall\, x \ge \hat{x}. \tag{42}$$

We calculate each term in the above separately. For the local term $(\mathcal{A} - r\mathbb{I})w$ we have (taking into account that $w(x) = x - K$ and the local form of the operator) that

$$(\mathcal{A} - r\mathbb{I})w(x) = (\mu - r)x + rK.$$

We now consider the nonlocal operators $\mathcal{T}$ and $\mathcal{G}$. These have to be treated carefully because their action involves the function w calculated at $x(1 + \gamma(z)) \le \hat{x}$ (since $-1 \le \gamma(z) \le 0$) so that, for these terms, we need to involve the form of w which is valid in D. In particular, define

$$B(x) := \{z \in \mathbb{R} \,:\, x(1 + \gamma(z)) \le \hat{x}\} \subset \mathbb{R}.$$

and note that

$$w(x(1+\gamma(z))) - w(x) - \frac{dw}{dx}(x)\gamma(z)x = \begin{cases} \frac{K}{\beta-1}(\frac{x}{\hat{x}})^{\beta} - (1+\gamma(z))x + K & \text{for } x > \hat{x}, \; z \in B(x), \\ 0 & \text{for } x > \hat{x}, \; z \in B(x)^c \end{cases}$$

Hence,

$$\mathcal{T}w(x) = \int_{B(x)} \left(\frac{K}{\beta - 1}\left(\frac{x}{\hat{x}}\right)^{\beta} - (1 + \gamma(z))x + K\right)\mathrm{d}\nu(z) = \int_{B(x)} (L(x,z) + K)\mathrm{d}\nu(x).$$

We note that this term is bounded as follows:

$$0 \le \mathcal{T}w(x) \le \int_{B(x)} K\mathrm{d}\nu(z) \le K\int_{\mathbb{R}} \mathrm{d}\nu(z) = K\|\nu\|, \tag{43}$$

where we used the auxiliary result in eq. (41).

We now estimate the term $\mathcal{G}w$. We first note that

$$(\mathcal{B}w)(x)(z) = w(x+\gamma(z)x) - w(x) = \begin{cases} \frac{K}{\beta-1}(\frac{x}{\hat{x}})^\beta - (1+\gamma(z))x & \text{for } x > \hat{x},\ z \in B(x), \\ \gamma(z)x & \text{for } x > \hat{x},\ z \in B(x)^c \end{cases}$$

so that

$$\begin{aligned} \mathcal{G}w(x) &= -\theta\left(k_1\sigma x + k_2 \int_{B(x)} \left|\frac{K}{\beta-1}\left(\frac{x}{\hat{x}}\right)^\beta - (1+\gamma(z))x\right| d\nu(z) + k_2 \int_{B(x)^c} |\gamma(z)| x \mathrm{d}\nu(z)\right) \\ &= -\theta\left(k_1\sigma x + k_2 \int_{B(x)} |L(x,z)| \mathrm{d}\nu(z) + k_2 \int_{B(x)^c} |\gamma(z)| x \mathrm{d}\nu(z)\right) \end{aligned} \tag{44}$$

Note that $\mathcal{G}w(x) \le 0$ for any $x > \hat{x}$.

Combining all the above

$$\begin{aligned} \big(\mathcal{A} - r\mathbb{I} + \mathcal{T} + \mathcal{G}\big)w(x) = & (\mu - r)x + rK + \underbrace{\int_{B(x)} (L(x,z)+K)\mathrm{d}\nu(x)}_{\text{jump contribution}} \\ & \underbrace{-\theta\left(k_1\sigma x + k_2 \int_{B(x)} |L(x,z)| d\nu(z) + k_2 x \int_{B(x)^c} |\gamma(z)| \mathrm{d}\nu(z)\right)}_{\text{uncertainty contribution}} \end{aligned} \tag{45}$$

Note that the jump contribution in the above is positive, while the uncertainty contribution is negative. Hence, uncertainty is facilitating the term $\big(\mathcal{A} - r\mathbb{I} + \mathcal{T} + \mathcal{G}\big)w(x)$ for $w(x) = x - K$ to be negative, whereas the jumps are acting in the opposite way.

We now find sufficient conditions under which the threshold $\hat{x}$ obtained by the smooth-fit principle indeed leads to an optimal stopping policy. We estimate (45) as

$$\begin{aligned} \big(\mathcal{A} - r\mathbb{I} + \mathcal{T} + \mathcal{G}\big)w(x) &\le (\mu - r)x + rK + \int_{B(x)} (L(x,z)+K)\mathrm{d}\nu(x) \\ &\le (\mu - r)x + rK + K\|\nu\| \\ &\le (\mu - r)\hat{x} + rK + K\|\nu\|, \quad \forall x > \hat{x}, \end{aligned} \tag{46}$$

where we first used the fact that $\mathcal{G}w(x) \le 0$ then estimate (43) and finally we used the fact that $\mu - r < 0$ and $x \ge \hat{x}$. By (46) if

$$(\mu - r)\hat{x} + rK + K\|\nu\| < 0, \tag{47}$$

then, $\big(\mathcal{A} - r\mathbb{I} + \mathcal{T} + \mathcal{G}\big)w(x) \le 0$ for all $x > \hat{x}$. In this case, $\hat{x}$ can be understood as an exercise threshold. Note that while the term $\mathcal{G}w(x) \le 0$ has been dropped when performing the above estimates, the role of uncertainty is still present in the condition stated in section D because $\hat{x}$ depends on θ (through its dependence on β).

The condition in section D leads to

$$\hat{x} = \frac{K\beta}{\beta - 1} \ge \frac{r + \|\nu\|}{r - \mu} K \iff \beta \le \beta_{cr} := \frac{r + \|\nu\|}{\mu + \|\nu\|}$$

Hence for $\hat{x} = \frac{\beta}{\beta-1}K$ to be an optimal threshold policy, β must not exceed β_{cr}. We now resort to Lemma 1 which provides explicit bounds for β in terms of θ. Setting $M = \beta_{cr}$ in Lemma 1, we see that $\beta \leq \beta_{cr}$ as long as

$$\theta < \theta_{cr} = -\frac{r - \frac{1}{2}\sigma^2\beta_{cr}(\beta_{cr}-1) - \mu\beta_{cr} - \varphi(\beta_{cr})}{k_1\sigma\beta_{cr} + k_2\psi(\beta_{cr})}$$

This concludes the proof. QED

Remark 2. We will also obtain a lower bound for $(\mathcal{A} - r\mathbb{I} + \mathcal{T} + \mathcal{G})w(x)$. Since $|\gamma(z)| \leq 1$ for all $z \in \mathbb{R}$ and by (41) it holds that $|L(x,z)| \leq K$ in $B(x)$, we can estimate

$$\begin{aligned} 0 \leq -\mathcal{G}w(x) &\leq \theta\left(k_1\sigma x + k_2 K \int_{B(x)} d\nu(z) + k_2 x \int_{B(x)^c} \mathrm{d}\nu(z)\right) \\ &\leq \theta(k_1\sigma x + k_2 K\|\nu\| + k_2 x\|\nu\|). \end{aligned}$$

Using this bound, we estimate

$$(\mathcal{A} - r\mathbb{I} + \mathcal{T} + \mathcal{G})w(x) \geq (\mu - r - \theta k_1\sigma - \theta k_2\|\nu\|)x + rK$$

If $\mu - r - \theta k_1 - \theta k_2\|\nu\| \geq 0$ then

$$(\mu - r - \theta k_1\sigma - \theta k_2\|\nu\|)x + rK \geq (\mu - r - \theta k_1\sigma - \theta k_2\|\nu\|)\hat{x} + rK \geq 0, \ \forall x > \hat{x},$$

so $(\mathcal{A} - r\mathbb{I} + \mathcal{T} + \mathcal{G})w(x) \geq 0$ for all $x \geq \hat{x}$ and we cannot have a threshold policy. Note that in this case we have that

$$\mu_\theta - r := \mu - \theta k_1 - \theta k_2\|\nu\| - r \geq 0,$$

where μ_θ is the drift perceived by the decision maker on the worst case scenario, so this corresponds to the condition $\mu \geq r$ for the non ambiguity case, where the value function is not defined (the decision maker never stops).

Appendix E: Proof of Corollary 2

We want to show that v decreases with β, i.e., if $\beta_1 < \beta_2$ then $v(x,\beta_2) \leq v(x,\beta_1)$ for all $x \in \mathbb{R}_+$. First, $\hat{x}$ is decreasing with β because $\mathrm{d}\hat{x}/\mathrm{d}\beta = -K(\beta-1)^{-2} < 0$. Now consider $\beta_1 < \beta_2$. From the above, $\hat{x}(\beta_2) \leq \hat{x}(\beta_1)$. We will compare $v(x,\beta_1)$ and $v(x,\beta_2)$. We consider three regions for x:

$\underline{x < \hat{x}(\beta_2)}$. Here, $v(x,\beta_i) = \frac{K}{\beta_i - 1}(\frac{x}{\hat{x}(\beta_i)})^{\beta_i}$, $i = 1, 2$. We calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\beta}v(x,\beta) &= K\frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{1}{\beta-1}\exp\left(\beta\ln x - \beta\ln\hat{x}(\beta)\right)\right) \\ &= K\left(-\frac{1}{(\beta-1)^2} + \frac{1}{\beta-1}\frac{\mathrm{d}}{\mathrm{d}\beta}(\beta\ln x - \beta\ln\hat{x}(\beta))\right)\left(\frac{x}{\hat{x}(\beta)}\right)^\beta \\ &= \frac{K}{\beta-1}\ln\left(\frac{x}{\hat{x}(\beta)}\right)\left(\frac{x}{\hat{x}(\beta)}\right)^\beta < 0, \end{aligned}$$

since $\beta > 1$ and $x < \hat{x}(\beta)$. Hence for $x < \hat{x}(\beta_2)$, the value function decreases with β, and

$$v(x,\beta_2) < v(x,\beta_1), \ \forall\, x < \hat{x}(\beta_2).$$

$\underline{\hat{x}(\beta_2) < x < \hat{x}(\beta_1)}$. From the above, we already know that $v(x,\beta_1) = \frac{K}{\beta_1 - 1}\left(\frac{x}{\hat{x}(\beta_1)}\right)^{\beta_1} \geq x - K$ in $(0,\hat{x}(\beta_1)) \supseteq (\hat{x}(\beta_2),\hat{x}(\beta_1))$. It follows from $v(x,\beta_2) = x - K$ for $x \geq \hat{x}(\beta_2)$ that $v(x,\beta_2) \leq v(x,\beta_1)$ for all $\hat{x}(\beta_2) < x < \hat{x}(\beta_1)$.

$\underline{x \geq \hat{x}(\beta_1)}$. Here, we have that $v(x,\beta_i) = x - K$ so that the two value functions coincide.

Hence, $v(x,\beta_2) \leq v(x,\beta_1)$ for all $x \in \mathbb{R}_+$, i.e., the value function decreases with respect to β.

To complete the proof of Corollary 2, it remains to note that $\theta \mapsto h(\beta)$ for $h(\cdot)$ given in eq. (23) is monotone increasing. So the root β of $h(\cdot)$ is monotone increasing in θ.

Appendix F: Proof of Proposition 2

We begin the proof by noting that $\mathcal{E}_0(\cdot) = \mathbb{E}^{\mathbb{Q}_{\hat{\lambda}}}(\cdot)$ (see Theorem 1). This means that we can calculate the variational expectation of the stopping time, working with the value process under the probability measure $\mathbb{Q}_{\hat{\lambda}}$ and calculating its expected exit time from $(0,\hat{x})$. The strategy of the proof is to consider the auxiliary expected exit-time problem from $[x_0,x_1] \subset \mathbb{R}_+$, viz.,

$$W(x) := \mathbb{E}^{\mathbb{Q}_{\hat{\lambda}}}\left[\tau \middle| X_0 = x\right] \quad \text{where } \tau := \inf\left\{t \geq 0 \,|\, X_t \notin (x_0,x_1)\right\}$$

and then set $x_1 = \hat{x}$ and take the limit $x_0 \to 0$, to obtain $T(x)$. We consider several steps.

Step 1. We use a heuristic approach to obtain W's functional representation. The boundary conditions at x_0 and x_1 are immediate: $W(x_0) = W(x_1) = 0$. For a small $h > 0$, we have $W(x) = h + E^{\mathbb{Q}_{\hat{\lambda}}}\left[W(X_h)\middle| X_0 = x\right]$. Subtracting $W(x)$ on both sides, dividing by h and taking the limit as $h \to 0$ yields a integro-differential equation with the operator $\mathcal{A} + \mathcal{T}$. In summary, the solution W to the above auxiliary problem solves the Dirichlet problem

$$\big(\mathcal{A} + \mathcal{T}\big)W(x) = -1, \quad W(x_0) = 0, \quad W(x_1) = 0, \tag{48}$$

with $W = 0$ for $v > v_1$. The solution of (48) consists of the general solution of the homogeneous plus a particular solution of the nonhomogeneous part.

We now look for a general solution of the homogenous part of (48). Let us try for solutions of the form $W(x) = x^\rho$, with an exponent ρ to be determined, supplemented with the above boundary conditions. Under the assumption $\gamma(\cdot) \leq 0$, we can easily see that such a function is a solution of (48). The exponent ρ must solve the eq. (29), namely

$$\frac{\sigma^2}{2}\rho(\rho-1) + \mu_\lambda \rho + \psi(\rho) = 0,$$

with the function ψ given in (23b). We can see that $\rho=0$ is always a solution. Denote by $\rho\neq 0$ the nonzero solution of (29). Hence, the general solution of the homogeneous is

$$W(x)=C_0+C_1x^\rho, \tag{49}$$

where C_0 and C_1 are constants to be determined.

We now look for a particular solution of the nonhomogeneous part of the form $W(x)=A\,\ln(x)$ for a constant A to be determined, with $W(x)=0$ for $x>x_1$. Substituting this ansatz into $\big(\mathcal{A}+\mathcal{T}\big)W(x)=-1$, and taking into account that $\gamma(\cdot)\leq 0$, we obtain that

$$\big(\mathcal{A}+\mathcal{T}\big)W(x)=A\left(\mathcal{A}+\mathcal{T}\right)\ln x=A[(\mu_\lambda-\frac{\sigma^2}{2})+\Gamma(\gamma)]=-1 \tag{50}$$

with

$$\Gamma(\gamma):=\int_{\mathbb{R}}[\ln(1+\gamma(z))-\gamma(z)]\,\nu(\mathrm{d}z), \tag{51}$$

from which we immediately obtain that

$$A=-\frac{1}{(\mu_\lambda-\frac{\sigma^2}{2})+\Gamma(\gamma)}. \tag{52}$$

Hence, the general solution of (48) is of the form

$$W(x)=C_0+C_1x^\rho+A\ln(x),$$

where the constants C_0 and C_1 are such that the boundary conditions in (48) are satisfied:

$$C_1=-A\frac{\ln\frac{x_1}{v_0}}{x_1^\rho-x_0^\rho}\quad\text{and}\quad C_0=-A\ln x_1+A\frac{x_1^\rho\ln\frac{x_1}{x_0}}{x_1^\rho-x_0^\rho};$$

hence, the solution of (48) becomes

$$W(x)=A\left\{\ln\left(\frac{x}{x_1}\right)+\frac{1-(\frac{x}{x_1})^\rho}{1-(\frac{x_0}{x_1})^\rho}\ln\left(\frac{x_1}{x_0}\right)\right\}. \tag{53}$$

Step 2. We now set $x_1=\hat{x}$ and pass to the limit as $x_0\to 0$ in (53). If $\rho<0$ then $(\frac{x_0}{x_1})^\rho\to\infty$ as $x_0\downarrow 0$, so then

$$\frac{1}{1-(\frac{x_0}{x_1})^\rho}\ln\left(\frac{x_1}{x_0}\right)\to 0$$

Therefore, in the limit as $x_0\downarrow 0$ we have

$$W(x)=A\ln\left(\frac{x}{\hat{x}}\right).$$

If $\rho>0$ the limit is infinite. Q.E.D.

Appendix G: Proof of Proposition 3

The proof proceeds in 5 steps.

Step 1. Expansion of $\psi(\beta)$ in terms of γ. Assume $\gamma(z) = \gamma\bar{\gamma}(z)$, with γ a small parameter. Then, using an expansion around $\gamma = 0$, we can write

$$\begin{aligned}\psi(\beta) &= \int_{\mathbb{R}} [(1+\gamma\bar{\gamma}(z))^{\beta} - 1 - \beta\gamma\bar{\gamma}(z)]\nu(\mathrm{d}z) \\ &= \int_{\mathbb{R}} [1 + \beta\gamma\bar{\gamma}(z) + \frac{1}{2}\gamma^2\beta(\beta-1)\bar{\gamma}^2(z) + \mathcal{O}(\gamma^3) - 1 - \beta\gamma\bar{\gamma}(z)]\nu(\mathrm{d}z) \\ &= \frac{1}{2}\gamma^2\beta(\beta-1)\int_{\mathbb{R}} \bar{\gamma}^2(z)\nu(\mathrm{d}z) + \mathcal{O}(\gamma^3), \end{aligned} \tag{54}$$

with $\mathcal{O}(\gamma^3)$ capturing terms of order γ^3 and above. Define $\bar{\gamma}^2 := \int_{\mathbb{R}} \bar{\gamma}^2(z)\nu(\mathrm{d}z)$ so that equation (54) becomes

$$\psi(\beta) = \frac{1}{2}\gamma^2\bar{\gamma}^2\beta(\beta-1) + \mathcal{O}(\gamma^3). \tag{55}$$

Step 2. Expansion of β in terms of γ. Substituting (55) into the function h in (23) yields

$$\frac{1}{2}\left(\sigma^2 + \gamma^2\bar{\gamma}^2\right)\beta(\beta-1) + \mu_\lambda\beta - r = 0 \tag{56}$$

We look for a solution of (56) in the form of $\beta = \beta_0 + \gamma^2\beta_1 + \mathcal{O}(\gamma^3)$, where β_1 is the correction in the expansion due to the jumps. Then (56) becomes

$$\begin{aligned}\frac{1}{2}\sigma^2\beta_0(\beta_0-1) + \frac{1}{2}\gamma^2\bar{\gamma}^2\beta_0(\beta_0-1) + \frac{1}{2}\sigma^2\gamma^2\beta_1(\beta_0-1) + \frac{1}{2}\sigma^2\gamma^2\bar{\gamma}^2\beta_1\beta_0 \\ +\mu_\lambda\beta_0 + \gamma^2\mu_\lambda\beta_1 - r = 0 + \mathcal{O}(\gamma^4)\end{aligned} \tag{57}$$

with $\mathcal{O}(\gamma^4)$ capturing terms of order γ^4 and above.

At order $\mathcal{O}(\gamma^0)$, one gets from (57) the "quadratic equation" (Dixit and Pindyck, 1994)

$$\frac{1}{2}\sigma^2\beta_0(\beta_0-1) + \mu_\lambda\beta_0 - r = 0 \tag{58}$$

where β_0 is the solution for the case with a GBM. At order $\mathcal{O}(\gamma^2)$, one gets from (57)

$$\left\{\frac{1}{2}\sigma^2(\beta_0-1) + \frac{1}{2}\sigma^2\bar{\gamma}^2\beta_0 + \mu_\lambda\right\}\beta_1 + \frac{1}{2}\bar{\gamma}^2\beta_0(\beta_0-1) = 0,$$

which yields

$$\beta_1 = -\frac{\frac{1}{2}\bar{\gamma}^2\beta_0(\beta_0-1)}{\frac{\sigma^2}{2}(\beta_0-1) + \frac{\sigma^2}{2}\bar{\gamma}^2\beta_0 + \mu_\lambda} < 0. \tag{59}$$

Hence, β *decreases* (from the GBM case) due to jumps, for small γ. This concludes the result (i) in Proposition 3.

Step 3. Expansion of the threshold $\hat{x}$ in terms of γ. We recall from Theorem 3 the threshold $\hat{x} = \frac{\beta}{\beta-1}K$. We try $\beta = \beta_0 + \gamma^2\beta_1 + \mathcal{O}(\gamma_0^3)$, with $\beta_1 < 0$ as established in the previous section. Using a Taylor series expansion of $\beta \mapsto \beta/[\beta-1]$ around $\gamma = 0$, it can be shown that

$$\frac{\beta}{\beta-1} = \frac{\beta_0 + \gamma^2\beta_1}{(\beta_0-1) + \gamma^2\beta_1} + \mathcal{O}(\gamma^3) = \frac{\beta_0}{\beta_0-1} - \gamma^2\frac{\beta_1}{(\beta_0-1)^2} + \mathcal{O}(\gamma^4). \tag{60}$$

Since $\beta_1 < 0$, the inclusion of jumps *increases* the threshold $\hat{v}$ to this order. This concludes the result (ii) in Proposition 3.

Step 4. Expansion of the value function $F(x)$ in terms of γ. The optimal threshold for the GBM is $\hat{x}_0 = \beta_0 K/[\beta_0 - 1]$. From (60), we can write

$$\hat{x} = \hat{x}_0 + \gamma^2 \hat{x}_1 \quad \text{with} \quad \hat{x}_1 = \tfrac{-\beta_1}{\beta_0(\beta_0 - 1)} \hat{x}_0 > 0 \tag{61}$$

since $\beta_1 < 0$. In the continuation region, $D = (0, \hat{x})$, the value function can be written as $v(x) = Cx^\beta$. Using the expansion, one gets

$$C = \frac{K}{\beta - 1} \frac{1}{\hat{x}^\beta} = C_0 \left\{ 1 - \beta_1 \gamma^2 \left[\frac{2}{\beta_0} + \ln \hat{x}_0 \right] \right\}, \tag{62}$$

with $C_0 = \hat{x}_0^{1-\beta_0}/\beta_0$, and, by Taylor expansion of $\beta \mapsto \exp(\beta \ln x)$ around $\gamma = 0$,

$$x^\beta = x^{\beta_0}(1 + \beta_1 \gamma^2 \ln x). \tag{63}$$

Putting (62)–(63) together yields

$$v(x) = Cx^\beta = C_0 x^{\beta_0} \left\{ 1 - \beta_1 \gamma^2 \left[\frac{2}{\beta_0} + \ln \frac{\hat{x}_0}{v} \right] \right\} = v_0(x) - \beta_1 \gamma^2 v_0(x) \left[\frac{2}{\beta_0} + \ln \frac{\hat{x}_0}{x} \right], \tag{64}$$

with $v_0(x) = C_0 x^{\beta_0}$. Given that $\beta_1 < 0$ and in the continuation region $\ln(\hat{x}_0/x) > 0$, the above provides the conclusion that the inclusion of jumps leads to an *increase* of the value function $v(x)$ for $0 < x < \hat{x}_0$.

A legitimate question is whether $v(\cdot)$ also increases, because of the jumps, in the region $\hat{x}_0 < x < \hat{x}_0 + \gamma^2 \hat{x}_1$, since in this region $\ln(\hat{x}_0/x)$ is no longer positive. To answer this, observe that in this region

$$x < \hat{x}_0 + \gamma^2 \hat{x}_1 \Leftrightarrow \frac{1}{x} > \frac{1}{\hat{x}_0}(1 - \gamma^2 \frac{\hat{x}_1}{\hat{x}_0}) \Leftrightarrow \ln \frac{\hat{x}_0}{x} > \ln(1 - \gamma^2 \frac{\hat{x}_1}{\hat{x}_0}) \approx -\gamma^2 \frac{\hat{x}_1}{\hat{x}_0}. \tag{65}$$

Therefore, in this region we have

$$\begin{aligned} v(x) &= F_0(x) - \beta_1 \gamma^2 v_0(x) \left[\frac{2}{\beta_0} + \ln \frac{\hat{x}_0}{x} \right] \\ &\geq v_0(x) - \beta_1 \gamma^2 v_0(x) \left[\frac{2}{\beta_0} - \gamma^2 \frac{\hat{x}_1}{\hat{x}_0} \right] \\ &\approx v_0(x) - \beta_1 \gamma^2 v_0(x) \frac{2}{\beta_0} + \mathcal{O}(\gamma^4) \end{aligned} \tag{66}$$

So there is also an increase in the value function in the region $\hat{x}_0 < x < \hat{x}_0 + \gamma^2 \hat{x}_1$ to this order. This concludes the result (iv) in Proposition 3.

Step 5. Expansion of $T(v)$ in terms of γ. We recall the term A introduced in (52) and look for

$$A = A_0 + \gamma^2 A_1 + \mathcal{O}(\gamma^3), \tag{67}$$

where $\mathcal{O}(\gamma^3)$ captures terms of order γ^3 and above. Substitute (67) into (50) to get

$$A_0\left(\mu_\lambda - \frac{\sigma^2}{2}\right) - \frac{\gamma^2}{2}\bar{\gamma}^2 A_0 + \gamma^2 A_1\left(\mu_\lambda - \frac{\sigma^2}{2}\right) = -1 \tag{68}$$

At order $\mathcal{O}(\gamma^0)$, it follows from eq. (68) that

$$A_0 = -\frac{1}{\delta_0} \quad \text{where} \quad \delta_0 := \mu_\lambda - \frac{\sigma^2}{2}. \tag{69}$$

At order $\mathcal{O}(\gamma^2)$, one gets

$$A_1 = -\frac{\bar{\gamma}^2}{2\delta_0^2}. \tag{70}$$

Hence,

$$A = -\frac{1}{\delta_0} - \gamma^2\frac{\bar{\gamma}^2}{2\delta_0^2} \tag{71}$$

We now look for a solution of the form $\rho = \rho_0 + \gamma^2\rho_1$. Substitute in eq. (29), expand, and use (55) to get

$$\frac{1}{2}\sigma^2(\rho_0 - 1) + \mu_\lambda + \frac{1}{2}\gamma^2\bar{\gamma}^2(\rho_0 - 1) + \frac{1}{2}\gamma^2\sigma^2\rho_1 = 0 \tag{72}$$

At the order $\mathcal{O}(\gamma^0)$, we have

$$\frac{1}{2}\sigma^2(\rho_0 - 1) + \mu_\lambda = 0 \Rightarrow \rho_0 = 1 - \frac{2\mu_\lambda}{\sigma^2} \tag{73}$$

At the $\mathcal{O}(\gamma^2)$ order, we get

$$\frac{1}{2}\sigma^2\rho_1 = \frac{1}{2}\bar{\gamma}^2(1 - \rho_0) \Rightarrow \rho_1 = \bar{\gamma}^2\frac{2\mu_\lambda}{\sigma^4} \tag{74}$$

Putting together (73) and (74) yields

$$\rho = 1 - \frac{2\mu_\lambda}{\sigma^2} + \gamma^2\bar{\gamma}^2\frac{2\mu_\lambda}{\sigma^4} \tag{75}$$

Thus, in the presence of jumps the exponent increases, however for small γ, ρ can be negative.

We then proceed in the expansion of $T(v)$ in Proposition 2 using the above expansions for A and ρ. Using the expansion $\hat{x} = \hat{x}_0 + \gamma^2\hat{x}_1$ to get

$$T(x) = T_0 + \gamma^2 T_1, \text{ where } T_0 = -\frac{1}{\delta_0}\ln\left(\frac{x}{\hat{x}_0}\right) \text{ and } T_1 = -\frac{1}{\delta_0^2}\gamma^2\ln\left(\frac{x}{\hat{x}_0}\right) + \frac{1}{\delta_0}\frac{\hat{x}_1}{\hat{x}_0} \tag{76}$$

Because $\delta_0 > 0$, in the continuation region where $x < \hat{x}_0 < \hat{x}_1$ we get $T_0 > 0$ and $T_1 > 0$, so we obtain the result (iii). The proof is complete. Q.E.D.

Appendix H: Proof of the expression in eq. (33)

We recall the function G in case of Hansen and Sargent's (2001) multipliers preferences in eq. (32). In this case, the function g defined in eq. (8) (where the time variable becomes irrelevant) becomes

$$g(\Lambda_W,\Lambda_J) = \inf_{(\lambda_W,\lambda_J)\in\mathbb{R}\times L^2_\nu}\left\{\left[\frac{1}{2\theta k_1}\lambda_W^2+\lambda_W\Lambda_W\right]+\int_{\mathbb{R}}\left[\frac{1}{2\theta k_2}\lambda_J^2(z)+\lambda_J(z)\Lambda_J(z)\right]\mathrm{d}\nu(z)\right\} \tag{77}$$

We can work each contribution separately. The first–order condition for the first term gives $\lambda_W=-\theta k_1\Lambda_W$, which yields

$$\inf_{\lambda_W}\left\{\frac{1}{2\theta k_1}\lambda_W^2+\lambda_W\Lambda_W\right\}=-\frac{\theta k_1}{2}\Lambda_W^2.$$

The first–order condition of the second term gives $\lambda_J(z)=-\theta k_2\Lambda_J(z)$, for all z, which yields

$$\inf_{\lambda_J}\left\{\int_{\mathbb{R}}\left[\frac{1}{2\theta k_2}\lambda_J^2(z)+\lambda_J(z)\Lambda_J(z)\right]\mathrm{d}\nu(z)\right\}=-\frac{\theta k_2}{2}\int_{\mathbb{R}}\Lambda_J^2(z)\mathrm{d}\nu(z).$$

Hence, the expression for eq. (77) becomes in eq. (33).

Appendix I: Proof of Proposition 4

Let $\epsilon := \frac{\theta}{2}$ and look for a value function in the form $v(x)=v_0(x)+\epsilon v_1(x)+O(\epsilon^2)$. We first look for a solution of $\big(\mathcal{A}+\mathcal{T}-r\mathbb{I}+\mathcal{G}\big)v(\cdot)=0$ of this form, so that substituting the above ansatz and expanding in orders of ϵ we obtain

$$O(\epsilon^0):\ \frac{1}{2}\sigma^2v^2v_0''(x)+\mu x v_0'(v)+\mathcal{T}v_0(x)-rv_0(x)=0$$

which admits a solution of the form $v_0(x)=\bar{A}x^{\beta_0}$ where $\bar{A}$ is a constant to be determined and β_0 is a root of $\beta\mapsto r-\frac{1}{2}\sigma^2\beta(\beta-1)-\mu\beta-\phi(\beta)$ for $\phi(\cdot)$ given in eq. (23b).

Proceeding to next order $O(\epsilon)$ we obtain

$$\begin{aligned}O(\epsilon^1):\ &\frac{1}{2}\sigma^2x^2v_1''(x)+\mu x v_1'(x)+\mathcal{T}v_1(x)-rv_1(x)\\ &-\left(k_1\sigma x^2 v_0'(x)^2+k_2\int_{\mathbb{R}}\big\{v_0(x+\gamma(z)x)-v_0(x)\big\}^2\mathrm{d}\nu(z)\right)=0.\end{aligned}$$

A simple calculation yields that

$$\left(k_1\sigma x^2 v_0'(x)^2+k_2\int_{\mathbb{R}}\big\{v_0(x+\gamma(z)x)-v_0(x)\big\}^2\mathrm{d}\nu(z)\right)=\bar{A}^2C_0x^{2\beta_0},$$

where

$$C_0:=k_1\sigma^2\beta_0^2+k_2\int_{\mathbb{R}}\big\{(1+\gamma(z))^{\beta_0}-1\big\}^2\mathrm{d}\nu(z)>0.$$

Hence, v_1 satisfies

$$\frac{1}{2}\sigma^2x^2v_1''(x)+\mu x v_1'(x)+\mathcal{T}v_1(x)-rv_1(x)=\bar{A}^2C_0x^{2\beta_0} \tag{78}$$

We look for a solution of (78) of the form $v_1(x) = \bar{A}_1 x^{2\beta_0}$ for $\bar{A}_1$ to be determined. Substituting this form into (78), we obtain that

$$C_1 \bar{A}_1 x^{2\beta_0} = \bar{A}^2 C_0 x^{2\beta_0},$$
$$C_1 = \frac{\sigma^2}{2} 2\beta_0(2\beta_0 - 1) + 2\mu\beta_0 - r + \int_{\mathbb{R}} \left\{(1+\gamma(z))^{2\beta_0} - 1 - 2\beta_0\gamma(z)\right\} \mathrm{d}\nu(z)$$

We express C_1 as

$$\begin{aligned}
C_1 =& \frac{1}{2}\sigma^2\beta_0(\beta_0 - 1) + \mu\beta_0 - r + \int_{\mathbb{R}} [(1+\gamma(z))^{\beta_0} - 1 - \beta_0\gamma(z)] \mathrm{d}\nu(z) \\
&+ \int_{\mathbb{R}} [(1+\gamma(z))^{2\beta_0} - 1 - 2\beta_0\gamma(z)] \mathrm{d}\nu(z) - \int_{\mathbb{R}} [(1+\gamma(z))^{\beta_0} - 1 - \beta_0\gamma(z)] \mathrm{d}\nu(z) \\
&+ \mu\beta_0 + \frac{1}{2}\sigma^2\beta_0(\beta_0 - 1) + \sigma^2\beta_0^2 \\
=& \int_{\mathbb{R}} [(1+\gamma(z))^{2\beta_0} - (1+\gamma(z))^{\beta_0} - \beta_0\gamma(z)] d\nu(z) + \frac{1}{2}\sigma^2\beta_0(\beta_0 - 1) + \sigma^2\beta_0^2,
\end{aligned}$$

so that

$$C_1 = \int_{\mathbb{R}} [(1+\gamma(z))^{2\beta_0} - (1+\gamma(z))^{\beta_0} - \beta_0\gamma(z)] d\nu(z) + \frac{1}{2}\sigma^2\beta_0(\beta_0 - 1) + \sigma^2\beta_0^2, \tag{79}$$

Since the function $\rho(z) := (1+z)^{2\beta_0} - (1+z)^{\beta_0} - \beta_0 z \geq 0$ for $z \in (-1, \infty)$ as can be shown numerically and ν is a positive measure, we see that $C_1 > 0$. Hence, $\bar{A}_1 = \frac{\bar{A}^2 C_0}{C_1} > 0$, and

$$v(x) = \bar{A}x^{\beta_0} + \epsilon \frac{\bar{A}^2 C_0}{C_1} x^{2\beta_0} + O(\epsilon^2) \tag{80}$$

Here we seem to get v increasing with θ. This seems to be the opposite effect that what we get for the Gilboa model. Thanasis: According to Theorem 4.1 of Quenez and Sulem (2014), we should be getting that higher θ decreases the value function. It may be that we have done something wrong in one of the signs. Please check as we also do so independently. It may also be that the second–order term in the expansion has the opposite behaviour and dominates this first–order term.

As (80) shows, we can obtain the value function in terms of a free constant $\bar{A}$. It remains to specify the constant $\bar{A}$ in (80). This can be done by employing the smooth pasting condition which at the same time will allow us to obtain the free boundary. We specify $\hat{x}$ so that

$$v(\hat{x}) = \hat{x} - K \quad \text{and} \quad v'(\hat{x}) = 1.$$

Using the representation (80), this yields

$$\bar{A}\hat{x}^{\beta_0} + \epsilon \frac{\bar{A}^2 C_0}{C_1} \hat{x}^{2\beta_0} + O(\epsilon^2) = \hat{x} - K \quad \text{and} \quad \beta_0 \bar{A}\hat{x}^{\beta_0 - 1} + \epsilon \frac{\bar{A}^2 C_0}{C_1} 2\beta_0 \hat{x}^{2\beta_0 - 1} + O(\epsilon^2) = 1. \tag{81}$$

We look for a perturbative solution of this system of the form $\bar{A} = A_0 + \epsilon A_1 + O(\epsilon^2)$ and $\hat{x} = \hat{x}_0 + \epsilon\hat{x}_1 + O(\epsilon^2)$. Upon substituting this expansion in (81) we obtain to zeroth order that

$$A_0\hat{x}_0^{\beta_0} = \hat{x}_0 - K \quad \text{and} \quad \beta_0 A_0 \hat{x}_0^{\beta_0-1} = 1 \tag{82}$$

which is identical to the conditions for the ambiguity free problem and lead to

$$\hat{x}_0 = \frac{\beta_0}{\beta_0 - 1}K \quad \text{and} \quad A_0 = \frac{1}{\beta_0}\hat{x}_0^{1-\beta_0},$$

which leads to the familiar representation

$$v_0(v) = \frac{K}{\beta_0 - 1}\left(\frac{x}{\hat{x}_0}\right)^{\beta_0}$$

We now proceed to the first order corrections in (81). This yields the linear system

$$\begin{aligned} \hat{x}_0^{\beta_0}A_1 + (A_0\beta_0\hat{x}_0^{\beta_0-1} - 1)\hat{x}_1 &= -\frac{A_0^2C_0}{C_1}\hat{x}_0^{2\beta_0}, \\ \beta_0\hat{x}_0^{\beta_0-1}A_1 + \beta_0(\beta_0-1)A_0\hat{x}_0^{\beta_0-1}\hat{x}_1 &= -2\beta_0\frac{A_0^2C_0}{C_1}\hat{x}_0^{2\beta_0-1} \end{aligned} \tag{83}$$

Solving this for A_1 and $\hat{x}_1$ will specify the solution and the free boundary to order $O(\epsilon^2)$.

Note that by (82) this simplifies to

$$\begin{aligned} \hat{x}_0^{\beta_0}A_1 &= -\frac{A_0^2C_0}{C_1}\hat{x}_0^{2\beta_0}, \\ \beta_0\hat{x}_0^{\beta_0-1}A_1 + (\beta_0-1)\hat{x}_1 &= -2\beta_0\frac{A_0^2C_0}{C_1}\hat{x}_0^{2\beta_0-1}, \end{aligned}$$

which immediately yields

$$\begin{aligned} A_1 &= -\frac{A_0^2C_0}{C_1}\hat{x}_0^{\beta_0} = -\frac{1}{\beta_0^2}\frac{C_0}{C_1}\hat{x}_0^{2-\beta_0}, \\ \hat{x}_1 &= -\frac{\beta_0}{\beta_0-1}\frac{A_0^2C_0}{C_1}\hat{x}_0^{2\beta_0-1} = -\frac{1}{\beta_0(\beta_0-1)}\frac{C_0}{C_1}\hat{x}_0. \end{aligned}$$

Hence, we obtain an approximation to the free boundary as

$$\hat{x} = \hat{x}_0 - \epsilon\frac{\beta_0}{\beta_0-1}\frac{A_0^2C_0}{C_1}\hat{x}_0^{2\beta_0-1} = \left(1 - \epsilon\frac{1}{\beta_0(\beta_0-1)}\frac{C_0}{C_1}\right)\hat{x}_0, \tag{84}$$

which leads to a reduction of the threshold with ambiguity.

Moreover, the value function admits the expansion

$$\begin{aligned} v(x) &= \bar{A}x^{\beta_0} + \epsilon\frac{\bar{A}^2C_0}{C_1}x^{2\beta_0} + O(\epsilon^2) \\ &= (A_0 + \epsilon A_1)x^{\beta_0} + \epsilon\frac{A_0^2C_0}{C_1}x^{2\beta_0} + O(\epsilon^2) \\ &= \left(A_0 - \epsilon\frac{A_0^2C_0}{C_1}\hat{x}_0^{\beta_0}\right)x^{\beta_0} + \epsilon\frac{A_0^2C_0}{C_1}x^{2\beta_0} + O(\epsilon^2) \\ &= \left(1 - \epsilon\frac{1}{\beta_0}\frac{C_0}{C_1}\hat{x}_0\right)\frac{\hat{x}_0}{\beta_0}\left(\frac{x}{\hat{x}_0}\right)^{\beta_0} + \epsilon\frac{C_0}{C_1}\frac{\hat{x}_0^2}{\beta_0^2}\left(\frac{x}{\hat{x}_0}\right)^{2\beta_0} + O(\epsilon^2) \\ &= \left\{1 - \epsilon\frac{\hat{x}_0}{\beta_0}\frac{C_0}{C_1}\left(1 - \left(\frac{x}{\hat{x}_0}\right)^{\beta_0}\right)\right\}\left(\frac{x}{\hat{x}_0}\right)^{\beta_0} + O(\epsilon^2) \end{aligned}$$

Note that since this solution is valid in the continuation region $x \leq \hat{x} = \hat{x}_0 - \epsilon \hat{x}_1$ the first term $\left(\frac{x}{\hat{x}_0}\right)^{\beta_0}$ will dominate over the second term $\left(\frac{x}{\hat{x}_0}\right)^{\beta_0}$ hence we observe a decrease of the value function as an effect of uncertainty in the limit of small ambiguity. Q.E.D.