Product life cycles and investment: A real options analysis

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This version: February 1, 2022

Abstract

Typically product demand follows a product life cycle (PLC). This means that after a product is introduced, demand for this product first starts to grow, which after some time is followed by a decline in demand. Moreover, in most cases demand is stochastic. This paper combines these two characteristics by employing a geometric Brownian motion process with a first increasing and afterwards decreasing trend. Our aim of the paper is to investigate the optimal investment decision of a firm in production capacity. The investment decision involves deciding about the timing and the size of the investment. We make a distinction between firms being a *product-life-cycle leader* and a *product-life-cycle follower*. For a PLC-leader the growth stage starts at the moment this firm invests. In case of a PLC-follower, the firm enters an existing product life cycle, implying that the decline can already start before this firm even has invested.

One of the interesting results is that a PLC-leader waits for a higher demand level before it invests with the same amount when the expected length of the growth interval is shorter. For the PLC-follower it holds that it may be optimal to invest earlier because of this probability that the decline could already start before the firm invests. In such a case the expected future demand is lower, which makes it optimal that the firm attracts less capacity. This makes the investment cheaper and then the firm does not need to wait for a high demand level to make the investment profitable.

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1 Introduction

Product Lifecycle (PLC) differentiates four main stages, i.e., (1) market introduction stage, (2) growth stage, (3) maturity stage, and (4) saturation and decline stage. Each stage can be identified by different levels of sales, cost, competition and profit. Product Lifecycle Management (PLM)[1] manages each stage. We introduce and add stage (0) to the lifecycle which deals with the market entry and plant size decisions. PLM deals with the engineering aspects and management during the lifecycle of a product aimed at increasing the efficiency and effectiveness of business strategies in the process. We however take the product life-cycle as given and investigate when to start the production process and we derive how large the plant should be in a market with stochastic demand. We do not investigate macro-economic business cycles which are re-occurring cycles contrary to product life-cycles.

We use real options methods to determine the most efficient and effective moment to enter the market and decide on the capacity of the plant. We consider an irreversible investment choice on the size of the plant. We focus on the decision that has to be made before the product lifecycle starts, i.e., stage zero. To the best of our knowledge, on the one hand, the product lifecycle literature focusses on the different and dynamic stages within the lifecycle without optimizing the best time to enter the market and optimizing the best size of the plant. On the other hand, the real options literature has only optimized the timing and size given a demand process with a positive trend, i.e., without allowing for a product lifecycle.

We find that the PLC-leader always invests later than the PLC-follower. Thus the fact that a decline cannot happen during the waiting period of the PLC-leader makes him to invest later as meanwhile he postpones the decline and actually causes the demand to grow, in other words, the follower needs to speed up entering the market in the fear of missing out. Moreover, The PLC-leader capacity decision is independent of λ. The optimal capacity of the PLC-leader and PLC-follower has the same functional form (thus difference only explained by difference in optimal X), see (20). For the PLC-leader, the λ cancels out. Several illustrations show that the PLC-follower's threshold can be both increasing and decreasing in λ.

Literature: TBA.

[1] PLM is one of the four cornerstones of a manufacturing corporation's information technology structure, among customer relationship management (CRM), supply chain management (SCM), and enterprise resource planning (ERP).

2 Model

We consider a monopoly, thus a setting in which there is one firm who has to decide when to enter the market and how much to produce. The price at time t in this market is given by the inverse demand function

$$P(t) \quad = \quad X(t)\big(1-\eta Q(t)\big), \tag{1}$$

where $Q(t)$ is the total market output/production/capacity, $\eta > 0$ is a constant and $X(t)$ follows a geometric Brownian motion

$$dX(t) \quad = \quad \begin{cases} \mu_1 X(t)dt + \sigma X(t)dW(t), & \text{before event that occurs with prob } \lambda \\ \mu_2 X(t)dt + \sigma X(t)dW(t), & \text{after event that occurs with prob } \lambda \end{cases} \tag{2}$$

where $\mu_1 > 0$ and $\mu_2 < 0$ are the drifts implying a positive expected trend in demand or negative, $\sigma > 0$ is the volatility and $W(t)$ is the Wiener process. With probability λdt a shock takes place, which starts a decline in demand representing a usual product's life cycle pattern.

The firm produces from the moment of investment onwards, so that

$$Q(t) = \begin{cases} 0 & \text{if } t < t_I \\ K & \text{if} t \geq t_I \end{cases}.$$

The investment costs are proportional to the capacity K and we assume that the firm produces up to capacity. Denoting I as investment, we thus have $I = \delta K$.

The investment problem that the firm is facing is to maximize the expected profit from the moment t_I that the investment is made. The control variables are thus the time at which the investment is undertaken, and the capacity level that the firm acquires at t_I.

$$\max_{t_I \geq 0, Q(t_I)=K \geq 0} V(X,Q) \quad = \quad \max_{t_I \geq 0, Q(t_I)=K\geq 0} \mathbb{E}\left[\int_{t=t_I}^{\infty} e^{-rt} Q(t_I)\cdot P(t)dt - e^{-rt_I}\delta Q(t_I) | X(0) = X\right], \tag{3}$$

where r is the discount rate with $r > \mu_1 > \mu_2$. We now transform the optimal t_I^* by the trigger point X^*. Let X^* be the value at which the firm is indifferent between investing and not. Thus for $X > X^*$, it is optimal to invest immediately, whereas for $X < X^*$ demand is still too low to undertake the investment and thus the firm waits. The optimal investment time t_I^* equals the first time that the stochastic process X reaches this level X^*.

We can solve the double maximization in two steps, first for a given X we maximize V w.r.t. Q. Both the dynamic programming and contingent claims approach value the real option that is present in the discussed optimization problem. The option of waiting is added to the net present valuation technique.

By $V(X)$ we denote the pure NPV value and let $F(X)$ be the option value. The boundary conditions are

$$F(X^*) = V(X^*, Q) - \delta Q \tag{4}$$

$$\frac{\partial F(X)}{\partial X}|_{X=X^*} = \frac{\partial V(X,Q)}{\partial X}|_{X=X^*}, \tag{5}$$

where (4) is the value-matching condition that states that when the firm invests at optimality the net payoff equals the option value. For $X < X^*$ the option value $F(X) > V(X) - \delta Q$ and thus it is better to wait until $F(X) = V(X) - \delta Q$ after which it is optimal to invest and receive the net value function. And (5) is the smooth pasting condition.

The event that causes a drop in demand occurs randomly and follows a Poisson process. At any time t the project's demand switches regime with probability λdt. The cumulative probability that this happens before t from the initial time zero onwards, equals $1 - e^{-\lambda t}$. And the density of the Poisson distribution is $\lambda e^{-\lambda t}$.

3 PLC-leader

In this section we consider the case in which the threat of a decline can happen only after the investor has entered the market and in the next section we consider the case in which the decline can also happen before he has entered the market. Hence these two scenarios differ in their continuation region. We call this the PLC-leader and PLC-follower setting respectively. In the first case the product lifecycle starts as soon as the investor is in the market while in the second case the product lifecycle is already running. [Add examples]

First we derive the optimal threshold for the PLC-leader when, with probability λdt, the demand's stochastic component switches from a geometric Brownian motion with an upward drift to a downward drift for a fixed capacity choice K. Next we consider both the optimal threshold and capacity choice for the PLC-leader.

3.1 Fixed capacity: PLC-leader

With probability λdt the demand's stochastic component switches from a geometric Brownian motion with an upward drift to a downward drift. Consider a project in which the drift parameter can change from $\mu_1 > 0$ to $\mu_2 < 0$ with probability λdt. Here we assume that the capacity K is fixed, implying that the investor only has to decide about when he enters the market, i.e. the optimal threshold X.

We assume that the PLC starts when the investor enters the market. Thus the threat of a shock that causes a decline in demand is introduced as soon as the market leader invests and not earlier: PLC-leader.

First we derive the stopping value $V^{(1)}(X,K)$, and the value of the option, $F^{(1)}(X,K)$. The superscript (1) represents the value when X starts in the μ_1 regime and switching to μ_2 with probability λdt while threshold (2) represents the value in the μ_2 regime only.

We solve this by decomposing the value of the project into two. First we consider the stage in which demand has a negative trend, i.e. the recession. The value of the project at time t in stage 2 can be expressed as the sum of the operating profit over the interval $(t, t+dt)$ and the continuation value beyond $t+dt$.

$$rV^{(2)}(X,K) = \text{Profit}(X,K) + \lim_{dt\downarrow 0}\frac{1}{dt}\mathbb{E}[dV^{(2)}] \tag{6}$$

where

$$\mathbb{E}[dV^{(2)}] = \frac{\partial V^{(2)}}{\partial X}\mu_2 X dt + \frac{1}{2}\frac{\partial^2 V^{(2)}}{\partial X^2}\sigma^2 X^2 dt. \tag{7}$$

Hence we get

$$rV^{(2)} = KX(1-\eta K) + \frac{\partial V^{(2)}}{\partial X}\mu_2 X + \frac{1}{2}\frac{\partial^2 V^{(2)}}{\partial X^2}\sigma^2 X^2 \tag{8}$$

Proposition 1. *The value function*

$$V^{(2)}(X,K) \quad = \quad \frac{XK(1-\eta K)}{r-\mu_2} \tag{9}$$

solves differential equation (8).

Proof. See Appendix A.2 □

Now we consider the stage in which demand has a positive trend but with probability λdt switches to $V^{(2)}$. The value of the project at time t in stage 1 can be expressed as the sum of the operating profit over the interval $(t, t+dt)$ and the continuation value beyond $t+dt$

$$rV^{(1)}(X,K) = \text{Profit}(X,K) + (1-\lambda)\lim_{dt\downarrow 0}\frac{1}{dt}\mathbb{E}[dV^{(1)}] + \lambda\lim_{dt\downarrow 0}\frac{1}{dt}\mathbb{E}[dV^{(2)}] \tag{10}$$

Rewriting gives

$$rV^{(1)} = KX(1-\eta K) + V_X^{(1)}\mu_1 X + \frac{1}{2}\sigma^2 X^2 V_{XX}^{(1)} + \lambda\left(-V^{(1)} + V^{(2)}\right) \tag{11}$$

Proposition 2. *The value function*

$$V^{(1)}(X,K) = \frac{K(1-\eta K)X(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} \tag{12}$$

solves differential equation (11).

Proof. See Appendix A.4 □

If the event of a switch in the demand process can only happen after investment, then the continuation region is defined by

$$rF^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 \tag{13}$$

Theorem 3.1. *The value matching and smooth pasting conditions imply that the optimal threshold for a fixed capacity for the PLC-leader is given by*

$$X^*_{KL} = \frac{\beta_{(1)}}{\beta_{(1)}-1}\frac{\delta(r-\mu_1+\lambda)(r-\mu_2)}{(1-\eta K)(r-\mu_2+\lambda)} \tag{14}$$

where $\beta_{(1)}$ is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta_{(1)}(\beta_{(1)}-1) + \mu_1\beta_{(1)} - r = 0 \tag{15}$$

Proof. See Appendix A.1 □

Sensitivities with respect to the probability of a decline and the upward and downward trend are provided in the following proposition.

Proposition 3.

$$\frac{\partial X^*_{KL}}{\partial\lambda} = \frac{\delta(r-\mu_2)(\mu_1-\mu_2)}{(1-\eta K)(r-\mu_2+\lambda)^2(\beta-1)} > 0 \tag{16}$$

$$\frac{\partial X^*_{KL}}{\partial\mu_1} = \frac{\delta(r-\mu_2)\beta\left((\beta-1)\sigma^2+2(\lambda+\beta\mu_1-r)\right)}{(r-\mu_2+\lambda)(\beta-1)^2(1-\eta K)(2\mu_1+(2\beta-1)\sigma^2)} = \begin{cases} >0 & (\beta-1)^2\sigma^2 < 2\lambda \\ <0 & (\beta-1)^2\sigma^2 > 2\lambda \end{cases} \tag{17}$$

$$\frac{\partial X^*_{KL}}{\partial\mu_2} = -\frac{\delta\beta\lambda(r+\lambda-\mu_1)}{(1-\eta K)(r-\mu_2+\lambda)^2(\beta-1)} < 0 \tag{18}$$

Some other comparative statics yield

$$\lim_{\lambda->\infty} X^*_{KL} = \frac{\beta_{(1)}}{\beta_{(1)}-1}\frac{\delta(r-\mu_2)}{(1-\eta K)} \neq X_{K2}, \tag{19}$$

where X_{K2} is the optimal threshold as in Huisman and Kort (2015) under a μ_2-regime for fixed K. Due to the fact that in the continuation region demand grows with μ_1, the optimal threshold depends on $\beta_{(1)}$ and will not converge to the threshold of an always μ_2-regime. Moreover,

$$X^*_{KL} > X_{K1},$$

implies that for fixed capacity the PLC-leader invests at a higher threshold compared to a situation where the decline stage never happens.

3.2 Capacity choice: PLC-leader

While considering the possibility of the change in demand to happen only once the investment has taken place, we now let the investor decide on both the timing and the capacity. We optimize both the market entrance moment and the capacity for the PLC-leader. When the investor enters the market he has to determine the size of the plant and thus set the capacity of production goods. We abstract from subsequent decisions on decreasing the production level temporarily or keeping inventory which take place within the lifecycle. The stage 0 that we add to the PLC stages takes the PLC as given and focusses on the question when the investor should start producing.

Proposition 4. *The optimal capacity is given by*

$$K^*(X) = \frac{1}{2\eta}\left(1-\frac{\delta(r-\mu_1+\lambda)(r-\mu_2)}{X(r-\mu_2+\lambda)}\right) \tag{20}$$

Proof. See Appendix A.11 □

If the event of a switch in the demand process can only happen after investment, then the waiting region is defined by

$$rF^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 \tag{21}$$

Theorem 3.2. *The value matching and smooth pasting conditions imply that the optimal thresh-*

old and capacity of benchmark 2 are given by

$$X_{CL}^* = \frac{\beta_{(1)}+1}{\beta_{(1)}-1}\delta\frac{(r-\mu_1+\lambda)(r-\mu_2)}{r-\mu_2+\lambda} \tag{22}$$

$$K_{CL}^* = \frac{1}{\eta(\beta_{(1)}+1)} \tag{23}$$

where $\beta_{(1)}$ is the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta_{(1)}(\beta_{(1)}-1)+\mu_1\beta_{(1)}-r=0 \tag{24}$$

Proof. See Appendix A.12 □

Some associated sensitivities are

Proposition 5.

$$\frac{\partial X_{CL}^*}{\partial\lambda} = \frac{\delta(r-\mu_2)(\mu_1-\mu_2)(\beta+1)}{(r-\mu_2+\lambda)^2(\beta-1)} > 0 \tag{25}$$

$$\frac{\partial X_{CL}^*}{\partial\mu_1} = \frac{\delta(r-\mu_2)\left(2\mu_1-\sigma^2+\frac{4\beta\lambda}{(\beta-1)^2}\right)}{(r-\mu_2+\lambda)\sqrt{8r\sigma^2+(-2\mu_1+\sigma^2)^2}} = \begin{cases} >0 & 2\mu_1-\sigma^2+\frac{4\beta\lambda}{(\beta-1)^2}>0 \\ <0 & 2\mu_1-\sigma^2+\frac{4\beta\lambda}{(\beta-1)^2}<0 \end{cases} \tag{26}$$

$$\frac{\partial X_{CL}^*}{\partial\mu_2} = -\frac{\delta\lambda(r-\mu_1+\lambda)(\beta+1)}{(r-\mu_2+\lambda)^2(\beta-1)} < 0 \tag{27}$$

In Figure 2 we plot X and K against λ, μ_1, μ_2 and r. The solid blue line is X_{CL}^*, the dotted blue line is $X_{KL}(K_0)$, the dotted red line is X_{C1}^* and the dotted orange line is X_{C2}^*, where the optimal decisions without PLC are – as in Huisman and Kort (2015):

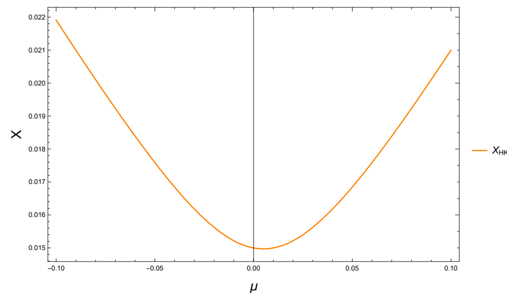
$$X_{Ki}^* = \frac{\beta_{(i)}}{\beta_{(i)}-1}\frac{\delta(r-\mu_i)}{(1-\eta K)} \tag{28}$$

$$X_{Ci}^* = \frac{\beta_{(i)}+1}{\beta_{(i)}-1}\delta(r-\mu_i) \tag{29}$$

$$K_{Ci}^* = \frac{1}{\eta(\beta_{(i)}+1)} \tag{30}$$

where i implies μ_i. When λ increases the project becomes less attractive and one would thus invest later. We observe that the capacity is unaffected by the potential decline.

Figure 1: Optimal X.



(a) $r = 0.1, \delta = 0.1, \eta = 0.05, \sigma = 0.1$

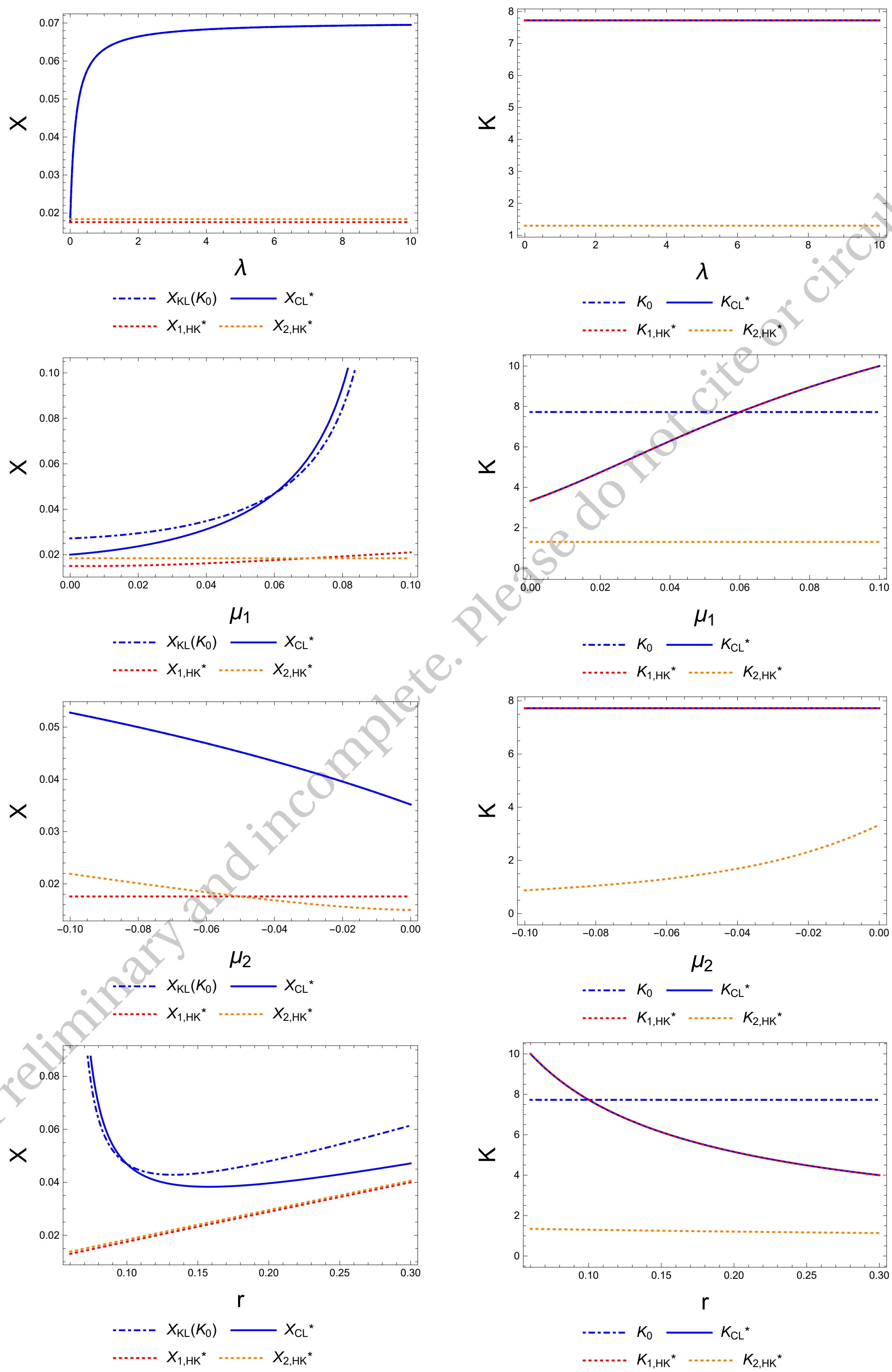
We observe that when μ_1 increases the project becomes more attractive but waiting implies an even higher μ_1 as this concerns the PLC-leader, therefore the threshold increases. This holds both for fixed capacity and also when the capacity can be freely chosen. In the latter case, the capacity increases in μ_1 causing a steeper increase in the threshold offsetting the higher investment costs. However, this does not necessarily mean that the investor enters later as a higher μ_1 also implies that the threshold is achieved earlier.

When μ_2 increases, i.e., becomes less negative, the decline regime is a less worse state and thus the project becomes more attractive causing the investor to enter at a lower threshold and here we can conclude earlier.

Several effects have an impact on the dependence of the threshold on the discount rate. If r increases then the NPV of the project decreases because of more heavily discounting and thus the project becomes less attractive causing an increase in the threshold to ensure a profitable project (*NPV effect*). When the discount rate is very low, the investor assigns higher weight to the future when the decline plays a role which he could postpone by delaying (*Delay effect*).Since the leader can achieve a higher X by waiting, he will do so when r is low as in those cases the potential decline is weighted heavily. When r increases the added value of the delay is discounted more heavily and thus diminishes. But at the same time a higher r leads to a lower NPV and thus makes the investor want to invest later. These two effects cause the non-monotone shape as depicted.

Similar as for fixed K, $\lim_{\lambda->\infty} X^*_{CL} \neq X_{C2}$. Thus, due to the fact that in the continuation region demand grows with μ_1, the optimal threshold depends on $\beta_{(1)}$ and will not converge to the threshold of an always μ_2-regime. Because X^*_{CL} is increasing in λ and decreasing in μ_2, it also follows that $X^*_{CL} > X_{C1}$.

Figure 2: PLC-leader. $K_0 = K_{CL}^*, r = 0.1, \mu_1 = 0.06, \mu_2 = -0.06, \delta = 0.1, \eta = 0.05, \sigma = 0.1, \lambda = 0.2$



4 PLC-follower

Contrary to the previous section, the decline in demand can now occur either before or after the investor entered the market, i.e., the investor is a PLC-follower. First we solve this for a fixed capacity, X^*_{KF}, and then for the optimal K^*_{CF} generating X^*_{CF}.

4.1 Fixed capacity: PLC-follower

The value in the continuation region under the μ_2 regime, similar as in the Huisman and Kort (2015) setting, is given by

Proposition 6. *The option value is*

$$F^{(2)}(X,K) = \begin{cases} A_2(K)X^{\beta_{(2)}} & \text{if } X < X_{K2} \\ V^{(2)}(X,K) - \delta K & \text{if } X \geq X_{K2} \end{cases} \tag{31}$$

when X follows a geometric Brownian motion with drift μ_2, where

$$X_{K2} = \frac{\beta_{(2)}\delta(r-\mu_2)}{(\beta_{(2)}-1)(1-\eta K)} \tag{32}$$

is the optimal threshold for a given capacity of K and $\beta_{(2)}$ is given by

$$\frac{1}{2}\sigma^2\beta_{(2)}(\beta_{(2)}-1) + \mu_2\beta_{(2)} - r = 0 \tag{33}$$

Proof. See Appendix A.3 □

If X is below the threshold, the investor should wait with investing, while if X is above the threshold he invests. The waiting period in case we start in the μ_1 regime is characterized by

$$rF^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 + \lambda\left(F^{(2)} - F^{(1)}\right) \tag{34}$$

We can now distinguish two cases, $X < X_{K2}$ and $X \geq X_{K2}$.

Proposition 7. *If $X \geq X_{K2}$ then (34) becomes*

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 + \lambda\left(V^{(2)}(X,K) - \delta K\right) \tag{35}$$

which solves for

$$F^{(c1)}(X,K) = C_1 X^{\beta^+_{(\lambda 1)}} + C_2 X^{\beta^-_{(\lambda 1)}} + \lambda\frac{K(1-\eta K)}{(r-\mu_2)(r+\lambda-\mu_1)}X - \frac{\lambda}{r+\lambda}\delta K \tag{36}$$

where $\beta^+_{(\lambda 1)}$ and $\beta^-_{(\lambda 1)}$ are the positive and negative root of

$$\frac{1}{2}\sigma^2\beta_{(\lambda 1)}(\beta_{(\lambda 1)}-1)+\mu_1\beta_{(\lambda 1)}-(r+\lambda)=0 \tag{37}$$

Proof. See Appendix A.5 □

Proposition 8. *If $X < X_{K2}$ then (34) becomes*

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X+\frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2+\lambda F^{(2)}(X) \tag{38}$$

which solves for

$$F^{(d1)}(X,K)=D_1 X^{\beta^+_{(\lambda 1)}}+\frac{\lambda}{r-\beta_{(2)}\left(\mu_1-\mu_2\right)}A_2(K)X^{\beta_{(2)}} \tag{39}$$

Proof. See Appendix A.6 □

The optimal threshold X_{KF}, C_1, C_2 and D_1 are unknown. For continuity, we need, at X_{K2}

$$F^{(c1)}(X_{K2},K) = F^{(d1)}(X_{K2},K) \tag{40}$$

$$\left.\frac{\partial F^{(c1)}(X,K)}{\partial X}\right|_{X=X_{K2}} = \left.\frac{\partial F^{(d1)}(X,K)}{\partial X}\right|_{X=X_{K2}} \tag{41}$$

And the value matching and smooth pasting conditions imply

$$V^{(1)}(X_{KF},K)-\delta K = F^{(1)}(X_{KF},K) \tag{42}$$

$$\left.\frac{\partial V^{(1)}(X,K)}{\partial X}\right|_{X=X_{KF}} = \left.\frac{\partial F^{(1)}(X,K)}{\partial X}\right|_{X=X_{KF}} \tag{43}$$

where either $F^{(c1)}$ or $F^{(d1)}$ can be used as $F^{(1)}$. For fixed K we thus have two system of equations that imply the optimal threshold. In both cases, there are four unknowns and four equations, i.e. C_1, C_2, D_1, X_{KF}.

Proposition 9. *For $X \geq X_{K2}$, the implicit function of X_{KF} is given by*

$$f(X_{KF}) =$$

$$\left((\beta^+_{(\lambda 1)}-1)\lambda\frac{K(1-\eta K)}{(r-\mu_2)(r+\lambda-\mu_1)}X_2(K)-\beta^+_{(\lambda 1)}\frac{\lambda}{r+\lambda}\delta K+\left(\beta_{(2)}-\beta^+_{(\lambda 1)}\right)a\left(\frac{X_2(K)K(1-\eta K)}{r-\mu_2}-\delta K\right)\right)\left(\frac{X_{KF}}{X_2(K)}\right)^{\beta^-_{(\lambda 1)}}$$

$$-\left(1-\beta^+_{(\lambda 1)}\right)\frac{K(1-\eta K)}{(r+\lambda-\mu_1)}X_{KF}-\beta^+_{(\lambda 1)}\delta K\frac{r}{r+\lambda}=0 \tag{44}$$

where $a=\frac{\lambda}{r-\beta_{(2)}\left(\mu_1-\mu_2\right)}$.

Proof. See Appendix A.7 □

Proposition 10. *For $X < X_{K2}$, the implicit function of X_{KF} is given by*

$$g(X_{KF}) = \left(\beta_{(2)} - \beta^{+}_{(\lambda 1)}\right) a \left(\frac{X_2(K)K(1-\eta K)}{r-\mu_2} - \delta K\right)\left(\frac{X_{KF}}{X_2(K)}\right)^{\beta_{(2)}} - \frac{K(1-\eta K)(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} X_{KF}\left(1-\beta^{+}_{(\lambda 1)}\right) - \beta^{+}_{(\lambda 1)}\delta K = \tag{45}$$

Proof. See Appendix A.8 □

Proposition 11. *The implicit function when $X \geq X_{K2}$ has no solution.*

Proof. See Appendix A.9 □

Theorem 4.1. *The optimal threshold $X_{KF} \leq X_{K2}$ is given by Proposition 10.*

Proof. See Appendix A.10 □

Under the assumption that $\mu_2 < \mu_1$, it follows that $X_{KF} \leq X_{K2}$. Hence, the firm invest earlier compared to a μ_2-regime and we conjecture later investment than in a μ_1-regime.

4.2 Capacity choice: PLC-follower

We now consider the setting in which both the decline in demand can also occur in the waiting period and the capacity has to be chosen by the investor. As we saw before, if the decline in demand can also occur in the waiting period, then $F(X)$ has to be composed into two. Let $F^{(2)}(X)$ be the option value when the drift of X is μ_2, then

$$rF^{(2)} = \frac{\partial F^{(2)}}{\partial X}\mu_2 X + \frac{1}{2}\frac{\partial^2 F^{(2)}}{\partial X^2}\sigma^2 X^2 \tag{46}$$

which solves for a form of $F^{(2)}(X) = A_2 X^{\beta_{(2)}}$ where $\beta_{(2)}$ is the positive root of

$$\frac{1}{2}\sigma^2\beta_{(2)}(\beta_{(2)}-1) + \mu_2\beta_{(2)} - r = 0 \tag{47}$$

After this waiting period, only $V^{(2)}(X)$ can happen, as the decline has already been set in motion. The value matching and smooth pasting conditions applied to $V^{(2)}(X)$ and $F^{(2)}(X)$ simply give the results as in Huisman and Kort (2015) with μ_2 as drift parameter.

$$X_{C2} = \frac{\beta_{(2)}+1}{\beta_{(2)}-1}\delta(r-\mu_2) \tag{48}$$

$$K_{C2} = \frac{1}{\eta(\beta_{(2)}+1)} \tag{49}$$

Thus if X is smaller than this optimal threshold, it is better to wait,and otherwise to invest.

$$F^{(2)}(X) = \begin{cases} A_2 X^{\beta_{(2)}} & \text{if } X < X_{C2} \\ V^{(2)}(X) - \delta K^*_{(2)}(X) & \text{if } X \geq X_{C2} \end{cases} \tag{50}$$

In case the switch from growth to decline has not happened yet, the waiting period is characterized by

$$rF^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 + \lambda\left(F^{(2)} - F^{(1)}\right) \tag{51}$$

We can now distinguish two cases, $X < X_{C2}$ and $X \geq X_{C2}$.

Proposition 12. *If $X \geq X_{C2}$ then (51) becomes*

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 + \lambda\left(V^{(2)}(X, K_2^*(X)) - \delta K_2^*(X)\right) \tag{52}$$

which solves for

$$F^{(c1)}(X) = C_1 X^{\beta^+_{(\lambda 1)}} + C_2 X^{\beta^-_{(\lambda 1)}} + \frac{\lambda}{4\eta(r-\mu_2)(r-\mu_1+\lambda)}X - \frac{\lambda\delta}{2\eta(r+\lambda)} + \frac{\lambda\delta^2(r-\mu_2)}{4\eta(r+\mu_1+\lambda-\sigma^2)}X^{-1} \tag{53}$$

Proof. See Appendix A.13 □

Proposition 13. *If $X < X_{C2}$ then (51) becomes*

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 + \lambda F^{(2)}(X) \tag{54}$$

which solves for

$$F^{(d1)}(X) = D_1 X^{\beta^+_{(\lambda 1)}} + \frac{\lambda}{\lambda - \beta_{(2)}\left(\mu_1 - \mu_2\right)} A_2 X^{\beta_{(2)}} \tag{55}$$

Proof. See Appendix A.14 □

The optimal threshold X_{CF}, C_1, C_2 and D_1 are unknown. For continuity, we need, at X_{C2}

$$F^{(c1)}(X_{C2}) = F^{(d1)}(X_{C2}) \tag{56}$$

$$\left.\frac{\partial F^{(c1)}(X)}{\partial X}\right|_{X=X_{C2}} = \left.\frac{\partial F^{(d1)}(X)}{\partial X}\right|_{X=X_{C2}} \tag{57}$$

Moreover, by the value matching and smooth pasting conditions

$$\begin{aligned} V^{(1)} - \delta K &= F^{(1)} && (58) \\ \frac{\partial V^{(1)}(X)}{\partial X} &= \frac{\partial F^{(1)}(X)}{\partial X} && (59) \end{aligned}$$

where either $F^{(c1)}$ or $F^{(d1)}$ can be used as $F^{(1)}$.

Theorem 4.2. *If $X \geq X_{C2}$ then X is implied by*

$$0 = (\beta^-_{(\lambda 1)} - \beta^+_{(\lambda 1)}) C_2 X_1^{\beta^-_{(\lambda 1)}} + a X_1 (1 - \beta^+_{(\lambda 1)}) + \beta^+_{(\lambda 1)} b - c X_1^{-1} (1 + \beta^+_{(\lambda 1)}) - e(K) X_1 (1 - \beta^+_{(\lambda 1)}) - \beta^+_{(\lambda 1)} \delta K \tag{60}$$

where a, b, c, d, e, C_2 can be found in the appendix.

Proof. See Appendix A.15 □

Theorem 4.3. *If $X < X_{C2}$, then X_{CF} is given by the implicit function*

$$\begin{aligned} 0 \;=\; & (\beta_{(2)} - \beta^+_{(\lambda 1)}) \frac{\lambda}{\lambda - \beta_{(2)} (\mu_1 - \mu_2)} \frac{\delta}{\eta(\beta^2_{(2)} - 1)} \left(\frac{X_{CF}}{X_2} \right)^{\beta_{(2)}} \\ & - \frac{1}{2\eta} \left(1 - \frac{\delta (r - \mu_1 + \lambda)(r - \mu_2)}{X_{CF}(r - \mu_2 + \lambda)} \right) \left(\frac{X_{CF} (1 - \beta^+_{(\lambda 1)}) \frac{1}{2} \left(1 + \frac{\delta (r - \mu_1 + \lambda)(r - \mu_2)}{X_{CF}(r - \mu_2 + \lambda)} \right) (r - \mu_2 + \lambda)}{(r - \mu_1 + \lambda)(r - \mu_2)} + \beta^+_{(\lambda 1)} \delta \right) \end{aligned} \tag{61}$$

Proof. See Appendix A.16 □

In Figure 3 we plot X and K against λ, μ_1, μ_2 and r. The solid green line is X^*_{CF}, the dotted green line is $X_{KF}(K_0)$, the dotted red line is X^*_{C1} and the dotted orange line is X^*_{C2} the latter two similar as in Figure 2.

For fixed capacity the optimal threshold now decreases in μ_1 reflecting that the project is more profitable and driven the fact that delaying no longer postpones the likelihood of a decline. However, for the optimal size, the threshold is increasing in μ_1. This can be explained by the fact that the capacity is increasing too.

The optimal threshold is in between X_{C1} and the X_{C2}. In the limit $\lambda \to \infty$ the decision converges to the capacity and threshold of the μ_2-regime. Since X_{Ci} in symmetric around the point $\mu_i = \frac{1}{2}\sigma^2$, this implies that $X_{C1} > X_{C2}$ when $\text{dist}(\mu_1) > \text{dist}(\mu_2)$ where $\text{dist}(x) = |x| - \text{sign}(x)\frac{1}{2}\sigma^2$ as can be seen in Figure 1. Thus

$$\lim_{\lambda -> \infty} X_{CF} \begin{cases} > X_{C1} & \text{if } \text{dist}(\mu_1) < \text{dist}(\mu_2) \\ < X_{C1} & \text{if } \text{dist}(\mu_1) > \text{dist}(\mu_2). \end{cases}$$

The quantity effect can cause a non-monotone effect of λ on the threshold as shown in Figure 4. The sharp decline in the optimal capacity causes the investor to invest earlier. Thereafter the capacity converges to the capacity of the μ_2-regime causing the threshold to increase in λ as the project becomes less attractive. Note that we considered three specific choice of μ_2 generating three different orderings due to the convex shape as shown in Figure 1.

Figure 3: PLC-follower. $K_0 = K^*_{CF}, r = 0.1, \mu_1 = 0.06, \mu_2 = -0.06, \delta = 0.1, \eta = 0.05, \sigma = 0.1, \lambda = 0.2$

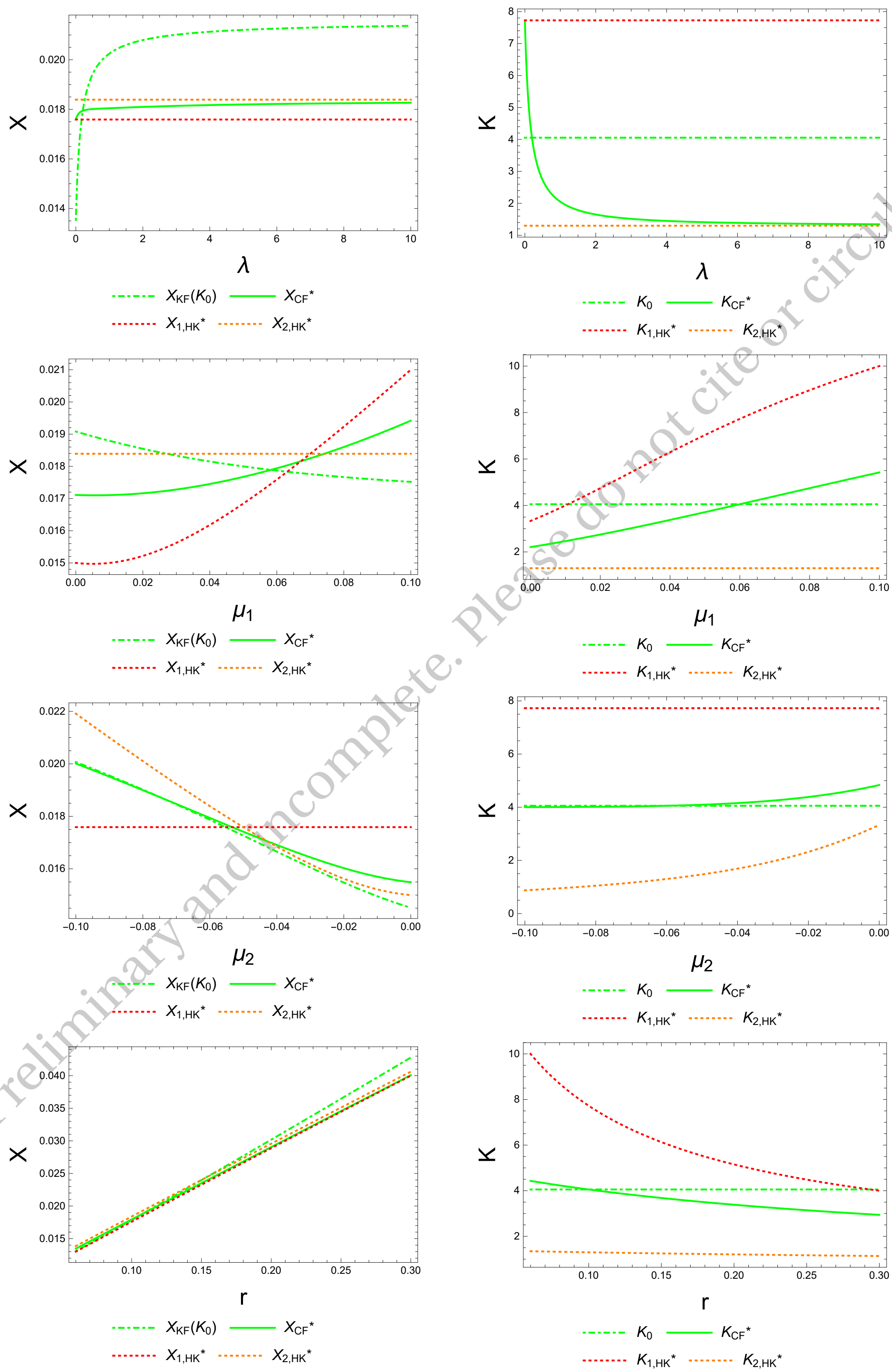
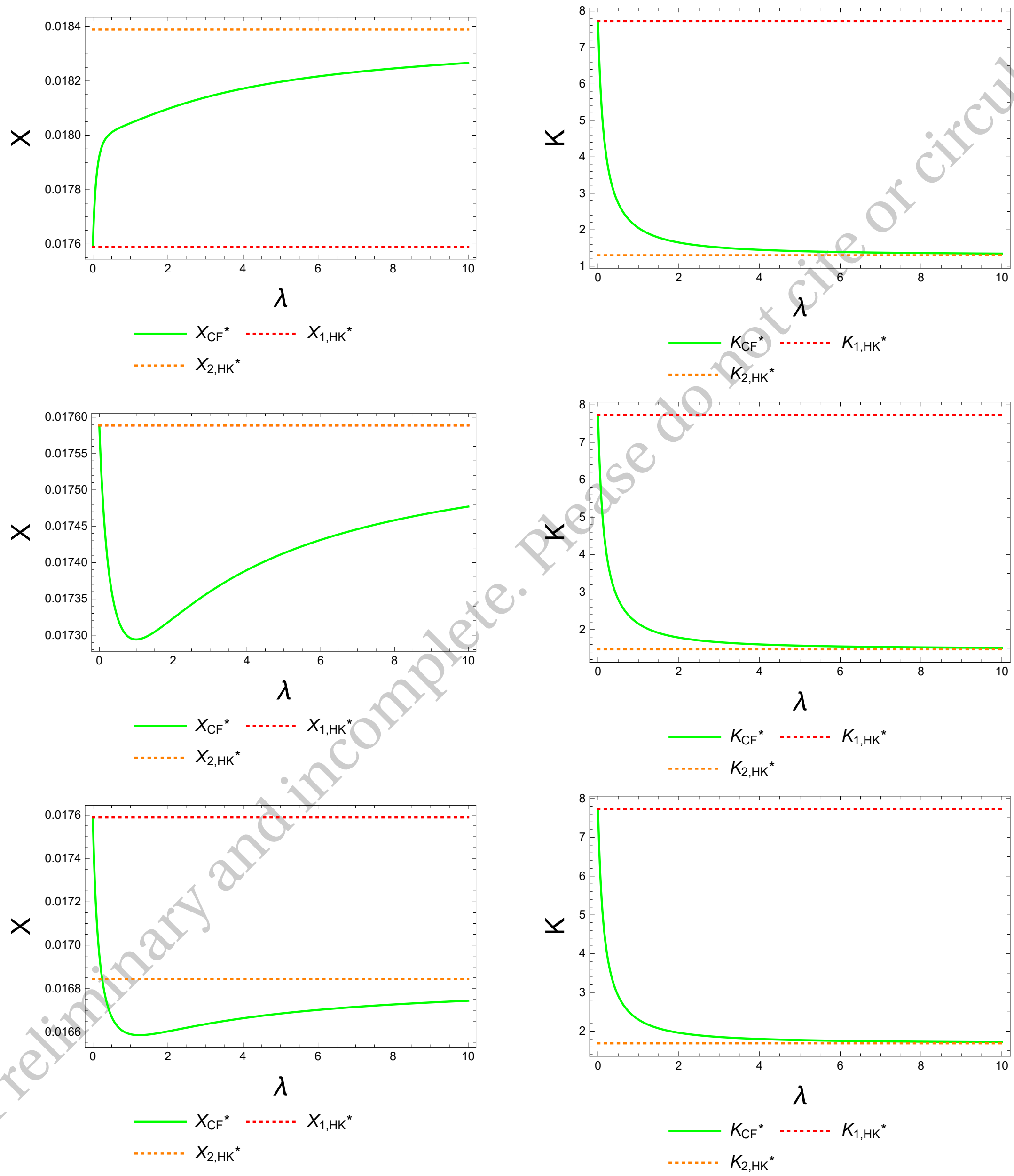


Figure 4: PLC-follower. $r = 0.1, \mu_1 = 0.06, \mu_2 = \{-0.06, -0.05, -0.04\}, \delta = 0.1, \eta = 0.05, \sigma = 0.1$



4.2.1 Probability

By Sarkar (2000) the probability of reaching the critical level X_1 (the probability of investing) within some time period t is given by (this can be derived from Harrison (1985), (pp.11-14))

$$P_1(X_1,t) = \Phi\left(\frac{\ln(X_0/X_1)+(\mu_1-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)+\left(\frac{X_1}{X_0}\right)^{2\frac{\mu_1}{\sigma^2}-1}\Phi\left(\frac{\ln(X_0/X_1)-(\mu_1-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \tag{62}$$

when X, conditional on before the event thus with a drift of μ_1, follows

$$dX(t) = \mu_1 X(t)dt + \sigma X(t)dW(t), X(0) = X_0. \tag{63}$$

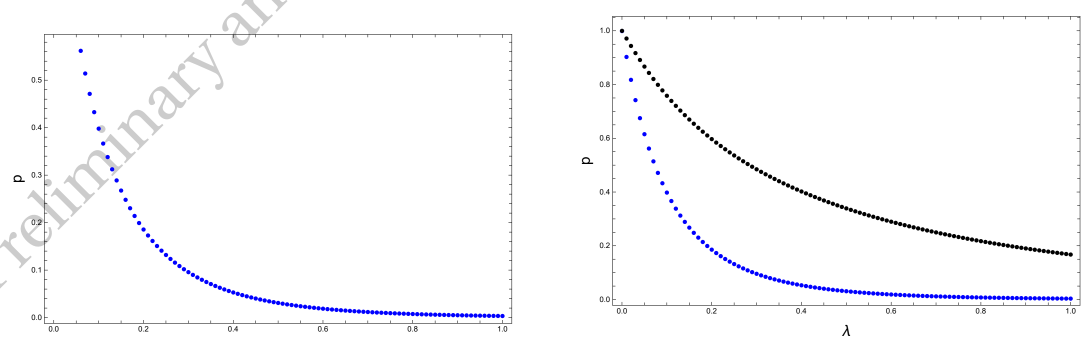
We call "the λ-event" the event that demand switches from a growth to a decline process. The event that causes a drop in demand occurs randomly and follows a Poisson process. At any time t the project's demand switches regime with probability λdt. The probability that this happens at time t is $\lambda e^{-\lambda t}$.

The probability that the firm has already invested while the event of a switch has not taken place yet, i.e. he invests before the decline phase starts, is given by

$$p = \int_0^\infty P_1(X_1,t)\lambda e^{-\lambda t}dt. \tag{64}$$

The threshold is obtained from the implicit function that is solved numerically in the previous section. For fixed parameter values and X_1 we observe in Figure 5 that if λ increase then p decreases and if X_0 increases then p increases. The probability depends on X_0.

Figure 5: Probability.



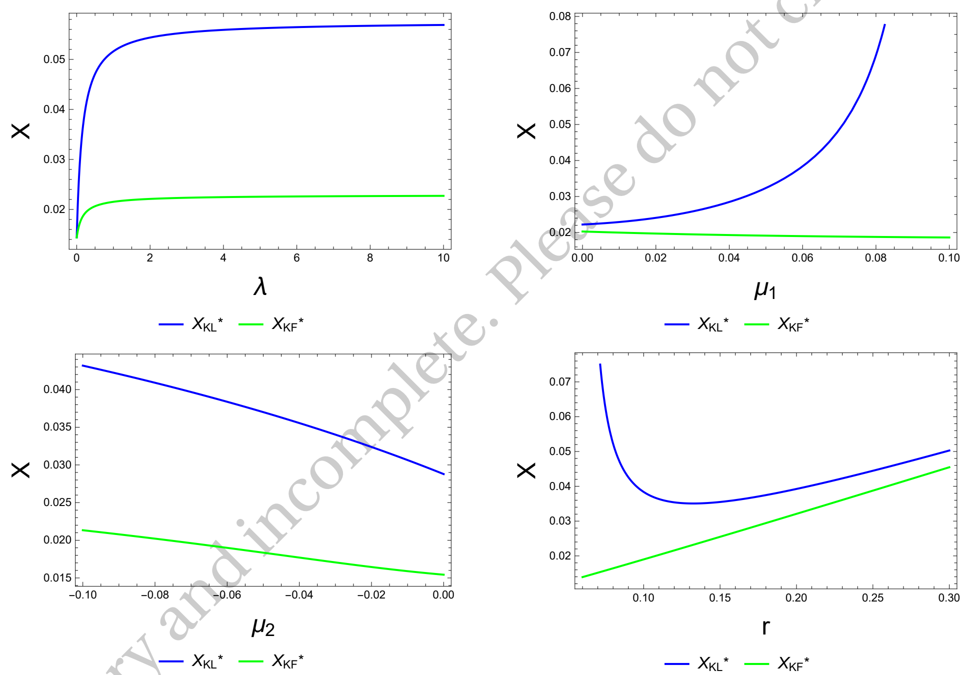
(a) $X_0 = 0.010, r = 0.1, \mu_1 = 0.06, \delta = 0.1, \eta = 0.05, \sigma = 0.1, \mu_2 = -0.06$

(b) Blue $X_0 = 0.010$ and black $X_0 = 0.015$, $r = 0.1, \mu_1 = 0.06, \delta = 0.1, \eta = 0.05, \sigma = 0.1, \mu_2 = -0.06$

5 Comparison PLC-leader and PLC-follower

First, wee compare the decisions of the PLC-leader and PLC-follower for a fixed capacity in Figure 6. We observe that the leader always invests later than the follower as the leader can let the market grow without any effort but simply by waiting. The two interesting differences are with respect to the μ_1 and r dependence. For a fixed capacity the threshold decreases when μ_1 increases because the project gets more attractive. As the leader can postpone the decline phase by waiting, he sets a higher threshold to get an even more attractive project. Since the follower has no such power, he invests earlier.

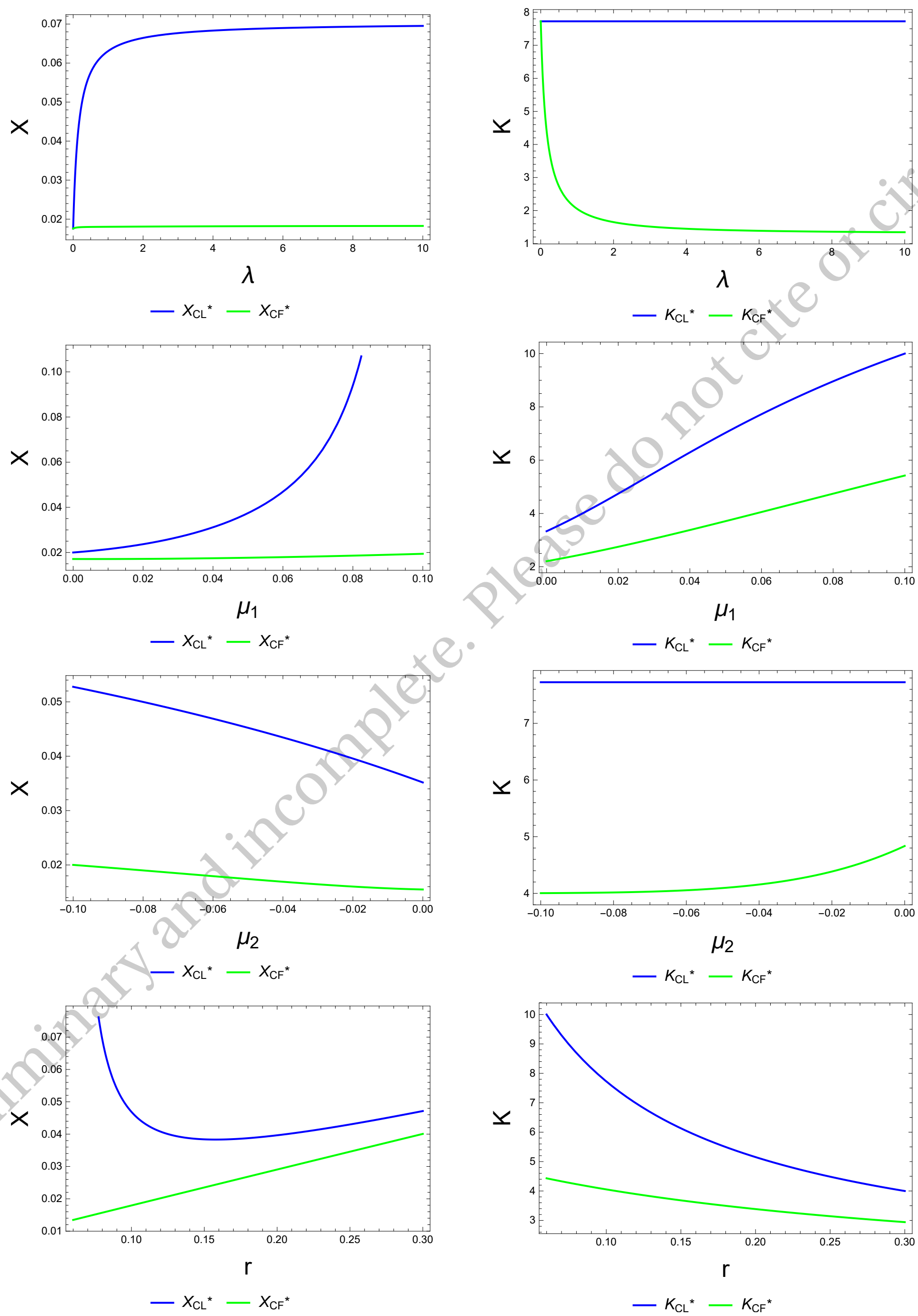
Figure 6: Fixed capacity PLC-leader and PLC-follower. $K_0 = 5, r = 0.1, \mu_1 = 0.06, \mu_2 = -0.06, \delta = 0.1, \eta = 0.05, \sigma = 0.1, \lambda = 0.2$



In Figure 7 we compare the PLC-leader and PLC-follower for their optimal capacity. If μ_1 increases then the project becomes more attractive and one can either invest earlier and/or more now that the investors also set the capacity. The fact that capacity increases in μ_1 could also imply that the threshold does not decrease. This is what we observe for the PLC-follower. We already explained that the possibility to benefit from an even higher regime makes the leader to set an even higher threshold.

Since the leader can achieve a higher X by waiting, he will do so when r is low as in those cases the potential decline is weighted heavily. When r increases the added value of the delay is discounted more heavily and thus diminishes. But at the same time a higher r leads to a lower NPV and thus makes the investor want to invest later. These two effects cause the non-monotone shape as depicted. The delay effect is only advantageous for the leader while delaying is disadvantageous for the follower. Thus only the NPV effect is present for the follower.

Figure 7: Fixed capacity PLC-leader and PLC-follower. $r = 0.1, \mu_1 = 0.06, \mu_2 = -0.06, \delta = 0.1, \eta = 0.05, \sigma = 0.1, \lambda = 0.2$



A Appendix

This appendix contains the proofs of the propositions.

A.1 Proof of Theorem 3.1

The differential equation solves for a form of $F^{(1)}(X) = A_1 X^{\beta_{(1)}}$. The value matching condition is

$$V^{(1)} - \delta K = F^{(1)} \tag{65}$$

and the smooth pasting

$$\frac{\partial V^{(1)}(X)}{\partial X} = \frac{\partial F^{(1)}(X)}{\partial X} \tag{66}$$

together these imply

$$\frac{\partial V^{(1)}(X)}{\partial X} = (V^{(1)}(X) - \delta K)\beta_{(1)} X^{-1} \tag{67}$$

and determine

$$X^*(K) = \frac{\beta_{(1)}}{\beta_{(1)} - 1} \frac{\delta(r - \mu_1 + \lambda)(r - \mu_2)}{(1 - \eta K)(r - \mu_2 + \lambda)} \tag{68}$$

A.2 Proof of Proposition 1

The homogeneous equation (terms involving value function) is

$$0 = \frac{\partial V^{(2)}}{\partial X}\mu_2 X + \frac{1}{2}\frac{\partial^2 V^{(2)}}{\partial X^2}\sigma^2 X^2 - r V^{(2)} \tag{69}$$

with solution

$$V^{(2)}(X) = B_1 X^{\beta_+} + B_2 X^{\beta_-} \tag{70}$$

where $\beta_+ > 1, \beta_- < 0$. For a particular solution of the total equation we propose

$$V^{(2)}(X) = aX + b \tag{71}$$

$$r(aX + b) = KX(1 - \eta K) + a\mu_2 X \tag{72}$$

$$a = \frac{K(1 - \eta K)}{r - \mu_2} \tag{73}$$

$$b = 0. \tag{74}$$

The total solution is the sum of the homogeneous solution and particular solution

$$V^{(2)}(X) = \frac{XK(1-\eta K)}{r-\mu_2} + B_1 X^{\beta_+} + B_2 X^{\beta_-} \tag{75}$$

The boundary conditions

$$V^{(2)}(0) = 0 \tag{76}$$

$$\lim_{X\to\infty} V^{(2)}(X) = wX \tag{77}$$

Since $\beta_- < 0$, X^{β_-} will go to infinity when X goes to zero. Thus (76) leads to $B_2 = 0$. And (77) refers to the exclusion of speculative bubbles, i.e. in the limit the value function is linear in X where w is a constant implying $B_1 = 0$.

A.3 Proof of Proposition 6

The optimal threshold is obtained by maximizing the value function of Proposition 1, $A_{(2)}$ is obtained by equality of the two cases at this threshold,

$$A_2(K) = \left(V^{(2)}(X_{K2}, K) - \delta K\right)(X_{K2})^{-\beta_{(2)}} \tag{78}$$

A.4 Proof of Proposition 2

Plug $V^{(2)}$ of Proposition 1 and use the Ansatz $V^{(1)}(X) = aX + b$.

A.5 Proof of Proposition 7

Plugging $F^{(2)}$ from Proposition 6 under the assumption that $X \geq X_{K2}$ leads to

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 + \lambda\left(\frac{XK(1-\eta K)}{r-\mu_2} - \delta K\right) \tag{79}$$

The homogeneous solution has form

$$F^{(1)} = C_1 X^{\beta^+_{(\lambda 1)}} + C_2 X^{\beta^-_{(\lambda 1)}} \tag{80}$$

solving

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 \tag{81}$$

For the particular solution we conjecture the functional form

$$F^{(1)}(X) = aX + b \tag{82}$$

Applied to the full equation gives

$$\begin{aligned}
(r+\lambda)(aX+b) &= a\mu_1 X + \lambda\left(\frac{XK(1-\eta K)}{r-\mu_2} - \delta K\right) && (83)\\
(r+\lambda)aX &= a\mu_1 X + \lambda\frac{XK(1-\eta K)}{r-\mu_2} && (84)\\
(r+\lambda)b &= -\lambda\delta K && (85)\\
a &= \lambda\frac{K(1-\eta K)}{(r-\mu_2)(r+\lambda-\mu_1)} && (86)\\
b &= -\frac{\lambda}{r+\lambda}\delta K && (87)
\end{aligned}$$

A.6 Proof of Proposition 8

Plugging $F^{(2)}$ from Proposition 6 under the assumption that $X < X_{K2}$ leads to

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 + \lambda A_2(K) X^{\beta_{(2)}} \tag{88}$$

The homogeneous equation (terms involving value function) is

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 \tag{89}$$

where the homogeneous solution has form

$$F^{(1)}(X) = D_1 X^{\beta^+_{(\lambda 1)}} + D_2 X^{\beta^-_{(\lambda 1)}} \tag{90}$$

Since $F^{(1)}(0) = 0$, $D_2 = 0$.

For the particular solution we conjecture

$$F^{(1)}(X) = aA_2(K)X^{\beta_{(2)}} + b \tag{91}$$

Plugging this into the ODE gives

$$
\begin{aligned}
(r+\lambda)\left(aA_2(K)X^{\beta_{(2)}}+b\right) &= a\beta_{(2)}A_2(K)X^{\beta_{(2)}}\mu_1+\frac{1}{2}a\beta_{(2)}(\beta_{(2)}-1)A_2(K)X^{\beta_{(2)}}\sigma^2+\lambda A_2(K)X^{\beta_{(2)}} \\
b &= 0 && (92)\\
(r+\lambda)a &= a\beta_{(2)}\mu_1+\frac{1}{2}a\beta_{(2)}(\beta_{(2)}-1)\sigma^2+\lambda && (93)\\
(r+\lambda-\beta_{(2)}\mu_1-\frac{1}{2}\beta_{(2)}(\beta_{(2)}-1)\sigma^2)a &= \lambda && (94)\\
a &= \frac{\lambda}{r+\lambda-\beta_{(2)}\mu_1-\frac{1}{2}\beta_{(2)}(\beta_{(2)}-1)\sigma^2} && (95)\\
a &= \frac{\lambda}{\lambda-\beta_{(2)}\left(\mu_1-\mu_2\right)} && (96)
\end{aligned}
$$

A.7 Proof of Proposition 9

At $X_{K2}=\frac{\beta_{(2)}\delta(r-\mu_2)}{(\beta_{(2)}-1)(1-\eta K)}$, (40) and (41) yield

$$C_1X^{\beta^+_{(\lambda 1)}}+C_2X^{\beta^-_{(\lambda 1)}}+\lambda\frac{K(1-\eta K)}{(r-\mu_2)(r+\lambda-\mu_1)}X-\frac{\lambda}{r+\lambda}\delta K=D_1X^{\beta^+_{(\lambda 1)}}+aA_2(K)X^{\beta_{(2)}} \tag{97}$$

$$\beta^+_{(\lambda 1)}C_1X^{\beta^+_{(\lambda 1)}-1}+\beta^-_{(\lambda 1)}C_2X^{\beta^-_{(\lambda 1)}-1}+\lambda\frac{K(1-\eta K)}{(r-\mu_2)(r+\lambda-\mu_1)}=\beta^+_{(\lambda 1)}D_1X^{\beta^+_{(\lambda 1)}-1}+\beta_{(2)}aA_2(K)X^{\beta_{(2)}-1} \tag{98}$$

Multiplying (98) by X and subtracting $\beta^+_{(\lambda 1)}$ times (97) leads to

$$C_2(K)=\frac{(\beta^+_{(\lambda 1)}-1)\lambda\frac{K(1-\eta K)}{(r-\mu_2)(r+\lambda-\mu_1)}X_2(K)-\beta^+_{(\lambda 1)}\frac{\lambda}{r+\lambda}\delta K+\left(\beta_{(2)}-\beta^+_{(\lambda 1)}\right)aA_2(K)X_2^{\beta_{(2)}}}{\left(\beta^-_{(\lambda 1)}-\beta^+_{(\lambda 1)}\right)(X_2(K))^{\beta^-_{(\lambda 1)}}} \tag{99}$$

At $X=X_{KF}$

$$C_1X^{\beta^+_{(\lambda 1)}}+C_2X^{\beta^-_{(\lambda 1)}}+\lambda\frac{K(1-\eta K)}{(r-\mu_2)(r+\lambda-\mu_1)}X-\frac{\lambda}{r+\lambda}\delta K=\frac{K(1-\eta K)X(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)}-\delta K \tag{100}$$

$$\beta^+_{(\lambda 1)}C_1X^{\beta^+_{(\lambda 1)}-1}+\beta^-_{(\lambda 1)}C_2X^{\beta^-_{(\lambda 1)}-1}+\lambda\frac{K(1-\eta K)}{(r-\mu_2)(r+\lambda-\mu_1)}=\frac{K(1-\eta K)(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} \tag{101}$$

Rewriting these gives

$$C_1 X^{\beta^+_{(\lambda 1)}} + C_2 X^{\beta^-_{(\lambda 1)}} - \frac{K(1-\eta K)}{(r+\lambda-\mu_1)} X + \frac{r}{r+\lambda}\delta K = 0 \tag{102}$$

$$\beta^+_{(\lambda 1)} C_1 X^{\beta^+_{(\lambda 1)}} + \beta^-_{(\lambda 1)} C_2 X^{\beta^-_{(\lambda 1)}} - \frac{K(1-\eta K)}{(r+\lambda-\mu_1)} X = 0 \tag{103}$$

Multiplying the SP condition by X and subtracting $\beta^+_{(\lambda 1)}$ times VM condition leads to

$$0 = \left(\beta^-_{(\lambda 1)} - \beta^+_{(\lambda 1)}\right) C_2(K) X_1^{\beta^-_{(\lambda 1)}} - \left(1 - \beta^+_{(\lambda 1)}\right) \frac{K(1-\eta K)}{(r+\lambda-\mu_1)} X_1 - \beta^+_{(\lambda 1)} \delta K \frac{r}{r+\lambda} \tag{104}$$

Plugging in C_2 leads to the implicit function.

A.8 Proof of Proposition 10

Consider, at $X = X_{KF}$,

$$D_1 X^{\beta^+_{(\lambda 1)}} + a A_2(K) X^{\beta_{(2)}} = \frac{K(1-\eta K)X(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} - \delta K \tag{105}$$

$$\beta^+_{(\lambda 1)} D_1 X^{\beta^+_{(\lambda 1)}-1} + \beta_{(2)} a A_2(K) X^{\beta_{(2)}-1} = \frac{K(1-\eta K)(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} \tag{106}$$

Multiplying the SP condition by X and subtracting $\beta^+_{(\lambda 1)}$ times VM condition leads to

$$\left(\beta_{(2)} - \beta^+_{(\lambda 1)}\right) a A_2(K) X_1^{\beta_{(2)}} - \frac{K(1-\eta K)(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} X_1 \left(1 - \beta^+_{(\lambda 1)}\right) - \beta^+_{(\lambda 1)} \delta K = 0 \tag{107}$$

A.9 Proof of Proposition 11

If $X_1 = 0$ then

$$f(0) = -\beta^+_{(\lambda 1)} \delta K \frac{r}{r+\lambda} < 0 \tag{108}$$

If $X_1 = X_2(K) = \frac{\beta_{(2)}\delta(r-\mu_2)}{(\beta_{(2)}-1)(1-\eta K)}$ then

$$f(X_2) = \left((\beta^+_{(\lambda 1)} - 1)\beta_{(2)} \frac{(r+\lambda-\mu_2)}{(\lambda+r-\mu_1)} - \beta^+_{(\lambda 1)}(\beta_{(2)}-1) + (\beta_{(2)} - \beta^+_{(\lambda 1)}) a \right) \frac{\delta K}{\beta_{(2)}-1} <?> 0 \tag{109}$$

Recall that a is given by (96)

$$a = \frac{\lambda}{\lambda - \beta_{(2)}\left(\mu_1 - \mu_2\right)} \tag{110}$$

We simplify the implicit function to

$$f(X_2) = -\frac{\delta K(\mu_1-\mu_2)\beta_{(2)}}{(\beta_{(2)}-1)(r+\lambda-\mu_1)(\lambda-\beta_{(2)}(\mu_1-\mu_2))}\left(\beta^+_{(\lambda 1)}\left\{\beta_{(2)}(\mu_1-\mu_2)+(r-\mu_1)\right\}+\left\{\lambda-\beta_{(2)}(r+\lambda-\mu_2)\right\}\right) \tag{111}$$

Let

$$f(X_2) = c\frac{T(\lambda)}{N(\lambda)} \tag{112}$$

where

$$T(\lambda) = -(\mu_1-\mu_2)\left(\beta^+_{(\lambda 1)}\left\{\beta_{(2)}(\mu_1-\mu_2)+(r-\mu_1)\right\}+\left\{\lambda-\beta_{(2)}(r+\lambda-\mu_2)\right\}\right) \tag{113}$$

$$N(\lambda) = (r+\lambda-\mu_1)(\lambda-\beta_{(2)}(\mu_1-\mu_2)) \tag{114}$$

$$c = \frac{\delta K\beta_{(2)}}{(\beta_{(2)}-1)} \tag{115}$$

The same applies here as in Appendix A.10, the proof of Theorem 4.1 so $f(X_2) > 0$. Thus $f(X_1) = 0$ for $X_1 \in (0, X_2(K))$ which contradicts that $X_1 \geq X_2(K)$.

A.10 Proof of Theorem 4.1

If $X_1 = 0$ then

$$g(0) = -\beta^+_{(\lambda 1)}\delta K < 0 \tag{116}$$

If $X_1 = X_{K2} = \frac{\beta_{(2)}\delta(r-\mu_2)}{(\beta_{(2)}-1)(1-\eta K)}$ then

$$g(X_{K2}) = \frac{X_{K2}K(1-\eta K)}{r-\mu_2}\left(\left(\beta_{(2)}-\beta^+_{(\lambda 1)}\right)a-\frac{(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)}\left(1-\beta^+_{(\lambda 1)}\right)\right)-(a(\beta_{(2)}-\beta^+_{(\lambda 1)})+\beta^+_{(\lambda 1)})\delta K \tag{117}$$

$$= \left(\beta^+_{(\lambda 1)}\left(-\frac{1}{\beta_{(2)}}a+\frac{(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)}-1+\frac{1}{\beta_{(2)}}\right)-\frac{(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)}+a\right)\frac{\beta_{(2)}\delta K}{\beta_{(2)}-1} \tag{118}$$

$$= \left(\frac{\beta^+_{(\lambda 1)}}{\beta_{(2)}}(1-a)+(\beta^+_{(\lambda 1)}-1)\frac{(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)}-(\beta^+_{(\lambda 1)}-a)\right) \tag{119}$$

$$= -\frac{(\mu_1-\mu_2)\left(\beta_{(2)}\beta^+_{(\lambda 1)}(\mu_1-\mu_2)+\beta^+_{(\lambda 1)}(r-\mu_1)+(\lambda-\beta_{(2)}(r-\mu_2+\lambda))\right)}{(\lambda-\beta_{(2)}(\mu_1-\mu_2))(r-\mu_1+\lambda)}\frac{\beta_{(2)}\delta K}{\beta_{(2)}-1} \tag{120}$$

Let

$$g(X_{K2}) = c\frac{T(\lambda)}{N(\lambda)} \tag{121}$$

where

$$T(\lambda) = -(\mu_1 - \mu_2)\left(\beta_{(2)}\beta^{+}_{(\lambda 1)}(\mu_1 - \mu_2) + \beta^{+}_{(\lambda 1)}(r - \mu_1) + (\lambda - \beta_{(2)}(r - \mu_2 + \lambda))\right) \tag{122}$$

$$N(\lambda) = (\lambda - \beta_{(2)}(\mu_1 - \mu_2))(r - \mu_1 + \lambda) \tag{123}$$

$$c = \frac{\beta_{(2)}\delta K}{\beta_{(2)} - 1} \tag{124}$$

The numerator and denominator are equal to zero at the same point x, i.e.,

$$x = \beta_{(2)}(\mu_1 - \mu_2) \tag{125}$$

$$T(x) = 0 \tag{126}$$

$$N(x) = 0 \tag{127}$$

The equations are not monotone. Both are convex. If $\lambda > x$ then $N(\lambda) > 0$ and if $\lambda < x$ then $N(\lambda) < 0$. The numerator $T(\lambda)$ is equal to zero when $\lambda = \beta_{(2)}(\mu_1 - \mu_2)$ since then $\beta_{(2)} = \beta^{+}_{(\lambda 1)}$. Furthermore

$$T'(\lambda) = (\mu_1 - \mu_2)\left(\frac{-r - \mu_1(\beta_2 - 1) + \mu_2\beta_2}{\sqrt{2(r+\lambda)\sigma^2 + \left(\mu_1 - \frac{1}{2}\sigma^2\right)^2}} + \beta_2 - 1\right) \tag{128}$$

so that $T''(\lambda) > 0$. Hence, since we know that $T(0) < 0$, it follows that $T(\lambda) = 0$ only for $\lambda = \beta_{(2)}(\mu_1 - \mu_2)$.

Both the numerator and denominator are convex in λ and are equal to zero when $\lambda = \beta_{(2)}(\mu_1 - \mu_2)$. Hence for $\lambda < \beta_{(2)}(\mu_1 - \mu_2)$ both are negative and for $\lambda > \beta_{(2)}(\mu_1 - \mu_2)$ both are positive implying that $g(X_{K2}) > 0$ since $c > 0$.

Let

$$f(\lambda) = \left(\beta_{(2)} - \beta^{+}_{(\lambda 1)}\right)a = \frac{\lambda\left(\beta_{(2)} - \beta^{+}_{(\lambda 1)}\right)}{\lambda - \beta_{(2)}(\mu_1 - \mu_2)}. \tag{129}$$

If $\lambda = x = \beta_{(2)}(\mu_1 - \mu_2)$ then $\beta_{(2)} = \beta^{+}_{(\lambda 1)}$, and by l'Hopital

$$f(x) = \lim_{\lambda -> x}\frac{\lambda\left(\beta_{(2)} - \beta^{+}_{(\lambda 1)}\right)}{\lambda - \beta_{(2)}(\mu_1 - \mu_2)} = \lim_{\lambda -> x}\frac{\beta_{(2)} - \beta^{+}_{(\lambda 1)}}{1} = 0. \tag{130}$$

If $\lambda > x$ then $\beta_{(2)} < \beta^+_{(\lambda 1)}$ because $\frac{\partial \beta^+_{(\lambda 1)}}{\partial \lambda} > 0$ and vice versa, if $\lambda < x$ then $\beta_{(2)} > \beta^+_{(\lambda 1)}$. This implies that $f(\lambda) < 0$ and thus $g(X)$ is concave in X. Since $g(X)$ is concave in X and $g(0) < 0$ and $g(X_{K2}) > 0$, there is a unique intersection of $g(X) = 0$ for $X \in (0, X_{K2})$.

A.11 Proof of Proposition 4

Derivative of $V^{(1)} - \delta K$ w.r.t. K yields

$$\frac{\partial V^{(1)} - \delta K}{\partial K} = 0 \tag{131}$$

$$\tag{132}$$

A.12 Proof of Theorem 3.2

The differential equation solves for a form of $F^{(1)}(X) = A_1 X^{\beta_{(1)}}$. The value matching condition is

$$V^{(1)} - \delta K = F^{(1)} \tag{133}$$

and the smooth pasting

$$\frac{\partial V^{(1)}(X)}{\partial X} = \frac{\partial F^{(1)}(X)}{\partial X} \tag{134}$$

together these imply

$$\frac{\partial V^{(1)}(X)}{\partial X} = (V^{(1)}(X) - \delta K)\beta_{(1)} X^{-1} \tag{135}$$

and determine

$$X^*(K) = \frac{\beta_{(1)}}{\beta_{(1)} - 1} \frac{\delta(r - \mu_1 + \lambda)(r - \mu_2)}{(1 - \eta K)(r - \mu_2 + \lambda)} \tag{136}$$

Together with Proposition 4 this leads to the decisions for benchmark 2.

A.13 Proof of Proposition 12

First, consider $X \geq X_{C2}$ then (51) becomes

$$(r + \lambda) F^{(1)} = \frac{\partial F^{(1)}}{\partial X} \mu_1 X + \frac{1}{2} \frac{\partial^2 F^{(1)}}{\partial X^2} \sigma^2 X^2 + \lambda \left(V^{(2)}(X, K^*_{(2)}(X)) - \delta K^*_{(2)}(X) \right) \tag{137}$$

where

$$K^{*}_{(2)}(X) = \frac{1}{2\eta}\left(1-\frac{\delta(r-\mu_2)}{X}\right) \tag{138}$$

and thus

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 + \lambda\left(\frac{(X-\delta(r-\mu_2))^2}{4X\eta(r-\mu_2)}\right) \tag{139}$$

The homogeneous solution has form

$$F^{(1)} = C_1 X^{\beta^{+}_{(\lambda 1)}} + C_2 X^{\beta^{-}_{(\lambda 1)}} \tag{140}$$

solving

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 \tag{141}$$

i.e.

$$\frac{1}{2}\sigma^2\beta_{(\lambda 1)}(\beta_{(\lambda 1)}-1) + \mu_1\beta_{(\lambda 1)} - (r+\lambda) = 0 \tag{142}$$

Since $X \geq X_{C2}$, the boundary condition $F^{(1)}(0) = 0$ does not hold and thus $C_2 \neq 0$. In the limit the waiting region can go infinity because there is a point in which investment becomes optimal (is that the reasoning?), therefore also $C_1 \neq 0$.

For the particular solution we conjecture the functional form

$$F^{(1)}(X) = aX + b + cX^{-1} \tag{143}$$

Applied to the full equation gives

$$
\begin{aligned}
(r+\lambda)(aX+b+cX^{-1}) &= (a-cX^{-2})\mu_1 X+2cX^{-3}\frac{1}{2}\sigma^2X^2+\lambda\left(\frac{(X-\delta(r-\mu_2))^2}{4X\eta(r-\mu_2)}\right) && (144)\\
&= (aX-cX^{-1})\mu_1+cX^{-1}\sigma^2+\lambda\left(\frac{X}{4\eta(r-\mu_2)}-\frac{2\delta(r-\mu_2)}{4\eta(r-\mu_2)}+\frac{\delta^2(r-\mu_2)^2}{4X\eta(r-\mu_2)}\right)\\
&= a\mu_1X-c\mu_1X^{-1}+c\sigma^2X^{-1}+\frac{\lambda}{4\eta(r-\mu_2)}X-\frac{\lambda\delta}{2\eta}+\frac{\lambda\delta^2(r-\mu_2)}{4\eta}X^{-1}\\
(r+\lambda)aX &= a\mu_1X+\frac{\lambda}{4\eta(r-\mu_2)}X\\
a &= \frac{\lambda}{4\eta(r-\mu_2)(r-\mu_1+\lambda)} && (145)\\
(r+\lambda)b &= -\frac{\lambda\delta}{2\eta} && (146)\\
b &= -\frac{\lambda\delta}{2\eta(r+\lambda)} && (147)\\
(r+\lambda)cX^{-1} &= -c\mu_1X^{-1}+c\sigma^2X^{-1}+\frac{\lambda\delta^2(r-\mu_2)}{4\eta}X^{-1} && (148)\\
c &= \frac{\lambda\delta^2(r-\mu_2)}{4\eta(r+\mu_1+\lambda-\sigma^2)} && (149)
\end{aligned}
$$

A.14 Proof of Proposition 13

Secondly, consider $X < X_{C2}$ then (51) becomes

$$
\begin{aligned}
(r+\lambda)F^{(1)} &= \frac{\partial F^{(1)}}{\partial X}\mu_1X+\frac{1}{2}\frac{\partial^2F^{(1)}}{\partial X^2}\sigma^2X^2+\lambda F^{(2)}(X) && (150)\\
&= \frac{\partial F^{(1)}}{\partial X}\mu_1X+\frac{1}{2}\frac{\partial^2F^{(1)}}{\partial X^2}\sigma^2X^2+\lambda A_2X^{\beta_{(2)}} && (151)
\end{aligned}
$$

The homogeneous equation (terms involving value function) is

$$
(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1X+\frac{1}{2}\frac{\partial^2F^{(1)}}{\partial X^2}\sigma^2X^2 \quad (152)
$$

where the homogeneous solution has form

$$
F^{(1)}(X) = D_1X^{\beta^+_{(\lambda 1)}}+D_2X^{\beta^-_{(\lambda 1)}} \quad (153)
$$

Since $F^{(1)}(0)=0$, $D_2=0$. And $\beta^+_{(\lambda 1)}$ is given by plugging

$$
F^{(1)}(X) = D_1X^{\beta^+_{(\lambda 1)}} \quad (154)
$$

into the homogeneous equation, giving

$$0 = \beta^{+}_{(\lambda 1)}\mu_1 + \frac{1}{2}\beta^{+}_{(\lambda 1)}(\beta^{+}_{(\lambda 1)} - 1)\sigma^2 - (r+\lambda). \tag{155}$$

For the particular solution we conjecture

$$F^{(1)}(X) = a_d A_2 X^{\beta_{(2)}} + b \tag{156}$$

where A_2 and $\beta_{(2)}$ are given by (50) and (47). Plugging this into the ODE

$$(r+\lambda)F^{(1)} = \frac{\partial F^{(1)}}{\partial X}\mu_1 X + \frac{1}{2}\frac{\partial^2 F^{(1)}}{\partial X^2}\sigma^2 X^2 + \lambda A_2 X^{\beta_{(2)}} \tag{157}$$

which gives

$$(r+\lambda)\left(a_d A_2 X^{\beta_{(2)}} + b\right) = a_d \beta_{(2)} A_2 X^{\beta_{(2)}}\mu_1 + \frac{1}{2}a_d\beta_{(2)}(\beta_{(2)}-1)A_2 X^{\beta_{(2)}}\sigma^2 + \lambda A_2 X^{\beta_{(2)}}$$

$$b = 0 \tag{158}$$

$$(r+\lambda)a_d = a_d\beta_{(2)}\mu_1 + \frac{1}{2}a_d\beta_{(2)}(\beta_{(2)}-1)\sigma^2 + \lambda \tag{159}$$

$$(r+\lambda-\beta_{(2)}\mu_1 - \frac{1}{2}\beta_{(2)}(\beta_{(2)}-1)\sigma^2)a_d = \lambda \tag{160}$$

$$a_d = \frac{\lambda}{r+\lambda-\beta_{(2)}\mu_1 - \frac{1}{2}\beta_{(2)}(\beta_{(2)}-1)\sigma^2} \tag{161}$$

$$= \frac{\lambda}{\lambda - \beta_{(2)}\left(\mu_1 - \mu_2\right)} \tag{162}$$

A.15 Proof of Theorem 4.2

At $X_{C2} = \frac{\beta_{(2)}+1}{\beta_{(2)}-1}\delta(r-\mu_2)$

$$C_1 X^{\beta^{+}_{(\lambda 1)}} + C_2 X^{\beta^{-}_{(\lambda 1)}} + \frac{\lambda}{4\eta(r-\mu_2)(r-\mu_1+\lambda)}X - \frac{\lambda\delta}{2\eta(r+\lambda)} + \frac{\lambda\delta^2(r-\mu_2)}{4\eta(r+\mu_1+\lambda-\sigma^2)}X^{-1}$$
$$= D_1 X^{\beta^{+}_{(\lambda 1)}} + a_d A_2 X^{\beta_{(2)}} \tag{163}$$

$$\beta^{+}_{(\lambda 1)} C_1 X^{\beta^{+}_{(\lambda 1)}-1} + \beta^{-}_{(\lambda 1)} C_2 X^{\beta^{-}_{(\lambda 1)}-1} + \frac{\lambda}{4\eta(r-\mu_2)(r-\mu_1+\lambda)} - \frac{\lambda\delta^2(r-\mu_2)}{4\eta(r+\mu_1+\lambda-\sigma^2)}X^{-2}$$
$$= \beta^{+}_{(\lambda 1)} D_1 X^{\beta^{+}_{(\lambda 1)}-1} + \beta_{(2)} a_d A_2 X^{\beta_{(2)}-1} \tag{164}$$

and if $X \geq X_{C2}$, then at X

$$C_1 X^{\beta^+_{(\lambda 1)}} + C_2 X^{\beta^-_{(\lambda 1)}} + \frac{\lambda}{4\eta(r-\mu_2)(r-\mu_1+\lambda)} X - \frac{\lambda\delta}{2\eta(r+\lambda)} + \frac{\lambda\delta^2(r-\mu_2)}{4\eta(r+\mu_1+\lambda-\sigma^2)} X^{-1}$$
$$= \frac{K(1-\eta K)X(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} - \delta K \tag{165}$$

$$\beta^+_{(\lambda 1)} C_1 X^{\beta^+_{(\lambda 1)}-1} + \beta^-_{(\lambda 1)} C_2 X^{\beta^-_{(\lambda 1)}-1} + \frac{\lambda}{4\eta(r-\mu_2)(r-\mu_1+\lambda)} - \frac{\lambda\delta^2(r-\mu_2)}{4\eta(r+\mu_1+\lambda-\sigma^2)} X^{-2}$$
$$= \frac{K(1-\eta K)(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} \tag{166}$$

Let

$$a = \frac{\lambda}{4\eta(r-\mu_2)(r-\mu_1+\lambda)} \tag{167}$$
$$b = \frac{\lambda\delta}{2\eta(r+\lambda)} \tag{168}$$
$$c = \frac{\lambda\delta^2(r-\mu_2)}{4\eta(r+\mu_1+\lambda-\sigma^2)} \tag{169}$$
$$e = \frac{K(1-\eta K)(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} \tag{170}$$

then we have two system of equations that imply the optimal threshold and capacity. In both cases, there are five unknowns and five equations, i.e. C_1, C_2, D_1, X_1, K and (20), (163), (164), (165), (166) and (20), (163), (164), (173), (174).

If $X \geq X_{C2}$ (c1), then the system of equations is obtained by(164) $\times X - \beta^+_{(\lambda 1)} \times$ (163) leading to C_2^* and $X \times$ (166) $- \beta^+_{(\lambda 1)} \times$ (165) leading to the implicit function for X_{CF}.

Thus solving

$$C_2^* = \frac{1}{\left(\beta^-_{(\lambda 1)} - \beta^+_{(\lambda 1)}\right) X_2^{\beta^-_{(\lambda 1)}}} \left(-aX_2(1-\beta^+_{(\lambda 1)}) - b\beta^+_{(\lambda 1)} + cX_2^{-1}(1+\beta^+_{(\lambda 1)}) + (\beta_{(2)} - \beta^+_{(\lambda 1)}) a_d A_2 X_2^{\beta_{(2)}}\right) \tag{171}$$

$$0 = (\beta^-_{(\lambda 1)} - \beta^+_{(\lambda 1)}) C_2 X_1^{\beta^-_{(\lambda 1)}} + aX_1(1-\beta^+_{(\lambda 1)}) + \beta^+_{(\lambda 1)} b - cX_1^{-1}(1+\beta^+_{(\lambda 1)}) - e(K)X_1(1-\beta^+_{(\lambda 1)}) - \beta^+_{(\lambda 1)}\delta K \tag{172}$$

A.16 Proof of Theorem 4.3

If $X < X_{C2}$, then at X,

$$D_1 X^{\beta^+_{(\lambda 1)}} + a_d A_2 X^{\beta_{(2)}} = \frac{K(1-\eta K)X(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} - \delta K \tag{173}$$

$$\beta^+_{(\lambda 1)} D_1 X^{\beta^+_{(\lambda 1)}-1} + \beta_{(2)} a_d A_2 X^{\beta_{(2)}-1} = \frac{K(1-\eta K)(r-\mu_2+\lambda)}{(r-\mu_1+\lambda)(r-\mu_2)} \tag{174}$$

The system of equations with unknowns C_1, C_2, D_1, X_1, K is obtained by (174) $X - \beta^+_{(\lambda 1)}$ (173) for X_{CF}.

Thus solving

$$0 = (\beta_{(2)} - \beta^+_{(\lambda 1)}) a_d A_2 X_1^{\beta_{(2)}} - e(K) X_1 (1 - \beta^+_{(\lambda 1)}) - \beta^+_{(\lambda 1)} \delta K \tag{175}$$

References

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