Convergence of 'Exercise Boundary Fitting' Least Squares Simulation Approach

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1 Introduction

Real option analysis (ROA) is recognized as a superior method to quantify the value of real-world investment opportunities where managerial flexibility can influence their worth, as compared to standard discounted cash-flow methods typically used in industry. ROA stems from the work of Black and Scholes (1973) on financial option valuation. Myers (1977) recognized that both financial options and project decisions are exercised after uncertainties are resolved. The ability for managers to react to uncertainties at a future time adds value to projects, and since this value is not captured by standard DCF methods, erroneous decision making may result (Trigeorgis (1996)).

An excellent empirical review of ex-post investment decisions made in copper mining showed that fewer than half of investment timing decisions were made at the right time and 36 of the 51 projects analyzed should have chosen an extraction capacity of 40% larger or smaller (Auger and Guzman (2010)). The authors were unaware of any mining firm basing all or part of their decision making on the systematic use of ROA and emphasize that the "failure to use ROA to assess investments runs against a basic assumption of neoclassical theory: under uncertainty, firms ought to maximize their expected profits". They make the case that irrational decision making exists within the industry due to a lack of real option tools available for better analysis. A number of surveys across industries have found that the use of ROA is in the range of 10-15% of companies, and the main reason for lack of adoption is model complexity (Hartmann and Hassan (2006), Block (2007), Truong, Partington, and Peat (2008), Bennouna, Meredith, and Marchant (2010), Dimitrakopoulos and Abdel Sabour (2007)).

Previously, we introduced a methodology based on exercise boundary fitting (EBF) in an effort to develop a practical Monte Carlo simulation-based real options approach (Bashiri, Davison, and Lawryshyn (2018)). We showed that our methodology converges in the case of simple Bermudan and American put options. More recently, we expanded on the model to solve a staged manufacturing problem (Fleten, Kozlova, and Lawryshyn (2021)). As we presented, utilizing boundary fitting allowed us to solve a computationally difficult problem. In another study we explored the use of the boundary fitting methodology for a number of cases, one being a build and abandon mining example (Davison and Lawryshyn (2021)). We showed that while the methodology provided good convergence on option value, under certain scenarios, where the optimal exercise boundaries occurred in regions

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where there were few Monte Carlo paths, the optimization algorithm struggled to converge. In other cases, convex optimization algorithms would get stuck in non-changing regions. The purpose of this paper is to explore accuracy and convergence issues related to the exercise boundary fitting methodology.

1.1 Motivation

As mentioned in Bashiri, Davison, and Lawryshyn (2018), our work is focused on developing a practical Monte Carlo simulation-based real options methodology as Monte Carlo simulation can be easily understood by managers and allows for the modelling of multiple stochastic factors (Longstaff and Schwartz (2001)). Realistic models that try to account for a number of risk factors can be mathematically complex, and in situations where many future outcomes are possible, many layers of analysis may be required. As a motivating example[1], consider the case of a greenfield mining site, where the life of the mine lease is 2 years, construction will take half a year, the ore price, S_t, follows geometric Brownian motion (GBM) and the per unit costs are K to construct and C_{ab} to abandon, and C_{op} is the operating cost rate. For a given set of parameters, the scenarios are depicted in Figure 1 in a binomial tree. The S_t process of the first panel is used to determine the operating cash-flow, calculated as $CF_t = S_t - C_{op}$. For this case, we assume that abandonment can occur at year 2 only, with cost C_{ab}. The real option can be valued in a recursive manner and the different scenarios are presented in Figure 2. Since it takes half a year for construction, the latest we would construct the mine is at year 1. In this case, only the cash-flows associated with the last period are of value and these are discounted twice to year 1 (relevant probabilities and discounting factor were used) to determine the expected value. At year 1, there are 3 possible values for S_t and thus three possible valuations for the cash-flows. Clearly, we would only invest if the total expected value of the cash-flows minus the investment cost, K, is greater than 0. As shown, only one of the three scenarios has a positive value, the others are set to 0. We continue to discount these expected values to reach a valuation of \$1.0 at year 0. Similar valuations are done for the case of building at years 0.5 and 0. Based on the analysis, we see that it is best to wait one period (half year) before constructing and then choosing to construct only if the price $S_{t=0.5} = \$10.7$ is realized. The overall project value at $t = 0$ is determined to be \$2.9. Note that even for this very simple problem, a separate binomial tree was required at each decision making time point. If we allowed for early abandonment, many more trees would be required. If we added a second stochastic factor, we would have another spatial dimension. Clearly, to value a complex real option the model's complexity increases substantially. This complexity leads us to the overall objective of developing a practical simulation based real options methodology that can model realistic decision-making scenarios encountered in industry.

The focus of this research is the development of a real options valuation methodology geared towards practical use with mining valuation as a context. A key innovation of the methodology that we presented previously was the idea of fitting optimal decision making boundaries to optimize the expected value, based on Monte Carlo simulated stochastic processes that represent important uncertain factors. Our specific emphasis in this work will be to explore accuracy and convergence issues related to the following cases:

- Bermudan put option

[1]Note that this example was presented in Bashiri, Davison, and Lawryshyn (2018).

Price Process (St)					Cash-Flow per Period				
0	**0.5**	**1**	**1.5**	**2**	**0**	**0.5**	**1**	**1.5**	**2**
				13.3					15.3
			12.4					11.8	
		11.5		11.5			7.6		6.6
	10.7		10.7			3.7		3.7	
10.0		10.0		10.0	0.0		0.0		-1.0
	9.3		9.3			-3.4		-3.4	
		8.7		8.7			-6.6		-7.6
			8.1					-9.6	
				7.5					-13.3

Figure 1: Price process and resulting cash-flow.

Build at Year 1					Build at Year 0.5					Build at Year 0				
0	**0.5**	**1**	**1.5**	**2**	**0**	**0.5**	**1**	**1.5**	**2**	**0**	**0.5**	**1**	**1.5**	**2**
				15.3					15.3					15.3
			11.6					23.4					23.4	
		3.1		6.6			16.4		6.6			24.0		6.6
	1.8		3.4			5.0		7.1			14.5		7.1	
1.0		~~0.0~~		-1.0	**2.9**		1.3		-1.0	0.6		1.3		-1.0
	0.0		-3.7			~~0.0~~		-7.1			-6.8		-7.1	
		~~0.0~~		-7.6			-11.9		-7.6			-18.5		-7.6
			-9.8					-19.4					-19.4	
				-13.3					-13.3					-13.3

Figure 2: Real option valuation based on different build times.

- Option to purchase a Bermudan put option
- Build / abandon real option
- Optimal investment rule in infinite time.

The Bermudan put option is the most basic of options and thus provides context regarding accuracy and convergence issues[2]. We then consider the option to purchase a Bermudan put option. This second case adds further complexity in that two exercise points need to be determined. The build / abandon real option case provides context in a more realistic practical valuation scenario. Convex optimization algorithms struggle to converge to the right solution and often get stuck in flat regions. Using constrained genetic algorithms (GA), we overcome this problem. Finally, we present the very simple case of the optimal investment rule in infinite time, first introduced by McDonald and Siegel (1986). The real options literature is rich in the development of analytical and pseudo-analytical models based on the optimal investment rule in infinite time and the results of these models provide important insights that can have policy implications. Because these models are analytical or pseudo-analytical, they provide the advantage that they can be solved almost instantaneously, allowing the practitioner to explore many scenarios. Arguably, a main disadvantage of these models is that they are based on simplifying assumptions – assumptions that may not be realistic. Utilizing the EBF method to test the simplifying assumptions and further explore more realistic scenarios is of value. We show that the EBF method provides similar results to the basic analytical model, providing confidence that the EBF method can be utilized to complement these models to allow for the valuation of more realistic scenarios.

2 Relevant Literature

We make the argument that a majority of real-world real options are either American or Bermudan type options – i.e. managers typically make strategic decisions either when there is a noticeable shift in important state variables (American), or decisions are made at predefined intervals (Bermudan). With this in mind, below we provide a brief review of relevant frameworks to estimate American / Bermudan options. Then, we provide a review of valuation methods utilized in mining as we see mining valuation in the real option context the leading use case example for our methodology.

A review of the valuation of American options was provided by Barone-Adesi (2005) where the LSMC of Longstaff and Schwartz (2001) was highlighted as "the most innovative", but other similar Monte Carlo based approaches have been proposed (Barraquand and Martineau (2007)) and the literature is abundant on the utilization of simulation and dynamic programming to value American options. There are many articles providing numerical or analytical approximations to an American exercise boundary (e.g. Barone-Adesi and Whaley (1987), Ju (1998), Tung (2016), Del Moral, Remillard, and Rubenthaler (2012)), however very few articles utilize a "forward" Monte Carlo approach, where the valuation method does not rely on the end result, but rather, the problem is worked forwards in time. Miao and Lee (2013) propose the use of forward Monte Carlo valuation, however the exercise boundary was estimated using the analytical method of Barone-Adesi and Whaley (1987), which negates the ability to develop a general model. Nasakkala and Keppo (2008) do utilize forward Monte Carlo simulation with a parametric boundary fitting approach in

[2]Note that this work was initially presented in Bashiri, Davison, and Lawryshyn (2018)

a hydropower planning problem utilizing two stochastic factors, namely the electricity price forward curve and random water inflow. However, in their approach the parameters are optimized for each path, an approach that is similar to that provided by (Broadie and Glasserman 1997) for calculating the upper bound for an American put option called the perfect foresight solution. In a more recent review of American options methods, Aydogan, Aksoy, and Ugur (2018) cited closed form analytical approximations as well as numerical methods based on the binomial model, partial differential equations and LSMC. We emphasize that in our approach we are proposing a general simulation approach to solve American / Bermudan models by optimizing a parameterized exercise boundary. We note that the accuracy of our approach will be based on the assumed parametric boundary equation. One reason why our proposed approach may not have been presented in the financial derivatives literature is that most works are focused on improving efficiency and accuracy of the pricing models. In the real options context, where many assumptions are required to estimate the cash-flows, accuracy is not as important – what is important is ease of implementation, comprehension by decision makers and a tool for better decision making.

Mining Context

The academic literature is very rich in the field of mining valuation. Mining projects are laced with uncertainty and many discounted cash-flow (DCF) methods have been proposed in the literature to try to account for the uncertainty (Bastante, Taboada, Alejano, and Alonso (2008), Dimitrakopoulos (2011), Everett (2013), Ugwuegbu (2013)). Several guidelines/codes have been developed to standardize mining valuation (CIMVAL (2003), VALMIN (2015)). The main mining valuation approaches are income (i.e. cash-flows), market or cost based and the focus of this paper is on income-based real option valuation, which resemble American (or Bermudan) type financial options. Earlier real option works focused on modelling price uncertainty only (Brennan and Schwartz (1985), Dixit and Pindyck (1994), Schwartz (1997)), however the complexity in mining is significant and there are numerous risk factors. Simpler models based on lattice and finite difference methods (FDM) are difficult to implement in a multi-factor setting (Longstaff and Schwartz (2001)) and, also, it is extremely difficult to account for time dependent costs with multiple decision making points (Dimitrakopoulos and Abdel Sabour (2007)). Nevertheless, the simpler models continue to merit attention (Haque, Topal, and Lilford (2014), Haque, Topal, and Lilford (2016)). Dimitrakopoulos and Abdel Sabour (2007) utilize a multi-factor least squares Monte Carlo (LSMC) approach to account for price, foreign exchange and ore body uncertainty under multiple pre-defined operating scenarios (states). However, the model only allows for operation and irreversible abandonment — aspects such as optimal build time, expansion and mothballing are not considered. Similarly, Mogi and Chen (2007) use ROA and the method developed by Barraquand and Martineau (2007) to account for multiple stochastic factors in a four-stage gas field project. Abdel Saboura and Poulin (2010) develop a multi-factor LSMC model for a single mine expansion.

A review of 92 academic works found that most real options research is focused on dealing with very specific situations where usually no more than two real options are considered (Savolainen (2016)). While the LSMC allows for a more realistic analysis, methods presented to date are applicable only for the case where changes from one state to another does not change the fundamental stochastic factors with time. For example, modular expansion would be difficult to implement in such a model if the cost to expand was a function of time and impacts extracted ore quality due to the changing rate of extraction – these issues were considered in Davison, Lawryshyn, and Zhang

(2015) and Kobari, Jaimungal, and Lawryshyn (2014). Also, modeling of multiple layers is still complex and will not lead to a methodology that managers can readily utilize.

3 Methodology

In Bashiri, Davison, and Lawryshyn (2018) we presented details of the theory and methodology of applying EBF for the following cases: 1) a Bermudan put option, 2) a Bermudan put option with a variable strike, 3) an American put option and 4) a build / abandon real option. In this section we present the general simulation framework, then provide details for each of the four cases discussed mentioned in the Introduction.

The EBF framework assumes that the (real) option valuation is based on a single or multiple stochastic processes, say $\vec{X}_t$, which are simulated using Monte Carlo simulation. Depending on the valuation model, these processes may be risk-neutral or actual and are general in that standard and non-standard processes can be used. We let $\vec{f}_B(\vec{x}, t; \vec{\theta})$ be a general function that represents N_B exercise boundaries parametrised by $\vec{\theta}$, where $\vec{x}$ represents possible realizations of the process $\vec{X}_t$. We note that $\vec{f}_B(\vec{x}, t; \vec{\theta})$ can be a single point, multiple points, a curve or multiple curves of fixed dimensional surfaces. Based on the path dependent journey of $\vec{X}_t$ we define appropriate first passage of time for the i-th path crossing the j-th boundary as

$$\tau_{B_j}^{(i)} \equiv \min\{t > 0, \vec{X}_t^{(i)} \geq \pm f_{B_j}(\vec{x}, t; \vec{\theta}) \mid \lambda^{(i)}(\vec{X}_t^{(i)})\}, \tag{1}$$

where we use $\pm$ to signify that the process could be hitting the exercise boundary from below or above, depending on the problem at hand, and state $\lambda^{(i)}(\vec{X}_t^{(i)}) \equiv \lambda_t^{(i)}$ with $j \in \{1, 2, ..., N_B\}$, where there could be N_S possible states, also dependent on the problem at hand.

For each simulated path $\vec{X}_t^{(i)}$, we define a cash-flow or payoff at time t as $CF^{(i)}(\vec{X}_t^{(i)}, \lambda_t^{(i)}) \equiv CF_t^{(i)}$ and thus the value generated by the i-th path can be determined by

$$V_0^{(i)}(\vec{X}_t^{(i)}, \lambda_t^{(i)}) \equiv V_0^{(i)} = \sum_{j=0}^{N_t} CF_{t_j}^{(i)} e^{(-rt_j)}, \tag{2}$$

where N_t is the number of time steps in the simulation such that $t \in \{t_0, t_1, ..., t_{N_t}\}$. We emphasize that $V_0^{(i)}$ is a function of $\lambda_t^{(i)}$ and is therefore a function of the exercise boundary parameters, $\vec{\theta}$. The overall option value becomes

$$V_0 = \frac{1}{N} \sum_{i=1}^{N} V_0^{(i)}(\vec{\theta}), \tag{3}$$

where N is the number of paths used in the simulation for $\vec{X}_t$. Our task reduces to maximizing $V_0^{(i)}$ by finding optimal exercise boundary parameters,

$$\vec{\theta}^* = \arg\max_{\vec{\theta}} \frac{1}{N} \sum_{i=1}^{N} V_0^{(i)}(\vec{\theta}) \tag{4}$$

and thus, the option value becomes

$$V_0^* = \frac{1}{N} \sum_{i=1}^{N} V_0^{(i)}(\vec{\theta}^*). \tag{5}$$

In the following subsections we reduce the general formulation presented above for our four specific cases.

3.1 Bermudan Put Option

As mentioned previously, we presented our formulation for the Bermudan put option in Bashiri, Davison, and Lawryshyn (2018). Here we summarize the methodology. For the Bermudan put option, we consider a GBM stock price process, S_t, as

$$dS_t = rS_t dt + \sigma S_t d\widehat{W}_t, \tag{6}$$

where r is the risk-free rate, σ is the volatility and $\widehat{W}_t$ is a Wiener process in the risk-neutral measure.

We assume the payoff of the option to be $\max(K - S_t, 0)$ and can be exercised at times $t = \tau$ and $t = T$ where $\tau < T$. We utilize Monte Carlo simulation to generate N GBM paths and we replace $X_t^{(i)}$ from the general formulation above, with $S_t^{(i)}$. The value of our option (cf. equation (2)) for the i-th path is

$$V_0^{(i)}(\theta) = \mathbb{1}_{S_\tau^{(i)} \leq \theta} \left(K - S_\tau^{(i)} \right) e^{-r\tau} + \mathbb{1}_{S_\tau^{(i)} > \theta} \max\left(K - S_T^{(i)}, 0 \right) e^{-rT}. \tag{7}$$

and the optimal exercise price (cf. equation (4)) is

$$\theta^* = \arg\max_\theta \frac{1}{N} \sum_{i=1}^N V_0^{(i)}(\theta), \tag{8}$$

leading to the optimal option value (cf. equation (5)) as

$$V_0^* = \frac{1}{N} \sum_{i=1}^N V_0^{(i)}(\theta^*). \tag{9}$$

In Bashiri, Davison, and Lawryshyn (2018) we showed that the *exact* value for θ^* can be determined by solving the equation

$$K - \theta^* = P_{BS_{put}}(\theta^*, \tau, T, r, \sigma, K), \tag{10}$$

where $P_{BS_{put}}(x, \tau, T, r, \sigma, K)$ is the Black-Scholes formula for the value of a European put option with current stock price x, maturity $T - \tau$, risk-free rate r, volatility σ and strike K, and $f_{S_\tau}(x|S_0)$ is the density for S_τ given S_0. Equation (10) can be solved using numerical methods and thus the option value simplifies to

$$V_0^{act} = e^{-r\tau} \left(\int_0^{\theta^*} (K - x) f_{S_\tau}(x|S_0) dx + \int_{\theta^*}^\infty P_{BS_{put}}(x, \tau, T, r, \sigma, K) f_{S_\tau}(x|S_0) dx \right), \tag{11}$$

which, too, can be solved using standard numerical methods. By observing equations (7), (9) and (11), it is easy to see that

$$\lim_{N \to \infty} V_0^* = V_0^{act}. \tag{12}$$

3.2 Option to Purchase a Bermudan Put Option

We now consider the option to purchase a Bermudan put option at a predetermined price K_1 at time τ_1 with strike price K_2 based on a GBM price process as presented in equation (6). We define τ_2 as the time where we have the option to exercise the put option early and T as the time the option expires, such that $0 < \tau_1 < \tau_2 < T$. The cash-flow for the i-th path for the option at $t = \tau_1$ is

$$CF_{\tau_1}^{(i)} = -\mathbb{1}_{S_{\tau_1}^{(i)} \le \theta_1} K_1, \tag{13}$$

where the option is purchased at the price K_1 if $S_{\tau_1}^{(i)}$ is below the exercise value θ_1. At $t = \tau_2$, the option is exercised with a payoff of $K_2 - S_{\tau_1}^{(i)}$ if the option had been purchased at $t = \tau_1$ and $S_{\tau_2}^{(i)}$ is below θ_2, i.e.,

$$CF_{\tau_2}^{(i)} = \mathbb{1}_{S_{\tau_1}^{(i)} \le \theta_1} \mathbb{1}_{S_{\tau_2}^{(i)} \le \theta_2} \left(K_2 - S_{\tau_2}^{(i)} \right). \tag{14}$$

At $t = T$, if the option had been purchased at $t = \tau_1$ and not exercised at $t = \tau_2$ then the payoff becomes $\max\left(K_2 - S_T^{(i)}, 0\right)$, or

$$CF_T^{(i)} = \mathbb{1}_{S_{\tau_1}^{(i)} \le \theta_1} \mathbb{1}_{S_{\tau_2}^{(i)} > \theta_2} \max\left(K_2 - S_T^{(i)}, 0 \right). \tag{15}$$

The value associated with the i-th path is given as (cf. equation (2))

$$V_0^{(i)}(\vec{\theta}) = CF_{\tau_1}^{(i)} e^{-r\tau_1} + CF_{\tau_2}^{(i)} e^{-r\tau_2} + CF_T^{(i)} e^{-rT}, \tag{16}$$

and the optimal exercise price is (cf. equation (4))

$$\vec{\theta}^* = \arg\max_{\vec{\theta}} \frac{1}{N} \sum_{i=1}^{N} V_0^{(i)}(\vec{\theta}), \tag{17}$$

leading to the optimal option value as (cf. equation (5))

$$V_0^* = \frac{1}{N} \sum_{i=1}^{N} V_0^{(i)}(\vec{\theta}^*), \tag{18}$$

where $\vec{\theta}^* = [\theta_1, \theta_2]'$.

We utilize numerical methods to determine the *actual* $\vec{\theta}^*$ and option value. To do so, we utilize equations (10) and (11) with $K = K_2$ to determine θ_2^* the value of the simple Bermudan put at $t = \tau_1$ as a function of S_t and then determine where this value equates to K_1 to find θ_1^*.

3.3 Build / Abandon Real Option

The build / abandon real option valuation methodology was presented in (Bashiri, Davison, and Lawryshyn 2018) and here we provide a summary. As above, we simulate N risk-neutral paths for S_t. We assume parametric functions $f_B(s, t; \vec{\theta}_B)$ for the construction (build) boundary and $f_A(s, t; \vec{\theta}_A)$ for the abandon boundary. Defining $\lambda_t^{(i)} = \{0, 1, 2, 3\}$ as the state variable of the i-th simulation such that $\lambda_0^{(i)} = 0$, where 0 denotes the state where no construction has taken place, 1 denotes state where the plant is under construction, 2 denotes the state where the plant is in operation and 3 denotes the state where the plant has been abandoned. We define the first passage of time when $S_t^{(i)}$ hits the build boundary,

$$\tau_B^{(i)} \equiv \min\{t > 0 \,:\, S_t^{(i)} \ge f_B(S_t^{(i)}, t; \vec{\theta}_B)\}. \tag{19}$$

Similarly, the first passage of time when $S_t^{(i)}$ hits the abandon boundary after construction has begun can be defined as

$$\tau_A^{(i)} \equiv \min\left\{t > 0 \,:\, S_t^{(i)} \leq f_A(S_t^{(i)}, t; \vec{\theta}_A),\, \lambda_t^{(i)} \in \{1,2\},\, T : \lambda_t^{(i)} \in \{1,2\}\right\}. \tag{20}$$

Thus, the state variable is set as follows,

$$\lambda_t^{(i)} = \begin{cases} 0, & \text{for } t < \tau_B^{(i)} \text{ or } \tau_B^{(i)} \in \emptyset, \\ 1 & \text{for } \tau_B^{(i)} \leq t < \tau_B^{(i)} + \tau_c, \\ 2 & \text{for } \left\{\tau_B^{(i)} + \tau_c \leq t < \tau_A^{(i)}\right\} \text{ or } \left\{\tau_B^{(i)} + \tau_c \leq t \text{ and } \tau_A^{(i)} \in \emptyset\right\}, \\ 3 & \text{for } t \geq \tau_A^{(i)}, \end{cases} \tag{21}$$

where τ_c is a constant representing the time required for construction.

The cash-flow for the i-th path for construction is

$$CF_{\tau_B}^{(i)} = -\mathbb{1}_{\lambda_t^{(i)}=1} K, \tag{22}$$

where K is the construction cost. Similarly, the abandonment cash-flow is given as

$$CF_{\tau_A}^{(i)} = -\mathbb{1}_{\lambda_t^{(i)}=3} C_{ab}, \tag{23}$$

where C_{ab} is the cost to abandon. Finally, the operating cash-flows are

$$CF_t^{(i)} = \mathbb{1}_{\lambda_t^{(i)}=2} \gamma \left(S_t^{(i)} - C_{op}\right) \Delta t, \tag{24}$$

where C_{op} is the per unit time operating cost, γ is the rate of extraction of the mineral and Δt is the time step in the simulation of S_t.

The value of the i-th path is (cf. equation (2))

$$V_0^{(i)}(\vec{\theta}) = \sum_{j=0}^{N_t} CF_{t_j}^{(i)} e^{(-rt_j)}, \tag{25}$$

where $\vec{\theta} = [\vec{\theta}_B, \vec{\theta}_A]'$ and N_t is the number of time steps used per simulation. Proceeding similarly as above, the optimal parameters defining the build and abandon exercise boundaries can be determined as (cf. equation (4))

$$\vec{\theta}^* = \arg\max_{\vec{\theta}} \frac{1}{N} \sum_{i=1}^{N} V_0^{(i)}(\vec{\theta}), \tag{26}$$

leading to the optimal option value as (cf. equation (5))

$$V_0^* = \frac{1}{N} \sum_{i=1}^{N} V_0^{(i)}(\vec{\theta}^*). \tag{27}$$

While it can be inferred that the negative of the right hand side of equation (26) is a convex function, it is not strictly convex, which leads to challenges in optimization. In the results section we show how robust multivariate convex optimization methods can stall. One solution to the problem is to utilize heuristic algorithms. We utilize a constrained genetic algorithm to improve accuracy and convergence.

3.4 Optimal Investment Rule in Infinite Time

Here we present the very simple case of the optimal investment rule in infinite time, first introduced by McDonald and Siegel (1986). We assume that the project value, X, follows a GBM,

$$dX_t = \alpha X_t dt + \sigma X_t dW_t \tag{28}$$

where α is the drift, σ is the volatility and W_t is a Wiener process. At some point in time, investors can invest in the project for a one time cost of I, and thus the optimal value of the investment is

$$V(X) = \max_t \mathbb{E}\left[(X_t - I)e^{-rt}\right] \tag{29}$$

where $r > \alpha$ is the discount rate.

If there is no termination time for the investment opportunity, then the optimal investment rule is

$$X^* = \frac{\beta}{\beta - 1} I \tag{30}$$

where

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{31}$$

We proceed similarly as in the above cases where we simulate N paths for V of equation (28). Knowing this is an infinite time problem, we set the boundary function to simply be

$$f_B(x, t; \theta) = \theta \tag{32}$$

with the first passage of time for the i-th path as

$$\tau_B^{(i)} = \min\{t > 0, X_t^{(i)} \geq \theta\} \tag{33}$$

and the cash-flow of the i-th path as

$$CF_{\tau_B}^{(i)} = X_{\tau_B}^{(i)} - I. \tag{34}$$

We proceed as above, applying equations (2) - (5) to find the optimal investment rule, X^*.

4 Results

In the following subsections we present some results of the simulation experiments that were performed for 1) the Bermudan put option, 2) the option to purchase a Burmudan put option, 3) the build / abandon real option and 4) the optimal investment rule in infinite time.

4.1 Bermudan Put Option

We presented the results of applying the EBF method to the Bermudan put option in detail in Bashiri, Davison, and Lawryshyn (2018). Here we provide a summary table of the results. For the Bermudan put option, we assume the following parameters:

- $S_0 = 5$
- $K = 5$
- $\tau = 1$
- $T = 2$
- $r = 3\%$
- $\sigma = 10\%$.

For these parameters the pseudo-analytical results, using equations (11) and (10), respectively, are:

- $V_0 = 0.1688$
- $\theta^* = 4.7571$.

In Table 1 we present the mean and standard deviation of 1000 simulation runs for increasing N for V_0 and θ^*. As expected, as N is increased, the values for V_0 and θ^* approach those of the pseudo-analytical solution. Based on both these numerical (simulated) results and equation (12), we have confidence in the applicability of the EBF method.

Table 1: Bermudan put option convergence for 1000 simulation runs; mean value and (standard deviation).

	100 Paths		1000 Paths		10,000 Paths		100,000 Paths		1,000,000 Paths	
V_0	0.1744	(0.0257)	0.1705	(0.0080)	0.1692	(0.0025)	0.1689	(0.0008)	0.1689	(0.0003)
θ^*	4.7052	(0.2324)	4.7353	(0.0819)	4.7538	(0.0334)	4.7564	(0.0157)	4.7572	(0.0070)

4.2 Option to Purchase a Bermudan Put Option

The EBF method proved to work well with the simple Bermudan put option, and, in Bashiri, Davison, and Lawryshyn (2018) we showed how the American put option value converged to the numerical value as the number of simulated paths increased. However, in Davison and Lawryshyn (2021) we showed how the build / abandon mining example struggled to converge to the optimal scenario where different initial conditions led to different results. To explore these issues further, we turned to the simpler case of the option to buy a Bermudan put option. This case is similar to the build / abandon case where a simulated path must first hit one boundary before the second can be considered, but is significantly less complex in that only two single point parameters need to be optimized.

Here we consider the scenario where at $\tau_1 = 1$y we have the option to buy a Bermudan put option for $K_1 = 0.1$ with a strike price $K_2 = 5$ expiring at $T = 3$y with the option to exercise early at $\tau_2 = 2$y. For the price process we have the following parameters:

- $S_0 = 5$
- $r = 3\%$
- $\sigma = 10\%$.

For these parameters, the numerical solution gives

- $V_0 = 0.1033$
- $\theta_1 = 5.2323$

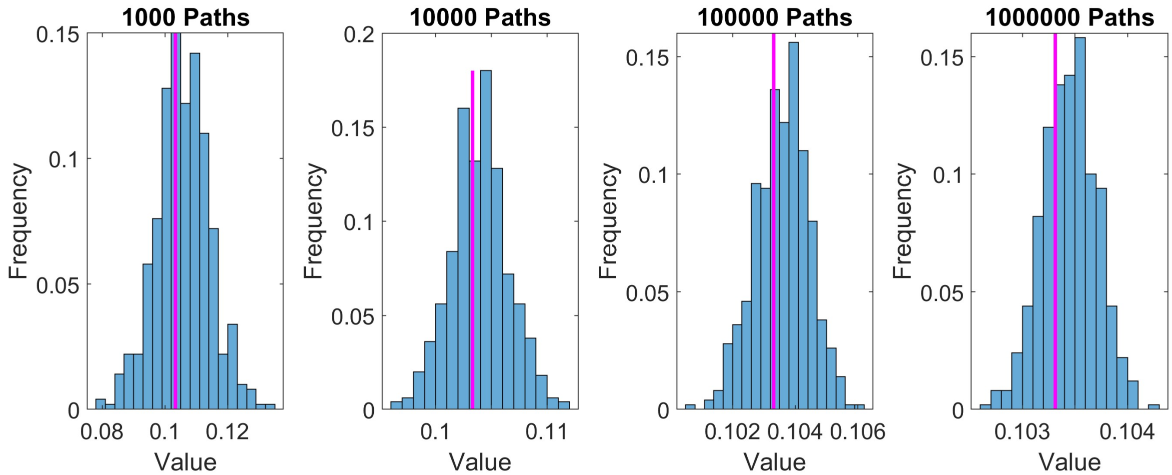

Figure 3: Histograms of V_0^* for the option to purchase a Bermudan put option (note that each case was simulated 500 times).

- $\theta_2 = 4.7571$.

In Figure 3 we plot histograms of V_0^* for varying N values. In each case, 500 simulations were run. The corresponding histograms for optimal exercise values, θ_1^* and θ_2^* are presented in Figure 4. The results are summarized in Table 2. As can be seen, as the number of paths is increased, the simulated results converge to those determined numerically and the standard deviation of the simulated results reduces. These results provide further confidence in the EBF methodology.

Table 2: Option to buy a Bermudan put option convergence for 500 simulation runs; mean value and (standard deviation).

	No. of Paths (N)							
	1,000		10,000		100,000		1,000,000	
V_0^*	0.1057	(0.00850)	0.1039	(0.00258)	0.1036	(0.00085)	0.1034	(0.00026)
θ_1^*	5.2240	(0.08127)	5.2314	(0.03548)	5.2321	(0.01768)	5.2324	(0.00769)
θ_2^*	4.7444	(0.09002)	4.7510	(0.03915)	4.7565	(0.01757)	4.7565	(0.00833)

4.3 Build / Abandon Real Option

The first two cases presented above explore simpler cases where the exercise boundaries were single points at predefined times. In Bashiri, Davison, and Lawryshyn (2018) we showed the convergence results for an American put option where we modelled the exercise boundary curve, f_B, as cubic splines and polynomials of varying order. Since, generally, the option values tend to be insensitive to small variations in the exercise boundaries, and, in an effort to provide significant degrees of freedom in the boundary, allowing for convexity and concavity in a given boundary while at the same time not making the boundary overly complex so that optimization routines have a relatively

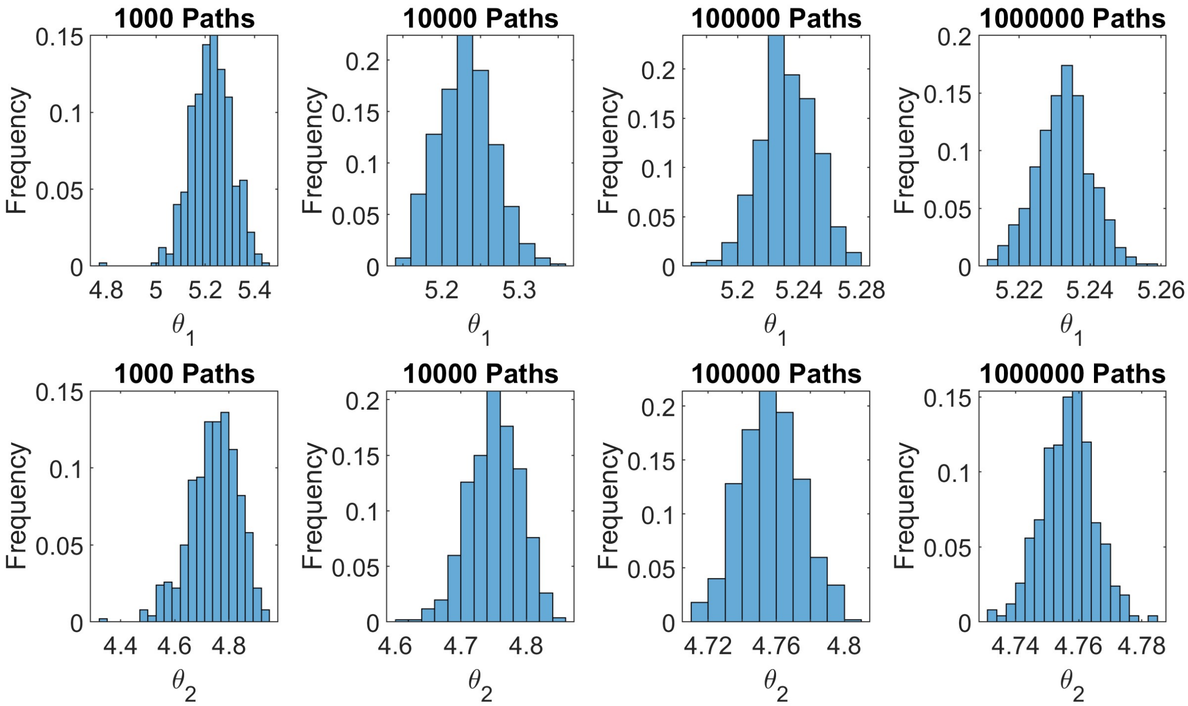

Figure 4: Histograms of θ_1^* and θ_2^* for the option to purchase a Bermudan put option (note that each case was simulated 500 times).

even chance of converging, in the results presented here we utilize piecewise linear functions for the exercise boundaries. We define the general boundary as

$$f_w(s,t;\vec{\theta},\vec{\eta}) = \begin{cases} \theta_1 & \text{for } 0 \le t < \eta_1, \\ \theta_i + (t-\eta_i)\frac{(\theta_{i+1}-\theta_i)}{(\eta_{i+1}-\eta_i)} & \text{for } \eta_i \le t < \eta_{i+1}, \\ \theta_n & \text{for } \eta_n \le t \le T, \end{cases} \tag{35}$$

where $w \in \{B, A\}$ represents the build and abandon scenarios, and $i \in \{1, 2, ..., n\}$ where n is the number of control points at times η_i determined by the analyst. For example, if we were to set the control points $\vec{\eta} = [2, 5, 9]'$ then for $0 \le t < 2$ the exercise boundary would be $f_w = \theta_1$, for $2 \le t < 5$ and for $5 \le t < 9$ linear interpolation would be used for the appropriate θ_i and θ_{i+1} values and for $9 \le t \le T$, $f_w = \theta_n$.

For the build / abaondon real option example, we assume the following parameters:

- $S_0 = 5$
- $T = 10$ years
- $K = 5$
- $C_{op} = 5$
- $C_{ab} = 5$
- $\tau_c = 0$
- $\gamma = 5$
- $r = 3\%$
- $\sigma = 10\%$.

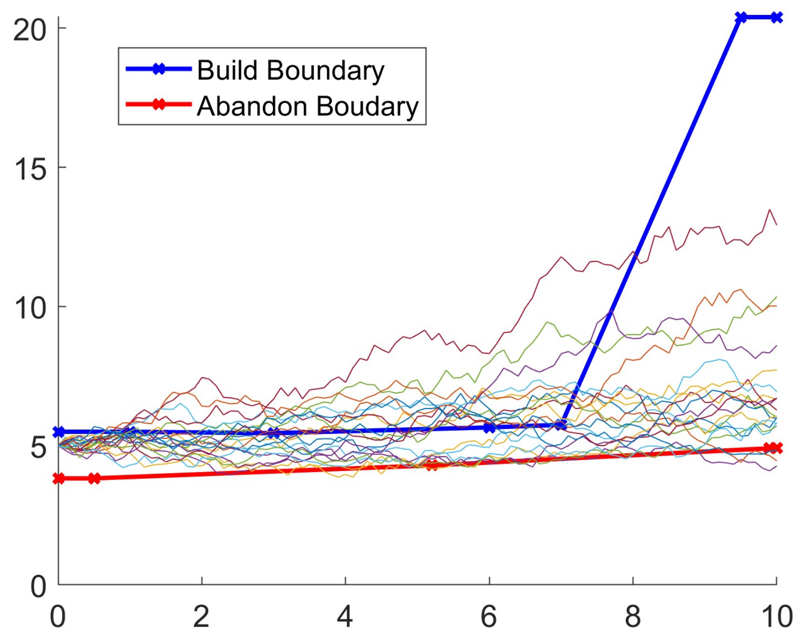

Figure 5: Build / abandon boundaries for the base case with $N = 10^6$ and $N_t = 100$.

Note that here we have used a zero time to construction (τ_c) however this parameter had no impact on convergence results. For the base case we assume build / abandon decisions can be made at half year increments. We set $\eta_B = [1, 3, 6, 7, 9.5]'$ and $\eta_A = [0.5, 5.2, 9.9]'$. From multiple numerical experiments using varying number of paths, N, and time steps, N_t, it was determined that results were stable using the constrained GA optimization with $N = 10^6$ and $N_t = 100$. Thus, with these given parameters, the value of the real option was determined to be $V_0^* = 28.1145$ The build / abandon boundaries, along with 20 random sample paths are plotted in Figure 5.

In Figure 6 we plot the histograms of V_0^* for varying N values and provide a tabulated summary in Table 3, where, in each case, 200 simulations were run. Again, as N is increased we see the expected convergence behaviour. In Figure 7 we plot a select number of build and abandon boundaries for varying N. Interestingly, even at higher levels of N there are some irregularities in the exercise boundary shapes. This result highlights the relative insensitivity of the option value to boundaries, especially in regions where the number of affected paths is low.

Table 3: Build / abandon real option convergence for 200 simulation runs; mean value and (standard deviation).

	No. of Paths (N)					
	1,000		10,000		100,000	
V_0^*	28.4265	(3.49904)	28.2280	(1.18566)	28.1441	(0.38221)

In Figure 8 we plot the histograms of V_0^* for varying N values determined using convex opti-

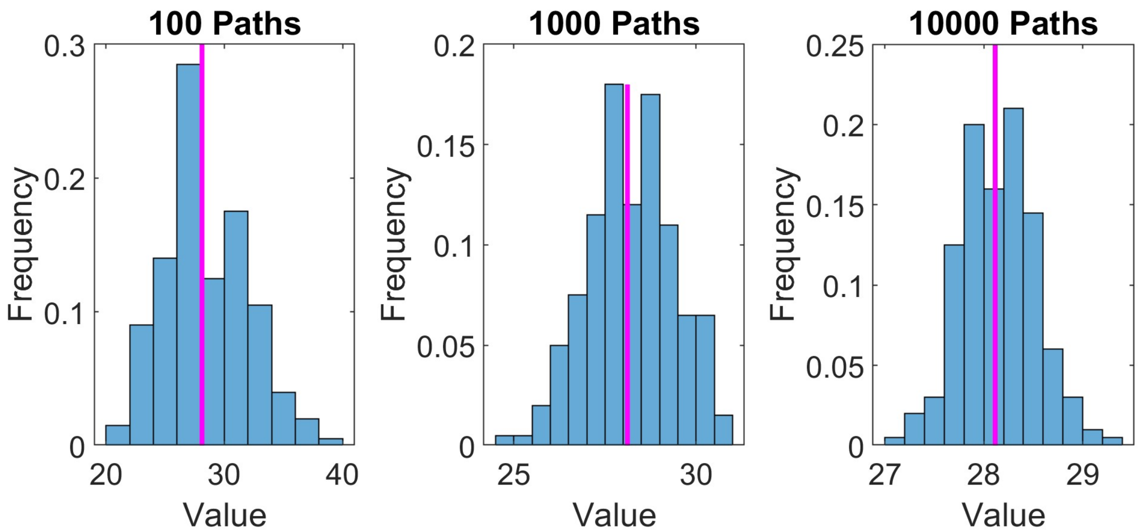

Figure 6: Histograms of V_0^* for the build / abandon real option (note that each case was simulated 200 times).

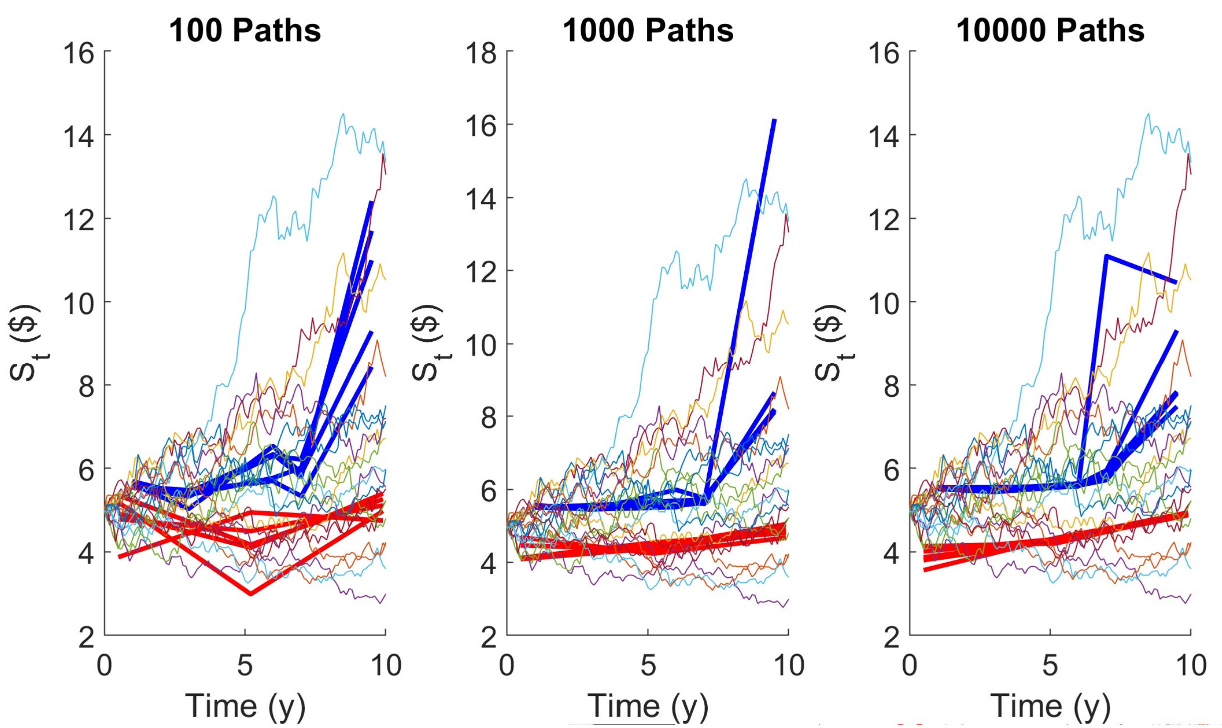

Figure 7: Build and abandon boundaries for varying number of paths, N.

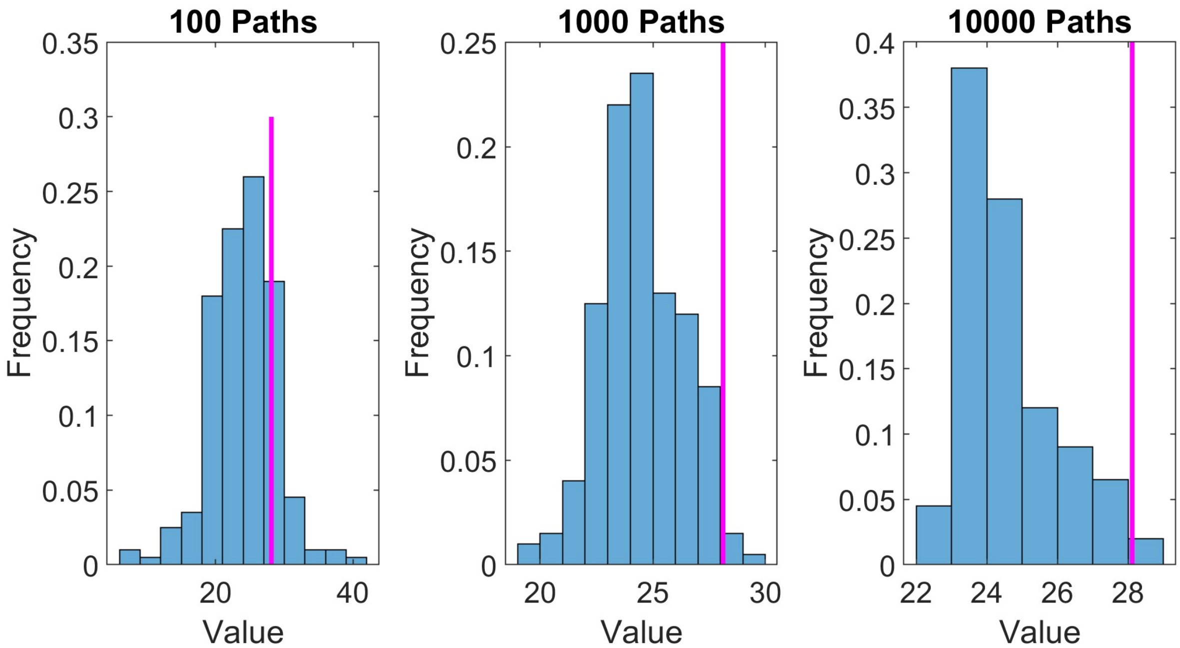

Figure 8: Histograms of V_0^* for the build / abandon real option using convex optimization (note that each case was simulated 200 times).

mization. We see that the optimization routine regularly converges to sub-optimal values. Table 4 presents the corresponding summary values for V_0^* and the standard deviation based on 200 simulations for varying number of paths N. In Figure 9 we plot a select number of build and abandon boundaries. As shown, the build boundaries are scattered and the optimization has clearly not converged to the global optimum. These results highlight the issues originally encountered when applying the EBF method for more complex problems. Using a genetic algorithm whose starting points span the domain of possible paths alleviates the problem.

Table 4: Build / abandon real option convergence using convex optimization for 200 simulation runs; mean value and (standard deviation).

	No. of Paths (N)					
	1,000		10,000		100,000	
V_0^*	23.9549	(4.75881)	24.5359	(1.79237)	24.6053	(1.37566)

4.4 Optimal Investment Rule in Infinite Time

For the optimal investment rule in infinite time we assume the following parameters:

- $I = 5$
- $r = 7\%$
- $\alpha = 3\%$

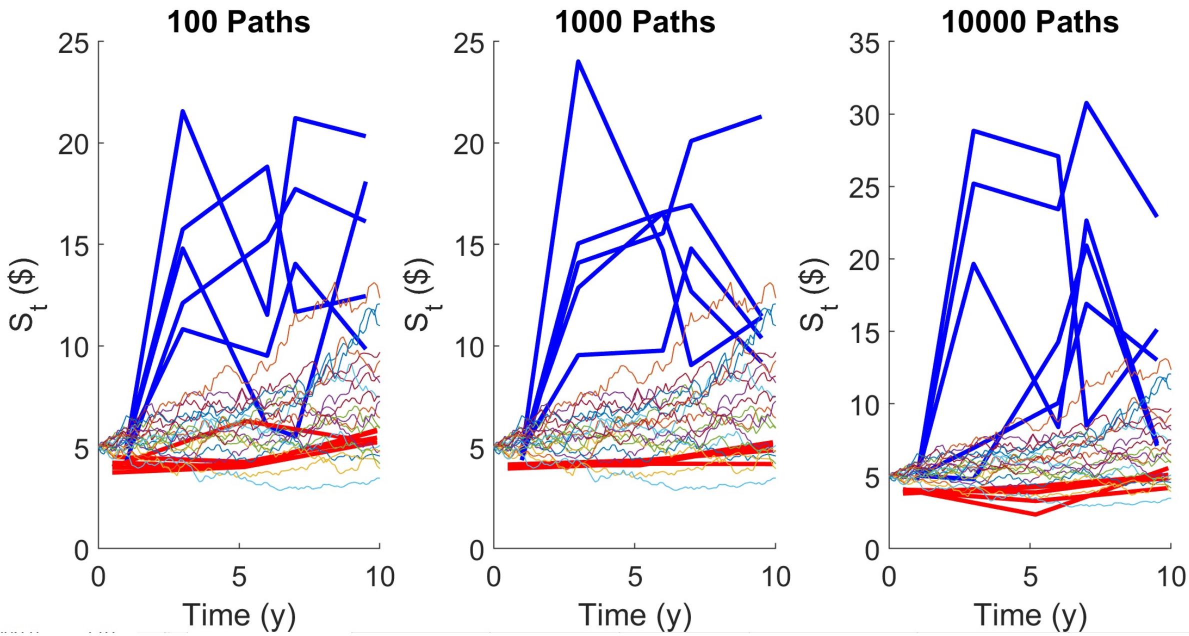

Figure 9: Build and abandon boundaries for varying number of paths, N.

- $\sigma = 10\%$.

With these parameters, the optimal investment rule gives $X^* = 10$. We simulate X_t with $X_0 = 5$. Note that increasing r improved convergence as a shorter time frame was required for simulation since future values approach discounted values to zero at earlier times. In the simulations presented here, we chose a terminal time $T = 200$ years. We found $N_t = 10,000$ to be more than sufficient in the analysis[3]. In this case θ consists of a single value, i.e., the optimal investment rule V^*, thus, convex optimization converged quickly and did not suffer the issues related to the build / abandon case.

In Figure 10 we plot the histograms of V^* for varying N values and the results are summarized in Table 5, where, in each case, 100 simulations were run. As N is increased we see the expected convergence behaviour, providing confidence that the EBF method can be applied to optimal investment rule problems in infinite time.

Table 5: Optimal investment rule in infinite time convergence for 100 simulation runs; mean value and (standard deviation).

	No. of Paths (N)					
	1,000		10,000		100,000	
V^*	9.6010	(1.22406)	9.9872	(0.51366)	9.9928	(0.26037)

[3]Note that when we set $r = 3\%$ we needed to use a terminal time of $T = 500$ years and needed to increase N_t proportionally to achieve converged results.

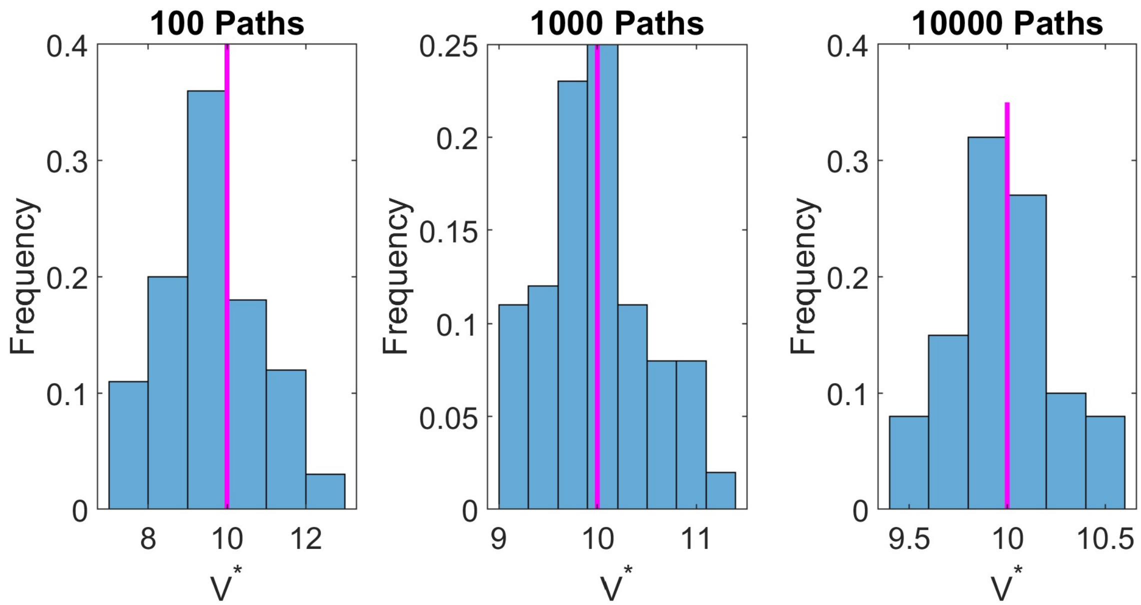

Figure 10: Histograms of V^* for the optimal investment rule in infinite time (note that each case was simulated 100 times).

5 Conclusions

The focus of this research was to present a real options valuation methodology geared towards practical use with an emphasis of exploring numerical accuracy and convergence issues. A key innovation of the EBF methodology is the idea of fitting optimal decision making boundaries to optimize the expected value, based on Monte Carlo simulated stochastic processes that represent important uncertain factors. We repeated our previous findings where we showed good accuracy and convergence results for the simple, single exercise parameter for the case of a Bermudan put option. We then explored the problem where one has the option to purchase a Bermudan put option. In this case we have two exercise parameters to determine. In both of these cases, standard convex optimization schemes provided good results. However, in our third case, the build / abandon option, which we had previously shown issues with convergence using standard convex optimization, good accuracy and convergence was achieved using a constrained genetic algorithm where we initiated the algorithm to search the entire stochastic process domain. Finally, we applied the EBF method to the very simple case of the optimal investment rule in infinite time, first introduced by McDonald and Siegel (1986). Again, the EBF method proved accurate and convergence was easily achieved using a standard convex optimization algorithm.

To value a realistic real option with multiple stochastic factors using current standard methods can lead to significant model complexity that may make the analysis intractable. Our theoretical and numerical presentation of EBF method shows how the complexity can be overcome through the use of Monte Carlo simulation. We emphasize that in a real options context, often many parameters can only be estimated. Errors associated with an approximate boundary fit may be significantly less than not modelling important complexities in the quest of a mathematically accurate solution. We feel that the EBF methodology is very tractable in an industry setting for it is simple enough for managers to understand, yet can account for important real world factors that make the real options model suitable for valuation.

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