# Forest Harvesting with Multi-Factor Uncertainty 1_15_11

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# Forest Harvesting with Multi-Factor Uncertainty


## Abstract

We provide a multi-factor real option model for a single harvest and then for perpetual forest harvesting, which extends the Faustmann (1849) approach to infinite rotations with multi-factor uncertainty. We examine separately forest growth and timber prices (and eventually credits/penalties for $CO_2$ uptake/release upon harvest), so the forester can focus on critical exogenous and partially endogenous factors. While the single harvest option and its value at optimal harvest time are expressible as products of the timber price and forest size, the replanting option is not, so the principle of similarity sometimes used to reduce the dimensions of two factor models does not apply. We provide a unique quasi-analytical solution for multi-factor uncertainty in forestry to determine the multiple rotation value and optimal harvest size.

# Forest Harvesting with Multi-Factor Uncertainty

## 1. INTRODUCTION

We provide a multi-factor real option model for a single harvest and then for perpetual forest harvesting (infinite rotations). In addition, we examine separately forest growth and timber prices, so the forester can focus on critical exogenous and partially endogenous factors. We provide a unique quasi-analytical solution for multi-factor uncertainty in forestry to determine the multiple rotation value and optimal harvest size.

There are several articles on single forest harvesting that are extensions of Tourinho (1979), except for renewable resources with both stochastic growth and timber prices. Chang (2005) and the Amacher, Ollikainen and Koskela (2009) book extend the Faustmann (1849) deterministic approach to stochastic multiple rotations. While the single harvest option and its value at optimal harvest time are expressible as products of the timber price and forest size, the replanting option is not, so the principle of similarity sometimes used to reduce the dimensions of two factor models does not apply.

We consider separately the components of harvesting gross profit H(P,Q), where P is the standard (per cubic meter, or per thousand board feet) net unit profit and Q is the forest standard size, assuming both P and Q may be affected by different factors. Also, eventually we allow for the possibility of $CO_2$ capture, but assuming no subsequent release, as a function of Q in terms of a constant yield while the forest is growing, similar to a "convenience yield". Correlation between P and Q may vary according to exogenous or local factors, where exceptional weather promotes growth with then excess timber supplies and price declines, or alternatively inhibits harvest with supply shortfalls and price increases.

Norstrøm (1975) provides a transition probability matrix reflecting the fluctuating price of timber, with some 44 state transitions, showing advantages over the Bierman (1968) deterministic approach for optimal harvest timing. Samuelson (1976) notes that forestry economist's "simple notion of stationary equilibrium needs to be …replaced by the notion of a perpetual Brownian motion" considering "the bouncings of the futures contracts for plywood on the organized exchanges" but he did not apparently reflect that the stochastic single harvest opportunity is similar to a perpetual American call option, solved in Samuelson (1965). According to Newman (2002) who surveyed hundreds of relevant articles, Miller and Voltaire (1980, 1983) were among the first to consider stochastic forest value, and infinite rotations, and provide a solution for a single factor forest stand evolving according to an arithmetic Brownian motion, extended to multiple harvests. Clarke and Reed (1989, 1990) study prices and age-dependent growth evolving according to geometric Brownian motion, aggregate forest value into a single variable, and note that when harvest costs are non-stochastic, the optimal stopping solution is given in Samuelson (1965). There are now many authors that have provided real option models of rotations, including Thompson (1992), Willassen (1998), Plantinga (1998), Sødal (2002), Insley (2002), Saphores (2003), and Shackleton and Sødal (2010), among others.

There are several deterministic models of forest growth, often based on empirics. Banks (1994) describes the Valentine (1983) logistic model and the Garcia (1983) Bertalanffy-Richards power law logistic model, also modified by adding an arithmetic Brownian motion process, fitted to heights and ages of radiate pine trees in the Kaingaroa Forest in New Zealand. Clarke and Reed (1989) use a cubic polynomial growth function fitted to the Clark (1976) data on net stumpage values for B.C. Douglas fir trees.

$CO_2$ is removed from the atmosphere and stored as carbon in forest biomass, but partially released upon harvest, particularly if wood is used as biomass fuel (or the forest burns). In order to model $CO_2$ uptake as a function of forest growth, van Kooten et al. (1995) use a simple power function for timber growth $f(t) = kt^a e^{-bt}$ which is fitted to coastal B.C. timber, and also to black spruce in the boreal forest of northern Alberta. The amount of release depends on the fraction of harvested timber that goes into long-term storage in structures and landfill, called "pickling". Reddy and Price (1999) also consider a deterministic model for carbon sequestration as a function of timber growth. Asante et al. (2010) propose complex deterministic models for carbon sequestration.

Several authors have proposed multi-factor real option models for growth, prices and/or carbon sequestration. Chladná (2007) assumes a mean-reverting process for prices, a deterministic function for forest growth, and a geometric Brownian motion for $CO_2$ prices, and apparently uses simulation to produce results. Morck et al. (1998) assume a geometric Brownian motion for both timber prices and inventory of timber in a leasehold, but modify the timber growth by subtracting the quantity of timber produced. With these separate functions in a partial differential equation (PDE), a numerical (Runge-Kutta) solution is used to evaluate the leasehold value at any time, and the optimal cutting rate (repeated partial single harvest, no benefit for the leaseholder from reforesting). Alvarez and Koskela (2003, 2006, 2007b) provide several multi-factor harvesting models covering interest rate variability, risk aversion and amenities. Alvarez and Koskela (2007a) is closest to our approach, considering resource stock and price uncertainty, typically geometric Brownian motion (and also considering quantity mean-reversion), but only for the single harvest or sequential case (where the part of the harvesting is postponed).

The next section describes a multi-factor harvest model for a single harvest, and illustrates some sensitivities. Section three extends this model to multiple harvests, with constant prices. Section four develops the multiple infinite rotation model under more realistic conditions, with stochastic prices and growth, and provides a quasi-analytical solution. Some sensitivity analyses similar to those for a single harvest using comparable parameter values are shown. The last section concludes and offers some suggestions for further research.

## 2. UNIT PROFIT and GROWTH UNCERTAINTY: SINGLE HARVEST

Suppose that both the profit per unit and the number of forest units follow different but possibly correlated geometric Brownian motion processes, following Paxson and Pinto (1995). Let P represent the profit per unit sold and Q the quantity of standing timber. Assume that each variable follows a geometric Brownian motion[1] of the form:

$$dP = \mu P\,dt + \sigma P\,dz_1 \tag{1}$$

$$dQ = \omega Q dt + \alpha Q\,dz_2 \tag{2}$$

where μ and ω are the expected multiplicative trends of P and Q, σ and α are the volatilities, and $dz_1$ and $dz_2$ the increments of a Wiener process. The two variables may be correlated with correlation coefficient ρ.

Consider a portfolio that consists of a long position in the option to harvest a given forest $H(P,Q)$, and a short position consisting of $\Delta_1$ and $\Delta_2$ units of P and Q, respectively. Assume that the forester is risk-neutral.[2] Applying Ito's lemma, the following PDE for a forest is obtained (where r = riskfree rate):

$$\frac{1}{2}\frac{\partial^2 H(P,Q)}{\partial P^2}\sigma^2 P^2 + \frac{1}{2}\frac{\partial^2 H(P,Q)}{\partial Q^2}\alpha^2 Q^2 + \frac{\partial^2 H(P,Q)}{\partial P \partial Q} PQ\rho\sigma\alpha + \mu P\frac{\partial H(P,Q)}{\partial P} + \omega Q\frac{\partial H(P,Q)}{\partial Q} - rH(P,Q) = 0 \tag{3}$$

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[1] There are several alternative growth functions for forests, including several deterministic models. Also ω in equation (2) could denote net growth in Q after subtracting a proportional carbon credit for $CO_2$ uptake. A constant proportional carbon credit implies that the absolute annual $CO_2$ sequestration increases over time and forest size without limit, which is not consistent with many empirical forest studies.

[2] The assumption of risk neutrality may be relaxed by adjusting the drifts of P and *Q* to account for a risk premium.

Equation (3) explains the movements in the value function of a forest with a single harvest opportunity (and no land value after harvest) and is subject to the usual boundary conditions. The first boundary condition is the value matching that gives the value of H(P,Q) at which the forester should harvest. The second boundary condition is the smooth pasting that assures that the derivatives of the two functions (before and after the harvest) are equal at the harvest point.

Let $X = PQ$ denote the total gross harvest profit (assuming no maintenance or reforesting costs) implying that $P(X) = H(P,Q)$, and assume that P and Q are linear homogenous to the degree one, so similarity arguments are valid[3]. After the appropriate substitutions, equation (3) can be re-written as:

$$\frac{1}{2}X^2\frac{d^2P(X)}{dX^2}\left[\sigma^2+\alpha^2+2\rho\sigma\alpha\right]+X\frac{dP(X)}{dX}\left[\rho\sigma\alpha+\mu+\omega\right]-rP(X)=0 \tag{4}$$

Equation (4) is an ordinary differential equation with the following characteristic quadratic function:

$$\frac{1}{2}(\sigma^2+\alpha^2+2\rho\sigma\alpha)\beta(\beta-1)+(\rho\sigma\alpha+\mu+\omega)\beta-r=0 \tag{5}$$

Equation (5) has two roots, a positive and a negative one, given by:

$$\beta_{1,2}=\frac{1}{z^2}\left(-\left(\rho\sigma\alpha+\mu+\omega-\frac{1}{2}z^2\right)\pm\sqrt{2rz^2+\left(\rho\alpha\sigma+\mu+\omega-\frac{1}{2}z^2\right)^2}\right) \tag{6}$$

where $z^2=\alpha^2+\sigma^2+2\rho\sigma\alpha$.

The solution of equation (4) is:

$$P(X)=AX^{\beta_1}+BX^{\beta_2} \tag{7}$$

We know that as X increases, the value function of the forest has to increase and that equation (7) has to be finite, thus B equals zero. Equation (7) is subject to the value-matching condition:

[3] See Paxson and Pinto (1995) appendix.

$$P(X^*) = X^* - K \tag{8}$$

where $X^*$ is the harvest trigger value, K is the harvest cost plus the value of any $CO_2$ release upon harvest, and is also subject to the smooth-pasting condition:

$$\frac{dP(X^*)}{dX} = 1 \tag{9}$$

Equations (7), (8) and (9) imply that:

$$X^* = \frac{K\beta_1}{\beta_1 - 1} \tag{10}$$

Thus the value function of the forest, H(P,Q), is given by:

$$H(P,Q) = \begin{cases} (X^* - K)\left(\frac{X}{X^*}\right)^{\beta_1} & X < X^* \\ X - K & X \geq X^* \end{cases} \tag{11}$$

Equation (11) describes the value function of the forest before and after the trigger is hit. Before the trigger X* is hit, the forest has not yet been harvested and its value function is a monopoly perpetual American option to harvest. At the trigger, the value function is the harvest net present value, assuming X is obtained and K incurred instantaneously.

Table 1

Base Case Data.

| **Description** | **Parameter** | **Value** |
|---|---|---|
| Timber Price per Unit Q | P | 1 |
| Quantity of Tree/Forest | Q | 100 |
| Price * Quantity | X | 100 |
| Harvest Cost | K | 100 |
| Price Volatility | $\sigma_P$ | 0.20 |
| Quantity Volatility | $\alpha$ | 0.05 |
| Correlation | $\rho$ | 0.00 |
| Relevant Discount Rate | r | 0.04 |
| Price Drift | $\mu$ | 0.01 |
| Forest Growth | $\omega$ | 0.02 |

Using the Table 1 data, H(P,Q) using equation (11) is 42.89, the optimal harvest size X* is 651.06 using equation (10), or over 6.5 times the current timber size times a unitary price. With the specified drifts, volatilities and correlation parameters, the gross profit volatility z=21% and $\beta_1$=1.18.

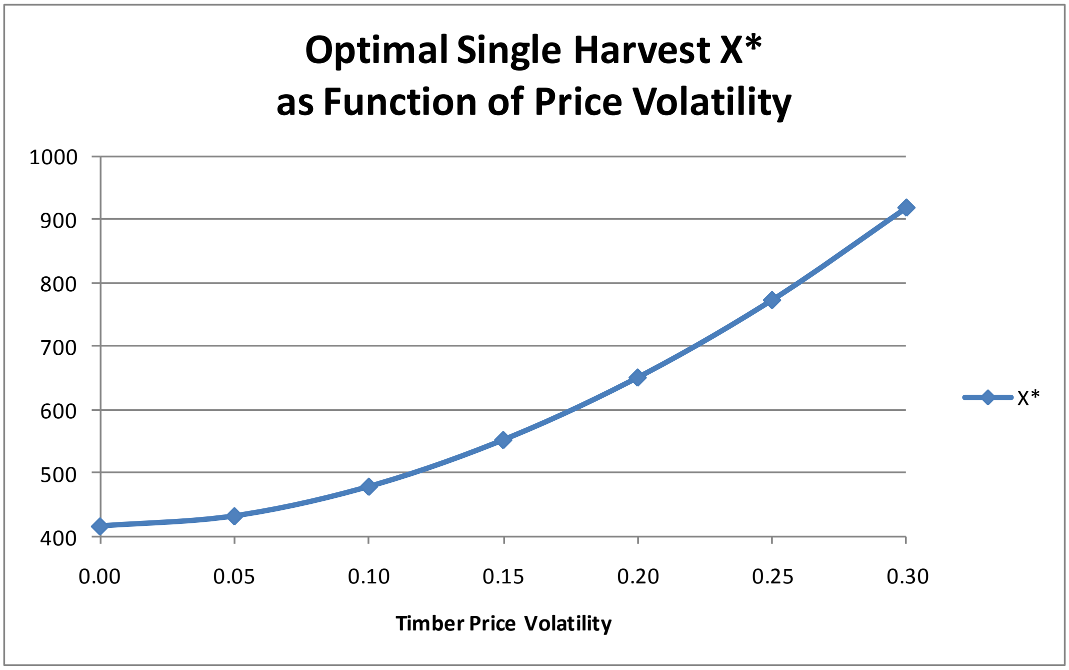
The first derivative (delta) of the value function of the forest, where P and Q are the state variables, is:

$$\frac{dH(P,Q)}{dX} = \begin{cases} \left(\frac{X}{X^*}\right)^{\beta_1 - 1} > 0 & X < X^* \\ 1 & X \geq X^* \end{cases} \tag{12}$$

Delta behaves as expected, that is as total gross profit increases the forest harvest option value also increases, until it reaches a constant 1 at X=X*.

Since P and Q are separate items, it is useful to examine the sensitivity of X* to changes in separate factors. Figure 1 shows the sensitivity of the optimal harvest size to changes in P volatility.

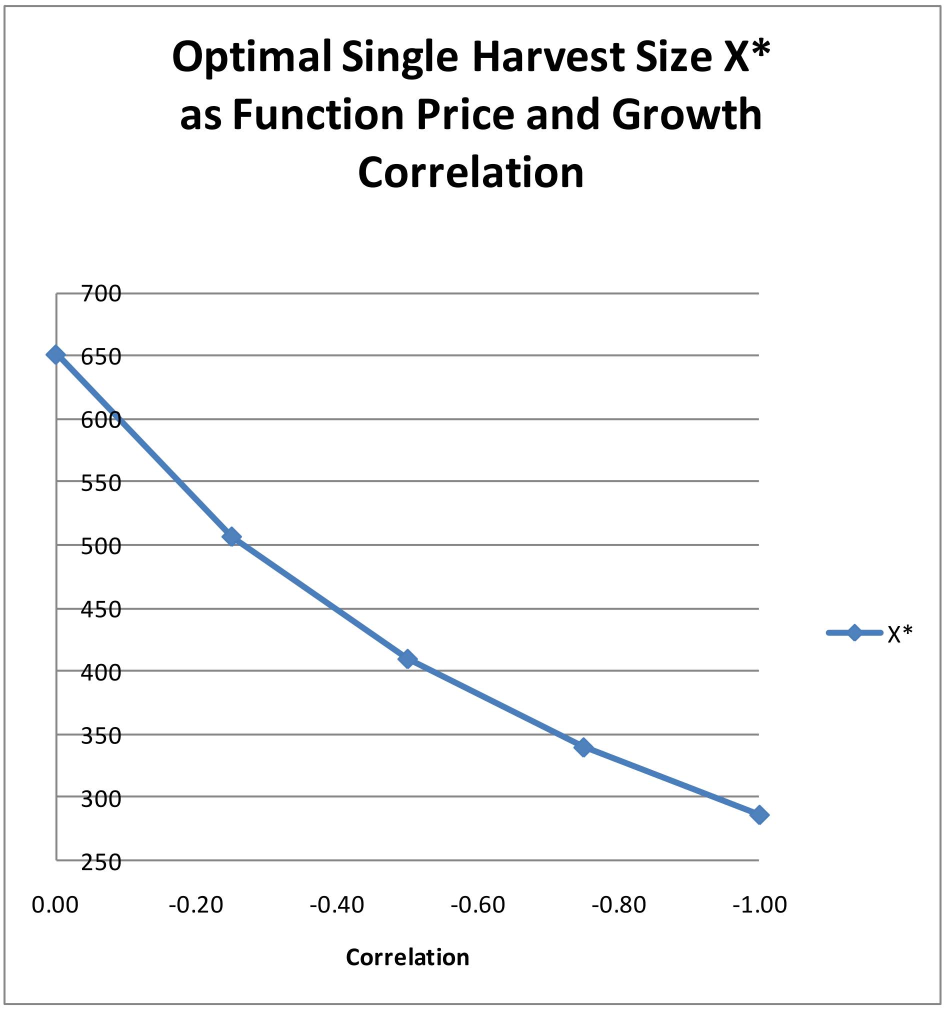
Figure 1



The optimal harvest size X* is 416 using equation (10), if the price volatility is zero (but the drift is still 1% per annum), but this is reduced to 212 if the price is constant with nil drift.

Figure 2 shows the sensitivity of the optimal harvest size to changes in P and Q correlation.

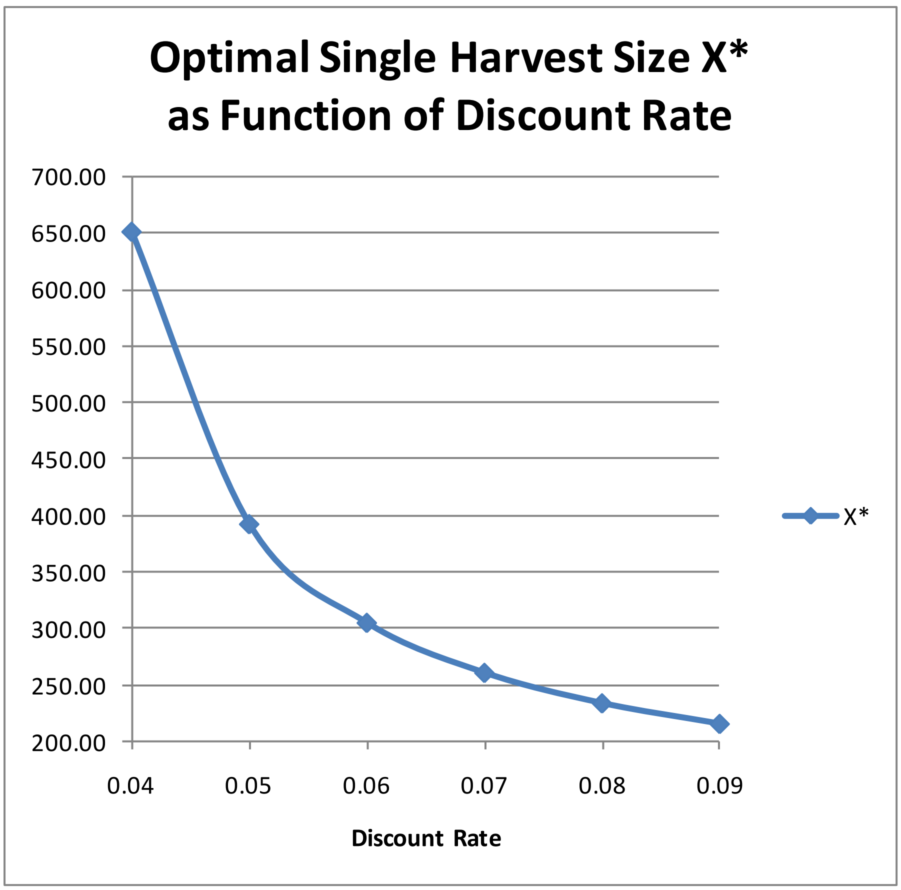
Figure 2




Perfect negative correlation of P and Q reduces X* to less than half of optimal harvest size if there is nil correlation, so clearly either local negative correlation (low prices associated with surplus timber production, or high prices associated with timber scarcity) could be critical in forestry harvest decisions. With high correlation (not shown), the harvest trigger high enough to discourage any immediate harvests.

It is well known that discount rates are an important factor in determining optimal forest harvest, even for deterministic models. Figure 3 shows the sensitivity of single harvest optimal size to changes in discount rates.

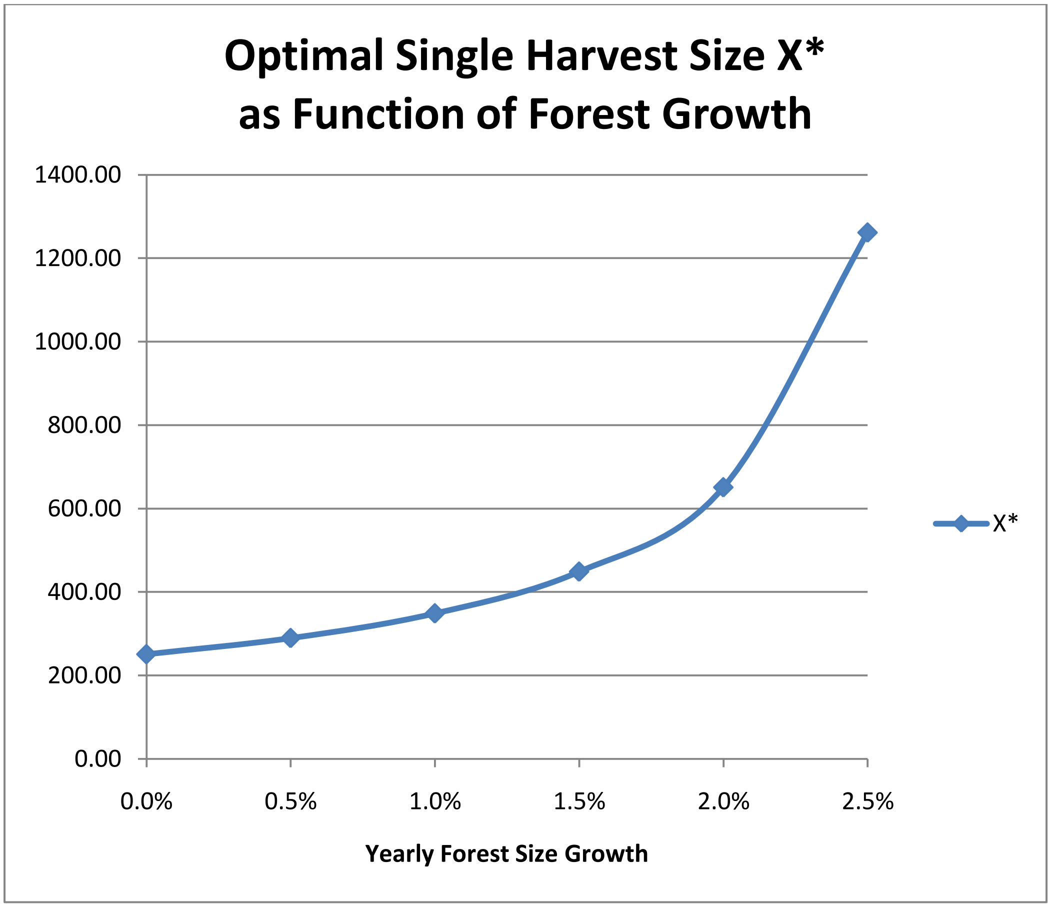
Figure 3




Note higher discount rates justify earlier harvesting, since the present value of the harvest is reduced with higher discount rates.

Finally, Figure 4 shows the sensitivity of a single harvest trigger to changes in the forest constant growth parameter value.

Figure 4




As in deterministic Jevons-Wicksell models, assuming constant timber volume growth, increases in the growth parameter value results in increased harvest triggers. Indeed, a high growth when combined with a positive price drift approaching r results in a harvest trigger indicating a very delayed harvest.

## 3. FOREST STAND UNCERTAINTY: MULTIPLE HARVESTS

Faustmann (1849) showed that the single harvest model could be extended to a multiple infinite rotation model by assuming essentially that the single harvest plus reforesting costs could be considered a perpetual recurring annuity, assuming harvest cash flows occur at equal optimal rotation times. Willassen (1998) solved the stochastic rotation problem using the theory of impulse control, but the general solution in Chang (2005) is similar to an extension of the Tourinho (1979) optimal extraction problem, dividing the single harvest option value by:

$$[1-(\frac{X_0}{X**})^{\beta_1}] \tag{13}$$

where $X_0$ is the forest size upon replanting.

The Chang (2005) one-factor representation of tree-cutting and rotation belongs to the broad group of real-option replacement models, see Ye (1990), Mauer and Ott (1995), and Dobbs (2004), and many of its results can be derived from these works and by using the standard real-option method. We define $X$ to denote tree size, whose growth follows a geometric Brownian motion process. When $X$ attains its optimal threshold for cutting, denoted by $\hat{X}$, the tree is cut at a cost $K$ and a sapling with size $X_0$ is replanted. The tree value is denoted by $F\left(X\right)$. The value matching relationship then requires that:

$$F\left(\hat{X}\right)=F\left(X_0\right)+X-K \tag{14}$$

assuming that the timber price is constant and that $K$ is defined as the cost of cutting divided by the price (unity in this article).

The valuation function, the solution to the one-factor risk-neutral valuation relationship is given by:

$$F=AX^{\beta}, \tag{15}$$

where $\beta$ is the positive root of the characteristic equation (also equation 6). So:

$$A\hat{X}^{\beta}=AX_0^{\beta}+\hat{X}-K, \tag{16}$$

with smooth-pasting condition expressed as:

$$\beta A\hat{X}^{\beta} = \hat{X}\,. \tag{17}$$

Combining (17) with (16) yields:

$$\frac{\hat{X}}{\beta}\frac{X_0^{\beta}}{\hat{X}^{\beta}} = \frac{\hat{X}}{\beta} - \left(\hat{X} - K\right) \tag{18}$$

Note that (18) is identical to equation (6) in Chang (2005). The left hand side of (18) is the value of planting a sapling with size $X_0$. The right hand side is the value of sustaining the tree less foregone value of harvesting. Re-arranging (18) yields:

$$\begin{aligned}\frac{\hat{X}}{\beta} &= \left(\hat{X} - K\right)\left[\frac{1}{1 - \frac{X_0^{\beta}}{\hat{X}^{\beta}}}\right] \\ &= \left(\hat{X} - K\right)\left(1 + \frac{X_0^{\beta}}{\hat{X}^{\beta}} + \frac{X_0^{2\beta}}{\hat{X}^{2\beta}} + \dots\right)\end{aligned} \tag{19}$$

The value of a tree at the harvest threshold is equal to the value from harvesting adjusted upwards by a factor exceeding one, which reflects the value of the recursive replanting option.

This result requires a constant tree price. It is not possible to extend the Chang (2005) analysis to two factors, timber price and tree size, because $X_0$ is treated as a constant in the one-factor model, but for the two-factor model, $X_0$ is a product of the timber price at harvest and the sapling size.

Thus the value function of the multiple forest rotation, MH(X), is given by:

$$MH(X) = \begin{cases}[(X^{**} - K)\left(\frac{X_t}{X^{**}}\right)^{\beta_1}]/[1 - (\frac{X_0}{X^{**}})^{\beta_1}] & if \quad X < X^{*} \\ \end{cases} \tag{20}$$

where X** is the solution to $X_0^{\beta} X^{**(1-\beta_1)} - \beta_1 K + (\beta_1 - 1) X^{**} = 0$

(21)

Equation (20) describes the value function of the forest before and after the trigger is hit. Before the trigger X** is hit, the forest has not yet been harvested and its value function with the first X=X(t) is a monopoly perpetual American option to harvest repeatedly. At the trigger, the value function is the harvest net present value, assuming X is obtained and K (which now includes reforesting costs) is incurred instantaneously, plus the infinite renewal option with $X=X_0$, that is $Q(t)=Q_0$.

Using the parameter values in Table 1, X**=187.6 (if $X_0$ =60, K=100 and price is constant), or 12% less than the optimal single harvest size if the price is constant.

Figure 5 shows the sensitivity of the multiple harvest trigger to changes in the forest size volatility. Note that size volatility is generally not high, but the optimal harvest size is anyway not very sensitive to size volatility changes.

Figure 5

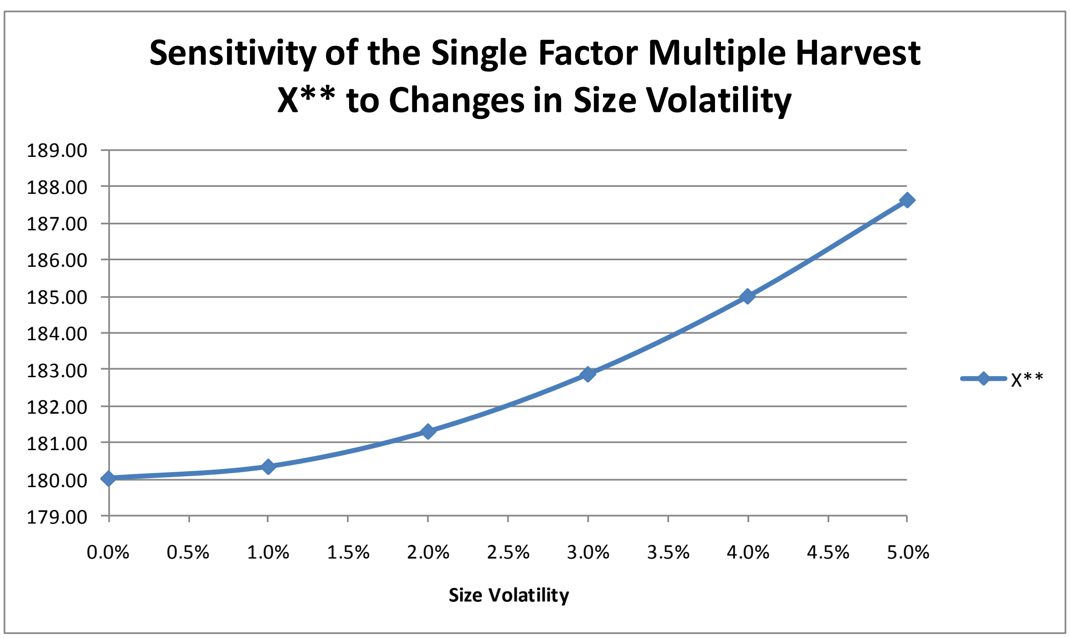
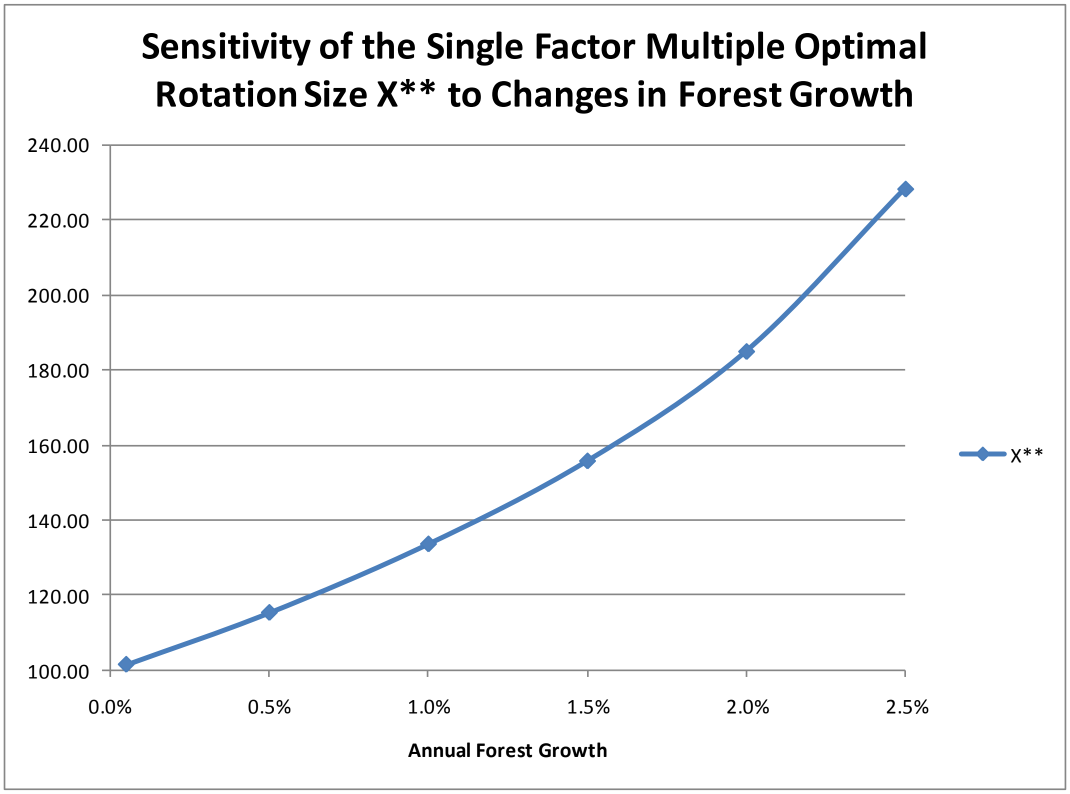


Figure 6 shows the sensitivity of single factor multiple rotation optimal size to changes in forest growth.

Figure 6




Forest growth relative to the interest rate of 4% is quite important for determining the optimal rotation size in infinite rotation. As in deterministic Jevons-Wicksell models, assuming constant timber volume growth, increases in the growth parameter value results in increased harvest triggers. Very low growth, with constant prices, implies that the traditional net present value rule (almost) holds, if replanting costs are ignored. A high growth approaching r (not shown) results in a harvest trigger indicating delayed harvests, but not so delayed as for the single harvest model. At all growth levels, the multiple single factor harvest triggers are lower for multiple compared to single harvest models.

## 4. SEPARATE UNIT PROFIT and GROWTH: MULTIPLE HARVEST

Although the principle of similarity conveniently transforms the two-factor valuation relationship (3) to a one-factor expression, applying this transformation relies on each of the terms in the economic boundary conditions being expressible as a product of price and tree size. If this condition is not met, then this transformation is unavailable. Alternatives are required for solving the valuation relationship. This can be demonstrated by reformulating the two-factor tree-cutting problem by stipulating that when a tree is harvested, it is replanted immediately by a sapling of size $q_0$. We now redefine the value of a forest stand with replanting as:

$$\tfrac{1}{2}\sigma_p^2 p^2 \frac{\partial^2 F}{\partial p^2} + \tfrac{1}{2}\sigma_q^2 q^2 \frac{\partial^2 F}{\partial q^2} + \rho\sigma_p\sigma_q pq \frac{\partial^2 F}{\partial p \partial q} + \theta_p p \frac{\partial F}{\partial p} + \theta_q q \frac{\partial F}{\partial q} - rF = 0 \quad (22)$$

The generic function satisfying the homogenous part of (22) is:

$$F_2(p,q) = A_2 p^{\beta_2} q^{\eta_2} \quad (23)$$

Adkins and Paxson (2011) provide a solution to such a two-factor valuation relationship. The generic characteristic root equation associated with (23) is:

$$Q_2(\beta_2,\eta_2) = \tfrac{1}{2}\sigma_p^2\beta_2(\beta_2-1) + \tfrac{1}{2}\sigma_q^2\eta_2(\eta_2-1) + \rho\sigma_p\sigma_q\beta_2\eta_2 + \theta_p\beta_2 + \theta_q\eta_2 - r = 0 \quad (24)$$

Unlike a one-factor model whose characteristic root equation yields the solution to the option parameter, the parameters $\beta_2$ and $\eta_2$ are not uniquely obtainable from (24), but have to be evaluated by incorporating additional information available from the economic boundary conditions. Since a function of the form $G_2(\beta_2,\eta_2) = 0$ is obtainable from these conditions, the solution to $\beta_2$ and $\eta_2$ can be evaluated from the intersection of $Q_2 = 0$ and $G_2 = 0$. Assuming that a point of intersection can occur in each quadrant, $\beta_2$ and $\eta_2$ can adopt any of the four following possibilities:

| | |
|---|---|
| Quadrant I | $\beta_{21} \geq 0, \eta_{21} \geq 0$ |
| Quadrant II | $\beta_{22} \geq 0, \eta_{22} < 0$ |
| Quadrant III | $\beta_{23} < 0, \eta_{23} < 0$ |
| Quadrant IV | $\beta_{24} < 0, \eta_{24} \geq 0$ |

This suggests that the generic solution (23) takes the extended form:

$$F_2\left(p,q\right) = A_{21} p^{\beta_{21}} q^{\eta_{21}} + A_{22} p^{\beta_{22}} q^{\eta_{22}} + A_{23} p^{\beta_{23}} q^{\eta_{23}} + A_{24} p^{\beta_{24}} q^{\eta_{24}}.$$

Now, since the value of a tree stand increases with tree size and lumber price, then we can expect both $\beta_2$ and $\eta_2$ to be positive, so $A_{22} = A_{23} = A_{24} = 0$, and the specific form of (23) becomes:

$$F_2\left(p,q\right) = A_{21} p^{\beta_{21}} q^{\eta_{21}}. \tag{25}$$

The value-matching relationship describes the value conservation that has to be observed at the optimal tree cutting event. When it is optimal to cut the tree, the thresholds for the timber price and tree size are denoted by $\hat{p}_2$ and $\hat{q}_2$, respectively. If the tree cutting option is exercised, then the stand value, which is given by $F_2\left(\hat{p}_2, \hat{q}_2\right)$, has to compensate the net value rendered from cutting the tree, given by $\hat{p}_2 \hat{q}_2 - K_2$ where $K_2$ denotes the cutting and replanting cost, and the option value of planting a sapling, given by $F_2\left(\hat{p}_2, q_0\right)$. The value-matching relationship becomes:

$$A_{21} \hat{p}_2^{\beta_{21}} \hat{q}_2^{\eta_{21}} = A_{21} \hat{p}_2^{\beta_{21}} q_0^{\eta_{21}} + \hat{p}_2 \hat{q}_2 - K_2. \tag{26}$$

In (26), we observe that while the option to cut the tree and its rendered value are expressible as products of the timber price and tree size, the replanting option is not, and so the principle of similarity does not apply. The two smooth-pasting conditions associated with (26) for $p_2$ and $q_2$ can be respectively expressed as:

$$\beta_{21} A_{21} \hat{p}_2^{\beta_{21}} \hat{q}_2^{\eta_{21}} = \beta_{21} A_{21} \hat{p}_2^{\beta_{21}} q_0^{\eta_{21}} + \hat{p}_2 \hat{q}_2, \tag{27}$$

$$\eta_{21} A_{21} \hat{p}_2^{\beta_{21}} \hat{q}_2^{\eta_{21}} = \hat{p}_2 \hat{q}_2. \tag{28}$$

Unless $A_{21} = 0$ and the replanting option has a zero value, the option parameters $\beta_2$ and $\eta_2$ are not equal, as they would have been under the similarity principle. Two reduced-form value-matching relationships are obtainable by substituting (27) and (28) in (26) to yield:

$$\hat{p}_2\hat{q}_2 = \frac{\beta_{21}}{\beta_{21} - 1} K_2, \tag{29}$$

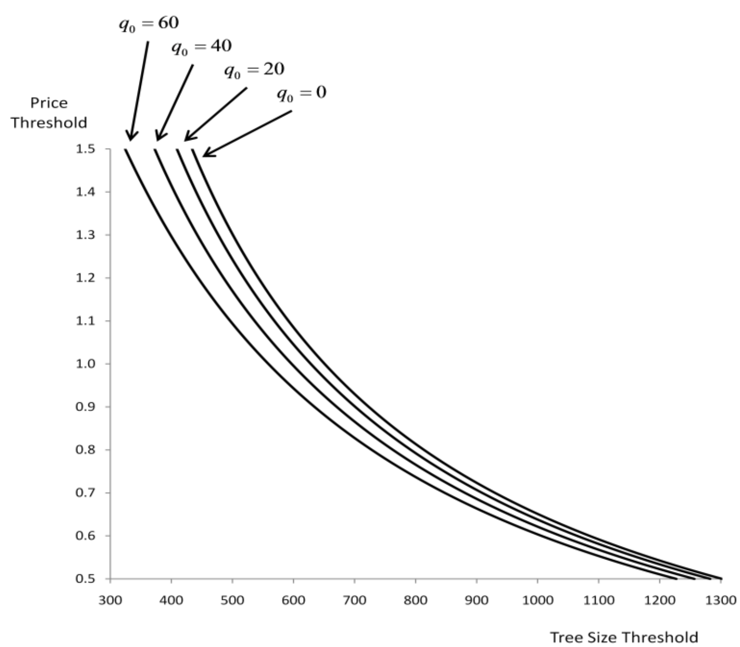
$$\hat{p}_2\hat{q}_2 = \frac{\eta_{21} K_2}{\eta_{21} - 1 + \left(\frac{q_0}{\hat{q}_2}\right)}. \tag{30}$$

From (29) and (30), the parameters $\beta_{21}$ and $\eta_{21}$ are greater than one. This means that the value rendered by harvesting has to exceed the cutting and replanting cost, which is the two-factor equivalent finding of the standard real-option result. Further, from (27) and (28), $\beta_{21}$ is greater than $\eta_{21}$, so timber prices exert a slightly greater influence over the harvesting option than tree size, given these parameter values. By setting the price threshold $\hat{p}_2$ to its prevailing level, the optimal tree size threshold $\hat{q}_2$ is obtainable from the simultaneous solution of the characteristic root equation $Q_2(\beta_{21}, \eta_{21}) = 0$ (24), and the two reduced-form value-matching relationships, (29) and (30). The sensitivities of the optimal tree size threshold $\hat{q}_2$ are illustrated for a price threshold range (the comparison to the multi-factor single harvest, and single-factor multiple harvest is made at the price threshold of 1.0).

Varying Replanting Sapling Size

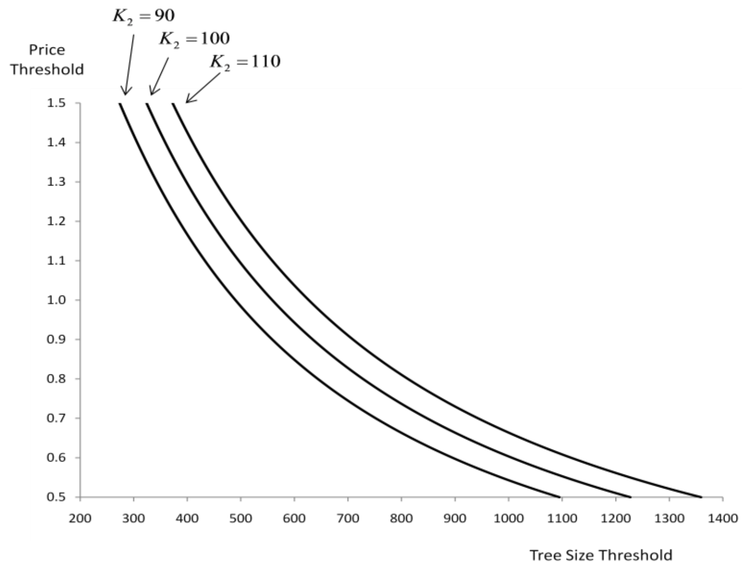
Figure 7 illustrates the effect of varying the sapling size $q_0$ on the harvesting policy, assuming $q_0$ = 60 and $K_2$ = 100/p. This figure reveals that a negative trade-off exists between the thresholds for the lumber price and the tree size such that a fall in the lumber price has to be compensated by a rise in the tree size. When the sapling size is set to equal zero, this compensation is complete in the sense that the revenue $\hat{p}_2\hat{q}_2$ remains constant at 651.06 for all harvested tree sizes[4]. However, the compensation is only partial for a positive sapling size because of the revenue decline accompanying a lumber price increase, and this effect is most pronounced for higher lumber prices. Further, the harvesting policies for the various sapling sizes are nested. This means that irrespective of the harvesting price threshold, the threshold tree size decreases for increases in the sapling size. On average, the expected time until harvesting is shorter for higher sapling sizes not only because of starting at a greater size but also because of a lower harvesting size threshold. However, we have to temper this finding by the fact that saplings of a larger size tend to command higher prices that leads to an increased replanting cost.

Figure 7




[4] As $q_0$ tends to zero, the parameters β and η become close and the results seems to approach the similarity solution for the single harvest model. The multiple rotation model form constrains the value of $q_0$ to between 0 and K/p.

Figure 8




The effect of variations in the cost of cutting and replanting $K_2$ is illustrated in Figure 8. This reveals that an increase in cutting and replanting cost produces a replanting boundary more distant from the origin. This is in line with expectations, since greater value has to be captured from the tree in the form of a greater lumber price or tree size in order to compensate the increased $K_2$. Clearly, a trade-off exists between the sapling size and $K_2$ that has to be acknowledged in deciding the appropriate $q_0$ level.

Variations in Volatility

A standard real-option finding concerning volatility is that an increase produces a rise in the threshold and the greater tendency for deferring exercise. In Figures 9

(a) and (b), we present illustrations of the effects of variations on the volatilities for lumber price and tree size, respectively, on the harvesting policy, whose profiles agree with the standard finding. However, the magnitudes of the two effects are very dissimilar. A change in tree size volatility produces a very modest impact on the harvesting policy, which is only visible in Figure 9 (b) because of a scale adjustment.

The difference in magnitudes is due to their asymmetrical roles, since the lumber price influences both the harvesting and replanting option values while the tree size only influences the harvesting option value. Accordingly, the harvesting policy responds more significantly to a change in the lumber price volatility than the tree size volatility.

The volatility of the tree revenue depends not only on the two constituent volatilities, but also on the correlation between the lumber price and tree size. Figure 10 illustrates the effect of negative variations in the correlation coefficient on the harvesting policy. We have chosen to select only negative variations in the correlation coefficient, since a period of above normal tree growth is likely to be accompanied by a lumber price fall. This figure reveals that a rise in the correlation coefficient has the effect of deferring the harvesting decision, since there is greater justification for cutting a tree earlier for greater negative correlation. Further, we know from the model using the similarity principle that the focal volatility measure affecting the harvesting decision is positively related to the correlation coefficient, so a negative correlation increase lowers the tree revenue volatility and advances the harvesting decision.

Figure 9 (a)

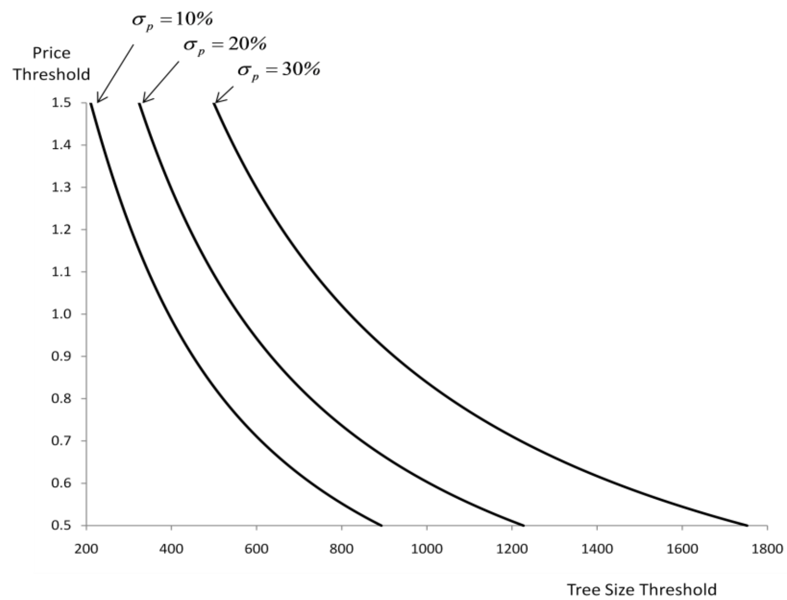



Figure 9 (b)

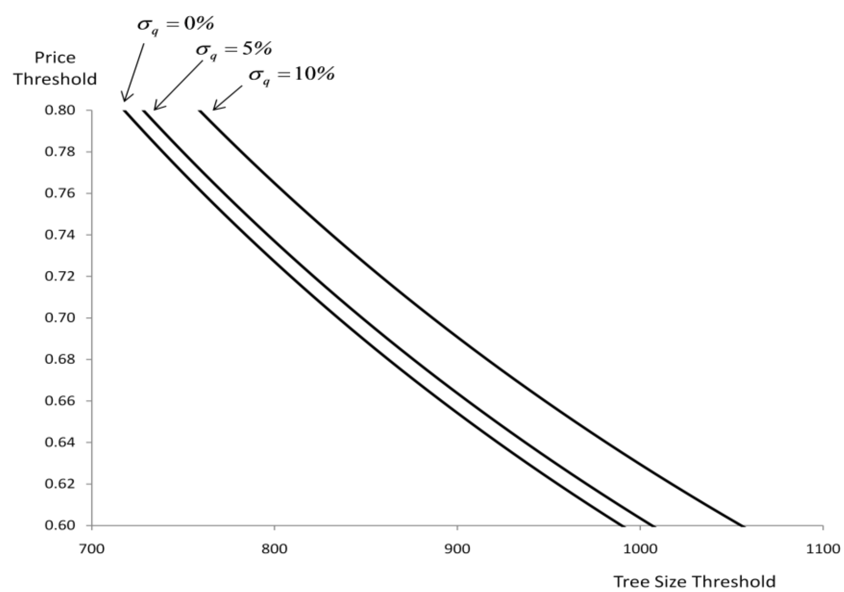
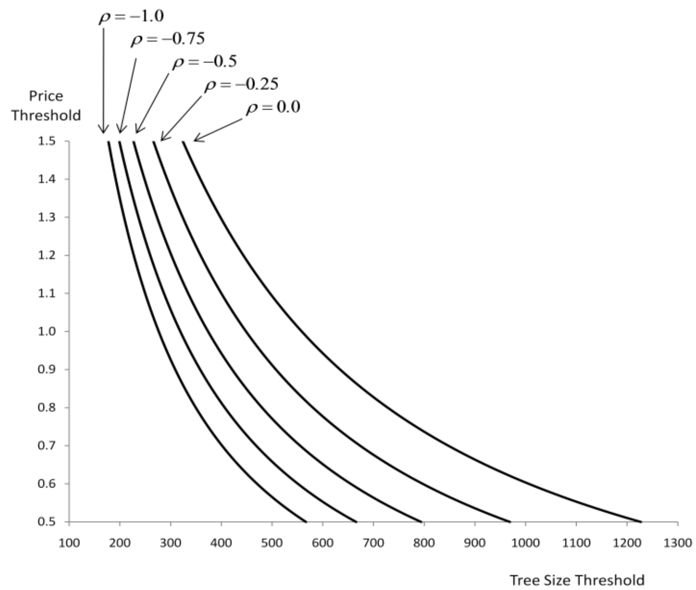


Figure 10

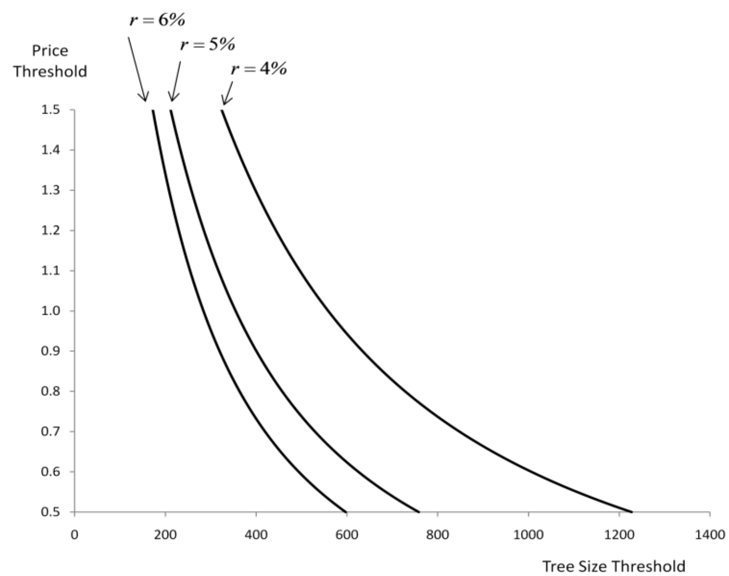



Variations in Discount Rate

Variations in the discount rate $r$ are illustrated in Figure 11. This reveals that for a specific lumber price, the tree size at harvest varies inversely with the discount rate so trees are cut earlier, at smaller sizes for higher discount rates. An increase in the discount rate leads to a rise in the parameter values $\beta_{21}$ and $\eta_{21}$, which in turn reduces the revenue threshold through the reduced-forms of the value-matching relationship.

Now, a discount rate increase makes future cash flow from tree cutting less valuable, so for tree harvesting to be economically justified, any rise in the discount rate has to be compensated by an increase in the tree revenue at harvest. Accordingly, a rise in the discount rate advances the harvesting decision.

Figure 11




Variations in Drift Rates

The effects of variations in the two drift rates, $\theta_p$ and $\theta_q$, are separately illustrated in Figures 12 (a) and (b), respectively. These profiles show that the tree size at harvest for a specific lumber price varies inversely with either drift rate, since an increase in either $\theta_p$ or $\theta_q$ leads to reductions in the values of $\beta_{21}$ and $\eta_{21}$, which consequently, produce a fall in the tree revenue at harvest. Now, since a rise in the drift rates for either the lumber price or the tree size entails an increased value for deferring the harvest, it becomes economic to allow trees to grow for a longer time so that their tree revenue becomes higher. Faster growing trees or higher rising lumber prices imply greater tree revenue at harvest because of their increased value for waiting.

Figure 12 (a)

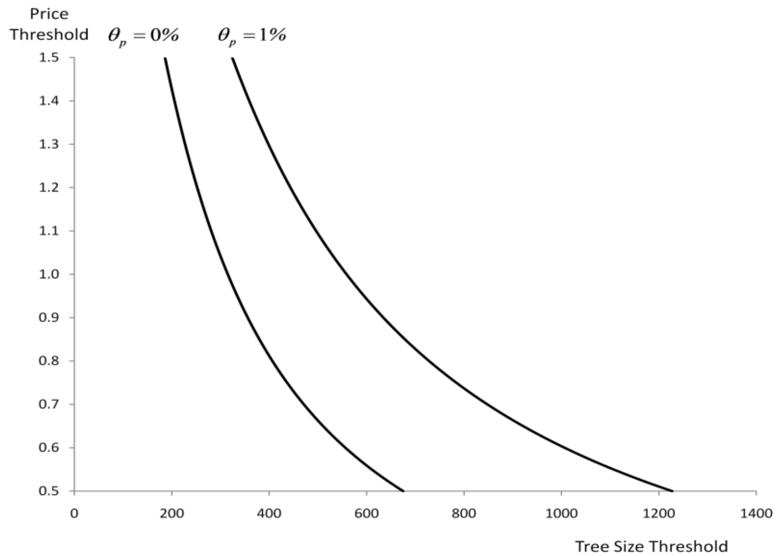
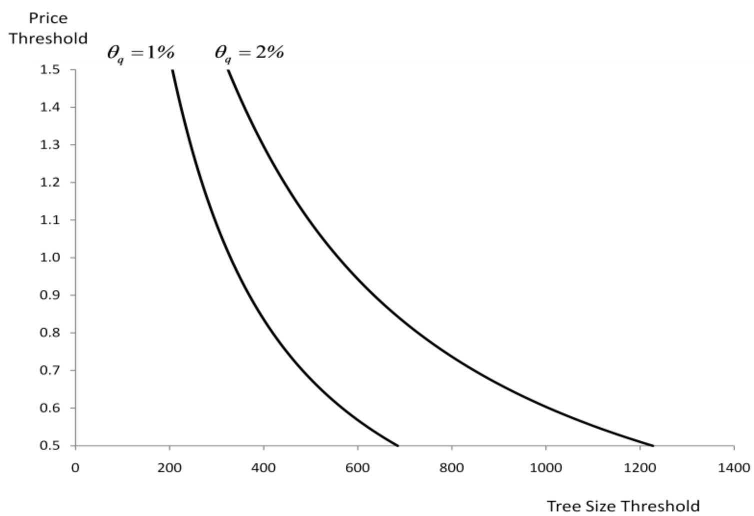



Figure 12 (b)




Generally, the sensitivities to changes in the parameter values are similar in sign but not necessarily magnitude for all three forest harvesting models. We assume the forester has a matrix of current prices and forest size (including replanting size and cost) so that at any point of time, using expected forest price and forest size drifts, volatilities and correlation, the value of the forest stand and the optimal forest harvest size can be determined.

## 5. SUMMARY and CONCLUSION

This paper presents real perpetual American multi-factor forest harvesting option models. A forester maximises the value of the harvest decision not when the present value of the cash flows equals the harvest cost, but when P*Q/K is much greater than one, unless there is little volatility in either timber prices or forest growth. The multi-factor single harvest optimal time X* and the single factor multiple harvest optimal time X** are derived as solutions to ordinary differential equations.

While the option to harvest a forest and its value at harvest are expressible as products of the timber price and forest stand size, the replanting option is not, so the principle of similarity sometimes used to reduce the dimensions of two factor models does not apply.  We provide a unique quasi-analytical solution involving the simultaneous solution of three equations for multi-factor uncertainty in forestry to determine the multiple rotation value and optimal harvest size.

Multi-factor models are able to cover estimations of several state variables, the volatilities of those variables, and correlations among the variables, if warranted. Replicating these real harvest options along a time frame might be attempted using a variety of real, financial and commodity securities, or eventually synthetic or virtual products created by imaginative enterprises.

This quasi-analytical approach can plausibly be used to incorporate more realistic forest growth models, especially reflecting the slower growth of older stands, and to consider the hypothetical subsidies and/or taxes on the other forest products (amenities, or the value of the uptake of $CO_2$ during growth and partial release upon harvesting or burning).  Obviously the application of such models should be made considering the vast amount of empirics on forest growth in different areas and for different species, for different $CO_2$ uptake for different species, as well as including the traded carbon emission prices as a separate stochastic process.

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