Optimal Capital Structure with Sequential Options and Finite Horizon

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May 2010

Abstract

A binomial lattice based framework for the analysis of finite investment options with finite operational phase is developed. Solutions for European and American type finite horizon investment options with optimal capital structure and a multi-stage investment setting with multiple debt issues are discussed. The analysis shows that optimal leverage ratios are not affected by option moneyness at the investment trigger, confirming earlier literature results in perpetual horizon. Sensitivity results show that leverage ratios are lower when the operational phase is longer. Long term debt maturity is optimal when principal payments exist, while the reverse is true in the absence of principal payments. Leverage ratios are higher for longer debt horizons for the case with principal payments, while this result is reversed when no principal payments exist. Sensitivity results with respect to model parameters enhance our intuition about the impact of several parameters on the firm investment and default policy and firm value.

1. Introduction

The purpose of this paper is to develop a binomial lattice based framework for the analysis of finite investment options with finite operational phase. Several modeling issues arise with finite horizon, since the model becomes path dependent and, in particular, the numerical solution requires a forward-backward algorithm to have a proper treatment of the optimal capital structure choices. Our lattice framework extends Broadie and Kaya (2007), who presented a binomial framework for the finite version of Leland's (1994) model. Firstly, our framework is explicitly specified with revenues as the stochastic variable. Secondly, we allow for different frequencies of the investment and default decisions, ranging from yearly to an instantaneous interval. Thirdly, Broadie and Kaya (2007) do not propose an approach for selecting an optimal capital structure using the binomial tree and do not model investment option stages. Both are introduced in our paper, where the solution is proposed for both European and American type investment options. Finally, our framework extends to multiple stages and allows us to study different classes of debt, seniority rules and debt covenant rules, in a way that has not been tackled within this methodology so far.

We first employ a simple lattice model based on backward induction that includes a first-stage investment and optimal default decision. Then, in a more general model we study a forward-backward lattice-based algorithm with optimal capital structure choice starting from one-stage European and American options and moving to a multi-stage framework. Using the one-stage investment setting several issues are explored: the optimal capital structure and credit spreads at investment maturity, the shape of the investment and default trigger as a function of investment maturity level of revenues, and the debt maturity choice. Finally, in the more general multi-stage framework, interactions between investment and financing decisions (e.g. staging investments versus accelerated investments and financing choices) and debt seniority can be studied.

Our work relates to Sundaresan and Wang (2007), Mauer and Sarkar (2005), Leland (1994) and (1998), Leland and Toft (1996). Leland (1994) explores the determination of capital structure in a contingent claims model with a trade-off between tax benefits and bankruptcy costs but without an investment option. Mauer and Sarkar (2005) extend Leland's model by adding a single investment option stage and investigate agency issues caused by asset substitution between equity and debt holders. Leland and Toft (1996) extend Leland's model to the case where the firm can choose both the amount and the maturity of its debt. They show that firms will choose to finance investments using long term debt unless there are agency costs caused by asset substitution. They study finite maturity debt but do not model the investment option stage. Leland (1998) analyses the average maturity choice by allowing for choice of the debt amortization rate and shows that average debt maturity decreases in the presence of agency costs. He also shows that hedging benefits (i.e., the option to switch to a low risk mode of assets) is more important for short term debt. Sundaresan and Wang (2007) is a paper very closely related to our work. Our paper provides a finite version implementation of their framework, including several investment option stages, the maturity of several debt issues, and a finite project (firm) life. Sundaresan and Wang (2007) provide new insights on the interaction between investment and financing decisions. Firstly, they show that when the firm issues debt in the first stage then, because of the absolute priority rule (APR), there exists a debt overhang problem that induces equity holders to delay the exercise of the second investment option. Secondly, they show that firms anticipating future growth prospects will initially choose low leverage ratios. The lattice model allows us to examine revisit these issues and expand their insights by exploring alternative investment strategies (expand or contract operations and abandonment options) and their interactions with debt financing choices.

Within our finite maturity framework, we investigate whether leverage and debt maturity are affected by the option moneyness, the horizon the firm intends to operate, the volatility of revenues, competitive erosion, expected bankruptcy costs and the tax rate. The default trigger boundary shape in the operational phase, and the investment trigger and leverage choices along the trigger for American options, are also investigated. This part of the analysis is considered as an extension of the Leland and Toft (1996) framework to allow for the investment option (with and without optimal timing) since our framework allows us to calculate optimal leverage, credit spreads, equity and debt values at the investment trigger. Some of the main results are outlined below. The analysis shows that optimal leverage ratios are not affected by option moneyness at investment maturity. Interestingly, this is also demonstrated for American options, where it is shown that leverage ratios remain constant along the investment trigger. Sensitivity results show that leverage ratios are slightly higher for shorter operational phase horizons and tend to converge as the horizon becomes larger. Sensitivity results with respect to model parameters enhance our intuition about the economics of such complex settings. In particular, it is shown that a lower investment trigger exists for the cases of lower volatility, higher opportunity cost, lower bankruptcy cost and lower tax rate. Leverage ratios are higher when the volatility, opportunity cost and the bankruptcy costs are lower and are reduced at lower tax rates. It is shown that leverage ratios are affected by the opportunity cost and volatility that exists in the operational phase and not the investment stage. We have investigated the default trigger shape, debt maturity choice and the connection between leverage ratios and debt maturity choice both under debt principal payments and in their absence. For the case were no principal exists at the end of the horizon, the default trigger boundary was shown to have an upward sloping shape for long horizons and downward sloping for short horizons. The results show a choice of short horizon is preferable assuming the firm can borrow heavily with coupon levels exceeding the revenue levels. In the case where coupon levels constraints exist so that coupon cannot exceed revenue levels, it shown that medium term horizons are optimal. Finally, leverage ratios are shown to be higher at shorter debt horizons. Some results are significantly different in the presence of principal payments. First, we observe that default triggers will always be upward sloping both for short and long debt horizons. Secondly, the optimal debt maturity choice will be to select the longest term horizon. Finally, leverage ratios are now shown to be positively associated with debt horizon choice. These results are consistent with the Leland and Toft (1996) framework[1].

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[1] Empirical evidence on the subject of maturity choice shows mixed results. Stohs and Mauer (1996) show that larger and less risky firms with longer term assets use long-term debt. Additionally, they show that

In the last part of the applications the paper focuses on multi-stage applications. Trigeorgis (1993) shows that interactions among combined real options make their values non-additive (see also Agliardi, 2007). The framework developed in the paper is used to obtain firm, equity and debt values in a multi-stage framework with such interactions. We focus on the impact of options to expand or contract and their impact on firm leverage choices over time and investigate the impact of investment option exercise and abandonment options on leverage choices. The paper intends to provide some predictions for firms facing alternative investment opportunities regarding investment and default policy and leverage choices over time.

2. The model

2.1. Extending the Broadie and Kaya (2007) lattice framework

In this section we extend Broadie and Kaya (2007), who propose a finite lattice implementation of the Leland (1994) model. Our lattice-based backward solution algorithm has the following extended features in comparison with Broadie and Kaya (2007):

1. A finite investment option stage and finite operational phase. The framework thus provides a finite maturity solution (for both the investment option stage and debt-operational phase) of the Mauer and Sarkar (2005) model and nests Broadie and Kaya as a special case.

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larger earnings surprises and the level of effective tax rate vary negatively with debt maturity, while they where not able to show clear evidence between growth opportunities and debt maturity. A non-monotonic relationship between bond ratings and debt maturity emerges: highly rated firms and low rated firms borrow short term. Guedes and Opler (1996) on the other hand show empirical evidence that large firms with investment grade rating borrow either short-term or long-term while risky firms are in the middle of the maturity spectrum. A large number of papers have studied whether actual debt ratios deviate from a target level (see, for example, the survey in Parsson and Titman, 2008). Hennessy and Whited (2005) have shown that there is no target leverage ratio with leverage being path dependent and decreasing in lagged liquidity

2. In contrast to Broadie and Kaya (2007), who model the present value of cash flows as the underlying stochastic variable and cash flows as proportional using a dividend-like parameter, we explicitly model price (or revenues) as a stochastic variable and allow for fixed costs.
3. A method of increasing the accuracy for in-between stages involving no cash inflows and the payment of debt interest. It also allows for arbitrary frequency in the time interval between cash inflows and outflows.
4. The difficulties involved in optimizing the capital structure on the tree for finite horizon problems are discussed. Section 2.2 proposes an alternative forward-backward solution methodology that intends to resolve these difficulties as well as extend the model in other dimensions (multiple investment stages, multiple debt issues etc)

This section provides a backward algorithm based on the binomial lattice tree showing how the value of unlevered assets, the tax benefits, the bankruptcy costs, equity, and debt can be calculated on the tree. We test the accuracy of the model against the known analytic solutions of Leland (1994) and Mauer and Sarkar (2005).

Let us assume that yearly price (or revenue) follows a geometric Brownian motion of the form:

$$\frac{dP}{P} = adt + \sigma dZ \qquad (1)$$

where α , σ >0 are constant parameters and dZ is the increment of a standard Wiener process. The firm pays an operational cost C per period so that total earnings before interest and taxes (EBIT) is $P - C$. In this simple setting EBIT coincides with the firm's unlevered cash flows since there are no additional costs that need to be incurred, no changes in working capital or other changes in the firm's cash flows.

The firm holds an investment option to obtain the present value of the above cash flows by paying an irreversible cost *I*. The maturity of this option is T_1. At the investment maturity the equity holders invest when the sum of the present value of unlevered cash flows (V^U) and the tax benefits of debt (*TB*) net of bankruptcy costs (*BC*) and the irreversible investment cost I are positive.

After investment, the firm will have a useful life (firm maturity) of T_F years and can use debt that demands a tax-deductible coupon payment R per period and a final principal debt (face value) F at maturity. Coupon levels will be a choice variable to determine capital structure. Let c denote the coupon rate, so $F = \frac{R}{c}$. With a single issue, debt maturity is specified by T_{D_1} with $T_{D_1} \leq T_F$. The firm pays annual taxes based on an annual tax rate τ. In the event of bankruptcy - which will be endogenously chosen by equity holders - proportional bankruptcy costs b need to be incurred by debt holders in order to liquidate the firm's unlevered asset value. Cash inflows (revenues) and outflows (costs and interest payments) occur every Δt. Δt can be controlled by a variable N_{dec} that specifies the number of decision-cash points within each year. Thus, $\Delta t = \frac{1}{N_{dec}}$. For example, yearly cash inflows-outflows will occur if $N_{dec} = 1$ whereas if cash flows occur every six months then $N_{dec} = 2$, etc . For accuracy each Δt interval will be approximated by a sub-tree $N_{\Delta t}$.

Starting from the operational stage, the lattice steps are determined by the frequency of decision-cash points and the approximation steps between decisions, so that $N_F = N_{\Delta t} \cdot N_{dec} \cdot T_F$. To maintain consistency, $N_1 = \left(\frac{T_1}{T_F}\right) \cdot N_F$. In the following section this approach will necessarily need to be altered, since to account for path dependency different sub-lattices will be emerging from the terminal states of each earlier stage lattice approximation.

In the time periods where there are no cash inflows or outflows involved, all variables are calculated and discounted from next stage values using the interval $dt = \frac{T_F}{N_F}$ (which is the same as in the investment stage where $\frac{T_1}{N_1}$). For example, consider a firm with an investment horizon of $T_1 = 5$ years, an operational phase of $T_F = 10$ years, with $N_{dec} = 1$ that implies Δt = 1, i.e., yearly cash inflows-outflows. Each year may approximated with $N_{\Delta t} = 12$ steps, i.e., one step per month. This means that the operational phase tree is $N_F = 12 \cdot 1 \cdot 10 = 120$ steps with $dt = \frac{10}{120} = \frac{1}{12}$ (one month). The investment stage will be approximated with $N_1 = \left(\frac{10}{5}\right) \cdot 120 = 60$ steps because it represents a period that is half of that of the operation phase.

Decisions in both the investment and the operational phase are undertaken every Δt. All decision points are then the ones included in the set:

$$t_{dec} = \{t_{N_{dec}} = T, t_{N_{dec}-1} = T - \Delta t, t_{N_{dec}-2} = T - 2\Delta t, ...t_0 = 0\}$$

with $T = T_F$ if in operational phase or $T = T_1$ if in the investment phase.

Note that the interval Δt will multiply the variables of price, cost and coupon inputs of the problem since it is standard to specify these variables on an annual basis. In the earlier example, $T_F = 10$ and $N_{dec} = 10$ would imply that the inputs for price, cost and coupon payments will remain as annual variables (multiplied by a $\Delta t = 1)$. In theory, the decisions can be made as dense as possible approximating the continuous decision limit when $N_{dec} \to \infty$. Perpetual analytic models like that of Leland (1994) can be approximated in our framework by letting T_F to be very high (e.g., 200 or 400 years) and allowing for decisions almost continuously by setting N_{dec} to be very high (e.g., 4000 decisions implying $\Delta t = 0.5\%$). In this case the input variables which are defined on a yearly basis would then be multiplied by Δt at each decision point. The model of Mauer and Sarkar (2005) can be approximated by setting both T_1 and T_F to be very large.

There are two ways we use to model the operational phase time horizon: first, one may assume that the time elapsed in the first stage is then deducted from the useful life in the operational phase, or, an alternative assumption is to use a “relative time” assumption which retain a fixed horizon T_F relative to the time that investment is initiated. In this section we model the operational phase using the first assumption. The relative time assumption is implemented in the next subsection and is used throughout our main numerical results. Since we allow for an investment option the investment timing $t_I \leq T_1$ this means that the number of operation years will range from a maximum of T_F periods (when $t_I = 0$) to a minimum of $T_F - T_1$ (when investment is delayed until maturity $t_I = T_1$). Furthermore, a constraint that $T_1 \leq T_F$ needs to be placed here. The firm’s opportunity cost of waiting is thus on foregone period cash flows, smaller number of operation years and delayed received present value of cash flows. Furthermore, competitive erosion is also taking place through the parameter δ. On the positive side, by waiting the firm lets more uncertainty to be revealed before committing to an irreversible investment. In Section 2.2. where a relative time assumption is used, the opportunity cost of waiting is only because of the delayed received cash-flows and competitive erosion.

The usual formulation of the lattice parameters for the up and down jumps and the up and down probabilities requires that:

$$
\begin{aligned}
u &= e^{\sigma dt} \\
d &= e^{-\sigma dt} = \frac{1}{u} \\
p_u &= \frac{e^{(r-\delta)dt} - d}{u - d} \\
p_d &= 1 - p_u
\end{aligned}
\tag{2}
$$

In contrast to Broadie and Kaya (2007), δ in our model is used to capture competitive erosion (and not to model the firm's cash flows which are explicitly modeled in our case).

We keep track of the following information at each node of the binomial tree:

- The value of unlevered assets (V^u): this is the present value of cash flows generated from the assets assuming no debt. If debt does not exist (or has expired), the value of unlevered assets will coincide with equity. In this case one may assume, similarly to Mauer and Sarkar (2005), that equity holders hold an abandonment option thus not allowing V^U (and thus equity) to become negative.

- The value of tax benefits of debt (TB): this is the present value of the tax shields of debt. The per period tax benefits are τR and are realized only if the equity holders decide to continue operations.

- The value of bankruptcy costs (BC): this is the present value of the costs of bankruptcy calculated as the product of the value of unlevered assets at the time of bankruptcy (V^B) times the proportion bankruptcy cost factor *b*. This is realized only in the event of bankruptcy, otherwise is set to zero.

- The value of shareholders equity (E): this is the present value of operational cash flows net of coupon payments and taxes. This cash flows are realized if the equity holders decide to stay in operational mode and coincides with the value the value of unlevered assets until default plus the value of tax benefits until default minus the present value of the coupon payments until default.

- The value of debt (*D*): this value includes the present value of the coupon payments until default plus the value of the firm's unlevered assets net of bankruptcy costs at the bankruptcy point.

- The value of the levered firm (V^L): this value is the sum of equity and debt value. Equivalently, it is the sum of value of unlevered assets plus the tax benefits of debt minus the bankruptcy costs.

Similarly to Leland (1994), bankruptcy is endogenously chosen by equity holders to maximize equity holders value. Starting backwards at the last operation point T_F, which is assumed here to coincide with the payment of the debt principal, equity and the other variables can be calculated as follows:

$$E_{T_2} = \max\left[(P - C - R)(1 - \tau)\Delta t - F,\ 0\right] \tag{3a}$$

If $E_{T_2} > 0$, then

$$\begin{aligned}
V_{T_2}^u &= (P - C)(1 - \tau)\Delta t \\
TB_{T_2} &= \tau R \Delta t \\
BC_{T_2} &= 0 \\
D_{T_2} &= R\Delta t + F \\
V_{T_2}^L &= E_{T_2} + D_{T_2},
\end{aligned} \tag{3b}$$

otherwise if $E_{T_2} = 0$ (i.e., bankruptcy occurs) and if $V^U > 0$

$$\begin{aligned}
V_{T_2}^u &= (P - C)(1 - \tau)\Delta t \\
TB_{T_2} &= 0 \\
BC_{T_2} &= bV^B = bV_{T_2}^u \\
D_{T_2} &= (1 - b)V_{T_2}^u \\
V_{T_2}^L &= E_{T_2} + D_{T_2}.
\end{aligned} \tag{3c}$$

In the case were $T_{D_1} < T_F$ the above boundary conditions should be adjusted so that there is no subtraction of coupon payments and the debt principal (since debt has expired). Furthermore, the condition that V^U cannot turn negative should also be investigated before calculating bankruptcy costs and debt values. For perpetual horizons, this condition does affect results significantly. In fact, not incorporating this condition allows for better accuracy of the analytic solutions that exist in this case (see Table 1 and discussion that follows). In the earlier steps at $t < T_F, t \neq t_{dec}$, i.e., for t not belonging to the set where cash flows accrue and decisions are undertaken, the values of each of these variables is simply the discounted present value of their expected value of the following step, i.e.,:

$$
\begin{aligned}
&V_t^u = (p_u V_{t+dt,u}^u + (1-p_u)V_{t+dt,d}^u)e^{-rdt} \\
&BC_t = (p_u BC_{t+dt,u} + (1-p_u)BC_{t+dt,d})e^{-rdt} \\
&TB_t = (p_u TB_{t+dt,u} + (1-p_u)TB_{t+dt,d})e^{-rdt} \qquad (4) \\
&E_t = (p_u E_{t+dt,u} + (1-p_u)E_{t+dt,d})e^{-rdt} \\
&D_t = (p_u D_{t+dt,u} + (1-p_u)D_{t+dt,d})e^{-rdt} \\
&V_t^L = E_t + D_t
\end{aligned}
$$

where $x_{t+1,u}$, $x_{t+1,d}$ denotes the high and low state of variable *x* in the next *dt* step.

In the steps before the maturity where a decision can be undertaken (t belongs to the t_{dec} set), the values of each of these variables are calculated as follows:

$$E_t = \max\left[(P - C - R)(1-\tau)\Delta t + \tilde{E}_t\,,\, 0\right] \tag{5a}$$

If $E_t > 0$, then

$$\begin{aligned} V_t^u &= (P - C)(1-\tau)\Delta t + \tilde{V}_t^u \\ BC_t &= 0 + \tilde{BC}_t \\ TB_t &= \tau R \Delta t + \tilde{TB}_t \\ D_t &= R\Delta t + \tilde{D}_t \\ V_t^L &= E_t + D_t\,, \end{aligned} \tag{5b}$$

whereas, if $E_t = 0$ and $V_t^u = (P - C)(1-\tau) + \tilde{V}_t^u > 0$, then

$$\begin{aligned} V_t^u &= (P - C)(1-\tau) + \tilde{V}_t^u \\ BC_t &= bV_t^u \\ TB_t &= 0 \\ D_t &= (1-b)V_t^u \\ V_t^L &= E_t + D_t\,, \end{aligned} \tag{5c}$$

where $\tilde{x}_t$ denotes the expected discounted value of variable x and equals $\tilde{x}_t = (p_u x_{t+dt,h} + (1-p_u)x_{t+dt,l})e^{-rdt}$. If V_t^u is negative then the value of all variables are set to zero.

The solution proceeds by backward induction until all the variables values are calculated at $t = 0$. It is important to note that operational phase variables are calculated back to time zero and not only until time T_1, i.e. the investment horizon. This is to allow for an early exercise of the investment option where the firm has the opportunity to obtain cash flows for more periods.

Backward induction is possible because of an implicit assumption that exists in these models[2] that at each decision step, equity holders deciding to continue operations when cash shortages exist need to inject new cash-equity contribution. Similarly, any cash surpluses at each point in time are distributed as dividends. Retaining cash within the firm would make the problem path-dependent because the cash flow stock variable should be retained at all time.

At the maturity of the investment option T_1, equity value is updated to include the investment paid and the amount of debt received:

$$\begin{aligned} E_{T_1}^I &= \max[E_{T_1} - (I - D_{T_1}),0] \\ &= \max[V_{T_1}^u + TB_{T_1} - BC_{T_1} - I,0] \\ &= V_{T_1}^L \end{aligned} \tag{6}$$

If $E_{T_1}^I > 0$ then all variables will take their values from the tree modeling the operation phase at T_1, otherwise will be set to zero ($E_{T_1}^I = 0$ and no investment is undertaken). With optimal investment timing, investment can be undertaken at each decision-cash point t_{dec} in the investment investment stage as:

$$E_t^I = \max[E_t - (I - D_t), \tilde{E}_t^I] \tag{7}$$

[2] See also Broadie and Kaya, 2007 for a discussion of this issue.

If early exercise is optimal then all variables at the investment stage are updated with the corresponding variables at time t existing in the lattice tree calculations used in the operation phase. If it is not optimal to make an early investment decision then the values of each variable are the discounted expected values of the variables of the following period of the investment stage. Note that at times $t \notin t_{dec}$ were no decision takes place the values are simply the expected discounted values of the following step within the investment stage (similarly with equation 4 but now using the variables in the investment stage).

In Table 1a we provide numerical results of the binomial tree model with decisions approximating the continuous limit ($\Delta t \to 0$) by increasing the N_{dec} variable. The solutions are contrasted to the closed form solution of Mauer and Sarkar (2005) in order to test the numerical accuracy of the model.

[Insert Table 1 here]

We analyze two cases: one with operational costs set to zero (panel A) and one with positive operational costs (panel B). In all cases we have used long horizons for the operational and investment phase $T_F = 400, T_1 = 200$[3]. In each panel, coupon levels are the optimal coupon levels according to the analytic model of Mauer and Sarkar (2005). In numerical models M1-M3 of panel A, unlevered values arepositive in all states of the tree since the operational costs are zero. However, in panel B, since operational costs are now positive, the value of unlevered assets may turn negative for low enough states of the revenue (P) level. For this reason models M4-M6 test for the accuracy of a numerical model were the value of unlevered assets is allowed to become negative, whereas models M7-M9 test for the accuracy of the models when the value of unlevered assets is not allowed to become negative.

[3] We have tested even longer horizons and the results are not materially different.

In panel A the numerical accuracy of all variables are less than 1% (except the case of bankruptcy costs in which there is a deviation of 2.6%). As pointed by Broadie and Kaya (2007) the accuracy is affected by the ability of the lattice model to approximate the default boundary. For the case of equity the approximation error is not important since the boundary is zero. For debt, however, the approximation error may be more significant because the boundary is the value of unlevered assets (or revenue level) at default. The tax benefits and bankruptcy costs exhibit similar oscillatory behavior at smaller steps because they are also affected more significantly by the approximation errors of the default boundary. In our case, the numerical accuracy will be further affected by the accurate approximation of the investment boundary. In numerical models M4-M6 the deviations are slightly higher because of the existence of the operational cost. The levered firm deviations from analytic ranges with maximum range around +/- 2.9% and minimum +/- 0.4%.

Table 1b provides solutions for the same problem using the numerical lattice model, however, assuming that decisions for investment timing and default are taken once a year (compared to almost instantaneous decisions of Table 1a). Table 1b produces a set of results varying the number of in-between lattice steps approximating each year between 5, 7 and 8. The results are not as accurately approximating the analytic solution in this case and solutions (in particular for debt values) exhibit larger oscillatory behavior. This was somehow expected given that perpetual models also assume instantaneous decisions.

Broadie and Kaya (2007) do not explicitly discuss finite horizon with optimal capital structure. The reason is that they focus on the accuracy of lattice method in approximating the perpetual limit of the Leland (1994). In that case, it is adequate to apply a single coupon level throughout the tree which is obtained from the analytic solution of Leland (1994). Similarly if one wants to approximate the Mauer and Sarkar (2005) solution, a single coupon applied uniformly at all lattice nodes is adequate. Thus the optimal coupon search is simplified (essentially avoided). To illustrate this we have performed a coupon grid search for the problem specified in panel A. The solutions reported in Figure 1 illustrate that the optimal coupon level is close to 11 (the actual perpetual limit is about 10.84). This approach cannot however be applied for truly finite horizon investment options since in this case the firm may optimally choose different coupon levels depending at ending nodes revenue levels of the investment horizon.

[Insert Figure 1 here]

Section 2.2. below discusses the optimization of optimal capital structure on the tree for finite maturity options. In section C this framework is generalized for multi-stage options.

2.2. A forward-backward algorithm and American type options with finite horizon and optimal debt maturity choice

The model of the previous section has the limitation that coupon levels cannot be different at different lattice nodes. In this section we present an extended model that accommodates the choice of possibly alternative coupon levels at each state of the revenue variable at the investment stage.

In order to achieve this, a forward-backward algorithm is now applied. The flexible formulation of controlling the frequency of decisions and the approximation of each decision interval is the same as in the previous section. Now, the approach starts by first creating the investment stage tree with $N_1 = N_{dec} \cdot N_{\Delta t} \cdot T_1$ steps. At the price level at the end nodes of the investment stage, several lattices are created that capture the operational phase and default decisions for each choice of the coupon levels. Then, the values of equity and debt are taken so that the highest equity value (which coincides with levered firm value) is selected (as can be seen in equation 6). Then optimization is performed, which selects the optimal coupon among the possible range of coupon levels. Figure 2 illustrates the procedure:

[Insert Figure 2 here]

In the case of optimal investment timing, new trees at each decision point ($t \in t_{dec}$) and node are created and the optimal coupon at that node is investigated. Optimal timing is investigated using equation 7. The investment trigger point is the minimum value at each state where exercise is triggered. It is of interest to investigate the shape of the investment trigger and whether leverage ratios and credit spreads change or remain constant at the trigger. Leland (1994) has demonstrated that leverage levels and credit spreads are not affected by the initial value of unlevered assets. However, his results were based on an assumption of perpetual horizon in the operational phase; the present model presents an opportunity to test it also for finite operational phase horizons.

Two approaches regarding coupon search are implemented. In the first approach the level of revenues P at each end node is discretized through the choice of n_C points and a maximum of $c_{\max}$ points. This implies a coupon grid of:

$$coupon = \{0, \frac{1}{n_c} \cdot P, \frac{2}{n_c} \cdot P, ... \frac{c_{\max}}{n_c} P\}$$

For example, a choice of $n_C = 10, c_{\max} = 10$ would mean that coupon levels will be within the range $coupon = \{0, \frac{1}{10} \cdot P, \frac{2}{10} \cdot P, ...P\}$. The coupon search process was a fraction of the level of revenues at each state. For most of our numerical results $n_C = c_{\max}$ is adequate, i.e., the maximum coupon does not exceed the level of revenues at that state (this constraint is not binding).

The second approach for selecting coupon specifies a denser grid for high revenue levels and a minimum specified grid $n_{\min}$ for low revenue levels. One way to achieve this is to allow coupons grid to be a function of the state of revenues. A linear discretization scheme would specify that at state i (where $i = 0$ is the highest revenue level) of lattice step N_1 coupons will be:

$$n_C(i) = n_C^{\min} + n_C^{\min} \cdot (N_1 - i)$$

$$c_{\max}(i) = c_{\max} \cdot n_C(i)$$

Based on this discretization scheme the coupon grid will be a function of the state at N_1 and would take the following values:

$$coupon(i) = \{0, \frac{1}{n_C(i)} P, \frac{2}{n_C(i)} P, ... \frac{c_{\max}(i)}{n_C(i)}$$

It should be emphasized that both approaches produce similar results. Based on our numerical simulations we note that the coupon levels are always selected as a proportion of the revenue level at the debt issuing time. This ratio is very close to the leverage ratio of the firm at that stage. This observation may allow significant reduction in computational time.

In order to model maturity choices a horizon discretization parameter n_D can be selected which specifies a set $T_D = \{\frac{T_F}{n_D}, 2\frac{T_F}{n_D}, ... T_F\}$ for possible debt maturity choices. For optimizing both maturity and coupon levels, a double loop search process is implemented. This process optimizes the coupon for each maturity choice and then selects the maximum firm value from the alternative optimal maturity choices.

2.3. Multi-stage extensions with multiple classes of debt

In this section we extend the model to multiple investment stages and multiple debt issues. The model builds around the assumptions of Sundaresan and Wang (2007) and generalizes their framework to multiple stages. In comparison with Sundaresan and Wang (2007) our framework allows for greater flexibility, with debt maturity potentially overlapping with investment stages before the end of the operational phase. Furthermore, both the absolute priority and the pari passu assumptions can be incorporated, extending their simplified assumptions that were needed for analytical tractability. The first investment has a time horizon T_1. Following, other investments may take place with horizons $T_2, T_{3,...}T_{N_I}$. Debt issues maturities are denoted by $T_{D_1}, T_{D_2}, ...T_{D_{N_D}}$. Figure 3 illustrates (using a two-stage example) how the previous section algorithm can be extended for multiple investment stages and multiple debt issues. The operational phase is initiated at the time of the first investment maturity. It is assumed to have a duration of T_F periods. The operational period may however be terminated if operational costs cause the firm to abandon or default if coupon payments exist. Operation may also be terminated at the subsequent investment stages if the firm decides not to proceed with new investment[4]. At the end of the first investment horizon a first debt issue can be made. At this stage a coupon selection process can start going forward with new lattice trees being created. Depending on the maturity of the first debt issue, the coupon payments may continue to run after the second investment stage, the third and so on. They may of course expire before the start of the second investment stage; their only restriction is that the maximum debt horizon is bounded by the firm's operational phase. At the time of the second investment stage, the firm may decide a new debt issue. At this stage a new coupon search process will start *conditional* on the earlier coupon selection. Similarly, the debt maturity of the second option may or may not overlap with other stages and should have a horizon of less than the operational phase of the firm.

[4] The framework is flexible enough to accommodate alternative assumptions. For example, setting the investment costs of some stages to zero allows that only debt choices are made at that stage. Furthermore, the coupon search process may be terminated at certain stages so that no new debt issue take place.

[Insert Figure 3 here]

Investment stages are approximated by lattices with sizes that are defined relative to the tree used for the first investment stage which has a size $N_1 = N_{dec} \cdot N_{\Delta t} \cdot T_1$. The size of the i investment stage will thus be $N_i = \left(\frac{T_i}{T_1}\right) \cdot N_1$. The last period (after T_{N_I}) is approximated by $N_F = \left(\frac{T_F - (T_1 + T_2 + ...T_{N_I})}{T_1}\right) \cdot N_1$.

Moving from one investment stage to another, the firm may achieve expanded revenue levels, which can be modeled as $e_1, e_2 ... e_{T_I}$ expansion factors multiplying the revenue variable. The same variables can be used to model contraction options ($e_i < 1$ in this case and the firm recovers part of the initial investment). Priority rules for debt holders in case of default need also to be specified. One reasonable assumption is that debt seniority is specified by the order of debt issuance with earlier debt issues having priority over following issues. In some cases, subsequent issues may have equal priority, i.e., the pari passu assumption.

As discussed earlier, default is triggered when equity value drops below zero. Under such a scenario equity holders declare bankruptcy. With positive operational costs one has to check that the value of unlevered assets is positive at that state of revenues. If not, then obviously all debt values will be zero since there is no value to be recovered. Under the absolute priority rule debt holders will receive the following, in case of default at any default time t when $V^U > 0$:

$$D_{1B} = \min\left[(1-b)V^u, D_{1t}\right]$$

$$D_{2B} = \min\left[(1-b)V^u - D_{1B}, D_{2t}\right]$$

$$D_{3B} = \min\left[(1-b)V^u - D_{1B} - D_{2B}, D_{3t}\right]$$

$$\dots$$

$$D_{N_D B} = \min\left[(1-b)V^u - D_{1B} - D_{2B} - \ldots D_{N_D - 1B}, D_{N_D t}\right]$$

The following observations can be made. First, since $V^U > 0$ then all debt issues are bounded by zero (cannot take negative values). Secondly, the rule specified in the paper is much more general than the one specified in Sundaresan and Wang (2007) who specify debt holders recovery value on the face value of debt and not based on the value of debt (D_{it}) at the default date. Thirdly, V^U in this case captures the continuation unlevered value of all subsequent stages irrespective of equity holders continuation decision. Finally, note that it is possible that when $(1-b)V^u$ is high relative to the debt issues a residual value is left even after full repayment of all debt issues. The question which naturally arises is where this value should be allocated. It is possible that this residual value is allocated to equity holders, however, under standard bankruptcy rules the debt holders will have full control and may distribute this value on a value weight base to all debt holders.

In the case of pari passu, similarly to Sundaresan and Wang (2007) any debt value *j* in case of bankruptcy will be determined as[5]:

$$D_{jB} = \left(\frac{D_{jt}}{\sum_{i=1}^{N_D} D_{it}} \right) \cdot (1-b)V^u$$

3. Applications

3.1. Finite investment horizon and optimal debt maturity choice

In this section a European type investment option is used which is computationally less intensive. The goal of this section is produce numerical results for: 1) Finite investment

[5] Sundaresan and Wang (2007) specify the rule in terms of coupon value weight which will have similar results.

horizon with finite operational phase 2) Leverage ratios and credit spreads at investment maturity as a function of state variable *P* values at investment maturity 3) The shape of the default trigger at different values of state variable *P* investment maturity 4) The choice of debt maturity as a function of state variable *P* values at investment maturity[6].

Table 2 first shows various values at $t = 0$ with sensitivity with respect to the operational phase horizon. In particular, the levered firm value, the unlevered firm value, the tax benefits (TB), bankruptcy costs, equity and debt values and the expected investment cost (Inv) are reported. The results show that for projects with short horizons analytic solutions using perpetual horizon will largely overstate true values. As the horizon becomes larger, the solution gradually converges to one solution which would reflect the perpetual case.

[Insert Table 2 here]

Calculated leverage ratios at investment maturity show that leverage remains constant at each end node for each particular case of T_F horizon. Interestingly, leverage ratios for shorter operational phases are slightly higher starting at 72% at $T_F = 10$ and then gradually reduced for longer horizons with $T_F = 15$ about 68% and then about 66% for horizons larger or equal to $T_F = 20$. One possible interpretation of this result is that at shorter operational phase horizon project values are not high enough to induce investment in some states; equity holders will thus prefer to borrow more heavily to allow the firm to proceed to the operational phase. A similar pattern exists between leverage ratios and debt maturity (results follow), i.e., leverage ratios are decreasing in debt maturity, at least for the parameters considered. This result contradicts the results of Leland and Toft (1996) who show that there is a positive relationship between leverage and debt maturity. As is discussed later, this result is driven by the non-existence of principal payments in the case considered here. When principal payments are included the results of Leland and Toft (1996) are replicated.

[6] The results of this section are based on the assumption that default is triggered when equity value gets equal to zero. The value of unlevered assets may be positive or negative at that point.

Table 3 shows numerical results for a finite horizon case both in the investment and operational phase. The operational phase is fixed at $T_F = 20$ years while the investment horizon is varied between $T_1 = 1, T_1 = 3, T_1 = 7$ and $T_1 = 10$ years. Panel A assumes yearly decisions ($\Delta t = 1$). The first sub-panel provides solutions using a uniform coupon at the end nodes of the investment stage that were selected based on the closed form solution with infinite investment horizon and infinite operational phase[7]. Optimal solutions for this type of options with optimal capital structure are hard to obtain using closed form solutions. The second sub-panel presents the value of the firm using a forward-backward algorithm discussed in the previous section and coupon search of 100 increments at each price level. The table provides the results for different approximation accuracy per year $N_{\Delta t} = 1, N_{\Delta t} = 6, N_{\Delta t} = 12, N_{\Delta t} = 18$ and $N_{\Delta t} = 24$.

The results show that the solutions based on a coupon level obtained from the optimal perpetual horizon model uniformly applied at each end node of the tree understate the true optimal. However, the solutions do not deviate substantially from the optimal. Of course, such a behavior is based on averaging out errors at the end of the investment horizon; at a particular node at the investment horizon such a naïve approach will result in gross errors. A further remark is that the lattice based solution seems to converge rather fast to a solution with only minor oscillations as the number of steps approximating each year increases.

[7] The closed form solution assumes an American type investment option so the differences between this solution and the optimal are understated.

[Insert Table 3 here]

In panel B, we provide the solutions for levered firm values when decisions about continuation or default in the operation phase are taken at different frequencies than a year. The results show that the solutions are uniformly lower for all cases compared to panel A corresponding solutions. At higher frequency levels the results seem to converge and the convergence is similarly to the first panel, i.e., very fast and not very oscillatory.

Figure 4a focuses on a selected case of investment time to maturity of 5 years with yearly decisions (panel A) and optimal coupon selection. It shows optimal coupon levels at maturity for different states of price where investment takes place (the value of levered firm exceeds investment cost). It is observed that the firm optimally adjusts its coupon downwards in connection with the realization of the price state variable. The results confirm that optimal coupon levels are always a fraction of the price level at the particular state. Figure 4b shows the values of equity, debt and the levered firm together with its components, unlevered asset value, tax benefits and bankruptcy costs. Interestingly, leverage ratios, calculated as the ratio of debt value over gross (before subtracting investment cost) levered firm value is fixed at about 65%. This creates a uniform credit spreads across all states of 4.28%. Debt yields where calculated using a simple division of the coupon level over debt value. This is not exact for finite horizon and when default is in place. The credit spreads calculated represent upper bounds. True yields can be calculated by solving the implicit equation at each state:

$$
\begin{aligned}
D = {} & R + R \cdot e^{-y\Delta t} \cdot prob(P_{\Delta t} > P_B^{\Delta t})) + R \cdot e^{-y2\Delta t} prob(P_{2\Delta t} > P_B^{2\Delta t})) + \\
& ...(R+F) \cdot e^{-yT_F \Delta t} \cdot prob(P_{T_F} > P_B^{T_F}))
\end{aligned}
$$

D, *R* values and the default trigger points at each decision point are known so the equation can be solved for the yield to maturity of debt. The probability to avoid hitting the boundary may be estimated using the cumulative bivariate standard normal.

The result that leverage ratios remain constant and are invariant to the underlying asset is particularly important since it shows that the result of Leland (1994) is preserved in a finite horizon environment. It should also be noted that the leverage ratio obtained in this example is very close to the optimal leverage ratio at the investment trigger obtained using the closed form solution under the perpetual horizon assumption which was around 63%. Such observation is useful in further expanding the numerical coupon search process to reduce computational time.

[Insert Figure 4 here]

The following figures show the default trigger at selected levels of terminal revenue value at the maturity starting from a deep in-the-money case and moving progressively to lower level of moneyness. As mentioned earlier, a challenge in working with binomial trees is reducing the approximation errors of the trigger boundaries. The in-between lattice steps for each Δt interval (here 12 steps are used) help reduce these errors since a denser set of scenarios can be created for each year. Using an even number of in-between steps ensures that the previous years steps are included in the following years' approximation tree.

The results show that the default trigger follows an upward sloping shape with upward jumps following as the time to expiry of the operational phase progresses for long debt maturities. The result confirms the intuition proposed by Dixit (2001) (see p.50) that at the early stages the firm has an incentive to delay default because there are still opportunities for a negative situation to be reversed. As the time to maturity progresses, default is triggered at a higher level since the flexibility that the situation is reversed gets reduced; at maturity default is triggered when the firm cannot cover its interest obligations (so terminal default triggers are equal to coupon levels initiated at the start of the horizon). The result also expands the insights in Leland and Toft (1996) who argue that for long term debt issues the default trigger will be set at low levels. In their case a unique trigger is defined based on simplifying assumptions on debt rebalancing that allows a stationary debt structure, whereas here we show that the default trigger will be low at the beginning and subsequently increasing as time progresses. Leland and Toft

(1996) analyze the case with debt principal at the end of the horizon. We have performed the same analysis here and have found an upward sloping default trigger for all debt maturities. In the case were a principal exists the default is upward sloping until the maturity of debt and exhibits a large jump at the last date of the balloon payment.

[Insert Figure 5 here]

The shape of the default trigger, however, may downward sloping for short debt horizon maturities (results not shown for brevity). This will be the case only if no principal payment exists at the end. Our numerical results show that for short horizon debt maturities the firm prefers to borrow heavily with optimal coupon exceeding the revenue level at the investment maturity. With heavy borrowing, the firm's equity holders will no longer have room for delayed default at the early years and default will be triggered at higher levels of revenues. This is because at the beginning of investment the firm faces high coupon payments for the next few years. If the firm survives the first stages the remaining payments are reduced substantially and given that the firm has a relatively long horizon ahead it may have better chances that the situation is reversed. The numerical results show that the high coupon levels are used so as to raise as much more debt possible, exploit better the tax advantage of debt and raise the investment trigger, i.e., investment in states that would not have been possible if the firm did not borrow heavily. Table 4 illustrates the case with $T_D = 5$ (with $T_F = 20$ like before). The results show clearly that coupon levels at 100% of revenues may be binding the firm from exploiting higher tax benefits, raising more debt and increasing the states were investment is possible. It should be emphasized that all values are the expected discounted values at investment maturity. Leverage ratios (last column) are the ones at investment maturity (and since they are the same at each state only one number is reported). The higher values of investment cost and the value of unlevered assets and tax benefits may thus reflect the fact the firm may have extend the states where investment takes place if it can borrow heavily. In the case were principal exists, our results have shown that equity holders will no longer have the ability to use high coupon levels because that will be connected to high principal values necessary to be paid in a short time horizon. For this reason, the result is reversed and the firm borrows more heavily at longer debt horizons (confirming the results of Leland and Toft, 1996). With respect to the default trigger in the existence of principal payments, the results show that the default trigger at shorter debt horizons is smaller than the corresponding period default trigger for longer debt horizon and only exceeds that of longer term horizons at the maturity of the short horizon debt (where as we have mentioned before the default trigger jumps upwards). Thus, we show that the result of Leland and Toft (1996) that the default trigger for short horizons is higher than that of long horizons is averaging the true results (with latter effect dominating).

[Insert Table 4 here]

Our final set of results for this subsection relate to the investigation of optimal debt maturity choice. The following table shows the values of (levered) firm at t = 0 under alternative assumptions about the choice of debt maturity at the maturity of the investment horizon T_1 . The firm's operational phase is $T_F = 20$ years and we allow for debt maturity choice among 4 discrete choices: $T_D = 5, T_D = 10, T_D = 15$ or $T_D = 20$. The last row shows the results when optimal choices are allowed among these 4 alternative maturity choices at each end node at the investment maturity. The results in parenthesis for the debt horizon of 5 years are firm values when coupon levels are restricted to be at 100% of the revenue level at maturity (in this case the constraint of coupons at 100% level is binding). The results show that with unconstrained coupon levels, the optimal debt maturity is to select a short horizon. This result holds for different model parameters, f.e., lower opportunity costs and lower volatility levels. The results differ from Leland and Toft (1996). The reason is the presence of the face value of debt in their numerical simulation. We have performed the same simulations using a positive face value of debt which is connected to the selected coupon level and we indeed show that it is always optimal to select the longer term maturity (in our case $T_D = 20$ years). This comparison may reveal potential differences in maturity selection between regular bond issues with principal payments and bank loans were the payments are generally fixed throughout the horizon. In the latter case, it is possible that constraints that coupon levels cannot exceed the current revenue levels of the firm may exist, and in that case the firm may select a medium term debt maturity (in this case 10 years).

[Insert Table 5 here]

The results show that optimal leverage ratios (case of no principal) are decreasing in debt maturity choice: leverage ratios are as high as 84% for 5 year horizon debt, 71% for 10 year horizon debt, 70% for 15 years horizon debt and 65% for 20 year horizon debt. In the case of a positive principal payment at the end of the debt horizon we have observed that this result is reversed (consistently with Leland and Toft, 1996).

3.2. Optimal investment timing

In this subsection we investigate the shape of the optimal investment trigger for finite maturity investment options and finite horizon operational phase at different model parameters. A particularly interesting case relates in allowing for separate levels of the erosion parameter δ and σ before and after the investment trigger and investigating the impact on values and the investment trigger. Furthermore, we investigate the optimal choice of leverage along the investment trigger and optimal values of coupon, equity, debt, the value of unlevered assets, the tax benefits and bankruptcy costs along the investment trigger.

Table 6 provides a set of results based on an American option with $T_1 = 5$. The operational phase is assumed to be $T_F = 20$. It is assumed that decisions are taken every year, i.e., $\Delta t = 1$ (including the investment timing and operation or default decision in the operational phase). An approximation of $N_{\Delta t} = 12$ lattice steps is used.

[Insert Table 6 here]

The results show that for a lower opportunity cost both before and after the investment enhances, the firm, equity and debt values values and expected investment are increased.

Furthermore, it increases the tax benefits of debt substantially despite a small increase in bankruptcy costs. A lower volatility before investment decreases option value and thus firm value drops; on the other hand a lower volatility after investment enhances firm value by enhancing the unlevered firm value, the tax benefits and reducing bankruptcy costs. Effectively, a lower volatility after the investment allows more debt to be raised. Similarly, smaller bankruptcy costs increase firm value by increasing expected unlevered firm value, tax benefits while there is a small increase in bankruptcy costs. A lower bankruptcy costs also allows for more debt to be raised. As expected, a lower tax rate enhances firm value mainly by enhancing the value unlevered assets despite the fact that tax benefits are reduced substantially. This result would not feasible in models concentrating on the modeling of the value of unlevered assets as the stochastic variable.

Figure 6 shows the investment trigger for different model parameters uniformly applied before and after the investment. The case analyzed is with an initial value P = 10. In years where no trigger is presented the algorithm signaled that "delay" is optimal for all range of possible values produced by the lattice[8]. The following figure shows that compared to the base case a lower opportunity cost will result in a higher investment trigger in all years prior to maturity and a lower volatility results in a decrease in the investment trigger. These results confirm that the results are consistent with what one would expect from option theory[9]. Lower bankruptcy costs and lower taxes will cause the firm to invest earlier than the base case.

[8] The investment trigger can be calculated by running alternative P values at each year assuming the remaining horizon is left. This could make the trigger even more accurate but would increase computational time. The triggers produced here suffice to illustrate the main insights.

[9] Note that the fact that investment trigger point for lower dividend at the maturity of the investment is lower than the base case is also what one should expect. This is because at maturity levered firm value with lower opportunity cost is higher than the base case and this allows the firm to invest earlier.

[Insert Figure 6 here]

Figure 7 focuses on a comparison between the base case and alternative parameters for the opportunity cost and volatility before and after the investment.

[Insert Figure 7 here]

It is observed that a lower opportunity cost before investment produces the opposite effect on the trigger compared to a lower opportunity cost after. A lower opportunity cost before investment acts on the investment trigger similarly to the result of a dividend yield effect in standard call options: the firm looses little by waiting and this creates a tendency to delay investment more. On the other hand however, a lower opportunity cost after the investment has the effect of an enhanced levered value for the firm and this means that the firm may now invest at lower values of P. A lower volatility before produces the result one would expect from option pricing theory: the option value to wait gets reduced and the firm invests earlier. A lower volatility after the investment produces a similar drop in the investment trigger, but now for different reasons. This result is now driven by an effect similar to the effect of a lower dividend yield after the investment, i.e., a lower volatility after the investment trigger causes the levered firm value to increase (see table 6) and allows the firm to invest earlier.

Figure 8 shows the leverage ratios at the investment trigger. These values where calculated as the value of debt at the investment trigger over the gross value of levered firm (before subtracting the investment cost)[10]. The results show that leverage ratios remain constant along the investment trigger for any given parameter. The result that leverage ratios are constant for American perpetual options was known (e.g., see Koussis and Martzoukos, 2009); the results here show that this is so even for finite horizon investment options with finite operational phase. As expected, leverage ratios are higher when the opportunity cost, volatility and bankruptcy cost are lower—situations where the

[10] In the cases where investment is delayed the expected value of costs was added to the net levered firm value.

probability of default may be reduced, debt tax benefits are enhanced or bankruptcy costs are reduced. Leverage ratios get reduced when the tax rate is lower because the tax benefits of using debt are reduced.

[Insert Figure 8 & Figure 9 here]

In Figure 9 the sensitivity to the opportunity cost and the volatility before and after the investment is investigated. The results show that what matters for capital structure decisions are the parameters in the operational phase. It is observed that leverage ratios remain the same when these parameters are different in the investment stage. Given that the investment trigger changes when these parameters change (see Figure 6) it must be that adjustments in the default trigger are such that debt and equity values create constant leverage ratios.

4. Conclusions

Finite horizon investment options with finite operational phase generally require numerical solutions. An intuitive binomial lattice based framework for the analysis of finite investment options with finite operational phase is developed. Solutions for European and American type finite horizon investment options with optimal capital structure and a multi-stage investment setting with multiple debt issues are discussed. The analysis shows that optimal leverage ratios are not affected by option moneyness at the investment trigger confirming earlier literature results in perpetual horizon. Sensitivity results show that leverage ratios are lower when the operational phase is longer. Long term debt maturity is optimal when principal payments exist while the reverse is true in the absence of principal payments. Leverage ratios are higher for longer debt horizons for the case with principal payments while this result is reversed when no principal payments exist. Sensitivity results with respect to model parameters enhance our intuition about the impact of several parameters on the firm investment and default policy and firm value. In particular, it is shown that a lower investment trigger exists for the cases of lower volatility, higher opportunity cost, lower bankruptcy cost and lower tax rate. Leverage ratios are higher when the volatility, opportunity cost and the bankruptcy costs are lower and are reduced at lower tax rates. It is shown that leverage ratios are affected by the opportunity cost and volatility that exists in the operational phase and are not affected by the parameters in the investment stage.

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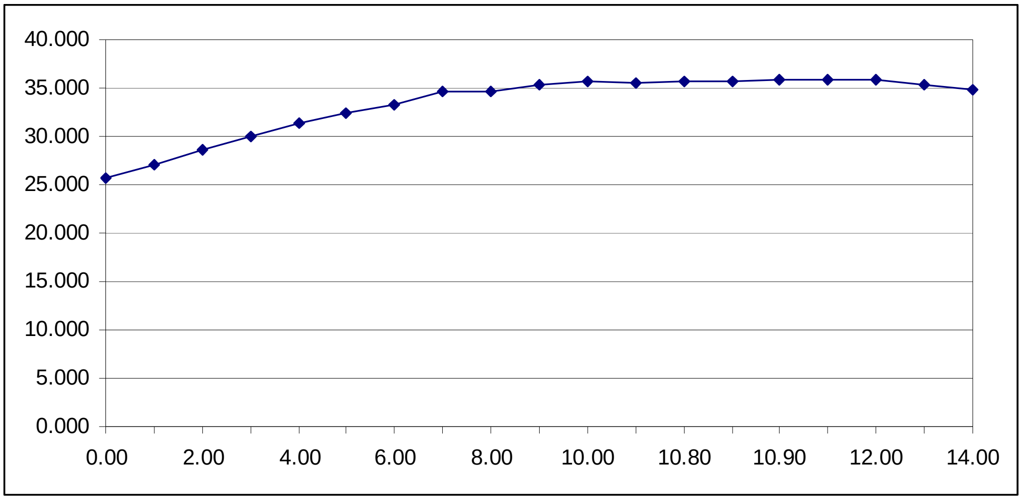
Figure 1: Optimal coupon selection using the numerical lattice model



Note: Parameters are those of panel A of table 1: P = 9.2308 (that corresponds to value-unlevered of 100 for panel A), C =0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35. In panel A: P = 9.9308 (that corresponds to value-unlevered of 100). Solution provided for coupons ranging from 0-14 with increments of 1. For coupon levels close to the perpetual solution of 10.84 denser choices of increments of 0.1 where performed. The optimal solution of the numerical model was found at R = 11 and resulted in value of 35.908. The analytic solution with optimal coupon is 35.420 while a coupon of 10.84 applied in the numerical model results in 35.700.

Figure 2: A graphical illustration of the forward-backward algorithm for one-stage investment options

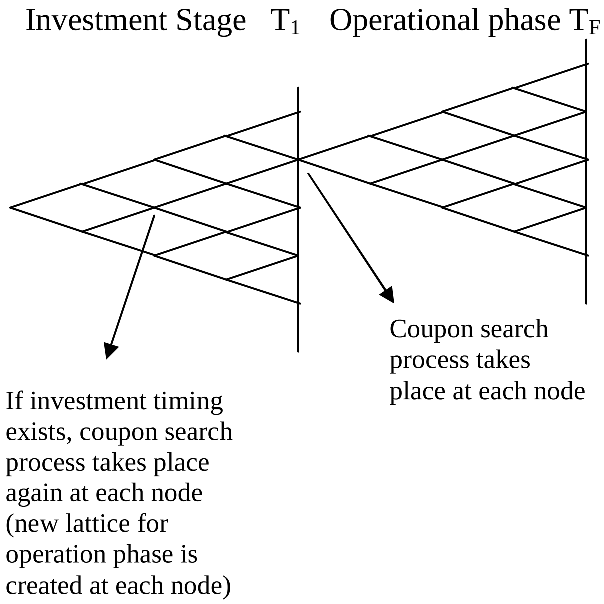


Figure 3: A graphical illustration of the forward-backward algorithm for multi-stage investment issues with multiple debt issues

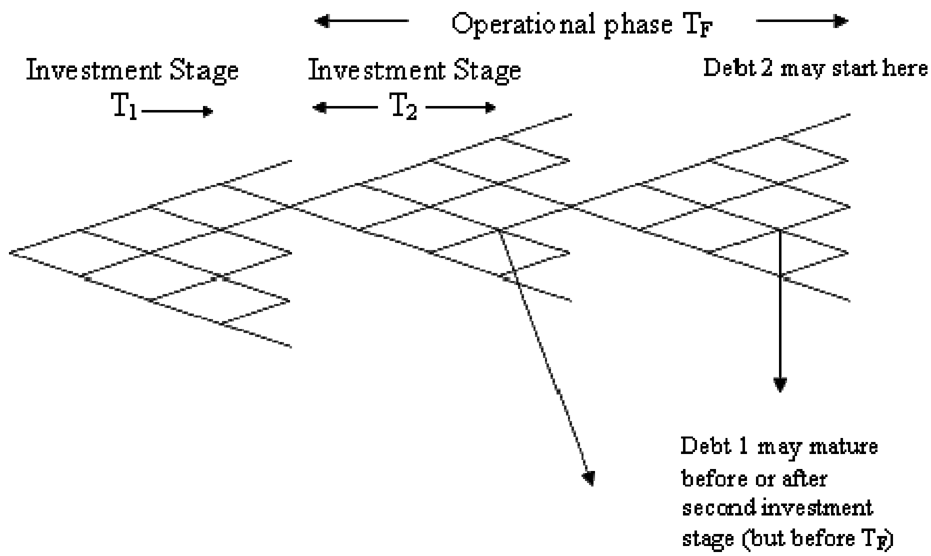


Figure 4a: Optimal coupon levels at different prices at investment maturity

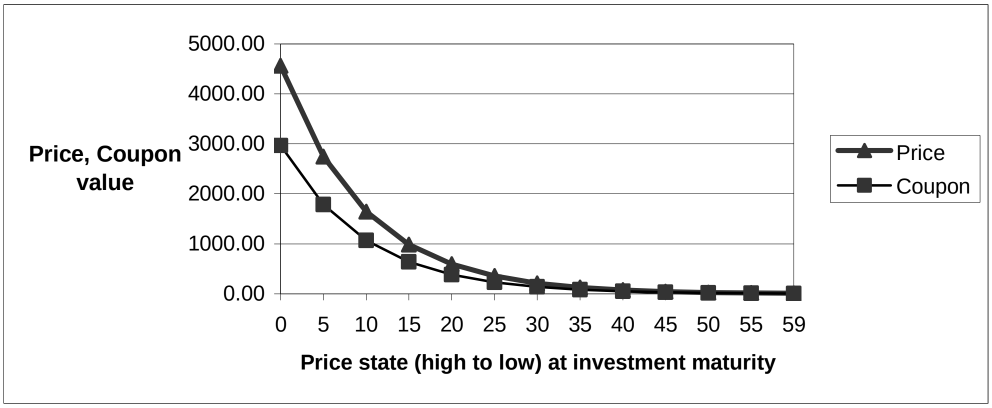
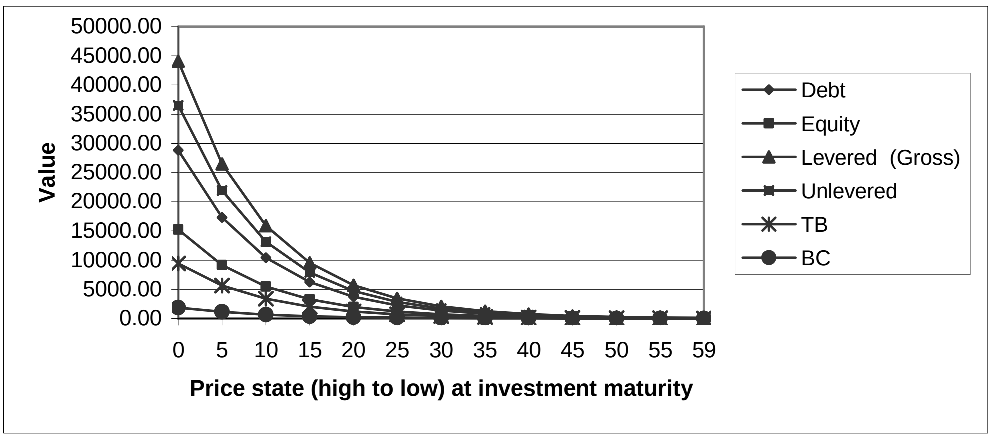


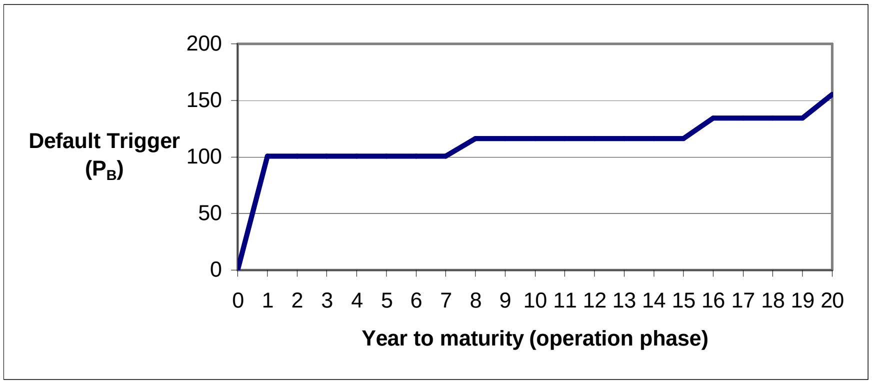
Figure 4b: Optimal values of equity, debt, levered firm and its components at different price level states at the maturity of the investment horizon



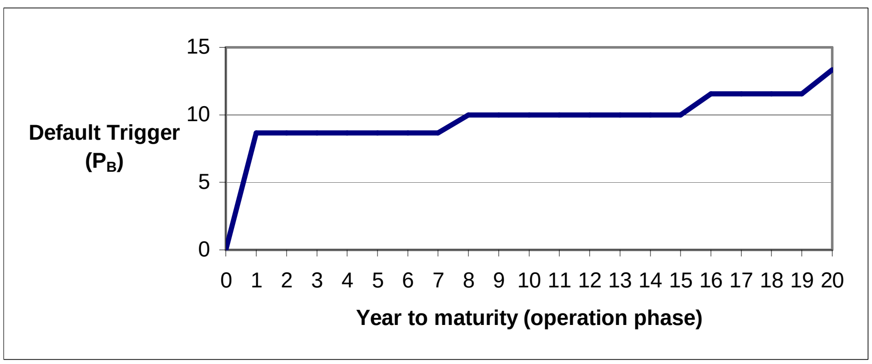
Note: Parameters are: P =10, C =0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and T_1 = 5, T_F = 20. Optimal coupon is chosen among a grid of 100 points of each price level are used (n_c =100) with maximum coupon level equal to the revenue level of the state (c_{max} = 100). The diagram shows the states where investment is exercised at maturity starting from highest (state 0) to state 59 among a total of 120 states. The diagram was produced from the case where N_{dec} = 1 and $N_{\Delta t}$ =24 so that N_1 = 120 and N_F = 600.

Figure 5: Default trigger functions at selected values of *P* at maturity

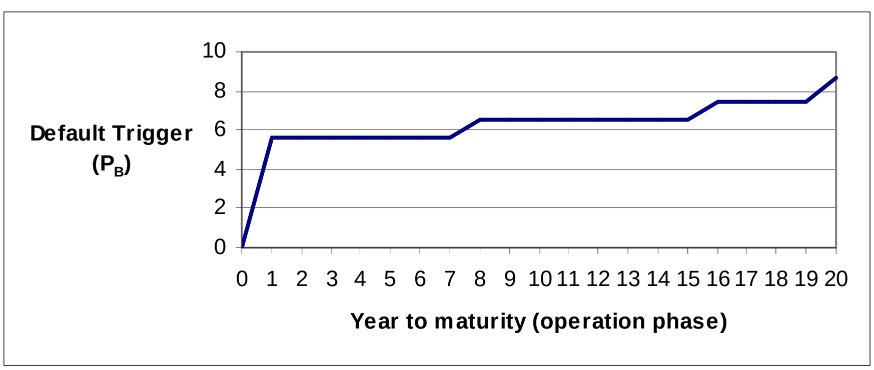
Price (P) = 239.37, Value levered (net of investment cost) =2209.95



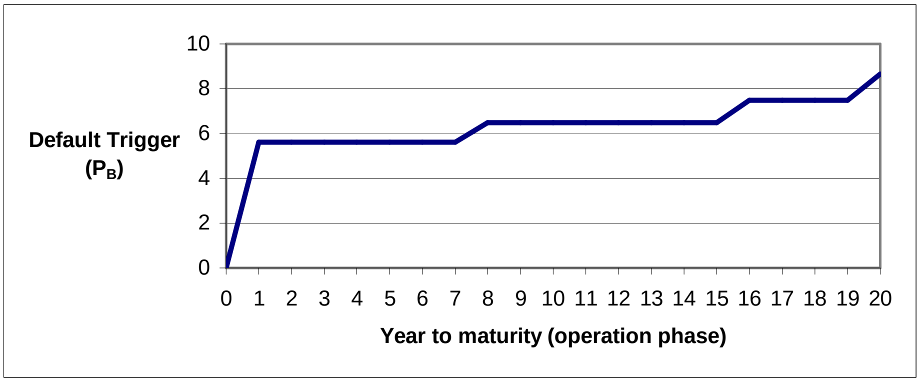
Price (P) = 20.58, Value levered (net of investment cost) =98.59



Price (P) = 13.35, Value levered (net of investment cost) = 28.80

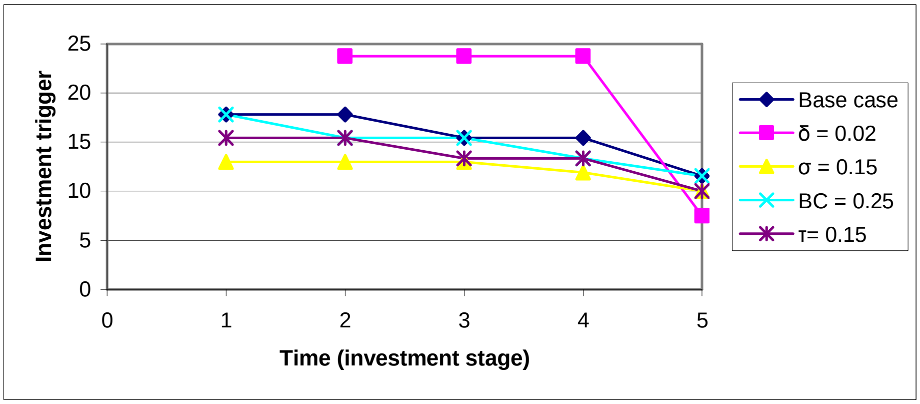


Price (P) = 11.55, Value levered (net of investment cost) = 11.49



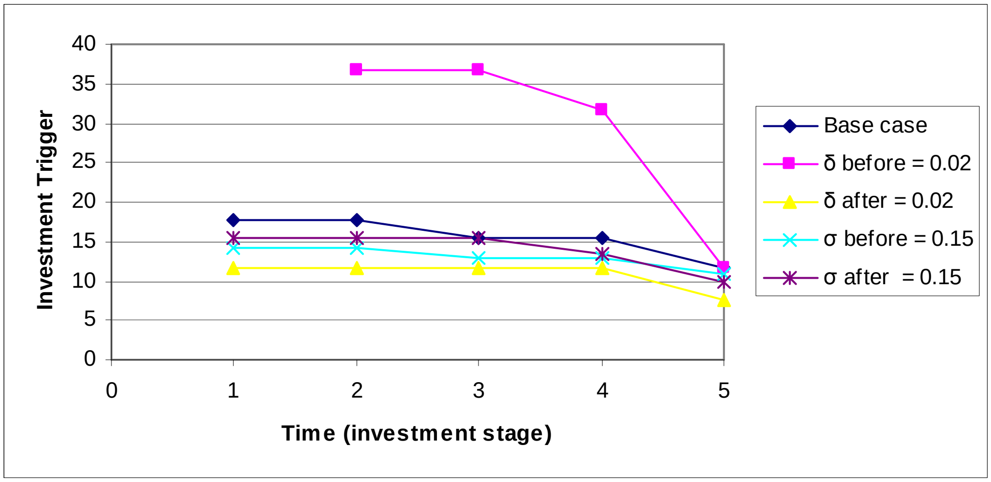
Note: Parameters are: P =10, C =0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and T_1 = 5, T_F = 20. Optimal coupon is chosen among a grid of 100 points of each price level are used (n_c =100) with maximum coupon level equal to the revenue level of the state (c_{max} = 100). The diagram shows the default trigger for selected terminal values at the investment maturity. The diagram was produced from the case where N_{dec} = 1 and $N_{\Delta t}$ =12 so that N_1 = 60 and N_F = 240.

Figure 6: Investment trigger for different model parameters



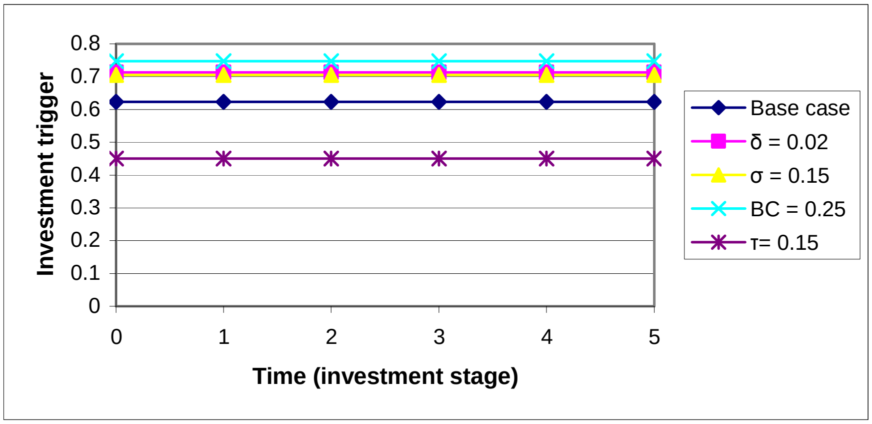
Note: Parameters are: P =10, C = 0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and T_1 = 5, T_F = 20. Optimal coupon is chosen among a grid of 20 points of each price level and (n_c =20) with maximum coupon level equal to double the revenue level of the state (c_{max} = 40). The figure results assume N_{dec} = 1 and $N_{\Delta t}$ =12 so that N_1 = 60 and N_F = 240.

Figure 7: Investment trigger for different model parameters before and after investment



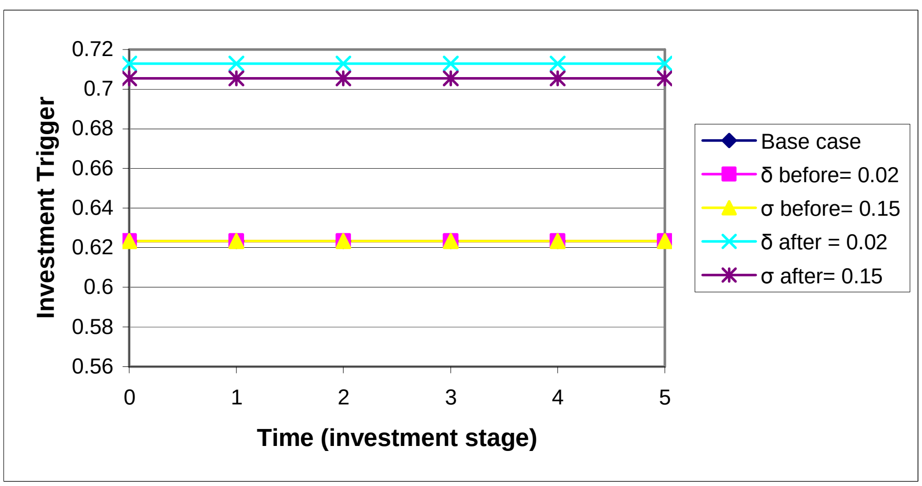
Note: Parameters are: P =10, C = 0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and T_1 = 5, T_F = 20. Optimal coupon is chosen among a grid of 20 points of each price level and (n_c =20) with maximum coupon level equal to double the revenue level of the state (c_{max} = 40). The figure results assume N_{dec} = 1 and $N_{\Delta t}$ =12 so that N_1 = 60 and N_F = 240.

Figure 8: Leverage ratios at the investment trigger for different model parameters



Note: Parameters are: P =10, C = 0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and T_1 = 5, T_F = 20. Optimal coupon is chosen among a grid of 20 points of each price level and (n_c =20) with maximum coupon level equal to double the revenue level of the state (c_{max} = 40). The figure results assume N_{dec} = 1 and $N_{\Delta t}$ =12 so that N_1 = 60 and N_F = 240.

Figure 9: Leverage ratios at the investment trigger for different model parameters before and after investment



Note: Parameters are: P =10, C = 0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and T_1 = 5, T_F = 20. Optimal coupon is chosen among a grid of 20 points of each price level and (n_c =20) with maximum coupon level equal to double the revenue level of the state (c_{max} = 40). The figure results assume N_{dec} = 1 and $N_{\Delta t}$ =12 so that N_1 = 60 and N_F = 240.

Table 1a: Numerical accuracy of the numerical lattice model

Panel A: Zero operational costs (C = 0, R = 10.84)

		Numerical Model (without abandonment option)					
		M1	M2	M3			
		T_F=400, T_1 = 200	T_F=400, T_1 = 200	T_F=400, T_1 = 200			
	Analytic (A)	N_F =1,000, N_1 = 500	N_F =2,000 N_1 = 1,000	N_F=3,000 N_1 = 1,500	%Diff (A-M1)	%Diff (A-M2)	%Diff (A-M3)
Equity	25.791	25.885	25.890	25.855	-0.004	-0.004	-0.002
Debt	44.098	37.645	43.197	43.739	0.171	0.021	0.008
V Unlevered	59.138	54.464	58.285	58.849	0.086	0.015	0.005
Tax benefits	14.220	12.110	13.999	14.126	0.174	0.016	0.007
Bankr. Costs	3.469	3.044	3.198	3.380	0.140	0.085	0.026
V Levered	35.420	34.939	35.960	35.700	0.014	-0.015	-0.008

Panel B: Positive operational costs (C = 7, R =17.7)

		Numerical Model (without abandonment option)					
		M4	M5	M6			
		T_F=400, T_1 = 200	T_F=400, T_1 = 200	T_F=400, T_1 = 200			
	Analytic (A)	N_F =1,000, N_1 = 500	N_F =2,000 N_1 = 1,000	N_F =3,000 N_1 = 1,500	%Diff (A-M4)	%Diff (A-M5)	%Diff (A-M6)
Equity	11.002	10.747	10.799	11.165	0.024	0.019	-0.015
Debt	18.622	19.592	20.103	17.849	-0.050	-0.074	0.043
V Unlevered	24.769	25.641	25.397	24.252	-0.034	-0.025	0.021
Tax benefits	6.087	6.297	6.639	5.862	-0.033	-0.083	0.038
Bankr. Costs	1.231	1.600	1.135	1.101	-0.231	0.085	0.118
V Levered	19.344	19.159	19.927	19.421	0.010	-0.029	-0.004

		Numerical Model (with abandonment option)					
		M7	M8	M9			
		T_F=400, T_1 = 200	T_F=400, T_1 = 200	T_F=400, T_1 = 200			
	Analytic (A)	N_F =1,000, N_1 = 500	N_F =2,000 N_1 = 1,000	N_F =3,000 N_1 = 1,500	%Diff (A-M7)	%Diff (A-M8)	%Diff (A-M9)
Equity	11.002	10.747	10.799	10.674	0.024	0.019	0.031
Debt	18.622	19.872	20.375	20.490	-0.063	-0.086	-0.091
V Unlevered	24.769	26.202	25.942	26.331	-0.055	-0.045	-0.059
Tax benefits	6.087	6.297	6.639	6.565	-0.033	-0.083	-0.073
Bankr. Costs	1.231	1.880	1.407	1.732	-0.345	-0.125	-0.290
V Levered	19.344	19.440	20.200	19.675	-0.005	-0.042	-0.017

Note: Parameters for both panels: P = 9.2308 (that corresponds to value-unlevered if investment takes place today of 100 for panel A), risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35. In panel A, solution obtained using model without abandonment option on unlevered assets. The same results are obtained by using the model with abandonment option (since operation costs are zero). The coupon levels in both panels correspond to the optimal coupon levels according to the Mauer and Sarkar (2005) perpetual model (Analytic solution).Ndec = N_F in all models and $N_{\Delta t}$ = 1

Table 1b: Yearly decisions with different accuracy of lattice steps per year

Panel A: Zero operational costs (C = 0, R = 10.84)

	Analytic (A)	Numerical 1 T_F=400, T_1 = 200 $N_{\Delta t}$ =5, N_F=2000, N_1 = 1,000	Numerical 2 T_F=400, T_1 = 200 $N_{\Delta t}$ =7, N_F=2,800, N_1 = 1,400	Numerical 3 T_F=400, T_1 = 200 $N_{\Delta t}$ =8, N_F=3,200, N_1 = 1,600	%Diff (A-1)	%Diff (A-2)	%Diff (A-3)
Equity	25.791	26.142	25.976	25.686	-0.013	-0.007	0.004
Debt	44.098	45.074	49.843	55.189	-0.022	-0.115	-0.201
V Unlevere	59.138	60.394	63.844	67.733	-0.021	-0.074	-0.127
Tax benefit	14.220	14.492	16.027	17.716	-0.019	-0.113	-0.197
Bankr. Cos	3.469	3.669	4.052	4.573	-0.054	-0.144	-0.241
V Levered	35.420	36.852	37.350	37.592	-0.039	-0.052	-0.058

Panel B: Positive operational costs (C = 7, R =17.7)

	Analytic (A)	Numerical 1 T_F=400, T_1 = 200 $N_{\Delta t}$ =5, N_F=2000, N_1 = 1,000	Numerical 2 T_F=400, T_1 = 200 $N_{\Delta t}$ =7, N_F=2,800, N_1 = 1,400	Numerical 3 T_F=400, T_1 = 200 $N_{\Delta t}$ =8, N_F=3,200, N_1 = 1,600	%Diff (A-1)	%Diff (A-2)	%Diff (A-3)
Equity	11.002	10.746	11.023	10.995	0.024	-0.002	0.001
Debt	18.622	20.838	19.369	19.576	-0.106	-0.039	-0.049
V Unlevere	24.769	26.134	25.423	25.617	-0.052	-0.026	-0.033
Tax benefit	6.087	6.815	6.310	6.360	-0.107	-0.035	-0.043
Bankr. Cos	1.231	1.366	1.340	1.406	-0.099	-0.082	-0.125
V Levered	19.344	20.148	19.887	19.862	-0.040	-0.027	-0.026

	Analytic (A)	Numerical 4 T_F=400, T_1 = 200 $N_{\Delta t}$ =5, N_F=2000, N_1 = 1,000	Numerical 5 T_F=400, T_1 = 200 $N_{\Delta t}$ =7, N_F=2,800, N_1 = 1,400	Numerical 6 T_F=400, T_1 = 200 $N_{\Delta t}$ =8, N_F=3,200, N_1 = 1,600	%Diff (A-4)	%Diff (A-5)	%Diff (A-6)
Equity	11.002	10.746	11.023	10.995	0.024	-0.002	0.001
Debt	18.622	21.136	19.632	19.846	-0.119	-0.051	-0.062
V Unlevere	24.769	26.730	25.948	26.158	-0.073	-0.045	-0.053
Tax benefit	6.087	6.815	6.310	6.360	-0.107	-0.035	-0.043
Bankr. Cos	1.231	1.664	1.603	1.676	-0.260	-0.232	-0.266
V Levered	19.344	20.446	20.150	20.132	-0.054	-0.040	-0.039

Note: Parameters for both panels: P = 9.2308 (that corresponds to value-unlevered if investment takes place today of 100 for panel A), risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35. In panel A, solution obtained using model without abandonment option on unlevered assets. The same results are obtained by using the model with abandonment option (since operation costs are zero). The coupon levels in both panels correspond to the optimal coupon levels according to the Mauer and Sarkar (2005) perpetual model (Analytic solution).Ndec = 1 in all models (yearly decisions with Δt = 1) and $N_{\Delta t}$ varied between 5, 7, and 8 per year.

Table 2: Sensitivity of results on the operational phase horizon

	Firm	Unlverered	TB	BC	Equity	Debt	Inv
T_F =10	4.5049	14.3773	4.1835	0.7874	5.0331	12.7404	13.2686
T_F =15	9.7618	25.6429	6.9571	1.3641	9.9943	21.2417	21.4742
T_F =20	14.7991	34.0524	8.8098	1.7488	14.1937	26.9197	26.3143
T_F =25	19.0896	42.1125	10.8133	2.3435	17.3438	33.2386	31.4927
T_F =30	22.4604	49.5776	12.6012	2.8656	20.4442	38.8690	36.8528
T_F =35	25.1912	51.9467	13.2539	3.1566	21.0190	41.0250	36.8528
T_F =40	27.1697	53.7018	13.6689	3.3482	21.6204	42.4022	36.8528

Note: Parameters are: P =10, C =0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and investment horizon T_1 = 5 . An optimal coupon is chosen among a grid of 100 points of each price level are used (n_c =100) with maximum coupon level equal to the revenue level of the state (c_{max} = 100). Table results were produced using N_{dec} = 1 and $N_{\Delta t}$ =24 so that N_1 = 120 and $N_F = (T_F/T_1)\, N_1$.

Table 3: Levered firm values for European type investment option with finite horizon operational phase (T_F = 20)

Panel A: Yearly decisions

	Coupon based on infinite horizon solution				
	T1 = 1	**T1=3**	**T1 = 5**	**T1=7**	**T1 = 10**
$N_{\Delta t}$ = 1	9.5397	13.4663	15.0437	15.5177	14.6966
$N_{\Delta t}$ = 6	7.0776	12.1421	14.1406	14.8514	14.7165
$N_{\Delta t}$ = 12	7.0968	12.1635	14.1536	14.8594	14.7207
$N_{\Delta t}$ = 18	6.4834	11.7823	13.8849	14.6586	14.5832
$N_{\Delta t}$ = 24	6.6746	11.8905	13.9542	14.7068	14.6136
	Optimal coupon level				
	T1 = 1	**T1=3**	**T1 = 5**	**T1=7**	**T1 = 10**
$N_{\Delta t}$ = 1	10.2260	14.1111	15.8170	16.4309	15.6282
$N_{\Delta t}$ = 6	7.7494	12.7383	14.8348	15.6700	15.6858
$N_{\Delta t}$ = 12	7.7361	12.6966	14.7887	15.6244	15.6436
$N_{\Delta t}$ = 18	7.7335	12.6950	14.7879	15.6239	15.6435
$N_{\Delta t}$ = 24	7.7399	12.7056	14.7991	15.6349	15.6536

Panel B: Decisions on more frequent intervals

	Coupon based on infinite horizon solution				
	T1 = 1	**T1=3**	**T1 = 5**	**T1=7**	**T1 = 10**
N_{dec} = 1	9.5397	13.4663	15.0437	15.5177	14.6966
N_{dec} = 6	5.3688	10.2473	12.3305	13.1763	13.2583
N_{dec} = 12	4.7698	9.8394	12.0063	12.9059	13.0445
N_{dec} = 18	4.8194	9.8563	12.0066	12.8973	13.0296
N_{dec} = 24	4.8485	9.8481	11.9913	12.8802	13.0127
	Optimal coupon level				
	T1 = 1	**T1=3**	**T1 = 5**	**T1=7**	**T1 = 10**
N_{dec} = 1	10.2260	14.1111	15.8170	16.4309	15.6282
N_{dec} = 6	6.0279	10.8178	12.9736	13.9240	14.1386
N_{dec} = 12	5.6924	10.5362	12.7265	13.7046	13.9537
N_{dec} = 18	5.5272	10.4151	12.6255	13.6176	13.8824
N_{dec} = 24	5.4167	10.3360	12.5601	13.5615	13.8367

Note: Parameters are: P =10, C =0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35. For panel A the numerical method uses a coupon obtained from the perpetual horizon solution of R = 10.842 at all end nodes of the investment horizon.. In panel B an optimal coupon is chosen among a grid of 100 points of each price level are used (n_c =100) with maximum coupon level equal to the revenue level of the state (c_{max} = 100).

Table 4: Firm value and other information for a short debt maturity horizon with coupon levels varied between 100%-300% of the revenue level at maturity

Coupon level	Firm	Unlevered	TB	BC	Equity	Debt	Inv	Lev
100%	15.4617	34.0524	7.7304	0.0068	19.6824	22.0936	26.3143	0.53
150%	19.2056	38.1928	12.7363	0.2309	14.0779	36.6204	31.4927	0.72
200%	20.6256	38.1928	14.9399	1.0145	8.4182	43.7000	31.4927	0.84
250%	20.6256	38.1928	14.9399	1.0145	8.4182	43.7000	31.4927	0.84
300%	20.6256	38.1928	14.9399	1.0145	8.4182	43.7000	31.4927	0.84

Note: Parameters are: P =10, C =0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and T_1 = 5, T_F = 20 and debt maturity T_D = 5. Optimal coupon is chosen among a grid of 100 points of each price level are used (n_c =100) with maximum coupon level equal to c_{max} = 100, 150, 200, 250 and 300. The table was produced for the case where N_{dec} = 1 and $N_{\Delta t}$ =24 so that N_1 = 120 and N_F = 480.

Table 5. Firm values with choice of debt maturity

	$N_{\Delta t}$ = 1	$N_{\Delta t}$ = 6	$N_{\Delta t}$ = 12	$N_{\Delta t}$ = 18	$N_{\Delta t}$ = 24
Debt horizon, T_D = 5	23.6135 (16.2412)	21.1161 (15.4127)	21.0236 (15.4533)	20.971 (15.4500)	20.6256 (15.4617)
Debt horizon, T_D = 10	18.5183	17.5631	17.4435	17.3423	17.4695
Debt horizon, T_D = 15	17.2595	16.2703	16.0316	16.0518	15.9579
Debt horizon, T_D = 20	15.8170	14.8348	14.7887	14.7879	14.7991
Optimal debt horizon	23.6135	21.1161	21.0236	20.971	20.6256

Note: Parameters are: P =10, C =0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and T_1 = 5, T_F = 20 and debt maturity varied between T_D = 5, 10, 15 and 20. Optimal coupon is chosen among a grid of 100 points of each price level are used (n_c =100) with maximum coupon level equal to c_{max} = 300. Solution in parenthesis for T_D = 5 are for the case where c_{max} = 100. The table was produced for the case where N_{dec} = 1 and $N_{\Delta t}$ =24 so that N_1 = 120 and N_F = 480.

Table 6: American option with finite investment horizon

	Firm	Unlevered	TB	BC	Equity	Debt	Inv
Base Case	16.027	37.688	9.356	1.595	17.121	28.328	29.422
Lower δ before (δ before = 0.02)	25.114	50.397	12.512	2.133	22.894	37.881	35.661
Lower δ after (δ after= 0.02)	42.638	86.209	24.743	4.911	30.437	75.605	63.403
Lower δ before and after (δ = 0.02)	56.281	92.360	26.508	5.261	32.608	80.998	57.326
Lower σ before (σ before = 0.15)	9.150	34.811	8.642	1.474	15.814	26.165	32.829
Lower σ after (σ after= 0.15)	18.426	44.900	13.481	1.568	16.729	40.084	38.387
Lower σ before and after (σ = 0.15)	11.356	42.159	12.658	1.472	15.708	37.637	41.989
Lower bankrupcy costs (b = 0.25)	17.365	39.770	11.195	1.631	12.455	36.879	31.968
Lower tax rate (τ = 0.15)	23.049	60.016	4.044	1.287	34.526	28.247	39.724

Note: Parameters are: P =10, C = 0, risk-free rate r = 0.06, competitive erosion δ = 0.06, volatility σ = 0.25, investment cost I = 100, b = 0.5, tax rate τ = 0.35 and T_1 = 5, T_F = 20. Optimal coupon is chosen among a grid of 20 points of each price level and (n_c =20) with maximum coupon level equal to double the revenue level of the state (c_{max} = 40). The table results assume N_{dec} = 1 and $N_{\Delta t}$ =12 so that N_1 = 60 and N_F = 240.

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