Real Options Game Models: A Review

Alcino F. Azevedo[1,2]* and Dean A. Paxson**

*Hull University Business School

Cottingham Road, Hull HU6 7RX, UK

**Manchester Business School

Booth Street West, Manchester M15 6PB, UK

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[1] Corresponding author: a.azevedo@hull.ac.uk; +44(0)1482463107.
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Abstract

The combination of game theoretic analysis with the real options theory has been an active area of research in the last decade. Game theory has been focus of great attention in the academic field over the last decades and has influenced the development of a wide range of research areas from economics, biology and mathematics to political science. Real options theory, on the other hand, emerged in the eighties as a valuation technique, especially appropriate for investments with high uncertainty, and is today taught in most of the universities' MBAs and Postgraduate courses. The attractiveness of the researchers for modeling competitive investment decisions by mixing concepts from both theories is because an investment decision in a competitive market can be seen, in its essence, as a "game" between firms, in the sense that in their decision firms implicitly take into account what they think it will be the other firms' reactions to their own actions, and they know that their competitors think the same way. Thus, as one of the game theory's goals is to provide an abstract framework for modeling situations involving interdependent choices, a merger between these two theories appears to be a logic step to do. In this paper, we discuss some of the most important details underlying the combined real options and game theory framework and review an extensive number of real options game models, summarizing its results, relating these results to the known empirical evidence, if any, and suggesting promising avenues for future research.

keywords: Continuous-time Real Options Game, Continuous-time Game of Timing, Pre-emption Game, War of Attrition Game.

1. Introduction

In recent years, a growing number of papers in the real option literature incorporate game theoretic concepts. The reason for this tendency, it has been argued, is that such approach is often desirable in terms of real applications since many industries are characterized by both uncertainty and strategic interactions. Game theory has been the focus of great attention in the academic field over the last decades and has influenced the development of a wide range of research areas from economics, biology and mathematics to political science. Real options theory, on the other hand, emerged in the eighties as a valuation technique, especially appropriate for investments with high uncertainty, and is today taught in any MBA and Postgraduate courses.

An investment decision in competitive markets can be seen, in its essence, as a "game" among firms, since in their investment decisions firms implicitly take into account what they think will be the other firms' reactions to their own actions, and they know that their competitors think the same way. Consequently, as game theory aims to provide an abstract framework for modeling situations involving interdependent choices, so a merger between these two theories appears to be a logic step.

The first paper in real options literature to consider interactions between firms was Smets (1993). Since Smets's (1993) work a new branch of real options models, taking into account the interactions between firms, arose, being Grenadier (1996), Smit and Trigeorgis (1997), Huisman (2001), Murto and Keppo (2002), Weeds (2002), Lambrecht and Perraudin (2003), Huisman and Kort (2003, 2004), Smit and Trigeorgies (2004), Paxson and Pinto (2005), Pawlina and Kort (2006) and Kong and Kort (2007) and Azevedo and Paxson (2009) good examples of this type of models.

In the real options literature, a "standard" real options game[3] (ROG) model can be described as a model where the value of the investment is treated as a state variable that follows a known process[4]; time is considered infinite and continuous; the investment cost is sunk, indivisible and fixed[5]; firms are assumed to have enough internal resources to undertake investments when it is optimal to do so; the investment game is played on a single project[6] (i.e., the investment problem is studied in

[3] In this paper, "real options game" (ROG) means a duopoly investment game where firms' payoffs are derived combining game theory concepts with the real options methodology.

[4] Typically, gBm, gBm with jumps, mean reverting and birth and dead processes, or combinations of these processes.

[5] There are papers, however, where this assumption is relaxed. See, for instance, Pindyck (1993), where it is assumed that due to physical difficulties in completing a project, which can only be resolved as the project proceeds, and to uncertainty about the price of the project inputs, the investment costs are uncertain; see also Dixit and Pindyck (1994), chapter 6, where, for a slightly different context, the same assumption is made.

[6] This aspect represents a weakness of the real options games in the sense that the full dynamics of the industry is not analyzed. Baldursson (1998) and Williams (1993) models, who analyse the dynamics of oligopolistic industries, are exceptions to this rule.

isolation as if it were the only asset on the firm's balance sheet); the number of firms holding the option to invest is usually two[7] (duopoly); and the focus of the analysis is the derivation of the firms' value functions and their respective investment threshold under the assumption that either firms are risk-neutral or the stochastic evolution of the variable(s) underlying the investment value is spanned by the current instantaneous returns from a portfolio of securities that can be traded continuously without transaction costs in a perfectly competitive capital market.

Furthermore, firms can only improve their profits by reducing the profits of its opponent (zero-sum game) and are assumed to be ex-ante symmetric and symmetric/asymmetric after the investment; the investment game is formulated as a "one-shot game"[8], where firms are allowed to invest (play) either sequentially or simultaneously, or both; cooperation between firms is not allowed; the market for the project, underlying the investment decision, is considered to be complete and frictionless; the two most common investment games are the "pre-emption game" and the "war of attrition game" [9], both usually formulated as a "zero-sum game"; and the firm's advantage to invest first/second is assumed to be partial[10].

In addition, the way the firm' investment thresholds are defined, in the firm's strategy space, depends on the number of underlying variables used. Thus, in models that use just one underlying variable, the firm's investment threshold is defined by a point; in models that use two underlying variables, the firm's investment threshold is defined by a line; and in models that use three or more underlying variables, the firm's investment threshold is defined by a surface or other more complex space structures, respectively. However, regardless of the number of underlying variables used in the real options model, the principle underlying the use of the investment threshold(s), derived through the real options valuation technique, remains the same: "a firm should invest as soon as its investment threshold is crossed the first time". In table 1 in the appendix we summarize our results regarding "non-standard" real options game models. By "non-standard" real options game models we mean models which, due to one or several of their characteristics, do not fit into the definition stated above.

[7] For an incomplete version of a real options model with three firms see Bouis, Huisman and Kort (2005).
[8] Firms are allowed to invest (play) only once.
[9] In the preemption game, it is assumed that there is a first-mover advantage that gives firms an incentive to be the first to invest; in the attrition game, it is assumed that there is a second-mover advantage that gives firms an incentive to be the second to invest.
[10] That is, the investment of the leader/follower does not completely eliminate the revenues of its opponent. One exception to this rule is the Lambrecht and Perraudin (1999) model, which is derived for a context of complete preemption.

According to game theory, the three most basic elements that characterize a game are the players and their strategies and payoffs. Translating these to a ROG we have that the players are the firms that hold the option to invest (investment opportunity), the strategies are the choices "invest"/"defer" and the payoffs are the firms' value functions. Additionally, to be fully characterized, a game still needs to be specified in terms of what sort of knowledge (complete/incomplete) and information (perfect/imperfect, symmetric/asymmetric) the players have at each point in time (node of the game-tree) and regarding the history of the game; what type of game is being played (a "one-shot" game, a "zero-sum" game, a sequential/simultaneous[11] game, a cooperative/non-cooperative game); and whether mixed strategies are allowed[12].

Even though, at a first glance, the adaptability of game theory concepts to real options models seems obvious and straightforward, there are some differences between a "standard" ROG and a "standard" game like those which illustrate basic game theory textbooks. Starting from the differences between a "standard" game in both theories, one difference that is immediately recognized regards the way the player's payoffs are given: in "standard" games used in most of the game theory textbooks, for instance, "the prisoners' dilemma", the "grab-the-dollar", the "burning the bridge" or the "battle-of-the-sexes" games, the player's payoffs are deterministic while in "standard" ROGs they are given by, sometimes, complex mathematical functions that depend on one, or more, stochastic underlying variables. This fact changes radically the rules under which the game equilibrium is determined, because, if the players' payoff depend on time and time is continuous, so the game is played in continuous-time, but, if the game is played in a continuous-time and players can move at any time, so what does the strategy "move immediately after" mean? In the real options literature, the approach used to overcome this problem is based on the Fudenberg and Tirole (1985) work, which, in its essence, develops a new formalism for modeling games of timing, which permits a continuous-time representation of the limit of discrete-time mixed-strategy equilibria[13].

In addition, other potential formal problems may also arise when we combine real options and game theories. For instance, the risk-neutral assumption commonly made in the real option literature, based on which firms' payoffs and their respective investment thresholds are derived, might not be coherent with the world under which the principle of Nash equilibrium works. In a further Section,

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[11] In real option sequential games these value functions depend on time and are usually called the "Leader" and the "Follower" value functions.
[12] This definition was guided, and is corroborated, by the information give in table 1 in the Appendix.
[13] Fudenberg and Tirole (1985) contribution to real options game models will be discussed in more detail in Section 2.

we discuss with more detail some of the most important differences, from the point of view of the mathematical formulation of the model, between continuous-time ROG and a discrete-time ROG played, as well as some potential time-consistency and formal and structure-coherence problems which may arise when we use a continuous-time framework. In such analysis we rely mainly on the works of Pitchik (1981), Kreps and Wilson (1982), Fudenberg and Tirole (1985), Dasgupta and Maskin (1986a, 1986b), Simon and Stinchcombe (1989), Stinchcombe (1992), Bergin (1992), Dutta and Rustichini (1995), and Laraki et al. (2005).

The main principle underlying game theory is that those involved in strategic decisions are affected not only by their own choices but also by the decisions of others. Game theory started with the work of John von Neumann in the 1920s, which culminated in his book with Oskar Morgenstern published in 1944. Von Neuman and Morgenstern studied "zero-sum" games where the interests of two players were strictly opposed. John Nash (1950, 1953) treated the more general and realistic case of a mixture of common interests and rivalry for any number of players. Others, notably Reinhard Selten and John Harsanyi[14], studied even more complex games with sequences of moves and games with asymmetric information.

With the development of game theory, a formal analysis of competitive interactions became possible in economics and business strategy. Game theory provides a way to think about social interactions of individuals, by bringing them together and examining the equilibrium of the game in which these strategies interact, on the assumption that every person (economic agent) has his own aims and strategies. It characterizes a game in four main dimensions: the players, the actions available to them, the timing of these actions and the payoff structure of each possible outcome. The players are assumed to be rational[15] and their rationality is accepted as a common knowledge[16,17]. Once the structure of a game understood and the strategies of the players set, the solution of the game can be determined using Nash (1950, 1953) research, which uses novel mathematical techniques to prove the existence of equilibrium in a very general class of games and paved the way for practical applications of game theory.

[14] See, for instance, Harsanyi and Selton (1988) for a good summary of these games.

[15] Although recently game theory models also have been extended to cases where players are assumed to behave irrationally.

[16] That is, each player is aware of the rationality of the other players and acts accordingly.

[17] Note that, although game theory assumes rationality on the part of the players in a game, people may act in imperfectly rational ways. There are many unexplained phenomena, assuming rationality. There are other phenomena we do not currently understand at all and many times psychology can take dominance over rationality. However, in business and economic decisions, the assumption of rationality may be a good start for gaining a better understanding of what is going on around us.

Game-theoretic models can be divided into games with or without "perfect information" and with or without complete information. "Perfect information" means that the players know all previous decisions of all the players in each decision node; "complete information" means that the complete structure of the game, including all the actions of the players and the possible outcomes, is common knowledge[18]. In real-life, most of the times, it may be unclear to each firm where its rival is at each point in time and so the assumption of complete information may not be realistic[19]. In addition, games can also be classified according to whether cooperation among players is allowed or not. In the former case, the game is called a "cooperative game", in the later, it is called a non-cooperative game. In non-cooperative games it is assumed that players cannot make a binding agreement. That is, each cooperative outcome must be sustained by Nash equilibrium strategies. On the other hand, in cooperative games, firms have no choice but to cooperate. Many real life investment situations exhibit both cooperative and non-cooperative features.

The Nash equilibrium is a concept commonly used in the real options literature. In its essence, and translated to real options game models, it means that if when competing for the revenues from an investments firms reach a point where there is a set of strategies with the property that no firm can benefit by changing its strategy while its opponent keep its strategies unchanged, then that set of strategies, and the corresponding firms' payoffs, constitute a Nash equilibrium. This notion captures a steady state of the play of a strategic game in which each firm holds the correct expectation about its rival's behavior and acts rationally. Although less used in the real options literature, the notion of a real option "mixed strategy Nash equilibrium" is designated, however, to model a steady state investment game in which firms' choices are not deterministic but regulated by probabilistic rules. In this case we study a real option Bayesian Nash equilibrium, which, in its essence, is the Nash equilibrium of the Bayesian version of the real option game, i.e., the Nash equilibrium we obtain when we consider not only the strategic structure of the real option game but also the probability distributions over the firms' different (potential) characters or types. Considering, for instance, a *N*-firm real option game, a Bayesian version of this game would consist of: i) a finite set of potential types for each firm $i = 1, \ldots, N$; ii) a finite set of perfect information games, each corresponding to one of the potential combinations of the firms' different types and, iii) a probability distribution over a firms' type, reflecting the beliefs of its opponents about its true type.

[18] The distinction between incomplete and imperfect information is somewhat semantic (see Tirole (1988), p. 174, for more details). For instance, in R&D investment games, firms may have "incomplete information" about the quality or success of each other's research effort and "imperfect information" about how much their rivals have invested in R&D.

[19] It is quite common, for instance, that a firm, before an investment decision, is uncertain about the strategic implications of its action, such as whether it will make its rival back down or reciprocate, whether its rival will take it as a serious threat or not.

A game can be represented in a "normal-form" or in an "extensive-form". In the normal-form representation, each player, simultaneously, chooses a strategy, and the combination of the strategies chosen by the players determines a payoff for each player; in the extensive-form representation we specify: i) the players in the game, ii) when each player has the move, iii) what each player can do at each of his or her opportunities to move, (iv) what each player knows at each of his/her opportunity to move, and (v) the payoff received by each player for each combination of moves that can be chosen by players[20,21].

In our review we select an extensive number of papers, published or are in progress, modeling investment decisions considering uncertainty and competition, developed over the last 17 years. As a complement we use contributions from other real options-related areas. Our goal is to synthesize all the contributions to the literature on real options game models, summarize their results, relate these results to the known empirical evidence, if any, and suggest new avenues for future research. We focus our discussions and analysis, mainly, on the game theory part underlying the real options model. For three main reasons: first, because, nowadays there are few monopolistic sectors remaining and so investment projects are usually involved, at least to some degree, by competition, hence, competition is and will continue to be one of the most important factors driving firm's investment behavior; second, because the mathematics/stochastic formulation of a real options game model is, in its essence, similar to that used in real options models developed for monopolistic markets, so it has been extensively discussed over the last 30 years; third, because, despite all progresses made in real options game models over the last 2 decades, there is an implicit agreement among researchers that the real options game models available, although much closer to the real-world, when compared to real options models derived for monopolistic contexts, are still too much deterministic and, to some extent, unsophisticated in the way competition is incorporated in the investment model.

This paper is organized as follow. In section 2, we introduce basic aspects of the real options game models framework, discuss the mathematical formulation, principles and methodologies commonly used in real options game models, such as the derivation of the firms' payoffs, and respective investment thresholds, and the determination of firms' dominant strategies and game equilibrium(a). In addition, we analyse, and contrast, the differences between discrete-time real options games and continuous-time real options games. In section 3, as a complement to our

[20] For a detailed description about game representation techniques see Gibbons (1992), pp. 2-12, for the normal-form representation, and pp. 115-129, for the extensive-form representation.

[21] Thakor (1991) supplies a good overview of game theory and illustrate its application to some areas of finance theory.

discussions we briefly introduce two real options-related literatures, namely, the literatures on "continuous-time games of timing" and the literature on "non-stochastic (deterministic) investment game models". In section 4, we review more than 17 years of academic research, published or still in progress, on real options game models. In tables 1 and 2, in the appendix, we present a summary of our results. In section 5, we conclude the work and suggest new avenues for new research.

2. Real Options Game Framework

In the real options literature there are models concerned with an exclusive project, in the sense that when one of the firms invests, the opportunity is completely lost for the others, and models concerned with non-exclusive projects, leading usually to sequential investments (leader/follower models). In the former case, there is no investment game, in the sense that competition is absent; in the later, we are in the presence of the standard ROG described in the introduction. For a proper analysis of this ROG, the choice about the leadership in the investment should be endogenously taken into account in the derivation of the firms' value functions and investment thresholds. However, the mathematics for doing so are quite challenging and, consequently, in the real options literature, so far, the approach that has been followed in this regard has been to assign, deterministically or by flipping a coin, the leader and the follower roles[22].

2.1 Monopoly Market

The prototype of a standard real option model for a monopoly market can be described as follows: there is a single firm with the possibility of investing *I* in a project that yields a flow of income X_t, where X_t follows a gBm process given by equation (1).

$$dX_t = \mu_X X_t dt + \sigma_X X_t dz \tag{1}$$

where, μ_X is the instantaneous conditional expected percentage change in X_t per unit of time (also known as the *drift*) and σ_X is the instantaneous conditional standard deviation per unit of time in X_t (also known as the *volatility*). Both of these variables are assumed to be constant over time and the condition $\mu_X < r$ holds, where *r* is the riskless interest rate. *dz* is the increment of a standard Wiener process for the variable X_t. Given the assumptions above, using standard real options procedures the derivation of the firm's value function and investment threshold is straightforward.

[22] Grenadier (1996) and Huisman (2001) are among the few exceptions to this rule.

To save space, we provide the solution and refer the interested reader to McDonald and Siegel (1986) and Dixit and Pindyck (1994), chapter 5, for further details.

The firm's value function is given by (for simplicity of notation we neglect the subscript t in the variable X):

$$F(X) = \begin{cases} AX^{\beta} & \text{if } X \le X^* \\ X - I & \text{if } X > X^* \end{cases} \tag{2}$$

With

$$A = \left(\frac{\beta - 1}{\beta}\right)^{\beta-1} \frac{1}{\beta}\left(\frac{1}{I}\right)^{\beta-1} \tag{3}$$

And β is the positive root of the following quadratic function:

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r = 0 \tag{4}$$

More specifically:

$$\beta = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left[\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} \quad > 1 \tag{5}$$

With $\delta = r - \mu_X$.

The firm's optimal investment strategy consists in investing as soon as X_t first crosses X^*, where X^* is given by equation (6):

$$X^* = \frac{\beta}{\beta - 1} I \tag{6}$$

Since $\beta > 1$, so the investment rule says that the firm should not invest before the value of the project has exceeded *I*, the investment cost, by a certain amount.

This is the fundamental result from irreversible investment analysis under uncertainty. The essence of the investment timing strategy is to find a critical project's value, X^*, at which the value from postponing the investment further equals the net present value of the project $X - I$. As soon as such value (investment threshold) is reached, the firm should invest. Since this is the solution for a monopoly market, the investment threshold, X^*, is sometimes referred in the literature as the "non-strategic investment threshold", recognizing the fact that it is the firm's optimal threshold value on the assumption that its payoff is independent of other firms' actions[23].

2.2 Duopoly Market

Now suppose we have two firms competing for the same investment opportunity. More specifically, consider an industry comprised of two identical firms where each firm possesses an option to invest in the same (and unique) project that will produce a unit of output[24]. Furthermore, let assume that the cost of the investment is *I* and irreversible and the cash flow stream from the investment is uncertain. In such context the payoff of each firm is affected by the actions (strategy) of its opponent. Now consider the extreme case where not only the project is unique but also as soon as one firms invests, it becomes worthless for the firm which has not invested, i.e., at time *t* when one firm triggers its investment, the investment opportunity is completely lost for the other firm. Consequently, due to the fear of losing the investment opportunity, each firm has a strong incentive to invest before its opponent as long as its payoff is positive. Intuitively, we can see that in such contexts firms have an incentive to invest earlier than what is suggested by the monopoly solution (Equation 6).

Dixit and Pindyck (1994), chapter 9, Huisman (2001), Paxson and Pinto (2005), among others, developed real option models for leader/follower competition settings. In these models, at a first moment of the investment game only one firm invests and becomes the leader, achieving a (perhaps temporary) monopolist payoff; in a subsequent moment, a second firm is allowed to invest if that becomes optimal, and becomes the follower, sharing both firms thereafter the payoff of a duopoly market. More specifically, let assume that the price of a unit of output, P_t, fluctuates stochastically over time according to equation (7),

$$P_t = X_t D\left[Q_t\right] \tag{7}$$

where, the variable *D* is the inverse demand function, X_t is an exogenous shock process to demand and Q_t is the industry supply process. The inverse demand function is assumed to be downward

[23] Note, however, that investments in large projects in monopoly markets can have an effect on the value of the monopolistic firm similar to that related with the entrance of a new competitor. For instance, Keppo and Lu (2003) derived a real options model for a monopolistic electricity market where due to the size of the new electricity plant, its operation will affect the market supply and the path of the electricity prices, and consequently, the value of the firm's currently active projects.

[24] In this section we rely on Smets (1993).

sloping, $D'<0$ to ensure a "first-mover advantage". The market demand is uncertain and the shock to the industry demand, X_t, evolves as a gBm process given by (equation 1). Each firm contemplates two choices, whether it should be the first to exercise (becoming the leader) or the second to exercise (entering the market as a follower), having, for each of these strategies, an optimal time to act. To save space, we provide the solution and refer the interested reader to Dixit and Pindyck (1994), chapter 9, for further details.

For simplicity, let start the analysis by taking the positions of the firms in the investment as given. Thus, we derive the equilibrium set of exercise strategies letting the firms to choose their roles, starting from the value of the follower and then working backwards in a dynamic programming fashion to determine the leader's value function. Denoting $F_F(X_t)$ as the value of the follower and assuming that firms are risk-neutral, $F_F(X_t)$ must solve the following equilibrium differential equation:

$$\frac{1}{2}\sigma_X{}^2 X^2 \frac{\partial^2 F_F(X_t)}{\partial X^2} + \mu_X X \frac{\partial F_F(X)}{\partial X} - rF_F(X) = 0 \tag{8}$$

The differential equation (8) must be solved subject to the boundary conditions (9) and (10), which ensure that the follower chooses the optimal exercise strategy:

$$F_F(X_F^*) = \frac{X_F^* D(2)}{r-\mu} - I \tag{9}$$

$$F_F{}'(X_F^*) = \frac{D(2)}{r-\mu_X} \tag{10}$$

Where $D(2)$ is the industry output when both firms are active and X_F^* is the follower's investment threshold.

According to the real option theory, the optimal strategy for the follower is to exercise the first moment that $X_t > X_F^*$. The boundary condition (9) is the value-matching condition. It states that at the moment the follower's option is exercised its net payoff is $X_F^* D(2)/(r-\mu_X)-I$ (the discounted expected present value of the duopoly cash flow in perpetuity). The boundary condition (10) is called the "smooth-pasting" or "high-contact" condition, and ensures that the exercise trigger is chosen so that to maximize the value of the option. Through this procedure we get closed-form solutions for the leader's and the follower's value functions, $F_F(X_t)$ and $F_L(X)$, respectively, and for the follower's investment threshold, X_F^*. These solutions are given below:

$$F_F(X) = \begin{cases} \left(\frac{I}{\beta - 1}\right)\left(\frac{X}{X_F^*}\right)^{\beta} & \text{if } X < X_F^* \\ \frac{XD(2)}{r - \mu_X} - I & \text{if } X \geq X_F^* \end{cases} \tag{11}$$

$$X_F^* = \left(\frac{\beta}{\beta - 1}\right)\left(\frac{r - \mu_X}{D(2)}\right) I \tag{1 2}$$

And,

$$F_L(X) = \begin{cases} \frac{XD(1)}{r - \mu_X} + \frac{D(2) - D(1)}{D(2)} \frac{\beta}{\beta - 1} I \left(\frac{X}{X_F^*}\right)^{\beta} - I & \text{if } X < X_F^* \\ \frac{XD(2)}{r - \mu_X} & \text{if } X \geq X_F^* \end{cases} \tag{13}$$

The expression for the leader's investment threshold, X_L^*, is derived by equalizing, for $X \geq X_F^*$, expressions (11) and (13), replacing variable X by X_L^* and solving the resulting equation in order to X_L^*.

Finally, when both firms invest simultaneously they will share the duopoly cash flow in perpetuity given by equation (14).

$$S(X) = \frac{X\ D(2)}{r - \mu_X} - I \tag{14}$$

In the real options literature there are models for duopoly markets, such as Murto and Keppo (2002), where simultaneous investment is not allowed. On such cases, without any loss of insight, we can assume that "if the two firms want to invest simultaneously, then the one with the highest value, X, gets the project; if the project has the same value for both firms and both want to invest at the same time, the one who gets the project is chosen randomly using an even distribution. With few exceptions, in the literature it is generally assumed that both players can observe all the parameters of the model (drift, volatility, etc) and the evolution of the random variable *dz* given in Equation (1) [25].

2.2.1 Competition Setting

Smet (1993) was the pioneer to introduce the effect of competition in the real options analysis. His methodology, although not very sophisticated, was then followed by Dixit and Pindyck (1994), Huisman (2001), Paxson and Pinto (2005), among others, and it is today the standard approach used in the real options games literature. In its essence, Smet's (1993) framework consists in the (deterministic) definition of a certain number of competition factors, each assigned to a particular investment scenario, all governed by an inequality. These competition factors, and the respective inequality, are the key elements in the determination of the firms' dominant strategy at each node of the game-tree and the resultant equilibrium of the game.

2.2.2 Dominant Strategies and Game Equilibrium

For a standard duopoly pre-emption game, the formulation of the game setting can be described as follow: there are two idle firms, each with two strategies available "invest"/"defer" which can lead to three different game scenarios: i) both firms inactive; ii) one firm, the leader, active and the other firm, the follower, inactive; iii) both firms active being the leader the first to invest. To each of these investment scenarios corresponds a different firms' payoff conditioned by one (or several) competition factors governed by an inequality similar to the one below:

$$D_{1_i 0_j} > D_{1_i 1_j} > D_{0_i 0_j} \tag{15}$$

The competition factors are represented by $D_{k_i k_j}$, with $\mathrm{k} \in \{0,1\}$, where "zero" means inactive, "one" means active[26] and *i*, in this case, denotes the leader (L) and j denotes the follower (F). Following the notation above we can redefine inequality (14) for each of the firms. For the leader it would be:

$$D_{1_L 0_F} > D_{1_L 1_F} > D_{0_L 0_F} \tag{16}$$

[25] Two exceptions to that rule are, for instance, Décamps et al. (2002), who studied a competitive investment problem where firms have imperfect information regarding those variables, and Reiss (1998) who derived a real option model for a patent race where the actions of the investors are formulating in a non-game theoretic framework.

[26] Note that this notation allows to model a wider range of investment scenarios. For instance, in Azevedo and Paxson (2009), $D_{k_i k_j}$ is defined with $\mathrm{k} \in \{0,1,2,12\}$, with "0" and "1" meaning the same as above, and "2" and "12" representing investment scenarios where firms are active but with, respectively, technology 2 alone and both technologies at the same time.

The economic interpretation for the relationship between the first two factors, $D_{1_L 0_F} > D_{1_L 1_F}$, is that the leader's revenues market share is higher when operating alone than when operating with the follower; the economic interpretation for the relationship between the second and the third factors, is that the leader's market share is higher when it operates with the follower than when it is idle.

After the definition of the competition factors, their economic meaning and the inequality that govern the relationship between the competition factors, we can determine at each node of the investment game-tree the firms' dominant strategy and study the equilibrium of the game. Note that, the example used above regards a "zero-sum pre-emption game" with the two firms competing for a percentage of the market revenues and where, for each investment scenario, it is deterministically assigned to the leader and the follower a given proportion of the total market revenues. These deterministic competition factors can take, however, more sophisticated forms and different meanings, but, in its essence, the framework described above to derive the firms' payoffs, determine the dominant strategies at each node of the investment game-tree and study the equilibrium of the game is the same.

Below we present a figure which illustrates the relationship between the leader's competition factors and the firms' investment thresholds.

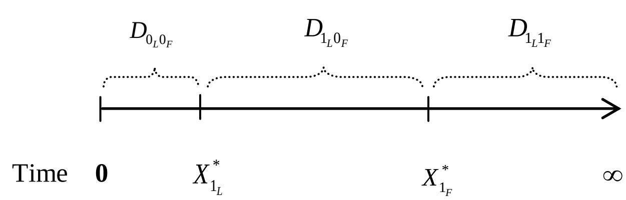

Figure 1 – Duopoly Pre-emption Game: Leader/Follower Investment Thresholds

2.2.3 The Firms' Payoffs

Using the general form for the representation of the firm's value as a function of *t*, with $t > 0$ at the beginning of the game, the firm' revenues flow is given by:

$$F_{i_k j_k} = X_t \left[D_{k_i k_j} \right] \tag{17}$$

where, X_t is the underlying variable (for instance, market revenues); $D_{k_i k_j}$ represents the competition factors, with $\mathrm{k} \in \{0,1\}$, where "0" means that the firm to which is assigned this competition factor is inactive and "1" means that the firm is active, with *i* denoting the leader (L) and *j* the follower (F).

The existence of a first mover's advantage (pre-emption game) is one assumption underlying the derivation of the real option model and so there is no need to make this assumption explicit in the inequality. However, in order to do so we just need to introduce a new pair of competition factors, $D_{1_L 1_F} > D_{1_F 1_L}$, and inequality (16) would become $D_{1_L 0_F} > D_{1_L 1_F} > D_{1_F 1_L} > D_{0_L 0_F}$ with the second and third competition factors ensuring that the market revenue share of the leader, $D_{1_L 1_F}$, is greater than that of the follower, $D_{1_F 1_L}$, when both firms are active.

This framework allows the treatment of any other types of investment games such as second mover's advantages settings (war of attrition game). In addition, we can set the first mover's advantage as temporary or permanent. If permanent, we assume that inequality (16) holds forever, i.e., as soon as the follower enters the market both firms will share the market revenues in a static and pre-defined way, governed by the competition factors and respective investment game inequality, with an advantage for the leader; if temporary, it is assumed that, at some stage of the game, with both firms active, a new market share arrangement will take place, reducing, or even eliminating, the leader's initial market share advantage. Entries or exits of players are not allowed.

The firms' value functions (payoffs) can incorporate one or several competition factors and, as mentioned earlier, a key parameter for the comparison of the firms' payoffs, at each node of the game-tree, is (are) the competition factor(s) from which depends the payoff assigned to each firms and investment strategy available. The information underlying each competition factors/game inequality is then transposed to the firms' payoffs and allows the determination of the firms' dominant strategy at each node of the game-tree. Doing so, for instance, for the leader and the case where the leader is active and the follower is idle, we would arrive at the following payoff function:

$$F_{L_1 F_0} = X_t \left[D_{1_L 0_F} \right] \tag{18}$$

Following similar procedures as those described above, the payoff functions for the leader and the follower when both firms are active are given, respectively, by:

$$F_{L_1 F_1} = X_t \left[D_{1_L 1_F} \right] \tag{19}$$

$$F_{F_1 L_1} = X_t \left[D_{1_F 1_L} \right] \tag{20}$$

Going back to inequality (16) we can see that $D_{1_L 0_F} > D_{1_L 1_F}$ and $D_{1_L 1_F} > D_{1_F 1_L}$, hence $F_{L_1 F_0} > F_{L_1 F_1}$ $D_{1_L 1_F} > D_{1_F 1_L}$ and $F_{L_1 F_1} > F_{F_1 L_1}$. Similar rationale is used to determine firms' dominant strategies at each node of the game-tree and the equilibrium of the game. Both firms are assumed to have common knowledge about inequality (16).

2.2.4 Two-Player Preemption Game

The preemption game is one of the most common games used in the real option literature, usually formulated as a two players game for economic contexts where investment costs are sunk, firms' payoffs uncertain and time is assumed to be continuous and the horizon of the investment game infinite. Real options theory shows that when an investor has the monopoly over an investment opportunity, where the investment cost is sunk and the revenues are uncertain, there is an option value to wait which is an incentive to delay the investment opportunity more than the net present value methodology suggests. The more uncertain are the revenues, the more valuable is the option to wait. However, when competition is introduced into the investment problem, for a *ceteris paribus* analysis, the intuition is that the value of the option to wait erodes. The higher the competition among firms, the less valuable is the option to wait (defer) the investment.

In modeling duopoly pre-emption investment games using the combined real options and games framework one key element which is common to almost all ROG models is the use of the Fudenberg and Tirole's (1985) principle of rent equalization. According to this principle, the erosion in the value of the option to defer the investment is caused by the fact that both firms fear to be preempted in the market by its rival due to the existence of a first mover-advantage. Consequently, both firms know that by investing a little earlier than its opponent, they will get a revenues advantage. When this advantage is sufficiently high firms will try to preempt each other, leading them to invest earlier than it would be the case otherwise.

Fudenberg and Tirole (1985), use the example of a new technology adoption game to illustrate the effect of preemption in games of timing, showing that the threat of preemption equalizes rents in a duopoly. In the real option literature their results, now called the Fudenberg and Tirole (1985) "principle of rent equalization", has been used to formulate duopoly pre-emption investment games. In Figure 2 we illustrate how this principle works in practice.

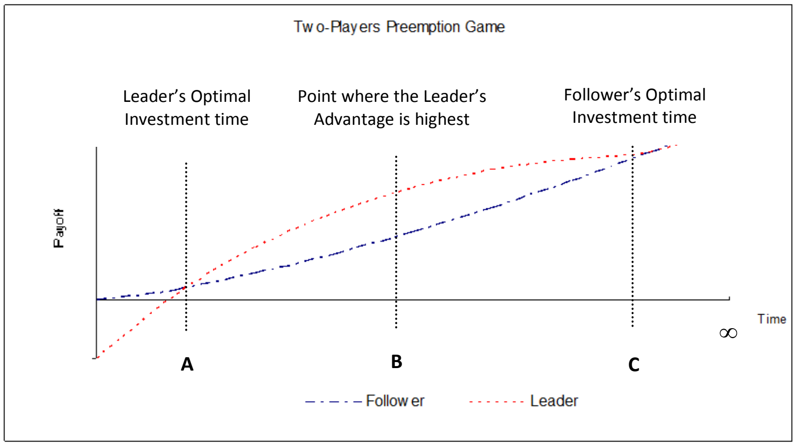

Figure 2 – Two-Player Preemption Game

In the Figure 2, we can identify three different regions on the time horizon: $[0, A)$, $[A, C)$ and $[C, \infty)$. In the interval $[0, A)$ the payoff of the follower is higher than that of the leader; in the interval $[A, C)$ the payoff of the leader is higher than that of the follower; and in the interval $[C, \infty)$ both players have the same payoff. In addition, we can see that point B is the point at which the leader's advantage reaches a maximum. In absence of the preemption effect, the optimal investment time for the leader would be point B, since its advantage over the follower at this point is the highest it could achieve. However, in a context where there is a first-mover advantage, because firms are afraid of being preempted, the leader invests at point A, a point where the payoffs (rents) from being the leader and the follower are equalized.

Note that, in the interval $(A, B]$ there are an infinite number of timing strategies that would lead to a better payoff for the leader than the strategy to invest at time A. However, in a game where firms have perfect, complete and symmetric information about the game, both firms know that, in the interval $(A, B]$, if they invest an instant before the opponent they will get a payoff advantage. This competition for preempt the rival leads both firms to target their investment at point A, point at which each firm has 50 percent chance of being the leader. In these cases, the leader is chosen by flipping a coin. As soon as one firm gets the leadership in the investment, for the other firm, the follower, the optimal time to invest is point C. From this point onward, both firms will share the market revenues in a pre-assigned way, i.e., according to the information given in inequality (16).

2.2.5 Discrete-time game *Versus* Continuous-time game

Real options game models are usually focused on symmetric, Markov, sub-game perfect equilibrium exercise strategies in which each firm's exercise strategy, conditional upon the other's exercise strategy, is value-maximizing. It is a Markov equilibrium in the sense that it is considered that the state of the decision process tomorrow is only affected by the state of the decision process today, and not by the other states before that; and it is a "sub-game perfect equilibrium" because the players' strategies must constitute a Nash equilibrium in every sub-game of the original game.

In continuous-time games with infinite horizon, the time index *t*, is defined in the domain $t \in [0, \infty)$. Hence, given the relative values of the leader and the follower for a given current value of X_t, we are allowed to construct the equilibrium set of exercise strategies for each of the firms. Real options games are usually formulated in continuous-time hence they can be classified as "continuous-time real options games". There is an obvious link between the literature on real options game models and the literature on continuous-time games of timing. Below we briefly introduce, discuss and illustrate the concept of continuous-time game and its relation with the real options games models. In such discussion we relying mainly on the works of Pitchik (1981), Kreps and Wilson (1982a,b), Fudenberg and Tirole (1985), Dasgupta and Maskin (1986a,b), Simon and Stinchcombe (1989), Stinchcombe (1992), Bergin (1992), Dutta and Rustichini (1995), and Laraki et al. (2005).

As discussed earlier, when we treat a sequential real options game in continuous-time, we have to be aware that in a game played in continuous-time there is no definition for "the last period" and the "next period"[27], and that this restricts the set of possible strategic game equilibria[28] and introduces potential time-consistency problems into the real options game model. The formulation of firms' investment strategies in continuous-time is complex. Fudenberg and Tirole (1985) highlight the fact that there is a loss of information inherent in representing continuous-time equilibria as the limits of discrete-time mixed strategy equilibria. To correct this they extend the strategy space to specify not only the cumulative probability that player *i* has invested, but also the "intensity" with which each player invest at times "just after" the probability has jumped to one. In their paper, an investor's strategy is defined as a "collection of simple strategies" satisfying an "inter-temporal consistency condition".

[27] See Fudenberg and Tirole (1985), Simon and Stinchcombe (1989) and Bergin (1992) for detailed discussions in this regard.
[28] For instance, the follower's strategy "invest immediately after the leader" cannot be accommodated.

More specifically, a simple strategy for investor *i* in a game starting at a positive level θ of the state variable is a pair of real-value functions $\left(G_i(\theta), \varepsilon_i(\theta)\right):\left(0,\infty\right)\times\left(0,\infty\right)\rightarrow\left[0,1\right]\times\left[0,1\right]$ satisfying certain conditions (see definition 1, p. 391, in the paper) ensuring that G_i is a cumulative distribution function, and that when $\varepsilon_i > 0$, $G_i = 1$ (i.e., if the intensity of atoms in the interval $\left[\theta, \theta + d\theta\right]$ is positive, the investor is sure to invest by θ). A collection of strategies for investor *i*, $\left(G_i^{\theta}(.), \varepsilon_i^{\theta}(.)\right)$, is the set of simple strategies that satisfy inter-temporal consistency conditions.

Although this formulation uses mixed strategies, the equilibrium outcomes are equivalent to those in which investors employ pure strategies. Consequently, the analysis will proceed as if each agent uses a pure Markovian strategy, i.e., a stopping rule specifying a critical value or "trigger point" for the exogenous variable θ at which the investor invests[29]. Fudenberg and Tirole (1985) work was developed for a deterministic framework. Extensions of their methodology have been developed, however, for a stochastic environment. The pioneer was Smets (1993), and then Dixit and Pindick (1994), Grenadier (1996), Huisman (2001), Thijssen, et al. (2002), Weeds (2002), Paxson and Pinto (2005) and Azevedo and Paxson (2009), among others, followed his approach.

An investment game can be represented using one of the following techniques: i) a normal-form representation or ii) an extensive-form representation. The choice between these two types of representation depends on the type of investment game. In Table 1 and Figure 3, we present illustrations of a sequential investment game using a normal-form representation and an extensive-form representation, respectively.

		Firm j	
		Defer	Invest
Firm i	Defer	Repeat game	$\left[F_F(X_t), F_L(X_t)\right]$
	Invest	$\left[F_L(X_t), F_F(X_t)\right]$	$\left[F_S(X_t), F_S(X_t)\right]$

Table 1 – Normal-Form Representation: Sequential Real Option Duopoly Game

[29] Note, however, that this is for convenience only given that underlying the analysis is an extended space with mixed strategies (a good discussion about this issue can be found also in Mason and Weeds, 2001).

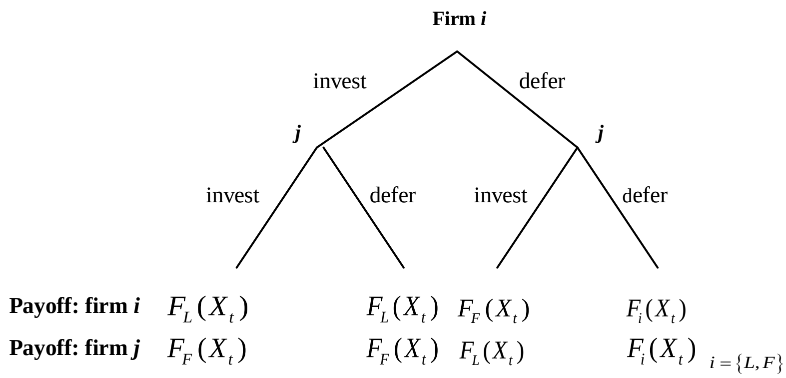

Figure 3 – Extensive-Form Representation: Sequential Real Option Duopoly Game

Comparing Table 1 with Figure 3, we can see that in table 1 the concept of "timing strategy", implicit in a sequential ROG, and the sequence of the players' moves is not as intuitive as in Figure 3, which explains the convenience of using the extensive-form representation to describe this type of game rather than the normal-form representation. In both of the representations above, however, the leader's and the follower's payoffs are represented by the same expressions $F_L(X_t)$ and $F_F(X_t)$, respectively, which were formally introduced in this paper in page 13, expressions (11) and (13), respectively. $F_S(X_t)$ and $F_S(X_t)$ are the leader's and the follower's payoff when both firms invest simultaneously.

The expressions $F_L(X_t)$ and $F_F(X_t)$ above, the subscript *t* recalls the fact that *X* is not static but varies over time, meaning that as time changes so do the firms' payoffs. Consequently, in practice, for each firm, Table 1 and Figure 3 will display different payoffs at each instant of the game. An intuitive view of the dynamic nature of the firms' payoffs, "timing strategy" and the Fudenberg and Tirole (1985) methodology of using the discrete-time framework as a *proxy* of the continuous-time approach is the elaborated representation of a duopoly ROG given in Figure 4.

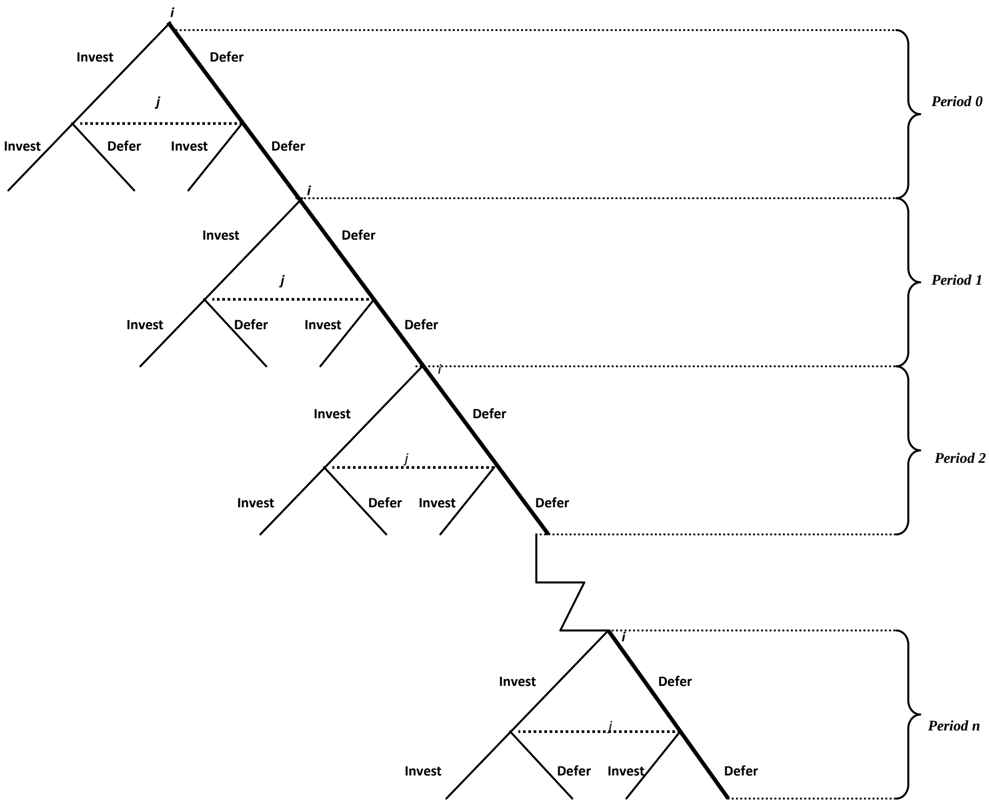

Figure 4 – Illustrative Extensive-Form: Continuous-Time Real Options Duopoly Game

An additional aspect that Figure 4 makes easier to see is the fact that in a duopoly sequential game where firms have two strategies available (invest/defer), although they can choose the strategy "invest" only once, they are allowed to choose the strategy "defer" an infinite number of times, since in a continuous time framework, in between any two instants of the game where firms did not chose the strategy "invest", that means they have chosen, theoretically, an infinite number of times the strategy "defer"[30].

ROG models usually assume that time is infinite. This assumption is of great mathematical convenience to derive the firms' payoffs and respective investment thresholds. However, it does not fit with most of the investment projects. From the point of view of the equilibrium of the game analyse there are differences between games where the option to invests matures at some particular point in time and game where the option to invest can be hold infinity. However, this problem has passed "unnoticed" because the focus of our analysis has been directed not to the "timing strategy",

[30] Note that this does not happen, for instance, in the "The Prisoner's Dilemma" game because it is a "simultaneous-one-shot" game, where players can choose only once either "confess" or "defeat".

chronologically speaking, but to the time at which the value of the investment (i.e., the underlying variable) reaches a threshold, regardless of what which chronological point in time such even occurs.

Using Expressions (11) and (13) we can plot the leader's and the follower's payoff functions, respectively, whose shape is standard (see Dixit and Pindyck, 1994).

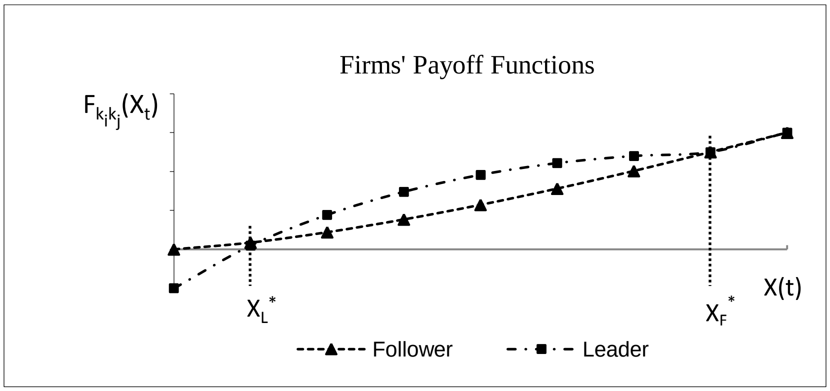

Figure 5 – The Leader and the Follower Payoff Functions and respective Investment Thresholds

From Figure 5 we can see that there exists a unique point $X_L^* \in \left(0, X_F^*\right)$ with the following properties:

$$
\begin{array}{ll}
F_L(X_t) < F_F(X_t) & \text{if } X_t < X_L^* \\
F_L(X_t) = F_F(X_t) & \text{if } X_t = X_L^* \\
F_L(X_t) > F_F(X_t) & \text{if } X_L^* < X_t < X_F^* \\
F_L(X_t) =_F (X_t) & \text{if } X_t \geq X_F^*
\end{array}
$$

which demonstrates that there is a unique value at which the payoffs to both the leader and the follower are equal. At any point below X_L^* each firm prefers to be the follower, at X_L^* the benefits of a potentially temporary monopoly just equal the costs of paying the exercise price earlier, at any point above X_L^* each firm prefers to be the leader; for $X_t \geq X_F^*$, the value of leading, following or simultaneous exercise are equal.

3. Real Options-Related Literature

3.1 Continuous-Time Games of Timing

We have a reach literature on continuous-time games of timing. As mentioned earlier, real options games models are usually formulated in continuous-time. Hence, many of the concepts developed within the literature of "continuous-time games of timing" applies to, and are used in, the literature of real options games. The application of game theory to continuous-time models is not well developed and can be quite challenging. To reduce complexity, one key assumption for modeling continuous-time games as the limit of discrete-time is not to allow firms to exit and reentry infinite times. However, such assumption is unrealistic for most of the investments in practice[31]. Stochastic nonzero-sum differential games, on the other hand, have not been extensively used in modeling investment decisions in competition settings as the mathematical tools required for doing so are quite complicated in the sense that the Hamilton-Jacobi-Bellmen equations do not lead easily to a qualitative analysis of their solutions. Only the non-cooperative feedback Nash-solution has been characterized for this class of games. To get the intuition for these and other important issues about continuous-time games, and their relation to the derivation of real option game models, we suggest the reading of the papers below.

Pitchik (1981), following Owen (1976), studies the necessary and sufficient conditions for the existence of a dominating equilibrium point in a "two-person non-zero sum game of timing" and the problem of preemption in a competitive race. Kreps and Wilson (1982a) propose a new criterion for equilibria of extensive-form games, in the spirit of Selten's perfectness criteria, and study the topological structure of the set of sequential equilibria. Kreps and Wilson (1982b) study the effect of *reputation* and *imperfect* information on the outcomes of a game, starting from the observation that in multistage games players may seek early in the game to acquire a reputation for being "tough" or "benevolent" or something else. Ghemawat and Nalebuff (1985) applied game theory concepts to the study of when and how a firm exits first from a declining industry where shrinking demand creates pressure for capacity to be reduced. Hendricks and Wilson (1985) investigate the relation between the equilibria of discrete and continuous-time formulations of the *War of Attrition* game and show that there is no analogue in continuous-time for the variety of types of discrete-time equilibrium and generally not a one to one correspondence between the equilibria of the continuous-time with the limiting distributions of the equilibria of discrete-time games, discussing extensively the reasons for such divergence and the relation to the results obtained by others.

Dasgupta and Maskin (1986a and 1986b) extend the previous literature by studying the existence of Nash equilibrium in games where an agent's payoffs functions are discontinuous, and Fudenberg and Levine (1986) provide a necessary and sufficient condition for equilibria of a game to arise as limit of ε-equilibria of games with smaller strategy spaces. As the smaller games are frequently more tractable, their results facilitate the characterization of the set of equilibrium. Hendricks and

[31] See Weyant and Yao (2005) for a good discussion on this issue.

Wilson (1987) provide a complete characterization of the equilibria for a class of “pre-emption” games, when time is continuous and information is complete, that allows for asymmetric payoffs and arbitrary time horizon. Hendricks, et al. (1988) present a general analysis of the *war of attrition* in continuous-time with complete information. Simon and Stinchcombe (1989) propose a new framework for continuous-time games that conforms as closely as possible to the conventional discrete-time framework taking the view that continuous-time can be seen as “discrete-time” but with a grid that is infinitely fine[32]. Huang and Li (1990) prove the existence of a Nash equilibrium for a set of continuous-time stopping games when certain monotonicity conditions are satisfied.

Following Hendricks and Wilson (1985) and Simon and Stinchcombe (1989), Bergin (1992) tackles the problem of the difficulties involved in modeling continuous-time strategic behaviour, since “time is not well ordered”, and develops a general repeated game model over an arbitrary time domain. Stinchcome (1992) defines the maximal set of strategies for continuous-time games, characterized by two conditions: i) a strategy must identify an agent’s next move time, and ii) agents’ only initiate finitely many points in time. Dutta and Rustichini (1993) studied a general class of stopping games with pure strategy sub-game perfect equilibria and show that there always exists a natural class Markov-perfect equilibria. Bergin and Macleod (1993) develop a model of strategic behaviour in continuous-time games of complete information, excluding conventional repeated games in discrete-time as special case. Stenbacka and Tombak (1994) introduce experience effects into a duopoly game of timing the adoption of a new technology which exhibit exogenous technological progress, concluding that a higher level of technological uncertainty increases the extent of dispersion between the equilibrium timings of adoption and that the equilibrium timings are even more dispersed when the leader takes the follower’s reaction into account. Dutta and Rustichini (1995) study a class of two-player continuous-time stochastic games in which agents can make (costly) discrete or discontinuous changes in the variables that affect their payoffs and show that in these games there are Markov-perfect equilibria of the two-sided (s, S) rule type. Laraki et al. (2005) address the question of the existence of equilibrium in general timing games with complete information. All these papers along with many others, paved the progress towards more sophisticated methodologies to treat games in continuous-time and have been implicit or explicitly used in modeling “continuous-time real options games”.

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[32] This is the approach that has been followed in the real options literature in continuous-time real option games.

3.2 Other Investment Game Models

There are also other branches of real options-related literature which although based on, sometimes, radically different theories and mathematical formulations have been used a good source of insight to developing new real option game models. Lucas and Prescott (1971), Mills (1988), Leahy (1993) and Baldursson and Karatzas (1997), as models derived for a wide range of investment contexts, and, Reinganum (1981a), Reinganum (1981b), Reinganum (1982), Gilbert and Newbery (1982), Reinganum (1983), Gilbert and Harries (1984), Jensen (1992), Hendricks (1992) and Stenbacka and Tombak (1994) as models derived, specifically, for investments on new technologies, are good examples of these real options-related literatures.

In Lucas and Prescott (1971) seminal paper, for instance, it is assumed that the actual and anticipated prices have the same probability distribution, or that price expectations are rational, and the social optimality of the equilibrium in a discrete-time Markov chain model is established and determines a time series behavior of investment, output, and prices for a competitive industry with stochastic demand. Mills (1988) examines timing and profits in investment-timing games where two or more firms vie to make an indivisible one-time investment, showing that the perfect-Nash equilibrium timing strategies eliminate rents only when it is costless for rivals to threaten preemption credibly. Leahy (1993), starting from the real options insight about the effect of irreversibility on a firm's investment decision, discovered that the equilibrium entry time under free entry is the same as the optimal entry time of a myopic firm who ignores future entry by competitors, even when we consider the effect that entry may have on the mean and variance of the output price process.

Following Leahy (1993), Baldursson and Karatzas (1997) establish the links between social optimum, equilibrium, and optimum of a myopic investor under a general stochastic demand process utilizing singular stochastic control theory. Their main focus is on a partial equilibrium model of a competitive industry. In Leahy (1993) model, the industry is composed of a continuum of infinitesimally small firms which incur irretrievable costs as they enter or exit. It is argued that each firm can be myopic as regards future investment in the industry and yet its decision will be optimal. The investment game is formulated in discrete-time and the model is applicable only to very specific industry in which demand is linear in the sense that the methodology does not work for more general investment game specifications. Based on the result of Leahy (1993) and Baldursson and Karatzas (1997), regarding the "optimality of myopia", Baldursson (1998) study an oligopoly, where firms facing a stochastic inverse demand function use capacity as strategic variable, using a fictitious social planner and the theory of irreversible investment under uncertainty for a context where production capacity can be adjusted continuously over time with linear cost. Firms are assumed to be non-identical and the investment game is formulated in continuous-time.

Reinganum (1981a) noted that the perfection of a new and superior technology confers neither private nor social benefit until that technology is adopted and employed by potential users, and, she added, in an industry with substantial entry costs, perfection and adoption of an innovation are not necessarily coterminous. She studied the diffusion of new technologies considering an industry composed of two firms, each using current best-practice technology, assuming that the firms are operating at Nash equilibrium output levels, generating a market price (given demand) and profit allocation. When a cost-reducing innovation is announced, each firm must determine when (if ever) to adopt it, based in part upon the discounted cost of implementing the new technology and in part upon the behaviour of the rival firm. Reinganum (1981b), investigates the issues related to industrial research and development, in particular, situations in which two firms are rivals in developing a new process or device. She notes that in such cases there is, sometimes, a distinct advantage to being the first to produce a new product or implement a new technology, but since only the first to succeed realizes this advantage, each firm's profits will depend upon the research efforts of its rival, which suggests a game-theoretic approach. In addition, she developed a theory of optimal resource allocation to R&D, under the assumption of uncertain technical advance and in presence of game-playing rivals, and found that the Nash equilibrium and the socially optimal rates of investment do not coincide.

Reinganum (1982) addresses the problem of resource allocation to R&D in an n-firm industry using differential games. Following Reinganum (1981a, 1981b, 1982), Gilbert and Newbery (1982) enquire whether institutions such as the patent system create opportunities for the firms with monopoly power to maintain their monopoly power. They show that, under certain conditions, a firm with monopoly power has an incentive to maintain its monopoly power by patenting new technologies before potential competitors and that this activity can lead to patents that are neither used nor licensed to others ("sleeping patents"). Reinganum (1983) reports an application of two-person, nonzero-sum game theory to a problem in the economics of technology adoption, extending previous papers by considering differentiable mixed-strategy equilibria. Gilbert and Harries (1984) develop a theory of competition in markets with indivisible and irreversible investments, noting that in markets with increasing returns to scale in investment, competition occurs over both the amount and timing of the new capital construction and that the consequences of competition depend on the strategies and information available to the competitors.

Jensen (1992) examines the welfare effects of adopting an innovation when there is uncertainty about whether it will succeed or fail, noting that the incentives of firms to adopt a new process need not coincide with maximum expected consumer surplus or social welfare if there is uncertainty before the process is adopted and if the only loss from failure is a fixed cost. Additionally, he finds that in some cases no firm will adopt an innovation likely to fail, although expected welfare is maximized if one adopts. There are cases where both firms will adopt an innovation likely to succeed, although expected welfare is maximized if only one firm adopts. Hendricks (1992) studies the effects of uncertainty on the timing of adoption of a new technology in a duopoly. Firms are assumed to be uncertain about the innovation capabilities of their rivals and the profitability of the adoption, which creates a richer and, in some respects, more plausible theory of adoption where rents from delayed adoption are always realized and returns are not equalized across adoption times. Stenbacka and Tombak (1994) introduce the effect of the experience into a duopoly game of timing of adoption of new technologies that exhibit exogenous technological progress. Their results show that a higher level of uncertainty increases the extent of dispersion between the equilibrium timings of adoption and that the equilibrium timings are even more dispersed when the leader takes the follower's reaction into account.

4. Real Options Game Models

The literature combining the real options valuation technique with game theory concepts started with Smets (1993), which derived, for a duopoly market, a continuous-time model of strategic real option exercise under product market competition. In this paper he assumes that entry is irreversible, demand stochastic and simultaneous investment may arise only when the leadership role is exogenously pre-assigned. Other pioneers in this literature were Smit and Ankum (1993) and Williams (1993). The former combines the real options approach of investment timing with basic principles of game theory and industrial organization. Using simple examples they illustrate the influence of competition on project value and investment timing. The real option game model formulation is "standard" (i.e., fits into the definition stated in the introduction), with exception of the variable "time" which is assumed to be discrete. In the real options literature very few papers use time as a discrete variable (see table 1 in the Appendix). Williams (1993), on the other hand, provides the first rigorous derivation of a Nash-equilibrium in a real options framework. He derives an equilibrium set of exercises strategies for real estate developers where equilibrium development is symmetric and simultaneous. More specifically, in equilibrium all developers build at the maximum feasible rate whenever income rises above a critical value and each developer conjectures correctly that each other developer currently builds at his optimal rate. The aggregate demand for the good or service and its supply by each developed asset are proportional to power functions of the income and the optimal building rate depends on an exogenous factor which changes stochastically through time and affects the aggregate demand. Additionally, it is assumed that the number and the owners of undeveloped assets is constant over time and identical, respectively, and that the identical owners have equal number of undeveloped assets. This model is very versatile in terms of market conditions to which it applies. Essentially, it provides investment thresholds which, in equilibrium, all market players, simultaneously, should use to optimize their investment, regardless of the type of market (monopoly, oligopoly or perfect competition) in which they operate.

Also for real estate markets, Grenadier (1996) develops an equilibrium game framework for strategic option exercise games for duopoly markets. He suggests having found a possible explanation for why some markets may experience building booms in the face of declining demand and property values. Contrary to Williams' (1993) model, where equilibrium real estate development is symmetric and simultaneous (i.e., all developers build at the maximum feasible rate whenever income rises above a critical value), in Grenadier's model, equilibrium real estate development may arise endogenously as either simultaneously or sequentially, depending on the initial conditions and the parameter values. More specifically, if at the beginning of the game, $t = 0$, the variable underlying the value of the real estate development, $X(t)$, is below the trigger value determined for the leader entry time, $X(0) < X_L$, one developer will wait until the trigger X_L is reached, and the other will wait until the trigger X_F is reached. Developers will be, therefore, indifferent between leading or following. If $X(0) \in \left[X_L, X_F\right)$, each will race to build immediately. The random winner of the race will then build, and the loser will wait until the trigger X_F is reached. If $X(0) \geq X_F$, any equilibrium will be characterized by simultaneous exercise. In the rest of the game characterization this model is standard, i.e., as defined in the introduction section.

Reiss (1998) derived a real options model for investments in innovation. Competition is considered but in a rather non-standard way. The framework is developed so that to determine whether, and when, a firm should patent and adopt an innovation if the arrival time of competitors is stochastic.

In terms of mathematical formulation the model is standard within the real options literature with the exception of the inclusion of the Poisson distribution to model the competitors' arrival time. More specifically, the innovation value change over time is defined by the following differential equation:

$$dCF = \alpha CFdt + \sigma CFdz - CFdq \tag{21}$$

Where dq is the increment of a Poisson process and assumed to be independent of dz:

$$dq = \begin{cases} 0 & \text{with probability } (1-\lambda dt) \\ 1 & \text{with probability } \lambda dt \end{cases} \tag{22}$$

The author finds four different option exercise strategies and respective investment threshold. The model applies to markets where there is competition, but does not specifies the number of market participants. Instead, the intensity of rivalry is specified through a constant hazard rate λdt which can be regarded as a measure of intensity of rivalry, since the expected arrival time of competition decreases with an increasing hazard rate, therefore, a flexible model in this regard. However, the characterization of the investment game is incomplete. For instance, if we assume the innovation game is played in a context where firms are ex-ante symmetric and have complete, perfect and symmetric information, then simultaneous investment may occur. So, why does this outcome is not allowed? In addition, the market is not explicitly characterized. Thus, we ay infer that the model applies to several types of competition and market structures. However, if we use it for oligopoly or perfect competition markets with complete, perfect and symmetric information, all market participants would be guided in their investment decision by the same, and unique, investment scenario thresholds, and, consequently, the option to invest in the innovation project would be simultaneously exercised by all market players and the value of the innovation project would decrease significantly for each player as a consequence. This scenario is not, however, discussed in the paper. Finally, the model is derived for a pre-emption investment game with competition exogenously set[33]. This later aspect is a weakness of the model and, to be accurate, a weakness shared by most of investment game models in the real options literature.

Kulatilaka and Perrotti (1998), provide a strategic rationale for the growth options under uncertainty and imperfect competition. According to their results, in a market with strategic competition, investment confers a greater capability to take advantage of future growth opportunities. This

[33] Note that, the best approach would be to model competition endogenously which would allow the analysis of the industry equilibrium under different market structures.

strategic advantage leads to the capture of a greater share of the market, either by dissuading entry or by inducing competitors to "make room" for the stronger competitor. When the strategic advantage is strong, increased uncertainty encourages investment in growth options; when the strategic effect is week the reverse is true. An increase in systematic risk discourages the acquisition of growth options. These results contradict the view that volatility is strong disincentive for investment. The authors analyse both the case where firms are ex-ante symmetric (i.e., firms are ex-ante identical) and the case where firms are ex-ante asymmetric. The former leads to simultaneous strategic entry by all market players, the latter leads to a pre-emption game where one firm enters the market first. Lambrecht (1999) and Joaquin and Butler (1999) present models where competing firms have opportunities to invest in discrete investment projects and where the investment game is played on the timing of these investments.

Boyer, et al. (2001) extend previous pioneering contributions while bringing to bear the older, and highly relevant, literature on strategic investment with a deterministic formulation and perfect foresight by firms, most notably Gilbert and Harris (1984), Fudenberg and Tirole (1985), and Mills (1988). They restrict their attention to duopoly on homogeneous product market with incremental indivisible capacity investments, paying also attention to the role of uncertainty and the speed of market development on investment strategies and competition. The mathematical formulation and game concepts used in the model are standard within the real options literature. The assumptions about the economic context to which the model applies to are very similar to those of Gilbert and Harries (1984) model. More specifically, the industry is assumed to face growing demand with indivisibilities in installing new capacity, firms are assumed to have access to the same technology and time is continuous. Building on Gilbert and Harries (1984) work, the authors avoid the commonly used, and not very sophisticated, technical assumption that gives first-mover advantage to one of the firms. The properties outlined in this paper suggest that, for the context underlying the model, collusion is more likely when the industry is made up of two active firms of equal size and market develops quickly and with much volatility, and that competition is more likely to be at work when only one firm operates.

Garlapy (2001) develops a large discrete-time nonzero-sum stochastic game for two all-equity financed single project firms competing in the development of a project that requires N phases to be completed. At each date before completion, the two firms must decide, simultaneously, whether to keep working on the project in the attempt to reach the next hurdle, or to wait. In making their decisions on whether to undertake a phase of the investment or not, the firms consider: i) their position in the investment race, i.e., the number of stages completed, and ii) a signal, δ, in the form of potential cash flows generated by the completed project, modeled as a geometric random walk; a random variable $n(t)$ that represents the number of phases completed by firm A at time t, and an analogous variable, $m(t)$, for the opponent of firm A, where, δ, $n(t)$ and $m(t)$ are common knowledge, i.e., at each stage of the game both firms know the potential cash flows generated by the completed project and the number of phases completed by the opponent. For mathematical tractability, simultaneous success in the investment race is not allowed. In addition, this paper also studies the effect of cooperation and preemption on the value of the investment race and the risk premium, i.e., the discount rate to be used in evaluating future uncertain cash flows. Although the intuition underlying this model is similar to that which guides most of the real options game models, the fact is that the mathematical formulation used is substantially different in order to accommodate the assumption that firms have an undefined number of options to invest.

Mason and Weeds (2001) demonstrate that strategic interactions can have important consequences for irreversible and uncertain investments. More specifically, the paper shows that preemption significantly decreases investment option values, externalities introduce, relative to the cooperative outcome, inefficiencies in the investment decisions, and both preemption and externalities combined can actually hasten, rather than delay, investment. The model is derived for a duopoly market with or without cooperation. The innovativeness of this model, compared with other models within the real options literature at the time, is that it does not impose exogenously an asymmetry between firms but, instead, allows the first-mover to be determined endogenously. The authors derive two versions of the model. In the first version, the roles of the leader and the follower are pre-assigned exogenously; in the second version, the roles are determined endogenously, i.e., the leader invests at the point at which it is indifferent between leading and following (at the point where the rents of the leader and the follower equalize[34].

Huisman's (2001) book supplies several real options models, for several different economic contexts, applied to new technology investments, namely, models for monopoly and duopoly markets, with constant and non-constant investment costs, with one, two or multiple new technology(ies) available with and without technological uncertainty[35]. The real option game formulation used is standard. Grenadier (2000a) provides a good summary of existing literature on

[34] That is, the leader invests according to the Fudenberg and Tirole's (1985) principle of rent equalization described in section 2.
[35] Huisman (2001) is an excellent introductory textbook for postgraduate students which aim to have a first contact with real options game models. It provides detailed and rigorous mathematical derivations and descriptions about the methodology used to derive real options investment models for a wide range of economic contexts.

game-theoretic option models. Grenadier (2000b) illustrates how intersection of real options and game theory provides powerful new insights into the behavior of economic agents under uncertainty, with examples from real estate development in an oligopoly and oil exploration investment decisions with symmetric information. Cottrel and Sick (2001), starting from their belief that fear of losing first-mover advantages causes managers to ignore real options analysis completely and simply go ahead with any project that they think has a positive net present value, they study first-mover advantage and find some "surprising results". They show that by considering the merits of a delayed-entry follower strategy the value enhancing managers will want to be suitably cautious before ignoring the real options analysis. The authors supplied and describe several illustrative practical examples to emphasize their results. Martzoukos and Zacharias (2001) developed a real options duopoly game model to study the optimization of R&D value enhancement in the presence of spillover effects. In their model, firms have the option to enhance value by doing R&D and/or acquiring more information about the project and due to information spillovers, they act strategically by optimizing their behavior conditional on the actions of their counterpart. The game set assumes that firms have incomplete information about the investment game.

Also for R&D investments, Weeds (2002) derived a real option game model to study an investment on a winner-takes-all patent system with irreversible investment cost and uncertain revenues. According to her framework, the technological success of the project is probabilistic and the economic value of the patent to be won evolves stochastically over time. She found that, comparing with the optimal cooperative investment pattern, investment is more delayed when firms act non-cooperatively as each holds back from investing in the fear of starting a patent race. In terms of game structure, the model is standard in all aspects except regarding the "winner takes all" assumption, since, as the information in table 1 in the appendix confirms, zero-sum real options games are much more frequent within the real options literature. Thijssen, et al. (2002) study pre-emption (i.e., first-mover advantage) games and war of attrition (i.e., second-mover advantage) games extending the strategy spaces and equilibrium concepts as introduced in Fudenberg and Tirole (1985). Note that Boyer, et al. (2001), the paper reviewed earlier in this section, had made similar attempt, but his adaptation is less suitable to modeling war of attrition games. The attempts to extend the firms' strategic space and equilibrium concepts was welcomed at the time because it tried to overcome one of the greatest weakness underlying real options models such as those of Grenadier (1996) and Weeds (2002) reviewed above, which, essentially, make the assumption that, at the preemption point, only one firm can succeed in investing. This assumption is unsatisfactory because firms are assumed to be ex-ante symmetric. Hence, if firms are ex-ante symmetric so there is no *a priori* ground for assuming that firms are not allowed to invest simultaneously even if it is optimal for both to do so.

Décamps et al. (2002) investigate the impact of incomplete information on firms' investment strategies. They study the optimal time to invest in an indivisible project whose value, while still perfectly observable, is driven by a parameter that is unknown to the decision maker ex-ante, i.e., there is a structural element of uncertainty besides the standard diffusion component of the value process. They argue that this captures in a simple way a variety of empirically relevant investment situations. For instance, a firm might ignore the exact growth characteristics of a market on which it contemplates investing, the owner of an asset who considers selling might ignore how the willingness to pay of potential buyers will evolve in the future, etc. By observing the evolution of the value, the decision maker can update his beliefs about the uncertain drift of the value process. However, this information is noisy, since it does not allow to distinguishing perfectly between the relative contributions of the drift and diffusion components to the instantaneous variations of the project's value. Consequently, they use filtering and martingale techniques, to show that the optimal investment region is characterized by a continuous and non-decreasing boundary in the value state space and that the decision maker always benefit from being uncertain about the drift of the value process, i.e., the decision maker prefers the option to invest in a project with unknown drift to that of investing in a project with constant drift equal to the prior expectation of the drift in the first option. According to this result one might expect the value of claims on structural uncertain assets (for instance in an emerging sector in which future growth prospects are uncertain) to be higher than that of claims on assets in more traditional sectors with otherwise identical risk characteristics. This is a real options game model for monopoly markets[36], i.e., an investment game with just one player playing against nature. However, its reach mathematical formation, originality of the economic context to which it applies to and insights for the development of real options for competition game settings justify its inclusion in this review.

Other paper assuming incomplete information, but considering competition, is that of Lambrecht and Perraudin (2003). They derived a full dynamic model of investment under uncertainty for first-mover advantage contexts. In addition, they assume that where firms have incomplete information about each other, i.e., firms observe their own investment cost, but knows only that the cost of its opponents is an independent draw from a distribution which has a continuous differentiable density with strictly positive support on an open interval. Their approach leads to a Bayesian Nash equilibrium where each firm invests strategically. The inclusion of incomplete information yields

[36] Hence, it is not included in Table 1 in the Appendix.

quite rich implications for the equity return distributions of companies holding real options subject to possible preemption, in particular, the model predicts that returns on such equities will contain jumps and that the volatility associated with that jumps will be negatively correlated across competing firms unlike more standard volatility attributable to news on the general prospects of the industry.

Grenadier (2000) provides a very general and tractable approach for deriving equilibrium investment strategies in a continuous-time Cournot-Nash equilibrium framework. According to his framework, each firm faces a sequence of investment opportunities and must determine an exercise strategy for its path of investment. The cash flows from investment are determined by a continuous-time stochastic shock process as well as the investment strategies of all firms in the industry. A symmetric Nash equilibrium in exercise policies is determined such that each firm's equilibrium exercise strategy is optimal, conditional on its competitors following their equilibrium exercise strategies. The resulting equilibrium is quite simple and shows that the impact of competition on exercise strategies is substantial. More specifically, he shows that competition drastically erodes the value of the option to wait and leads to investment at very near the zero net present value threshold. This work is in some way close to those of Williams (1993), Baldursson (1998) and Lambrecht and Perraudin (2003), however, these later works use a different solution approach and firms compete over a single investment opportunity, while, the former, describes an industry equilibrium with multiple active firms and does not use the simplified "myopic" solution approach, which is a key element, for instance, in Grenadier's (2002) model.

Baba (2001) derived a leader/follower real options model to optimize bank's entry decisions into duopolistic loan market in an attempt to shed light on the prolonged slump in the Japanese loan market in the 1990s. He gives special emphasis to the differences resulting from the alternative assumptions regarding whether the roles of leader and follower are interchangeable or pre-determined and shows that when the roles are pre-determined as in the case of the Japanese main bank system, both leader and follower banks have a greater incentive to wait until the loan demand condition improves sufficiently than when the roles are interchangeable. Murto and Keppo (2002) develop a game-theoretic model to study the competition for a single investment opportunity under uncertainty. This model combines real options and game theories for contexts where many firms compete for a single investment opportunity. The Nash-equilibrium of the game is characterized under the assumption that firms do not know each other's valuation for the project and so firms' strategies are defined as functions of all information they have on the stage of the game. They show that the information about each other's valuation for the project has an important effect on the equilibrium. More specifically, they conclude that, if there are at least two firms with the same valuation for the project, then the competition completely eliminates all profits; when one of the firms invests in the project, it is indifferent between investing and not investing and; if one of the firms has some advantage over the others (for instance, the investment cost is lower or the value of the project is higher for this firm than for the others), then, in equilibrium, that firm gets a positive payoff. Most of the assumptions underlying the derivation of this model are standard within the literature on real options games, except the “winner-takes-all” assumption and the existence of an “exit strategy”.

Nielson (2002), extend the oligopolistic industry result described in Dixit and Pindyck (1994) for investments with positive externalities and scenarios where the monopolist has multiple investment opportunities. His results show that, with decreased profit flow, a monopolist always makes its first investment later than the leader among two competitive firms would have done; that it makes no difference for the first investment whether the monopolist has access to one or two investment projects; and that a monopolist will make its second investment earlier than the follower if the profit loss, due to increased competition is larger than that due to increased supply. In terms of game characterization this investment game is standard, i.e., in line with that derived by Smets (1993) and then followed by Dixit and Pindick (1994) and Huisman (2001). Cottrel and Sick (2002), discuss the follower advantages, providing practical examples of successful delay in the context of a real option on innovation, such as the ability to learn more about a technology before irreversibly committing scarce resources, the advantage of observing market reaction to product design and features, and the avoidance of sunk investment in obsolete technology. Maeland (2002) combines real options theory with auction theory to develop a winners-takes-all investment model for markets with two or more players with asymmetric information about the cost of the investment, i.e., each investor has private information about its own costs but no private information about the competitors’.

Most of the real options game models reviewed above focus on the effect of competition on the value of the option to invest, but ignore the operating decisions that may arise once the investment is completed. Aguerrevere (2003), however, study strategic investment behavior in a real options framework that includes more realistic features of investment projects such as time to build, operating flexibility and capacity choice. Namely, he studied the effects of competitive interactions on investment decisions and the dynamics of the price of a storable commodity, in a model of incremental investment with time to build and operating flexibility. His work extends the classic real options models of irreversible investment and capacity choice (see for instance, Pindyck,1988, and He and Pindyck, 1992) by modeling together "time to build" and "competition". In his framework a firm must decide how much capacity to build initially and when to expand it later and has the option to not use any incremental unit of capacity if demand falls. His results show that with time to build, more uncertainty may encourage the firm to hold more capacity and that firms' optimal capacity may be larger under uncertainty than under certainty. This result contrasts with that from models of incremental investment which assume no "construction lags" and where it has been show that there is a negative effect of uncertainty on capacity choice. Other works close to this paper are those of Baldursson (1998) and Grenadier (2002). However, Baldursson (1998) assumes that investment is instantaneous and installed capacity is fully utilized, and the example analyzed indicates that qualitatively the price process will be the same in oligopoly and perfect competition, while Grenadier (2002) develops an approach to solving for investment equilibrium that is applicable to a more general specification of demand. Both models do not assume flexibility in the use of the installed capacity and their result show that the resulting output price behavior is the same for different numbers of firms in the industry.

In the traditional real options game framework, "ex post" losses are highly infrequent, i.e., since a monopolistic invests at a substantial premium, the likelihood for large asset value reversals is remote. Hence, it is very difficult for a standard real options framework to explain boom-and-bust markets such as real estate, where periodic bust of overbuilding result in waves of high vacancy and foreclosure rates. However, Grenadier (2002) suggests a model as a possible solution for this problem, and discusses this issue extensively. Bulan, et al. (2002) use 1,214 individual real estate projects built in Vancouver, Canada, between 1979 and 1998, and get empirical support for the argument that competition erodes option value[37].

Huisman and Kort (2003) examine a new technology adoption game for a duopoly market considering competition and the possible occurrence of better technologies in future. However, the mathematical formulation is deterministic[38], in line with that used by Reinganum (1981) and Fudenberg and Tirole (1985). More specifically, they considers a duopoly with two identical risk-neutral and value maximizing firms, where at $t = 0$ they can invest in a technology which is currently available, but, in future, at a known date, a new and more efficient technology will be available for adoption. Depending on the investment scenario, they arrive at several different game equilibrium strategies with both exogenous and endogenous firm role. Huisman and Kort (2004) addresses a new technology adoption game similar in many respects to that studied in Huisman and

[37] This paper is not included in Table 1 in the Appendix given that its contribution lies mainly in the empirical data used and results found and not in any particular game aspect underlying a real option model.
[38] Hence it is not included in Table 1 in the Appendix.

Kort (2003) paper, except that in this case they use a stochastic-real options approach, instead of a deterministic approach, and the arrival time of the second new technology is uncertain, instead of arriving a known future date. In addition, firms' role is set exogenously and the arrival of the second and more efficient technology is assumed to follow a Poisson process with parameter λ. The parameter λ, which represents the likelihood that a second new technology arrives in the market in the next instant, is a key element in this model, in the sense that it leads to different game equilibrium strategies.

Tsekrekos (2003), studies the effect of first-mover advantage on the strategic exercise of the options. Paxson and Pinto (2003) derived, for a duopoly market, firms' real value functions assuming that the leader's market share evolves according to an immigration (birth) and death process. In terms of game formulation both of these real options models are standard. Murto (2004) examines a declining duopoly market where firms must choose when to exit from the market, considering a Markov-perfect equilibrium. He finds that with low degree of uncertainty there is a unique equilibrium, where one of the firms always exits before the other, and, when uncertainty is increased, another equilibrium with the reverse order of exit may appear ruining the uniqueness. The occurrence of this event depends on the degree of asymmetry in the firm specific parameters. Murto, et al. (2004) present a modeling framework for the analysis of investments in an oligopoly market for homogeneous non-storable commodity, where the demand evolves stochastically and the firms carry out investment projects in order to adjust their production cost functions or production capacities. They use a discrete-time state-space game and assume that there are several large firms which move sequentially so that to ensure a Markov-perfect Nash equilibrium. Once the equilibrium has been solved they use Monte Carlo simulation to form probability distributions for the firms' cash flow patterns and accomplished investments, information which can be used to value firms operating in an oligopoly market. An important innovation in this paper is that it studies the timing of lumpy investment projects under uncertainty and oligopolistic competition, while "standard" real options game models, with the exceptions of Baldursson (1998) and Williams (1993), study investment games played on a single project, therefore, neglecting the full dynamics of the industry.

Décamps and Mariotti (2004), develop a duopoly model of investment in which each player learns about the quality of a common value project by observing some public background information, and possibly the experience of his rival. Investment costs are assumed to be private information and the background signal takes the form of a Poisson process conditional on the quality of the project being low. Their results show that the resulting "war of attrition" game has a unique symmetric equilibrium which depends on initial public beliefs. They determine the impact of changes in the cost and signal distributions on investment timing, and how equilibrium is affected when a first-mover advantage is introduced. Firms have incomplete but symmetric information about the value of the investment project, but asymmetric information about their investment costs. In addition, firms' payoffs incorporate both a common and a private value component. The return of the project is assumed to be the same for both players, and independent of whom invests first, and their opportunity cost of investment may differ. They assume that there are two sources of public information. A background signal provides free information about the value of the project, independently of firms' investment decisions and once a firm, acting as a leader, has sunk his investment, an additional signal is generated that may be used by the follower to optimally adjust his investment decision. Their aim is to study the learning externality due to the increase in the signal's quality generated by the leader's investment. By delaying investment, each firm tries to convince his rival that his own cost is high and thus that his rival should invest first. The difference with a "pure" common real option game model is that each player does not care about the information of his rival *per se*, but only in so far as it measures the likelihood of investing second and hence of benefitting from a better signal. As usual in the real options literature, the equilibria of the game is given in trigger strategies but, contrary to standard real options games, it is Markovian with respect to the common belief process, in the sense that the equilibrium trigger of each firm depends, besides his investment cost, on the initial belief about the value of the project. The more optimistic firms are ex-ante about the quality of the project, the higher their equilibrium trigger will be. As a benchmark the authors consider a complete information version of the investment game model in which firms know each others' investment cost.

Shackleton, et al. (2004) analyse for a duopoly market the entry decision of the competing firms when rivals earn different but correlated uncertain profitabilities from operating. Regarding the rest of the game structure this is a "standard" real options model, except that it allows each firm's decision to be subject to a firm-specific stochastic variable, as well as its competitor's. Their results show that, in the presence of entry costs, decision thresholds exhibit hysteresis and that the range of which is decreasing in the correlation between firms. They determine an explicit measure for the expected time of each firm being active in the market and the probability of both rivals entering within a finite time. An illustration of an application of their results is supplied using the well known rivalry case between Boeing and Airbus, namely, the Airbus's lunch of the A380 super carrier and Boeing's optimal response to that Airbus' strategic move.

Ziegler (2004) uses game theory to address several problems in finance, arguing that the payoff values of the economic agents can be obtained by using option pricing and that by inserting those payoffs into the strategic games between the agents it is possible to analyse, more realistically, the value of strategic decisions[39]. Smit and Trigeorgis (2004, chapter 7) use an integrated real options and game-theoretic framework for strategic R&D investments to analyze two-stage games where the growth option value of R&D depends on endogenous competitive reactions. In this model firms choose output levels endogenously and may have different (asymmetric) production costs as a result of R&D, investment timing differences or learning. Savva and Scholtes (2005) examine partnerships bilateral deals under uncertainty but with downstream flexibility. Their analysis is focused on the effect of options on the synergy underlying the deal, distinguishing between cooperative options, which are exercised jointly and in the interest of maximizing the total deal value, and non-cooperative options, which are exercised unilaterally in the interest of one partner's payoff. In this partnership game model, although not explicitly stated, it is assumed that firms are ex-ante asymmetric and share incomplete and imperfect information about the true intentions of each other regarding the deal.

Paxson and Pinto (2005), derive a real options model for a duopoly market using two stochastic underlying variables and show that the degree of correlation between the two variables results in different value functions and investment thresholds, especially for the follower and the case where firms invest simultaneously in a non pre-emption game. Mason and Weeds (2005) show that, in a duopoly market, greater uncertainty can actually hasten rather than delay investment. More specifically, they illustrate that in the presence of positive externalities greater uncertainty can raise the leader's value more than the follower's and so the leader must act soon, but that a switch in the pattern of equilibrium investment as uncertainty increases is also possible, which may hasten investment.

Bouis, et al. (2005) did attempt to derive a real option game model for markets with three firms. Weyant and Yao (2005) developed a real options game model for investments in R&D projects in contexts where market and technical uncertainty and competition hold. Since one characteristic of R&D projects is the fact that firms make investment decisions on an ongoing basis before the success of the project, they explore the assumption that these repeated strategic interactions may facilitate self-enforcement tacit collusion. Hence, they study the possibility of defining a collusion

[39] Ziegler's (2004) and Grenadier's (2000a) books were, to our knowledge, the first two books which tried to combine game theory concepts with finance theory and financial models. The former, discuss game theory applications to a wider range of topics in finance, the latter focus on the interception of real options and game theory.

(cooperative) equilibrium based on the use of a trigger strategy with information time lag and conclude that when time lag is long, a pre-emptive (non-cooperative) equilibrium emerges in which the option values of delay are reduced by competition and, when the information time lag is sufficiently short, a collusion (cooperative) equilibrium emerges in which investment is delayed more than the single-firm counterpart. Pawlina and Kort (2006) focus their attention on the study of the impact of investment cost asymmetry on the optimal real option exercise strategies and the value of firms in duopoly. Sources of potential investment cost asymmetries are, for instance, due to different liquidity constrains or organizational flexibility at implementing a new production technology, different real options embedded in the existing assets of the firm and different operating costs or other exogenous factors such as government regulations.

Wu (2006) explores the problem of firms' incentives to expand capacity using a continuous-time real options game model, where two *ex-ante* identical firms can choose capacity and investment timing regarding the entry into a new industry, whose demand grows until an unknown maturity date and declines thereafter until it disappears. The innovativeness of this paper is that firms are allowed to entry and exit when it is optimal to do so. Carlson, et al. (2006) identifies relationships between industry and individual firm risk that reflect the strategic interplay of option exercise by imperfect competition firms. The investment game set is standard in all its parameters except regarding the investment costs and salvage values, which are assumed to be asymmetric. They found counter-intuitive results. Their model adds an additional factor they name "industry factor". Many existing studies identify operating leverage and irreversibility as the two main channels that drive the risk dynamics in a market. In this paper, however, the results show that, the industry factor behaves differently, depending on whether the industry grows or shrinks.

Kong and Kwort (2007), examines strategic investment preemptive games for a duopoly market with uncertain revenues and asymmetric firms in terms of investment costs and revenue flows. Odening, et al. (2007), study investment decisions for markets where perfect competition holds. They assume that firms are risk neutral, price takers and produce with the same "constant returns to scale" technology at a constant variable cost per unit, investments are irreversible and infinitely divisible with capital stock subjected to depreciation at a given rate, and the demand shock follows a gBm diffusion process. Using a simulation experiment, they demonstrate that myopic planning may lead to non-optimal investment strategies. They quantify the degree of sub-optimality and propose measures to reduce the error. Their numerical results of real options models for non-exclusive investments support the argument that empirical applications tend to overestimate the reluctance to invest and show that "standard gBm estimators" of stochastic price processes are inconsistent with real world data stemming from a regulated gBm. Azevedo and Paxson (2009) developed a real options model for a duopoly market to optimize investment decisions on new technologies whose functions are complementary. They arrived at analytical and quasi-analytical solutions for the leader and the follower value functions and their respective investment thresholds. According to their model, at the beginning of the investment game firms have two technologies available, whose functions are complement, and the option to adopt both technologies at the same time or at different times, in a context where the evolution of the gains that can be made through the adoption of the technology(ies) and the cost of the technologies are uncertain. Their results contradict the conventional wisdom which says that "when a production process requires two extremely complementary inputs, a firm should upgrade (or replace) them simultaneously". They found that when uncertainty about revenues and the price of the two technologies is considered it might be optimal for the leader and the follower to adopt the two technologies asynchronously, first, the technology whose price is decreasing at a lower rate and then the technology whose price is decreasing more rapidly.

5. Summary and Conclusions

The 2007 financial crisis and subsequent economic shocks all over the world increased significantly the levels of uncertainty under which governments and private organizations do have to make their investment decisions. In addition, the globalization and subsequent emergence of some developing countries as important economic players in the world's economy has intensified competition among countries and multinational companies. Real options game models are, therefore, well placed to more adequately fit current and future investors' needs regarding new investment models for new and highly dynamic investment contexts. We expect that new and more sophisticated real options game models will arrive in the coming years given the high number of working papers we highlighted in this research and the high interest the real options topic still stir up among established and new academic researchers. Those models will fit necessarily a wider range of competition settings and economic contexts and making, possibly, better use of the powerful game theory mathematical tools.

For some industries, we expect that empirical data about the intensity of competition among market players and other important variables underlying real options models will be available, leading to an increase on empirical research and benchmark studies comparing the reliability of different type of real options models and the reliability of the real options methodology with that of other investment appraisal techniques. As highlighted in the research, there are very few empirical works within the currently available literature on real options games. Very few progresses have been made in this regard over the last two decades. We expect to see significant developments in this area in the coming years. Due to the high degree of innovation and intensity of competition, technological industries, and more specifically, investments in R&D projects, new technologies adoptions and the optimizations of market entrance of disruptive innovations will continue to be natural areas of application of real options game models.

The real options game models reviewed above address modern questions in investment analysis and provide new solutions to investment problems, contributing, therefore, to a better understanding of the complex nature of firms' investment behavior in markets where uncertainty and competition hold. We gave particular emphasis to the game theory aspects underlying each investment model reviewed, for four main reasons: first, because, nowadays few monopolistic sectors remain and so competition became one of the most important aspect driving firm's investment behavior; second, because the stochastic formulation of real options game models is, in its essence, similar to that used in real options models developed for monopolistic markets, hence it has been extensively discussed over the last 30 years; third, because despite all progresses made in real options game models over the last two decades, there is an implicit agreement among researchers that the investment models available, although more realistic in terms of assumptions when compared to those derived for monopolistic settings, are still too much deterministic and, to some extent, unsophisticated in the way the "competition factor" is incorporated in the model; fourth, because there is the intuition, or common believe among researchers, that it is possible to improve current real options game models by bringing together real options and game theory.

This literature review condenses the results of almost 2 decades of research in real options game models. It highlights new avenues for new research and summarizes the developments patterns followed, the problems we solved, the questions we answered and the questions we did not answer or answered inadequately. This is, therefore, an essential tool, especially, for new researchers aiming to have a first contact with real options games or for established researchers looking for new research directions within real options games.

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Appendix 1

Brief comments on the meaning of some of the information used in the organization of Table 2:

a) Complete/Incomplete information game: An investment game with complete information means that knowledge about other firms or players is available to all participants, i.e., every player knows its own payoffs (or payoff functions) and strategies available and the payoffs and strategies available to the other players.

b) Perfect/Imperfect information game: Complete and perfect information are not identical. Complete information refers to a state of knowledge about the structure of the game and objective functions of the players, while not necessarily having knowledge of actions. The distinction between incomplete and imperfect information is somewhat semantic. For instance, in R&D investment games, firms may have "incomplete" information about the quality or success of each other's research effort and "imperfect" information about how much their rivals have invested in R&D. For instance, in the classical example of the Prisoners' Dilemma game, the prisoners have complete information about the payoffs and strategies available of the other player but not about the action of the other player.

c) Symmetric/Asymmetric information game: Symmetric information means that all players participating in a game share the same information about the game, i.e., there are no players with more or less, better or worst information than other players.

d) Ex-ante Symmetric/Asymmetric game: For the context of this literature review, when firms are ex-ante symmetric this means that, before the game starts, they are symmetric about all parameters of the investment game, such as their competence to carry out the investment project, the access to funds to finance the project, the access to technologies and all the real option game model parameters, i.e., investment cost, uncertainty about the futures revenues, revenues drift, etc..

e) One-shot/Large game: In the context of this literature review, for simplicity, we define a "large" real options game as a game where players have two or more options to invest and a "one-shot" real option game as a game which ends as soon as the option to "invest" is exercised. In game theory, however, a larger game can be a game with many players, with one or several strategies available each; a game with one player with a larger number of strategies available; or a game with one player with one strategy available but which can be exercised a large number of times. Note however that, as highlighted in the review, as in a real options game played in a continuous-time framework in between two instants where the investor does not "invest it, indeed, exercises the option to "defer" the investment, theoretically, an infinite number of times. Hence, this should type of game could also be classified as a "large" game as well. However, for the sack of the simplicity of the organization of the information in table 2 in the appendix and the clarity of our analysis, and without losing any insight, we assume real options games where investors have only one option to invest are "one-shot" games.

f) Zero-sum/winner-takes-all game: A zero-sum game is a game where the player(s)' gain/loss is exactly balanced by the loss/gain of the other participant(s) in the game. In a winner-takes-all game, there is no payoff for the loser(s). Translating this to a first-mover advantage leader/follower investment game, it means that there is no payoff (revenues) for the firm which invests second.

g) **Monopoly game against nature**: Investment decisions in monopolistic settings can also be modeled as a game of one player against nature. These games are not, however, the main object of this literature review and they are only mentioned when the insights we can take from them are good illustration for describing investment game concepts for real options game models contexts or some particularity of those models can be used as benchmark for ROG models with at least two players. This is for instance the case of the Décamp, et al.'s (2002) paper which we mention in the review (not in table 1 in the appendix) because we believe its framework is a good illustration modeling investment games for contexts where players have incomplete information.

h) **Sequential/Simultaneous game**: A sequential game is a game where one or several players move (invests) first initiating a necessarily sequential game. A simultaneous game is a game where at least two players invest at the same time. Most of real options game models are derived for duopoly markets, i.e., they use two player.

i) **Cooperative/Non-cooperative game**: In non-cooperative games it is assumed that players cannot make a binding agreement, i.e., each cooperative outcome must be sustained by Nash equilibrium strategies. In cooperative games, firms have no choice but to cooperate.

j) **Endogenous/Exogenous leadership game**: In standard real options models sequential moves is allowed and the leadership in the investment is usually set exogenously. One advantages of the duopoly models, compared to those which cover a wider range of economic contexts, namely oligopolies and perfect competition, is that they allow us to model sequential investments games more easily and to determine explicit investment thresholds for each player. In other branches of literature we can find more general frameworks, like that of Leahy (1993), however, they have the disadvantage of determining the optimal investment behavior of all market players but without specifying what they should do in case one, or several, players move first initiating a necessarily sequential investment game. Essentially, these models advise firms about the adjustments they should perform over time assuming that all of them will react to market shocks, necessarily, at the same time. Both of these frameworks do carry some practical inconsistencies, but these are the two most popular ways used in the literature of real options games to deal with the competition factor.

Appendix 2

In Table 1 below we summarize the “non-standard” real options game models reviewed in this paper.

“Non-Standard” Real Option Game Models (Game Formulation)			
Game Information		**Type of Game**	
Incomplete	**Imperfect**	**Winner-Takes-All**	**Large Game**
1. Décamps and Mariotti (2004) 2. Hsu and Lambrecht (2003) 3. Lambrecht and Perraudin (2003) 4. Maeland (2002) 5. Martzoukos and Zacharias (2001) 6. Murto and Keppo (2002) 7. Savva and Scholtes (2005)	1. Hsu and Lambrecht (2003) 2. Maeland (2002) 3. Martzoukos and Zacharias (2001) 4. Savva and Scholtes (2005)	1. Maeland (2002) 2. Murto and Keppo (2002) 3. Weeds (2002)	1. Garlappi (2001) 2. Grenadier (2000a) 3. Martzoukos and Zacharias (2001) 4. Murto (2004) 5. Ruiz-Aliseda (2004) 6. Weyant and Yao (2005) 7. Wu (2006)
Type of Game		**N-Firms**	**Leadership**
Cooperative	**Firms: Ex-ante Asymmetric**	**N > 2**	**Endogenous**
1. Mason and Weeds (2001) 2. Savva and Scholtes (2005) 3. Thijssen (2004) 4. Weeds (2002)	1. Baba (2001) 2. Décamp and Mariotti (2004) 3. Grenadier (2000a) 4. Huisman (2001) 5. Hsu and Lambrecht (2003) 6. Kulatilaka and Perotti (1998) 7. Maeland (2002) 8. Mason and Weeds (2005) 9. Pawlina and Kort (2006) 10. Reiss (1998) 11. Ruiz-Aliseda (2004) 12. Shackleton, et al (2004) 13. Sparla (2004)	1. Aguerrevere (2003) 2. Bouis, et al. (2005) 3. Dixit and Pindyck (1994), ch. 8, 9 4. Grenadier (2000a) 5. Grenadier (2002) 6. Kulatilaka and Perotti (1998) 7. Lambrecht and Perraudin (2003) 8. Maeland (2002) 9. Murto and Keppo (2002) 10. Murto, et al. (2004) 11. Nielson (2002) 12. Odening, et al. (2007) 13. Reiss (1998) 14. Thijssen (2004) 15. Williams (1993)	1. Baba (2001) 2. Boyer, et al. (2001) 3. Grenadier (2000a) 4. Grenadier (2002) 5. Kuatilaka and Perotti (1998) 6. Mason and Weeds (2001) 7. Martzoukos and Zacharias (2001) 8. Mason and Weeds (2005) 9. Murto and Keppo (2002) 10. Murto, et al. (2004) 11. Odening, et al. (2007) 12. Shackleton, et al. (2004) 13. Sparla (2004) 14. Thijssen, et al. (2002) 15. Thijssen (2004) 16. Weyant and Yao (2005) 17. Williams (1993)

Table 1 – “Non-standard” Real Option Game Models

Game Theory Aspects underlying the most Relevant Literature on Real Options Games

	Table 2	**Formalism**		**Game Information**						**Type of Game**													**Firms**			**Leadership**		**Application**
	(Papers)	Discrete time	Continuous time	Complete	Incomplete	Perfect	Imperfect	Symmetric	Asymmetric	Simultaneous	Sequential	One -shot	Large	Winner Takes All	Zero-sum	Nonzero sum	**Time Horizon**		Cooperative	Non cooperative	Ex-ante Symmetric	Ex-ante Asymmetric	1	2	>2	Exogenous	Endogenous	**(Suggested/used in the article)**
																	Finite	Infinite										
1	Aguerrevere (2003)		x	x		x		x		x	x	x			x			x		x	x			x	x	x		Non-storable Commodities Projects
2	Azevedo and Paxson (2009)		x	x		x		x			x	x			x			x		x	x			x		x		New Technology Adoptions
3	Baba (2001)		x	x		x		x			x	x			x			x		x		x		x		x	x	Private Banking Lending Decisions
6	Bouis, et al. (2005)		x	x		x		x		x	x	x			x			x		x	x				x	x		Standard Investment Project
7	Boyer, et al. (2001)		x			x		x			x	x			x			x		x	x			x			x	Production Capacity Investment
8	Carlson, M. (2006)		x	x		x		x			x	x			x			x		x		x		x				Study of Industry Equilibrium
9	Cottrell and Sick (2001)		x	x		x		x			x	x			x			x		x	x			x		x		Patents and R&D Investments
10	Cottrell and Sick (2002)		x	x		x		x			x	x			x			x		x	x			x		x		Innovation: Second-mover Advantages
11	Décamp and Mariotti (2004)		x		x	x		x	x		x	x			x			x		x		x		x		x		Pre-emption and war of Attrition Games, Incomplete Information
12	Dixit and Pindyck (1994), Ch. 8, 9		x	x		x		x			x	x			x			x		x	x			x	x	x		**Textbook** Duopoly investment games
13	Garlappi (2001)	x		x		x		x			x		x		x			x		x				x		x		Patent Race, R&D ventures
14	Grenadier (1996)		x	x		x		x		x	x	x			x			x		x	x			x		x		Real Estate Market
15	Grenadier (2000a)		X	x		x		x		x	x	x	x		x			x		x	x	x		x	x	x	x	**Textbook**: Real Options Game Models
16	Grenadier (2000b)		x	x		x		x			x				x			x		x	x			x		x		Illustration of Real Options Investment Games Models
17	Grenadier (2002)		X	x		x		x		x	x	x			x			x		x	x				x		x	Real Estate Market
18	Huisman (2001)		x	x		x		x		x	x	x			x			x		x	x	x		x		x		**Textbook**: New Technology Adoptions

	Table 2 (cont.)	Formalism		Game Information						Type of Game													Firms			Leadership		Application
	(Papers)	Discrete time	Continuous Time	Complete	Incomplete	Perfect	Imperfect	Symmetric	Asymmetric	Simultaneous	Sequential	One -shot	Large	Winner Takes All	Zero-sum	Nonzero sum	Time Horizon: Finite	Time Horizon: Infinite	Cooperative	Non cooperative	Ex-ante Symmetric	Ex-ante Asymmetric	1	2	>2	Exosgenous	Endogenous	(Suggested/used in the article)
19	Huisman and Kort (2004)		x	x		x		x			x	x			x			x		x	x			x		x		New Technology Adoptions
20	Hsu and Lambrecht (2003)		x		x		x		x		x	x			x			x		x		x		x		x		Patent Race
21	Kong and Kwork (2007)		x	x		x		x			x	x			x			x		x	x			x		x		Standard Investment Project
22	Kulatilaka and Perotti (1998)	x		x		x		x		x	x	x			x			x		x	x	x			x		x	Strategic Growth Options
23	Joaquim and Butler (1999)		x	x		x		x			x	x			x			x		x	x			x		x		Standard Investment Project
24	Lambrecht, B. (1999)		x	x		x		x			x	x			x			x		x	x			x	x		x	Patent Race
25	Lambrecht and Perraudin (2003)		x		x	x		x	x		x	x			x			x		x	x			x	x	x		Investments Project with Competition and Incomplete Information about the Investment Costs.
26	Maeland (2002)		x		x		X		x		x			x				x		x		x			x	x		Standard Investment Project
27	Martzoukos & Zacharias (2001)		x		x		x	x		x	x		x		x			x		x	x			x			x	R&D Investment
28	Mason and Weeds (2001)		x	x		x		x			x	x			x			x	x	x	x			x			x	Standard Investment Project
29	Mason and Weeds (2005)		x	x		x		x		x	x	x			x			x		x	x	x		x			x	Standard Investment Project
30	Murto and Keppo (2002)		x	x	x	x		x	x	x		x		x				x		x	x				x		x	Declining Duopoly Market. Telecommunication Network
31	Murto (2004)		x	x		x		x		x	x	x	x			x		x		x	x			x		x		Exit strategy: Duopoly Market
32	Murto, et al. (2004)	x		x		x		x		x		x			x			x		x	x				x		x	Lumpy Investment: Non-storable Commodity
33	Nielson (2002)		x	x		x		x		x	x	x			x			x		x				x	x	x		Positive Externalities Software/hardware
34	Odening, et al. (2007)		x	x		x		x		x	x	x			x			x		x	x				x		x	Agriculture

	Table 2 (cont.)	Formalism		Game Information						Type of Game							Time Horizon						Firms			Leadership		Application
	(Papers)	Discrete time	Continuous Time	Complete	Incomplete	Perfect	Imperfect	Symmetric	Asymmetric	Simultaneous	Sequential	One -shot	Large	Winner Takes All	Zero-sum	Nonzero sum	Finite	Infinite	Cooperative	Non cooperative	Ex-ante Symmetric	Ex-ante Asymmetric	1	2	>2	Exosgenous	Endogenous	(Suggested/used in the article)
35	Pawlina and Kort (2006)		x	x		x		x		x	x	x			x			x		x		x		x		x		Standard Investment Project
36	Paxson and Pinto (2005)		x	x		x		x		x	x	x			x			x		x	x			x		x		Standard Investment Project
37	Paxson and Pinto (2003)		x	x		x		x		x	x	x			x			x		x	x			x		x		Telecommunication Sector
38	Reiss (1998)		x	x		x			x	x	x	x			x			x		x		x			x	x		Patent Race
39	Ruiz-Aliseda (2004)		x	x		x		x			x		x		x			x		x		x		x		x		Declining Markets
40	Savva and Scholtes (2005)	x		x	x		x		x	x	x	x			x	x		x	X	x	x			x		x		Partnership Deals Biotech and Pharmaceutical Industries
41	Shackleton, et al. (2004)		x	x		x		x			x	x			x			x		x	x	x		x			x	Aircraft Industry (Boeing/Airbus)
42	Smets (1993)		x	x		x		x			x	x			x			x		x	x			x		x		Foreign Direct Investment
43	Smith and Ankum (1993)	x		x		x		x		x	x	x			x			x		x	x			x		x		R&D Investment
44	Smit (2003)	x		x		x		x		x	x	x			x			x		x	x			x		x		Public Infrastructure (Airport)
45	Smit and Trigeorgies (2004)	x	x	x	x	x	x	x	x		x	x			x			x		x	x			x		x	x	**Textbook**: Strategic Investment models
46	Sparla (2004)	x		x		x		x		x	x	x			x			x		x	x	x		x			x	Duopoly Strong Strategic Externalities
47	Thijssen (2004)		x	x		x		x	x	x	x	x			x			x	x	x	x			x	x		x	New Products/Markets
48	Thijssen, et al. (2002)		x	x		x		x		x	x	x			x			x		x	x			x		x	x	Standard Investment Project
49	Tzekrekos (2003)		x	x		x		x		x	x	x			x			x		x	x			x		x		Leader/Follower Asymmetric Game after Investment
50	Weeds (2002)		x	x		x		x			x	x		x				x	x	x	x			x		x		Patent Race

	Table 2 (cont.) (Papers)	Formalism		Game Information						Type of Game													Firms			Leadership		Application (Suggested/used in the article)
		Discrete time	Continuous Time	Complete	Incomplete	Perfect	Imperfect	Symmetric	Asymmetric	Simultaneous	Sequential	One -shot	Large	Winner Takes All	Zero-sum	Nonzero sum	Time Horizon: Finite	Time Horizon: Infinite	Cooperative	Non cooperative	Ex-ante Symmetric	Ex-ante Asymmetric	1	2	>2	Exosgenous	Endogenous	
51	Weyant and Yao (2005)		x	x		x		x		x	x		x		x			x		x	x			x			x	R&D Market & Technical Uncertainty
52	Williams (1993)		x	x		x		x		x		x			x			x		x	x				x		x	Real Estate Development
53	Wu (2006)		x	x		x		x		x	x		x		x			x		x	x			x		x		Production Capacity Expansion