Reconciling Real Option Models: An Approach to Incorporate Market and Private Uncertainties

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Abstract

Several real options analysis techniques designed for practitioners exist in literature, but there is discrepancy in their underlying assumptions, mechanics and applicability. Within this paper, a review of approaches targeted towards practitioners is included, and a novel way of integrating market and private uncertainties is proposed. Market risk is incorporated into the dynamics of the project cash flow by assuming the success of the project is correlated to a traded index. The value of the real option on the project cash flows may be priced by traditional numerical methods or a simulation approach similar to the previously recommended Datar-Mathews method (Datar et al., 2007). A numerical example presents the proposed model within the simulation framework.

Keywords: Real options for practitioners, valuation under uncertainty, DM Method, correlated processes

Introduction

Traditional project valuation proceeds by a discounted cash flows analysis, namely a net present value (NPV) calculation, with the objective to maximize NPV. Unfortunately this measure assumes that cash flows follow a rigid and inflexible path, and is incapable of accommodating managements' responses to endogenous or exogenous uncertainties. Erroneous valuations occur; short term and less risky projects are favored due to an artificially high corporate discount rate. Consequently, literature recommends real options analysis (ROA) to value real-world investments where managerial flexibility can influence worth, especially when decisions need to be made among mutually exclusive opportunities.

The ROA technique builds on the seminal work of Black and Scholes (1973) in the area of financial option valuation. Myers (1977) recognized the analogy between financial options and project decisions; both are exercised after uncertainties are resolved. Since this connection, ROA has been popularized by business publications and valuation texts, and in the last decade, has transitioned from academic circles to heightened industry attention (Borison, 2005). Despite the theoretical appeal of ROA, a recent survey of the *Fortune 1000* largest companies found that only 14.3% of respondents use ROA (Block, 2007) in practice. Arguably, management realizes the added benefit of ROA, but the adoption of the technique within industry is limited due to the complexity of analytical solutions, the restrictive assumptions required, and the overall lack of intuition in the solution procedure.

To strengthen its appeal and gain its acceptance amongst practitioners, Copeland and Antikarov (2005) outline criteria for real options approaches. Seven requirements are discussed. An acceptable model should: intuitively dominate other valuation methods, capture the reality of the real world situation, eliminate arbitrage opportunities, properly incorporate risk, use market data, and be empirically testable while being mathematically transparent and computationally efficient. Ultimately, every real-world problem faces diverse uncertainties, and an appropriate ROA approach must have the ability to accommodate each unique situation.

The motivation for this research is to analyze ROA approaches designed for practitioners, and to reconcile valuations due to varied assumptions regarding uncertainty characteristics. Within the first section of this paper a review of literature building on the work of Borison (2005) summarizes ROA methods. In the second section, a new model is proposed, and the solution procedure is outlined. This model is integrated within the approach suggested by Datar, Matthews, and Johnson (2007), and demonstrated by a numerical example. A discussion and concluding remarks follow.

Review of Real Option Analysis Approaches

Varying ROA approaches designed for practitioners exist, but they differ in their assumptions regarding the nature of capital markets, the nature of uncertainties, and the source of data. Borison (2005) characterizes five approaches: the classical, the subjective, the Market Asset Disclaimer (MAD), the revised classical, and the integrated, and highlights their associated assumptions, applicability, and solution mechanics. These strategies are summarized and a review of ROA techniques extending these models is included.

Classical Approach

The classical real options approach assumes that real-world assets with cash flows that are highly dependent on market prices may be valued by replicating self-financing portfolios (Brennan & Schwartz, 1985). Capital markets are assumed to be complete and the underlying value of the project is represented by the market value of the equivalent portfolio. The project growth and project volatility matches the portfolio growth and portfolio volatility. For a specified maturity and strike price, the Black-Scholes equation determines the real option value of the underlying portfolio. The pitfalls of this scheme are the failure to incorporate risks not correlated to the market, idiosyncratic or private risk, and the inability to value compound options.

Due to the complexity of real assets, Copeland and Antikarov (2005) recognize the difficulty in finding a highly correlated traded portfolio. To better align the present value of a project with a traded portfolio, the authors recommend using an entity value versus a traded equity value before applying the Black-Scholes formula to price the real option. They argue that the entity value more closely resembles the total value of the firm, and is a better representation of the underlying asset.

Subjective Approach

As an alterative to using either the equity or entity value, the subjective ROA approach estimates the underlying project value and volatility variables from available indices or industry standards before using the Black-Scholes equation to determine option value (Luehrman, 1998a and 1998b). Mechanically the subjective approach is simple, but as Borison (2005) explains, there are inconsistencies in the use of subjective information with replicating portfolio techniques and no arbitrage assumptions.

Revised Classical Approach

Exogenous market factors such as market dynamics, regulatory or political uncertainty, and competitive forces drive uncertainty of a traded process, while endogenous risks, or private risks, including organizational capabilities and available resources are inherent to the firm and the project itself. As such, Dixit and Pindyck (1994) recognize the existence of two types of real projects; an investment dominated by exogenous market forces and one dominated by private risks.

Dixit and Pindyck (1994) suggest the aforementioned classical approach if the real-world investment is governed by market risks. Alternately, if the investment is primarily driven by private risks a decision tree analysis (DTA) is recommended. Using subjective probabilities of possible project outcomes an event tree represents changing project value through time. At terminal nodes, cash flow statements are constructed and the NPV calculated. A "roll back" procedure determines an initial expected NPV by use of the subjective probabilities.

Management has difficulty executing this method; projects are assumed to exhibit one of the two extreme market natures, but real-world applications typically encompass both types of uncertainties (Schneider et al., 2008). Difficulty also arises in DTA in the selection of an adequate discount rate and assignment of subjective probabilities (Borison, 2005).

Integrated Approach

Contrasting this dichotomous view of risk, the integrated ROA approach accommodates both private and market uncertainties by assuming markets are partially complete. In the integrated framework, Smith and Nau (1995) implement alternate binomial lattices with assigned risk-neutral probabilities and private event tree branches with subjective outcomes to manage both sources of risk.

The optimal investment strategy is again found by a dynamic programming procedure. A cash flow model is applied at terminal nodes. The expected value of the project from the private lattices replace the states of the previous binomial tree, and risk neutral probabilities compute the expect market value. All are discounted at the risk-free rate to find the option value.

Market Asset Disclaimer Approach

Unlike the classical ROA view, Copeland and Antikarov (2001) argue that it is pointless to search the economy for an adequate portfolio of twin securities to characterize real returns. Due to the incomplete nature of the economy, they assume that the best estimate of the market value of the project is the present value of the project itself, without flexibility. This is the Market Asset Disclaimer (MAD) assumption.

Copeland and Antikarov (2001) use Samuelson's proof, properly anticipated prices fluctuate randomly, to justify their assumption that the value of the project follows a geometric Brownian motion (GBM) process. The set of uncertainties affecting the project cash flows are assumed to be equivalent to those affecting the project through time; this justifies their consolidation of risks. A Monte Carlo simulation of the annual cash flows creates a distribution of possible project returns. The standard deviation of this distribution is the volatility of the project, and as the volatility of the GBM process remains constant over time this parameter is used to construct a recombining lattice representing the project value evolution. Option decision rules are applied at appropriate nodes of the lattice, and the current value of the project is found from risk neutral probabilities, discounting at the risk free rate.

As the MAD approach uses an inconsistent mixture of risk-free and risk-adjusted discount rates, Smith (2005) suggests a fully risk-neutral MAD approach. He proposes to risk adjust the underlying stochastic process containing uncertainty, namely the project cash flows, and find the NPV by risk-free probabilities. The risk-free NPV becomes the initial node in the value lattice.

Extensions of the Integrated and MAD Approaches

Brandao, Dyer, and Hahn (2005) propose an approach similar to the MAD method, but instead suggest a binomial decision tree versus a recombining binomial lattice to approximate the cash flow evolution. The authors argue that implementing options within the decision tree framework is more intuitive for practitioners and easily incorporate within current decision tree software. Scenarios with multiple uncertainties, complex options, and projects with heteroskedasticity are easily modeled. However, for larger problems with n periods the binomial decision tree has $2^{n+1} - 1$ nodes versus a total of $(n+1)(n+2)/2$ nodes for a recombining binomial lattice (Brandao et al., 2005); this leads to a solution that is more difficult to visualize and requires increased computations. Yet, Smith (2005) comments on this approach and shows that either tree or lattice yields similar numerical results if the implementation is correct within Excel. This negates the requirement for additional computational programs.

Schneider et al. (2008) extends the integrated and MAD approach by proposing an integrated multidimensional market and private lattice. Congruent with Copeland and Antikarov (2001), the underlying asset is the expected value of the project without flexibility. At nodes where the option to defer or switch production is introduced a new lattice layer is created; this allows for a management to model simultaneous paths contingent on previous options.

Once the lattice is developed the valuation procedure is similar to the integrated method; albeit more computationally intensive. Limiting this model's practical application is the complexity due to dimensionality. Unless a tool is made commercially available it is unlikely that practitioners would have the resources to manage the large data structures.

An alternative to lattice techniques, Monte Carlo methods may be applied to real options analysis. Datar et al. (2007) propose a simulation real options approach based on the MAD assumption entitled the DM Method. Management predicts optimistic, most likely, and pessimistic cash flows scenarios resulting in a triangular distribution of yearly profits. A Monte Carlo simulation produces random draws of annual cash flows, and after discounting at the corporate rate a distribution of present values result.

Arguing the initial investment is riskless and will occur only with favorable outcomes, Datar et al. (2007) discount the cost of the investment, the strike, at the risk-free rate. Simulated project values in excess of the strike are deemed successful. The maximum of the simulated values less the strike and zero is found; the real option value is the mean of this distribution.

The DM Method parallels the classical approach; as the number of simulations increase, the option value approaches the call option value obtained by Black-Scholes. Although this approach is straightforward and transparent for practitioners, the technique does not provide a strategic investment plan like lattice techniques. The DM method does not differentiate between the effect of market and private risks on the scenarios, but only includes an element of market risk by use of the corporate discount rate. Also, the mechanics of method are not congruent with the assumption that the cash flows follow a continuous GBM process. Instead, discrete distributions of the annual outcomes chart the simulated path, and cash flow distributions are related in risk analysis software, such as Crystal Ball and @Risk, by use of a rank correlation technique suggested by Iman and Conover (1982).

Proposed Model

Model Development

It is assumed that risk in venture is captured entirely by market and private uncertainty. Market risk is introduced into the model by a stock index, or stock S_t that follows a GBM process.

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

Where dS_t is the incremental change in the value of the index over the interval *dt;*
- μ is the growth rate of the index;
- σ is the volatility of the index; and
- dW_t is a standard Weiner process.

The cash flows of the project f_t also follow a GBM process subject to stochastic variations, but in this framework all drift and volatility parameters are considered constant over the duration of the project.

$$df_t = \nu f_t dt + \eta S_t dZ_t \tag{2}$$

Where df_t is the incremental change in the value of the cash flow;
- ν is the growth rate of the cash flows;
- η is the volatility of the cash flows; and
- dZ_t is a second Weiner process.

The process driving the cash flow variations dZ_t is decomposed into two motions: one correlated to the market, and one independent of market fluctuations.

$$dZ_t = \rho dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp} \tag{3}$$

Where $dW_t^{\perp}$ is a Weiner process uncorrelated to the market; and
- ρ is the degree of correlation to the market.

Assuming the market price of risk uncorrelated to the market is zero, the motion uncorrelated to the market is not risk adjusted. Hence, the cash flows are expressed by the following stochastic differential equation:

$$df_t = \bar{r} f_t dt + \eta f_t \left(\rho dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp} \right) \tag{4}$$

With the risk-neutral growth rate:

$$\bar{r} = \nu - \frac{\eta\rho}{\sigma}(\mu - r) \tag{5}$$

Where r is the constant risk-free rate.

The discrete form of this process is:

$$f_{t+\Delta t} = f_t e^{\left(\bar{r}-\frac{1}{2}\eta^2\right)\Delta t+\eta\sqrt{\Delta t}\left(\rho W_t+\sqrt{1-\rho^2}W_t^{\perp}\right)} \tag{6}$$

Next, the characteristics of the project value process are analyzed. The value of the project V_t is the discounted expectation of its future cash flows.

$$V_t = \mathrm{E}\left\{\int_t^{\infty} e^{-rs} f_{s,W_s,W_s^{\perp}}\, ds \,\middle|\, F_t\right\} \tag{7}$$

After integrating, a relation between the stochastic project value and project cash flows is determined:

$$V_t = \frac{f_t}{r-\bar{r}} \tag{8}$$

The value of the project is given by a function akin to the continuous time form of the Gordon growth model used to value a dividend paying stock (Gordon, 1959). If the project is held indefinitely, the cash flows act as a perpetuity of payouts to the investor. Similar to the requirement that the return on equity exceeds the growth rate of the dividend, the risk-free rate must be in excess of the risk-adjusted rate of cash flow growth. A similar relation was found by Berk, Green, and Naik (2004) when valuing staged R&D investment by assuming a continuous dividend payment.

Substituting this expression into Equation 3, it is realized that the volatility and growth of the project value is identical to the volatility and growth of the cash flows.

$$dV_t = \frac{\bar{r}}{r-\bar{r}} f_t dt + \frac{\eta}{r-\bar{r}} f_t\left(\rho dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp}\right) \tag{9}$$

If a venture exhibits market uncertainties an appropriate hedging strategy may be determined to mitigate the associated market risks. This strategy leads to the following partial differential equation representing an option on the underlying cash flows. This form is analogous to the Black- Scholes PDE; a second order, linear parabolic equation. The derivation of Equation 9 is found in Appendix A - Hedge Strategy.

$$rc_t = \frac{\partial c}{\partial t} + \left(\frac{\rho\eta r + \bar{r}\sigma - \bar{r}\rho\eta}{\sigma\left(r-\bar{r}\right)}\right) f_t \frac{\partial c}{\partial f_t} + \frac{1}{2}\frac{\eta^2}{\left(r-\bar{r}\right)^2} f_t^{\,2}\frac{\partial^2 c}{\partial f_t^{\,2}} \tag{10}$$

Model Implementation using Numerical Techniques

As a basis for comparison, the NPV without the influence of managerial flexibility is calculated by a naïve estimation of future cash flows discounted at the risk-adjusted rate. The risk-adjusted rate may be an industry standard or the WACC of the firm or investor.

Next the cash flow parameters, expected growth and volatility, are subjectively determined by management, or extracted from cash flow forecasts. Also a corresponding index or stock is found, and the growth and volatility are extracted from historical data. The degree of correlated between the market is subjectively chosen by executive based upon the type of investment.

Numerical methods used for financial option pricing can be applied to the derived PDE in Equation 9 to determine the real option value. A trinomial lattice can approximate the PDE, or a binomial lattice can represent the underlying stochastic cash flow process. Or, if appropriate boundary conditions are applied a closed form solution may result for a European call option. Appendix B – Numerical Solution Derivations contains the derivations for each numerical technique, and the parameterization for the log-transformed trinomial and binomial trees.

Model Implementation using a Simulation Procedure

This model can be integrated into a simulation technique similar to the one proposed by Datar et al. (2007), but with several methodological differences. Management forecasts most likely, or expected cash flows, and optimistic and pessimistic scenarios with a certain probability. These estimates reflect private uncertainty inherent to the project. Cash flows are assumed to follow a GBM process, and therefore lognormal distributions, as opposed to triangular distributions used within the DM Method, are used to match executive projections. Figure 1 visualizes the lognormal fits.

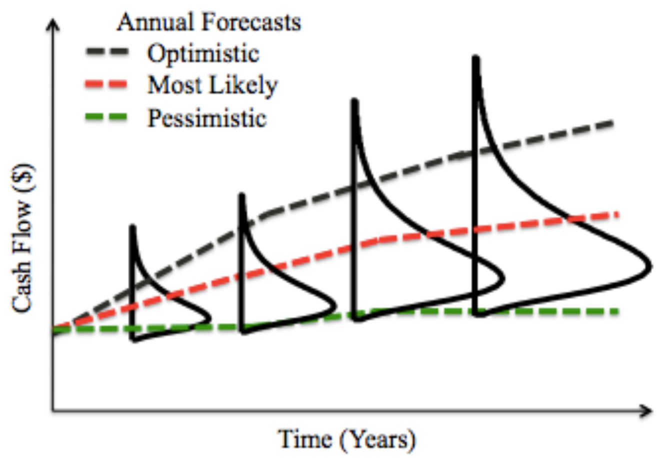


Figure 1: Lognormal distributions are used to fit management cash flow projections to determine the annual growth and volatility parameters of the process.

Mechanically the fitting technique proceeds by solving for the parameters of the lognormal probability density function v_t and η_t that describes the annual cash flow. The most likely cash flow is taken to the be expected value in that year; and the probability of the optimistic and pessimistic cash flows is less than or equal to the executive projection is known:

$$P\left(f_t \le x_{tj}\right) = \alpha_{tj} \tag{11}$$

Where f_t represents the cash flow in year t;

x_{tj} is the cash flow projection from management in year t and at level j; either optimistic or pessimistic; and
α_{tj} is the probability of the cash flow in year t and level j occurring.

Equivalently using the discrete form of the GBM cash flow process:

$$P\left(f_{t-\Delta t} e^{\left(v_t - \frac{1}{2}\eta_t{}^2\right)\Delta t + \eta_t \sqrt{\Delta t} Z_t} \leq x_{tj} \right) = \alpha_{tj}$$

(12)

Where v_t is the cash flow growth in year t;
η_t is the volatility in year t;
Z_t is a standard normal random variable ; and
Δt is the time duration between cash flows.

The fitting procedure is show graphically in Figure 2.

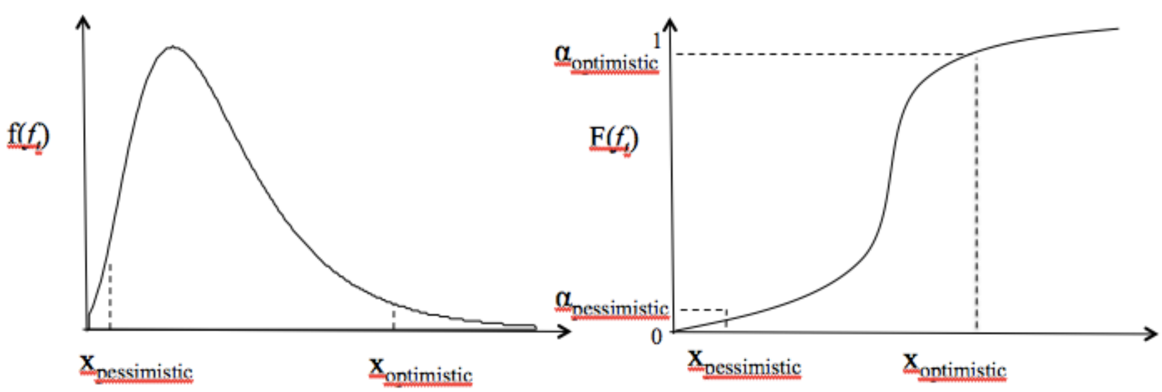


Figure 2: The parameters of the lognormal probability density function describing the annual cash flow is determined from knowledge of the cumulative density function and the expected value.

In the first year a system of three equations determines three unknowns: the growth, volatility and initial cash flow f_0. In subsequent years the risk-neutral cash flow from the previous year is known; this is an over determined system with two equations and three unknowns. Weightings may be applied to the three equations with greater emphasis on the most likely outcome.

Market risk is incorporated into the project cash flow dynamics by introducing the correlated index. The risk-neutral cash flow path, a function of both private and market risks, is simulated. Each trial evolves as unique path, and the annual the growth and volatility, is dependant upon the previous year's cash flow and distribution.

The cash flows are discounted at the risk-free rate, versus a corporate rate used by Datar et al. (2007), and summed to yield a distribution of present values. The real option value is found analogously to the DM Method.

Numerical Example

Datar et al. (2007) value an unmanned aerial vehicle (UAV) built by Boeing; a technology which promises increased efficiencies for several applications such as: monitoring remote areas, pipelines, and electrical transmission wires for the purpose of safety, forest health, and boarder security. An initial outlay of $15 million for R&D is required. Subsequently, product launch expected in the second year at a cost of $325 million. The launch cost is discounted at the risk free rate of 5%. Management forecasts three unique cash flows based upon optimistic, most likely, and pessimistic projections. Each scenario is also assigned a probability; the optimistic and pessimistic cases will each occur with likelihood of 10%. The cash flows are listed in

Table 1.

Table 1: The optimistic, pessimistic and most likely cash flow projections for the UAV project in $ million.

Scenario	1	2	3	4	5	6	7	8	9
Optimistic	0	0	80	116	153	177	223	268	314
Most likely	0	0	52	62	74	77	89	104	122
Pessimistic	0	0	20	23	24	18	20	20	22

Applying a corporate risk-adjusted discount rate of 15% to the most likely scenario cash flows, the total project is $-19 million. In contrast using the DM Method, with rank order correlations of 70% to link annual distributions, the real option is worth $23 million yielding a $8 million total project value, net of R&D. The success probability, the probability that the present value of the projected will be greater than the launch cost, is 42.1%. The overall recommendation is to undertake this venture.

Proposed Solution using the Simulation Procedure

For this example, the success of the UAV venture is assumed to be correlated to iShares Dow Jones U.S. Aerospace & Defense Index Fund, an exchange traded fund (ETF) on the New York Stock Exchange (NYSE: ITA.P). This ETF includes aerospace companies that manufacture, assemble, and distribute aircraft and aircraft parts. In April 2010, the fund was trading at $57.91, and since its inception the historical annual growth rate and volatility was 2.7% and 28.3%. The risk in the future price of the ETF represents the portion of market uncertainty inherent to the UAV venture.

The degree of correlation of the cash flows to the ETF is manipulated and simulations run for 10,000 trials. Figure 3 shows the real option value and the success probability for these simulations, and detailed numerical results are given in Appendix C – Detailed Numerical Example Results. Evidently there is value in the UAV project and the recommended action is to invest in this venture.

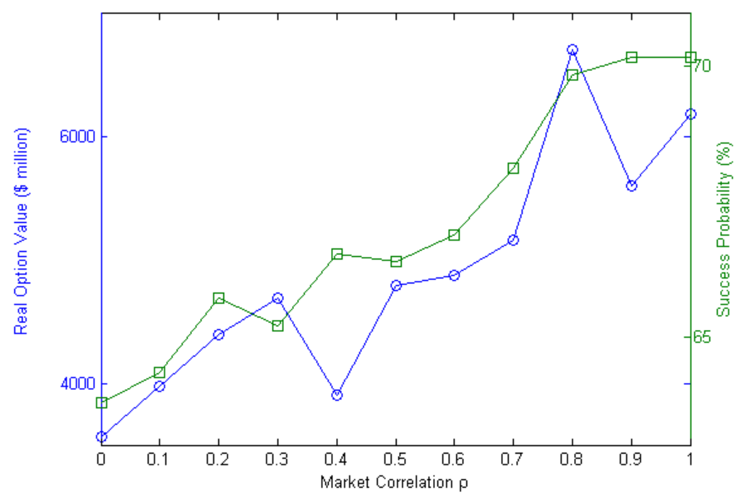


Figure 3: The real option value and success probability of the UAV venture with changing correlation to the ETF.

Inspection of the present value distributions reveals they are highly positively skewed. Figure 4 shows a sample of the simulated present value distributions.

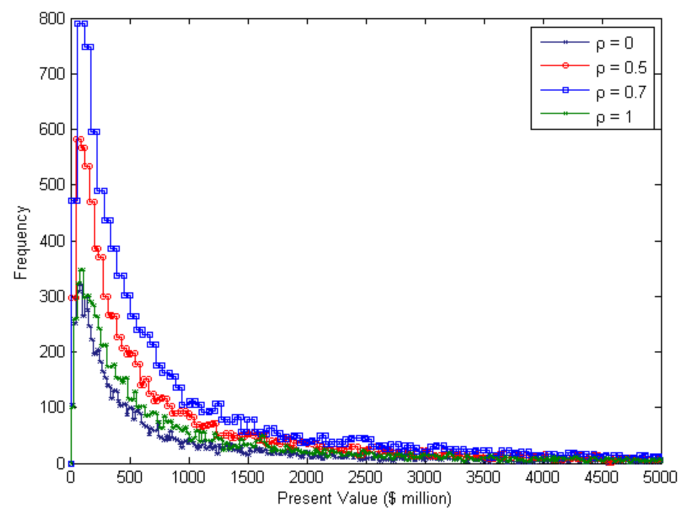
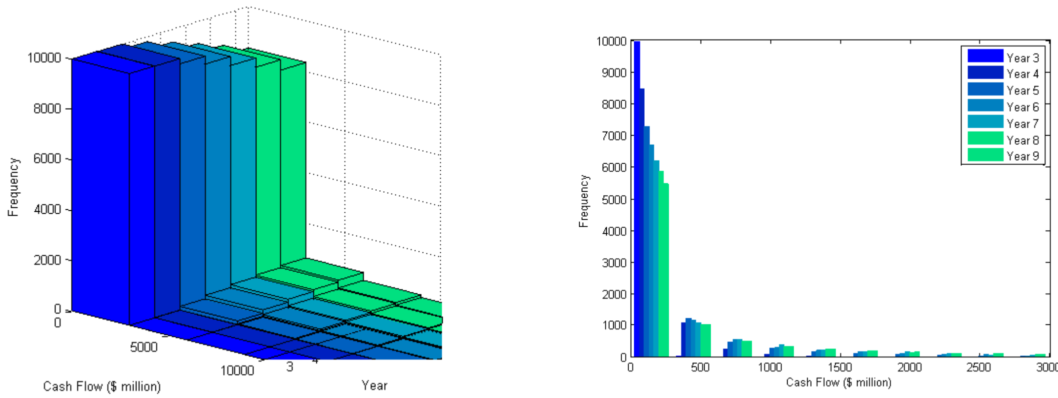


Figure 4: A sample of the simulated project value distributions for the UAV project with changing correlation to the ETF.

For the trial when cash flows are 70% correlated to the ETF the UAV project has a real option value of $5165 million, and a success probability of 68.1%. Figure 5 displays the annual cash flow distributions for this trial.



a) 3d distribution **b) 2d distribution**

Figure 5: The simulated cash flow distributions in a) 3d and b) 2d for the UAV project with 70% correlation to the ETF.

The evolution of 10,000 cash flows for this trial is displayed in Figure 6 a) and b) with difference in the scale of the vertical axis. This demonstrates the variability of the project outcomes under the GBM assumption while correlated to a volatile ETF process.

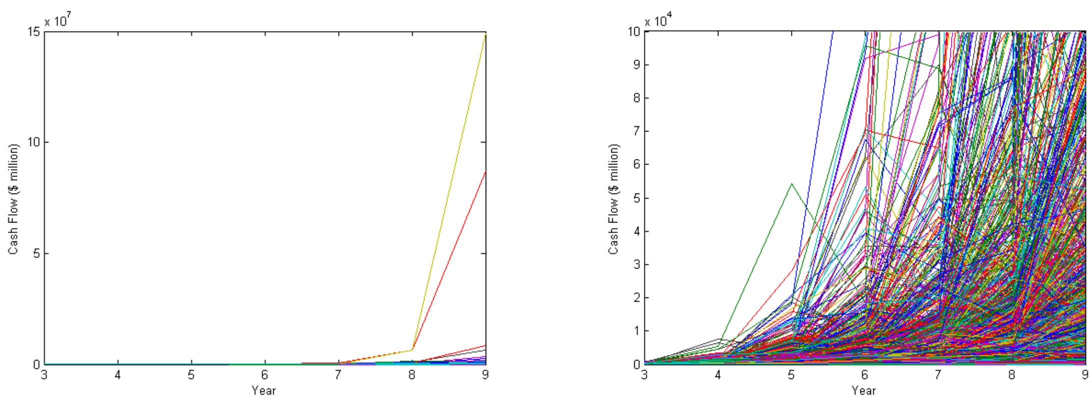
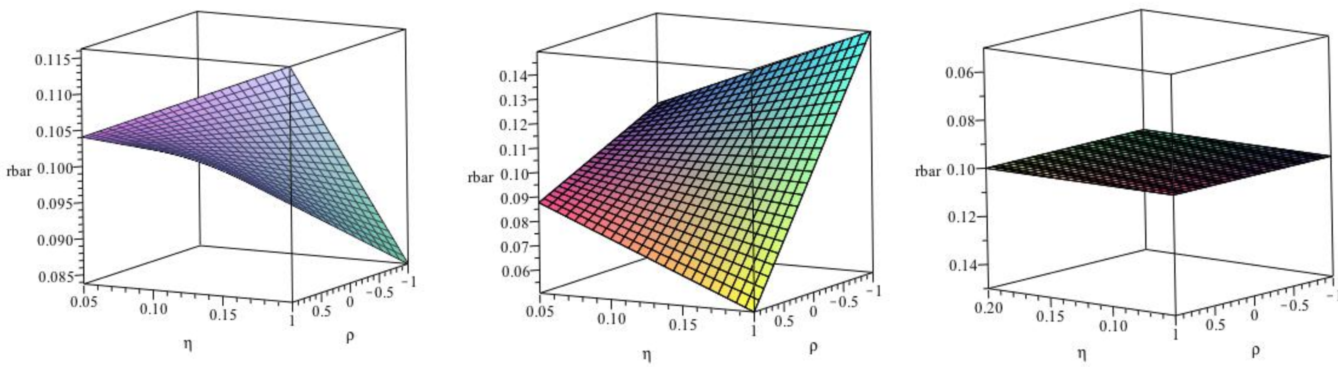


Figure 6: The simulated 10,000 cash flows grown at the risk-neutral rate for the UAV project with 70% correlation to the ETF.

Further analysis of the risk-neutral growth rate shows that the cash flow and index growth and volatility values have pronounced impact on this parameter. Most significant is the ratios of the two volatility parameters, and magnitude of index growth rate as compared to the risk-free rate.

The ETF growth and volatility parameters are analyzed with a cash flow growth rate of 10%; Figure 7 displays the surface formed by manipulating the market correlation and cash flow volatility. It is noticed that the risk-neutral growth rate increases with increasing correlation to the ETF. This affect is noticed in the previous calculations of the real option value: higher correlation to the ETF resulted in higher option values. Contrasting to this, if the index growth rate is larger than the risk-free rate than the risk-neutral growth rate decreases with increasing market correlation. Figure 7 b) demonstrates this inversed relationship; if these two parameters are equal a flat plane results in c). Finally, for the case when the cash flows are perfectly correlated to the market, and the cash flow and index volatility and growth variables are equivalent the risk-neutral growth rate is the risk-free rate.



a) μ= 2.7%

b) μ= 10%

c) μ= r =5%

Figure 7: The effect of manipulating the market correlation and the cash flow volatility on the risk-neutral growth rate from when a) μ < r, b) μ > r , and c) μ = r. For these trials r =5%, σ=28.3%, and ν=10%.

Discussion

The proposed model allows practitioners to capture the affect of market uncertainties on the value of the project by a assuming the cash flow process is correlated to a traded index. This model is similar to the classical ROA approach when the cash flows are perfectly correlated to the index and all risk is market related. Yet unlike the classical approach, difficulty in determining the underlying value is negated: the underlying value is the value of the cash flows of the project. In DTA, used for both the integrated and revised classical ROA methods, practitioners find it problematic to assign an appropriate discount rate and subjective probabilities, but in the proposed model the risk-free rate is always used. Likewise when a project is dominated by private risks, the uncertainty in the project is only related to the volatility of the cash flows; this replaces the requirement to estimate subjective outcome probabilities. The MAD assumption completes the incomplete market: the value of the project is the only asset in the market, and perfectly correlated with itself. When analyzed in the framework of the proposed model the correlation is one, and the index growth and volatility are equal to the volatility and the growth of the cash flows.

When comparing the results of the numerical example to the valuation by Datar et al. (2007), it is found that the use of rank order correlations to relate annual distributions do not accurately mimic the underlying GBM process. Furthermore the cash flows in the proposed model grow at a risk-neutral rate, and hence all cash flows are discounted at the risk-free rate. Again this avoids the inconsistent use of risk-free and risk-adjusted discount rates required for the DM Method and MAD approach.

Last, to reflect on the criteria outlined by Copeland and Antikarov (2005) it is realized that the proposed framework fits several of the necessary requirements outlined previously. The model accurately incorporates varying proportions of market and private risks by assuming the project cash flows are correlated to a traded index, and therefore too uses market data. The model may be implemented similarly to previously suggested ROA approaches, including trees or a simulation technique akin to the DM Method; this ensures that the model remains mathematically transparent and computationally efficient. Whether the model is empirically testable remains as future work for academics or practitioners.

Conclusion

In this paper a novel ROA model is proposed. The approach allows practitioners to incorporate both private and market uncertainties when valuing a real-world project by assuming the cash flow process is correlated to a stochastic traded index.

Numerical methods including Monte Carlo simulation, binomial tree, and trinomial tree techniques may used to value the real option on the project cash flows. Alternatively, as demonstrated within a numeric example, the model may be implemented within a simulation framework similar to the approach recommended by Datar et al. (2007).

Acknowledgements

The authors would like to thank the Canadian Social Science and Humanities Research Council (SSHRC) for the funding of this research, and members of the Centre for Management of Technology and Entrepreneurship at the University of Toronto for their ongoing contributions.

References

Berk, J. B. (2004). Valuation and Return Dynamics of New Ventures. *The Review of Financial Studies.* 17, 1, 1-35.

Black, F. & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities, *Journal of Political Economy,* 81, 637-659.

Block, S. (2007). Are "Real Options" Actually used in the Real World? *Engineering Economist.* 52, 3, 255-267.

Borison, A. (2005). Real Options Analysis: Where are the Emperor`s Clothes? *Journal of Applied Corporate Finance.* 17, 2, 17-31.

Brandao, L. E., Dyer, J. S. & Hahn, W. J. (2005). Using Binomial Trees of Solve Real Option Valuation Problems. *Decision Analysis.* 2, 2, 69-88.

Brennan, M. J., & Schwartz, S.S. (1985). Evaluating Natural Resource Investments. *Journal of Business.* 58, 2, 135-157.

Copeland, T. E. & Antikarov, V. (2005). Meeting the Georgetown Challenge. *Journal of Applied Corporate Finance.* 17, 2, 32-51.

Copeland, T. E. & Antikarov, V. (2001). *Real Options for Practitioners.* New York, NY: Texere

Cox, J. C., Ross, S. A. & Rubinstein, M. (1979). Option Pricing: A Simplified Approach. *Journal of Financial Economics.* 7, 3, 229-263.

Datar, V. T., Matthews, S. H., & Johnson, B. (2007). A Practical Method for Valuing Real Options. *Journal of Applied Corporate Finance.* 19, 2, 95-104.

Gordon, M, J. (1959). Dividends, Earnings and Stock Prices. *Review of Economics and Statistics.* 41, 99-105.

Iman, R. & Conover, W. (1982). A Distribution-Free Approach to Inducing Rank Correlation Among Input Variables. *Communication in Statistics – Simulation and Computation.* 11, 3, 311-334.

Kamrad, B. & Ritchken, P. (1991). Multinomial Approximating Models for Options with k State Variables. *Management Science.* 37, 12, 1640-1652.

Luehrman, T. (1998a). What's it worth? A general manager's guide to valuation. *Harvard Business Review*. 132-142.

Luehrman, T. (1998b). Investment Opportunities as Real Options: Getting Started with the Numbers. *Harvard Business Review*. 51-64.

Myers, S. (1977). Determinants of Corporate Borrowing. *Journal of Financial Economics.* 5, 147-175.

Rubenstein, M. (2000). On the Relation Between Binomial and Trinomial Option Pricing Models. *UC Berkeley Research Program in Finance, Working Paper Series, (1003).* Retrieved from: http://www.haas.berkeley.edu/groups/finance/WP/rpf292.pdf

Samuelson, P. A. (1956). Proof that Properly Anticipated Prices Fluctuate Randomly. *Industrial Management Review.* 6, 2, 41-49.

Schneider, M., Tejeda, M., Gabriel, D., Herzog, F., Keel, S. & Geering, H. (2008). Making real options work for practitioners: a generic model for valuing R&D projects. *R&D Management.* 38, 1, 85-106.

Smith, J. E. & Nau, R. (1995). Valuing Risky Projects: Option Pricing Theory and Decision Analysis. *Management Science.* 41, 5, 795-816.

Smith, J. E. (2005). Alternative Approaches for Solving Real Options Problems (Comment on Brandao et al. 2005). *Decision Analysis.* 2, 1, 89-102.

Appendix A - Hedge Strategy

To hedge market related risks associated with a project a contingent claim on project cash flows is considered.

$$c(f_t,t) \tag{13}$$

The Applying Ito's Lemma an SDE for the contingent claim is found.

$$dc_t = \left(\frac{\partial c}{\partial t} + \frac{\bar{r}}{r-\bar{r}} f_t \frac{\partial c}{\partial f_t} + \frac{1}{2}\frac{\eta^2}{\left(r-\bar{r}\right)^2} f_t^{\,2} \frac{\partial^2 c}{\partial f_t^{\,2}}\right) dt + \frac{\eta}{\left(r-\bar{r}\right)}\frac{\partial c}{\partial f_t}\left(\rho dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp}\right) \tag{14}$$

Assuming the market is complete, the option may be replicated by a portfolio Π consisting of the money market account and the stock market index.

$$\Pi = a_t S_t + b_t M_t \tag{15}$$

The self-financing condition is applied. As the behavior of the hedge portfolio must mimic the changing price of the contingent claim the following equation will hold.

$$d\Pi = dc_t = (a_t \bar{r} S_t + b_t r M_t)dt + a_t \sigma S_t dW_t \tag{16}$$

In order to hedge the tradable risks the appropriate holdings for the index and money market account are determined.

$$a_t = \frac{\rho\eta}{\sigma S_t\left(r-\bar{r}\right)} f_t \frac{\partial c}{\partial f_t} \tag{17}$$

$$b_t = \frac{1}{rM_t}\left(\frac{\partial c}{\partial t} + \frac{\bar{r}}{r-\bar{r}} f_t \frac{\partial c}{\partial f_t} - \frac{\rho\eta\bar{r}}{\sigma\left(r-\bar{r}\right)} f_t \frac{\partial c}{\partial f_t} + \frac{1}{2}\frac{\eta^2}{\left(r-\bar{r}\right)^2} f_t^{\,2}\frac{\partial^2 c}{\partial f_t^{\,2}}\right) \tag{18}$$

This leads to the following partial differential equation.

$$rc_t = \frac{\partial c}{\partial t} + \left(\frac{\rho\eta r + \bar{r}\sigma - \bar{r}\rho\eta}{\sigma\left(r - \bar{r}\right)} \right) f_t \frac{\partial c}{\partial f_t} + \frac{1}{2} \frac{\eta^2}{\left(r - \bar{r}\right)^2} f_t^{\,2} \frac{\partial^2 c}{\partial f_t^{\,2}} \tag{19}$$

Appendix B – Numerical Solution Derivations

Trinomial Tree Parameterization

As demonstrated by Brennan and Schwartz (1978), the explicit finite difference scheme can be used to approximate the value of a contingent claim on an underlying asset, namely the project cash flows. To simplify the numerical procedure, constant coefficients for the cash flow PDE are found by a logarithmic transformation of underlying variable.

$$X = \ln f_t \tag{20}$$

$$W(X,t) = c(f_t,t) \tag{21}$$

Where X is the logarithm of the cash flows at time t; and
W is the contingent claim on the transformed variable X.

This manipulation results in the next PDE.

$$rW = \frac{\partial W}{\partial t} + \left(\frac{\rho\eta r + \bar{r}\sigma - \bar{r}\rho\eta}{\sigma\left(r - \bar{r}\right)} - \frac{1}{2}\frac{\eta^2}{\left(r - \bar{r}\right)^2} \right) \frac{\partial W}{\partial X} + \frac{1}{2}\frac{\eta^2}{\left(r - \bar{r}\right)^2}\frac{\partial^2 W}{\partial X^2} \tag{22}$$

To approximate a numerical solution partial derivatives are replaced by finite differences.

$$\frac{\partial W}{\partial t} = \frac{W_{i+1,j} - W_{i,j}}{k} \qquad \frac{\partial W}{\partial X} = \frac{W_{i+1,j+1} - W_{i+1,j-1}}{2h} \qquad \frac{\partial^2 W}{\partial X^2} = \frac{W_{i+1,j+1} - 2W_{i+1,j} + W_{i+1,j-1}}{h^2} \tag{23}$$

For $i = 1 \ldots n-1$; and
$j = 1 \ldots m$.
Where h is a discrete increment in the underlying value of the cash flows; and
k the discrete time increment.

Upon manipulation this yields the corresponding difference equation.

$$W_{i,j} = \frac{1}{1+rk}\left(P_U W_{i+1,j+1} + P_M W_{i+1,j} + P_D W_{i+1,j-1} \right) \tag{24}$$

Where P_U is the probability of an up movement;
P_M is the probability that the claim remains the same value; and
P_D is the probability of a down movement over the increment k.

Figure 8 demonstrates these motions and corresponding probabilities.

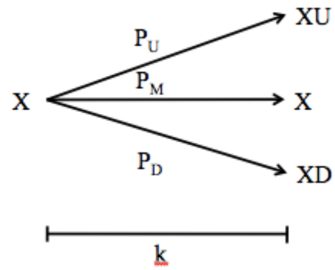


Figure 8: Motion of the cash flows under the trinomial tree parameterization.

The probabilities have the form:

$$P_U = \frac{k}{2}\left(\frac{\eta^2}{h^2\left(r-\bar{r}\right)^2} + \frac{\frac{\eta^2}{2\left(r-\bar{r}\right)^2} - \left(\frac{\rho\eta r + \bar{r}\sigma - \bar{r}\rho\eta}{\sigma\left(r-\bar{r}\right)}\right)}{h}\right) \quad P_D = \frac{k}{2}\left(\frac{\eta^2}{h^2\left(r-\bar{r}\right)^2} - \frac{\frac{\eta^2}{2\left(r-\bar{r}\right)^2} - \left(\frac{\rho\eta r + \bar{r}\sigma - \bar{r}\rho\eta}{\sigma\left(r-\bar{r}\right)}\right)}{h}\right) \quad P_M = 1 - \frac{k\eta^2}{h^2\left(r-\bar{r}\right)^2}$$

(25)

Or more simply,

$$P_U = \frac{k}{2}\left(\frac{\beta^2}{h^2} + \frac{\left(\alpha - \frac{\beta^2}{2}\right)}{h}\right) \quad P_D = \frac{k}{2}\left(\frac{\beta^2}{h^2} - \frac{\left(\alpha - \frac{\beta^2}{2}\right)}{h}\right) \quad P_M = 1 - \frac{k\beta^2}{h^2}$$

(26)

Where:

$$\alpha = \frac{\rho\eta r + \bar{r}\sigma - \bar{r}\rho\eta}{\sigma\left(r-\bar{r}\right)} \quad \beta = \frac{\eta}{\left(r-\bar{r}\right)}$$

(27)

The motion of the underlying asset has a mean and variance of:

$$\mathbf{E}\left[dX\right] = h\left(P_U - P_D\right) = \frac{-k\beta^2}{h} \quad \mathbf{V}\left[dX\right] = h^2\left(P_U + P_D\right) - \left(\mathbf{E}\left[dX\right]\right)^2 = \frac{k}{2h^2}\left(\left(\beta^2 - 2\alpha\right)h^3 - 2\beta^2 k\right)$$

(28)

Binomial Tree Parameterization

Researches have realized the equivalence of the trinomial tree to the binomial form provided correct parameterization (Rubenstein, 2000; Song & Ang, 2007). The trinomial technique has the same accuracy as a binomial tree with half of the required time steps with a reduction in computational time (Ahn & Song, 2007). However, practitioners with a background in financial option valuation or decision analysis are likely to be familiar with the binomial representation in Figure 9. Furthermore, binomial trees are typical in other established ROA approaches.

It is noticed that after one time increment the three states of the trinomial tree are equal to two steps of the binomial tree. Solving the recursive programming procedure of the trinomial method is parallel to pricing an option with the binomial tree for two time periods (Rubenstein, 2000). Consequently for parameterization, the time increment of the binomial tree is half that of the trinomial increment, and the spacing is the square root of the trinomial up and down factors. Using this, analogous parameters are determined for the binomial tree. The probability of an up movement for the binomial tree is the square root of that of the trinomial tree probability.

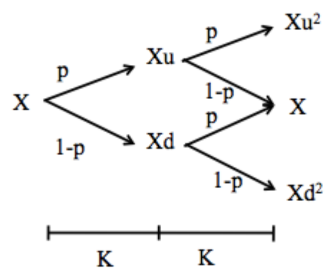


Figure 9: Motion of the cash flows under the binomial tree parameterization.

$$p = \frac{\sqrt{2}}{4}\sqrt{\frac{K\left(\beta^2 h - 2\alpha h - 2\beta^2\right)}{h^2}} \tag{29}$$

Similarly, the up factor and down factor for the binomial tree is found:

$$u = e^{\frac{\beta\sqrt{2}\sqrt{K}}{\sqrt{h}}} \qquad d = u^{-1} \tag{30}$$

Closed Form Solution

In the case of a European call option, or the right to purchase the set of cash flows at sometime in the future for a strike E*,* the boundary condition is simple.

$$c(f_t, T) = \max(f_T - E, 0) \tag{31}$$

Additional if the following conditions are applied

$$\begin{aligned} &c(0,t) = 0 \\ &c(f,t) \approx f \text{ as } f \to \infty \end{aligned} \tag{32}$$

a closed form solution, which again resembles Black-Scholes, is obtained.

$$c(f,T) = f\Phi(d_1) - Ee^{-rT}\Phi(d_2) \tag{33}$$

$$d_1 = \frac{\ln\left(f/E\right) + \left(\alpha + \frac{1}{2}\beta^2\right)T}{\beta\sqrt{T}} \tag{34}$$

$$d_2 = d_1 - \beta\sqrt{T} \tag{35}$$

Appendix C – Detailed Numerical Example Results

Table 2: The real option value and the success probability of the UAV project with changing correlation to the ETF.

Market Correlation	Real Option Value ($ million)	Success Probability (%)
0	3562.2	63.77
0.1	3972.3	64.35
0.2	4396.6	65.72
0.3	4688.0	65.20
0.4	3899.5	65.53
0.5	4789.7	66.39
0.6	4873.1	66.89
0.7	5164.5	68.11
0.8	6712.8	69.85
0.9	5599.1	70.17
1	6189.0	70.18