Production Flexibility and Investment under Uncertainty

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March 31, 2009

Abstract

This paper extends the theory of real options, which mainly considers the problem of timing of investment, by letting the firm choose also the quantity produced after the time of investment. Under the conditions of irreversible investment and uncertainty in future demand, the firm can adjust its output quantity at each point in time in the future. Our paper shows that changing the structure of the demand function from iso-elastic to linear, implicates the need of a suspension option. We see that flexibility in quantity choice makes the firm invest earlier and the gap between the different investment timings increases with uncertainty and is remarkably high. In the flexible case it can either happen that capacity is fully used or some capacity is left idle right at the moment of the investment. We identify how uncertainty in demand influences this decision.

1 Introduction

The theory of real options mainly considers problems where a firm must find the optimal time to invest in a certain project. Thus, in these models the only decision is the timing decision. We consider, besides the timing of the investment, the fact that the firm can also choose the quantity produced. In particular, after the time the firm has invested, it must also choose the quantity that it will produce at each point in time in the future. We suppose that the single discrete project, once installed, allows some flexibility in its operation at any instant, by varying some inputs, such as labor or raw materials, that do not require any irreversible commitments that extend over time. Then the optimal amounts will depend on the output price at that instant.

Paying attention to uncertainty in making investment decisions has led to an impressive new strand of literature. This strand based on real options theory, acknowledges partial irreversibility of investments and predicts that uncertainty delays investment. The real options theory is elaborately comprised in the book of Dixit and Pindyck [10]. The fact that the new literature has focused more on the timing of investment than on its intensity has been brought up already in a review of this book by Hubbard [12, p.1828]. He argues that "the new view models...do not offer specific predictions about the level of investment". Hubbard claims that in order to take this extra step "it requires the specification of structural links between the marginal profitability of capital and the desired capital stock (the usual research focus in the traditional, neoclassical literature)."

However, we can see first attempts towards this issue in chapter 6 of Dixit and Pindyck [10, Chapter 6.3]. Their model for a project with variable output is the starting point for our work. Dixit and Pindyck build a model where at every instant, the investor has the choice of some operating variables, which generate output according to a production function and entail variable cost. They keep it quite simple restricting their model to a Cobb-Douglas production function and constant cost. First, we develop their approach of a variable production model allowing for different inverse demand functions. In a second approach, the firm can furthermore, choose capacity when investing.

The generalization of the inverse demand function shows that, in contrast to the Cobb-Douglas production function leading to an always positive optimal quantity, there are inverse demand functions where it can be optimal to temporarily terminate production in case the optimal quantity falls to zero, after the investment decision has been made. This is true for a linear inverse demand function, where one has to consider the possibility of the optimal quantity falling to zero. First we examine a model with an isoelastic inverse demand function receiving an always positive optimal quantity. We obtain a similar result to Dixit and Pindyck [10, Chapter 6.3] and confirm the crucial result of the real options approach to investment under uncertainty, that increased uncertainty leads to deceleration of the investment. The second model considering a linear inverse demand function shows that the maximization of the profit function can lead to a negative optimal quantity. Therefore, we have to consider two scenarios that can happen after the investment decision has been made: for the case that the realisation of the stochastic process influencing the inverse demand is greater than the variable costs, a positive quantity produced is optimal. As soon as this falls below the cost value we have to consider temporary suspension. The arising necessity taking this additional assumption in account causes the crucial difference to the models considered before. We are not able to obtain an equation for the investment threshold analytically. Solving for the optimal trigger value numerically, the general real options result can be approved for this extended model of variable production.

There are few papers in the literature, which have brought up this issue. Dixit [9] picks up the capacity choice issue by evaluating a model with irreversible choice among mutually exclusive projects under uncertainty. He considers a project with output price uncertainty, sunk capital cost but no operating cost. Decamps, Mariotti and Villeneuve [8] renew this model, reducing it to a choice among two alternative investment projects of different scales. They provide parameter restrictions under which the optimal investment strategy is not a trigger strategy and the optimal investment region is dichotomous. Lee and Shin [14] determine the relationship between investment and uncertainty exploring the role of a variable input, labor, in striking a balance between a positive effect, due to the convexity of the profit function, and a negative effect, due to the option value of waiting.

Bar-Ilan and Strange consider both the timing and intensity of investment. Furthermore, they examine the evaluation of capital stock under incremental and lumpy investment. They put their main focus on their striking comparative results for the lumpy investment model, which say that an increase in output price uncertainty delays investment but lead to more intense investment when it eventually occurs. With incremental investment only the intensity of investment is chosen since the firm can invest at any point in time which results in an unambiguous evolution of the capital stock. For our purpose the model considering 'lumpy' investment is of more relevance. In the incremental investment model they focus on the intensity of investment but leave open the timing decision. In the 'lumpy' investment model of Bar-Ilan and Strange the level of capital is chosen at the instant that the investment occurs. By paying a fixed unit price the firm receives a production technology that allows it to produce forever at a certain production rate. We abandon this restriction of irreversible production decision and allow for flexibility in its operation at any instant after investment. Since the investment manager can decide at each instant after investment about the quantity considering the realisation of the stochastic process at that instant, the optimal quantity is not depending on the uncertainty.

In order to analyze how significant the value of flexibility is we have to derive the optimal values of the model with fixed production rate. We compare the results of the flexible model to the infexible one. For the inflexible model we eliminate the possibility of adapting quantity. This means that the firm has to choose for a certain level of quantity at which it will produce forever. Our results show that the flexibility in quantity choice makes the firm invest earlier. Numerical results show that the gap between the two investment thresholds increases with uncertainty and is remarkably high.

Since the capacity level of a firm is often restricted to major investment decisions, i.e. building a new factory or acquiring new machinery, most firms are not flexible in continuously adapting their capacity to the changing production level. Often this decision has to be made once and later capacity-adjustment of a ready build production facility or the installation of an additional project is not possible. Therefore, in a second step we add the restriction of limited capacity and let the firm choose the upper bound for capacity at the moment of investment. Unlike the production rate this capacity choice cannot not be adjusted later on.

The paper closest related to our work is Dangl [7]. As in Dangl's paper, the firm has to determine the optimal investment timing and optimal capacity choice at the same time. Dangl restricts to a linear inverse demand function, while we also consider the iso-elastic case. Dangl gets the known result that uncertainty in future demand leads to an increase in optimal installed capacity and causes investment investment to be delayed. We elaborate on his paper regarding the economic results. Considering the uncertain demand there are three regions which have to be contemplated for the linear inverse demand function model. In the region with a low realisation of the stochastic process the firm will wait to invest in capacity. So the firm can decide to invest in the remaining two regions, where the demand is high enough to cover the variable costs of production. Investing in the second region means, that the firm sets an upper bound for output at moment of investment but does not produce up to full capacity yet. It leaves some capacity idle. In the third case it starts producing up to full capacity after taking the investment and capacity choice decision. Later on it will adapt the production rate to the demand, while the maximum capacity level stays fixed. We want to investigate for which case the firm chooses to invest in region II and therefore leaves some capacity idle in the beginning and for which case it decides to invest in a capacity level which is optimal to be fully utilized after the moment of investment.

Value of flexibility considering capacity choice has been brought up in the literature mainly considering two- or three-stage decision models. We extend this idea regarding a continuous time setting. Anupindi and Jiang [2] consider a model where firms make decisions on capacity, production and price under demand uncertainty in a three-stage decision making framework. They focus on duoploy models and compare their competitive equilibrium solutions to the optimal solutions in a monopoly. This focus on competitive models initiates an interesting expansion of our models from a monopoly to a duopoly setting in a continuous time setting for future research.

Anupindi and Jiang consider different types of demand shocks and show that several of their results critically depend on the nature of demand shock. This approaches our attempt to look at different inverse demand function models. While the firm always decides about capacity before and price after demand realizations, there is a difference in the timing of the production decision. A flexible firm can postpone production decisions until the actual demand curve is observed, while the inflexible cannot. Furthermore, they show that capacity and profit increase both for flexible firms when the market is more volatile and that flexiblity allows a firm to increase investment in capacity and earn a higher profit while benefiting customers by keeping the price in a narrower range. A firm's strategic flexibility choice depends on the cost of technology as well as on the nature of demand shocks.

Goyal and Netessine [13] evaluate the strategic value of manufacturing flexiblity in an uncertain environment and study the impact of competition on this value of flexibility. Two firms compete with each other in two markets characterized by price-dependent and uncertain demand. The firms have to decide about technology and capacity investment while the demand curve is uncertain and about production quantities after the demand curve is revealed.

Literature that focuses more on monopoly are for instance Van Mieghem and Dada [16] and Chod and Rudi [5]. Chod and Rudi study two types of flexibility - resource flexibility and responsive pricing. The firm is selling two products where it faces linear demand curves for these products. They consider a situation in which a single flexible resource can be used to satisfy two distinct demand classes. They decide for a fixed decision structure, where the resource capacity must be decided based on uncertain demand function, while the resource allocation as well as the pricing decision are made based on the realized demand functions. They characterize the effects of demand variability and demand correlation. Interesting for our research are their results that the optimal capacity of flexible resource is increasing in demand variablity and that the value of flexibility is mostly significant if the demand levels are highly variable.

Also Van Mieghem and Dada [16] consider a two stage model where demand is represented by a linear curve with a stochastic intercept. The firm has to decide about capacity investment, production (inventory) quantity and price. They analyze several strategies which differ in the timing of the operational decisions (i.e. capacity, output and price) relative to the realization of uncertainty and show how the different strategies influence the strategic investment decision of the firm and its value. Relevant for us are the production postponment strategies which respond to our variable production model. More postponement increases firm value and this also makes the optimal capacity decision more sensitive to uncertainty. Reduced demand uncertainty decreases investment and inventories.

The paper is structured as follows: Beyond this introduction chapter we consider a model with an isoelastic inverse demand function in Section 2. In the following section the contemplation of a model with a linear demand function leads to the necessary assumption of a suspension possibility. The value of flexiblity is discussed in the third subsection of the respective model. In Section 4 simulations point out the relevant differences between the two models. Section 5 shows the extension for limited capacity and capacity choice. Section 6 concludes. All proofs of Section 2 and 3 are contained in Appendix A. Appendix B gives analytical details for Section 5.

2 Iso-elastic Inverse Demand Function

2.1 Flexible Model

In this section we present a model that is taking into account the timing and intensity decision considering investment under uncertainty. For the beginning we keep our model close to Dixit and Pindyck [10, p. 195ff]. Dixit and Pindyck build a model where at every instant the investor has the choice of some operating variables, denoted by the vector v, which generate output according to a production function, $h(v)$, and entail variable cost, $C(v)$. The uncertainty arises with the price, P_t, following a geometric Brownian motion. Therefore, they maximize the following profit function

$$\pi(P) = \max_v [Ph(v) - C(v)].$$

Dixit and Pindyck restrict their model to a Cobb-Douglas production function and constant cost. We modify their attempt and let the market uncertainty be reflected in our model by a stochastic process θ_t, which affects the firms decision through the inverse demand function. First we choose this inverse demand function to be iso-elastic

$$p = \theta q^{-\gamma} \quad with \quad 0 < \gamma < 1,$$

where

$$d\theta_t = \alpha\theta_t dt + \sigma\theta_t dW_t. \tag{1}$$

Variable costs of operation c are known and constant. We determine the optimal q^* by maximizing the profit flow

$$\max_q pq - cq = \max_q (\theta q^{-\gamma} - c)q.$$

This implies that at each point in time q satisfies

$$q^* = \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}.$$

For the considered parameter range this optimal quantity value is always positive. This implicates that after the investment decision was made, the firm will produce forever while the production rate will be attapted to the realisation of the demand process at every instant. Therefore, the profit flow is given by

$$\pi(\theta) = \theta^{\frac{1}{\gamma}}\gamma \left[\frac{1-\gamma}{c}\right]^{\frac{1-\gamma}{\gamma}}.$$

For brevity of notation we denote this as $\pi(\theta) = K\theta^\epsilon$, where $\epsilon = \frac{1}{\gamma} > 1$ (because $0 < \gamma < 1$) and $K = \gamma\left[\frac{1-\gamma}{c}\right]^{\frac{1-\gamma}{\gamma}}$. Then familiar steps produce the differential equation for the value of the project

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial V(\theta)^2}{\partial^2\theta} + \alpha\theta\frac{\partial V(\theta)}{\partial\theta} - rV(\theta) + K\theta^\epsilon = 0. \tag{2}$$

Solving equation (2) for $V(\theta)$ gives the following value of the project

$$V(\theta) = B_1\theta^{\beta_1} + B_2\theta^{\beta_2} + K\theta^{\epsilon}\frac{1}{r - \epsilon\alpha - \frac{1}{2}\epsilon(\epsilon-1)\sigma^2}, \tag{3}$$

where β_1 and β_2 are the roots of the fundamental quadratic equation

$$Q \equiv \frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0. \tag{4}$$

with the values

$$\begin{aligned}
\beta_1 &= \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left[\frac{\alpha}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} \\
\beta_2 &= \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left[\frac{\alpha}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}}.
\end{aligned}$$

Here $\beta_1 > 1$ and $\beta_2 < 0$. The particular solution of the stochastic differential equation (2), $K\theta^{\epsilon}\frac{1}{r-\epsilon\alpha-\frac{1}{2}\epsilon(\epsilon-1)\sigma^2}$, has a natural economic interpretation: it is just the expected present value of a profit stream $K\theta^{\epsilon}$ calculated using the appropriate risk-adjusted discount rate. The derivation of the expected present value can be found in Appendix A.

Since $\beta_2 < 0$, that power of θ goes to infinity as θ goes to zero. To prevent the value from diverging, we must set the corresponding coefficient $B_2 = 0$. Furthermore, we can also set the second unkown B_1 equal zero by ruling out bubble solutions. Therefore, we calculate the expected revenue of the project and see that it is equal to the value of the project. This derivations are shown in Appendix A.

For the solution of the option we have

$$F(\theta) = A_1\theta^{\beta_1},$$

where β_1 is the positive root of (4) and the unkown A_1 remains to be determined. Determining the usual conditions of value matching and smooth pasting

$$\begin{aligned}
F(\theta^*) &= V(\theta^*) - I, \\
F'(\theta^*) &= V'(\theta^*),
\end{aligned}$$

we can solve them in order to characterize the threshold of θ^* that triggers the investment

$$\theta^{*\epsilon}K\frac{1}{r - \epsilon\alpha - \frac{1}{2}\epsilon(\epsilon-1)\sigma^2} = \frac{\beta_1}{\beta_1 - \epsilon}I. \tag{5}$$

The left-hand side of equation (5) is the expected present value of the profits and the right-hand side is the so called 'option value multiple' times the cost of investment

In the following, we examine the effect of uncertainty on investment. One effect of an increase in σ is that it lowers β_1 and therefore increases the 'option value multiple' $\frac{\beta_1}{\beta_1-\epsilon}$ since $\beta_1 > \epsilon$ (derivation can be found in Appendix A). This contributes to increasing treshold θ^*. (Greater volatility raises the value of the option to invest, and therefore requires a higher current threshold of profitability to bring forth investment.) But there is a second effect of σ. If σ increases we see that the risk-adjusted rate $\delta' = r - \epsilon\alpha - \frac{1}{2}\epsilon(\epsilon-1)\sigma^2$ decreases. Then equation (5) shows that this added effect leads to a lower θ^* and therefore it contributes a greater inducement to invest. This is due to the socalled "Jensen's Inequality effect" explained in Dixit and Pindyck [10] p.199. The first effect dominates and therefore the threshold θ^* is increasing in uncertainty σ.

The following numerical example will illustrate that. Unless otherwise noted, we set $\alpha = 0$, $\sigma = 0.15$, $r = 0.04$, $c = 10$, $\epsilon = 1.9$, $I = 100$

Figure 1 shows $F(\theta)$ and $V(\theta) - I$ for $\sigma = 0$, 0.1, and 0.15. When $\sigma = 0$, the critical price is approx. 12.3, which makes the value of the project equal to its cost of 100. Both $V(\theta)$ and $F(\theta)$ increase as σ is increasing; θ^* is 16.8 for $\sigma = 0.1$, and 19.16 for $\sigma = 0.15$. This confirms the result that the threshold is increasing in σ.

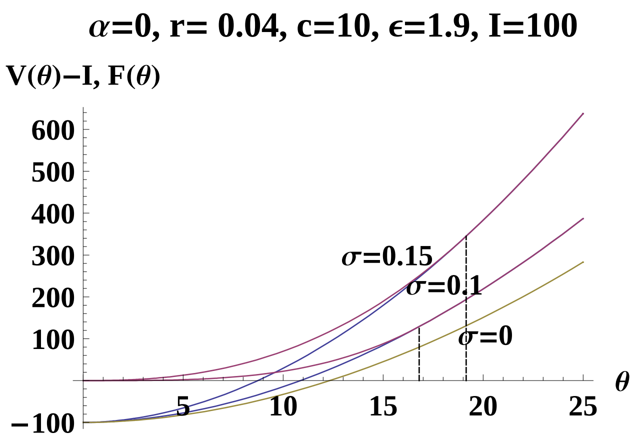

Figure 1: Value of Investment Opportunity, $F(\theta)$, and $V(\theta) - I$, for $\alpha = 0$, 0.005, and 0.01 ($\alpha = 0$, $r = 0.04$, $c = 10$, $\epsilon = 1.9$, and $I = 100$)

2.2 Inflexible Model

In this section we relieve the extension of flexibility in quantity. That means that the firm can just decide about the time to undertake an investment. After investment it will produce a fixed quantity.

The price at time t in this market is given again by the iso-elastic inverse demand function

$$p_t = \theta_t q_t^{-\gamma} \quad with \quad 0 < \gamma < 1,$$

where

$$d\theta_t = \alpha\theta_t dt + \sigma\theta_t dW_t.$$

The firm is risk neutral with a discount rate $r > \alpha$. Variable cost of operation c are known and constant. A firm can become active at the market by paying an investment cost I. The firm always produces up to capacity. The profit of this firm at time t is denoted by $\Pi(t)$ and is equal to

$$\Pi(t) = (p_t - c)q_t = (\theta_t q_t^{-\gamma} - c)q_t$$

We denote V as the value of the firm, where the investment problem the firm is facing is formalized as follows:

$$V(\theta_t, q) = \max_{T \geq t, q \geq 0} E\left[exp(-rT)\left(\int_{\tau=0}^{\infty} q(\theta_\tau q^{-\gamma} - c)exp(-r\tau)d\tau - I\right)\right]$$

Here T is the stopping time at which the investment is made and q the quantity that the firm chooses. The expectation is conditional on the information that is available at time t.

As usual we apply the dynamic programming approach. Solving the Bellman equation, familiar steps produce a differential equation, which we are able to solve. We get

$$V(\theta, q) = \frac{\theta q^{1-\gamma}}{r - \alpha} - \frac{cq}{r} - I, \tag{6}$$

for the expected value of the firm at the moment of investment given that the current level of θ_t is θ and the firm produces q units of quantity.

Maximizing with respect to q gives the optimal capacity size q^* for every given level of θ:

$$q^*(\theta) = \left[\frac{(1-\gamma)\theta r}{c(r-\alpha)}\right]^{\frac{1}{\gamma}} \tag{7}$$

For the solution of the option standard real options analysis shows that the value of the option to invest, denoted by F, is equal to

$$F(\theta) = A\theta^{\beta_1} \tag{8}$$

where β_1 is the positive root of the quadratic polynomial (4). For the optimal investment threshold θ^* we employ the value matching and smooth pasting conditions:

$$F(\theta^*) = V(\theta^*, q), \tag{9}$$

$$\frac{\partial F(\theta)}{\partial \theta}|_{\theta=\theta^*} = \frac{\partial V(\theta, q)}{\partial \theta}|_{\theta=\theta^*} \tag{10}$$

Substitution of (6) and (8) into (9) and (10) and solving for θ^* gives

$$\theta^*(q) = (r-\alpha)\left(\frac{\beta_1}{\beta_1 - 1}\right)\left[\frac{c}{r}q^\gamma + Iq^{\gamma-1}\right] \tag{11}$$

From (7) and (11) we obtain the following results

$$\theta^* = \left[I(r-\alpha)\left(\frac{\beta_1}{\beta_1\gamma - 1}\right)\right]^\gamma \left[\frac{(1-\gamma)r}{c(r-\alpha)}\right]^{\gamma-1}$$

$$q^* = \left(\frac{\beta_1}{\beta_1\gamma - 1}\right) I(1-\gamma)\frac{r}{c}$$

2.3 Value of Flexibility

In order to be able to analyze the value of flexibility we compare the result of the model without flexibility to that one where the firm has the flexibility to adapt quantity to the realization of process θ_t at every instant after the investment has been made.

We get the threshold for the flexible model by solving for θ^* in equation (5)

$$\theta_f^* = \left[\left(\frac{\beta_1}{\beta_1 - \epsilon}\right)\frac{I}{K}(r - \epsilon\alpha - \frac{1}{2}\epsilon(\epsilon-1)\sigma^2)\right]^\gamma,$$

where $K = \gamma\left[\frac{1-\gamma}{c}\right]^{\frac{1-\gamma}{\gamma}}$.

The following proposition states that the flexibility to adapt quantity makes the firm investing earlier. Since the firm is able to adapt the quantity produced to the demand it can invest earlier since it can avoid overproduction in case that the demand falls by simply reducing the quantity produced.

Lemma 1 $\theta_f^* < \theta^*$

In Figure 2 the thresholds are plotted as functions of the uncertainty parameter σ. In both the flexible and the inflexible model the investment threshold is increasing with uncertainty. The Plot shows that the gap between the two thresholds increases significantly with uncertainty. This means that an increasing σ has a stronger effect on the investment threshold of the inflexible model than on that one of the flexible model.

3 Linear Demand Function considering Temporary Suspension

3.1 Flexible Model

The following model extends the idea of an investment model with variable output. Our first model as well as Dixit and Pindyck's model deal with a situation where the optimal quantity, after the investment decision has been made, will always be positive. This means that the firm will continue producing forever. Now we generalize this idea and see what happens when we consider different inverse demand functions. Choosing for a linear inverse demand function we come upon the fact that the optimal quantity produced can also fall to zero. In the following we will see, that this makes it necessary to take into account a suspension option.

We consider the following inverse demand function

$$p = \theta - \gamma q$$

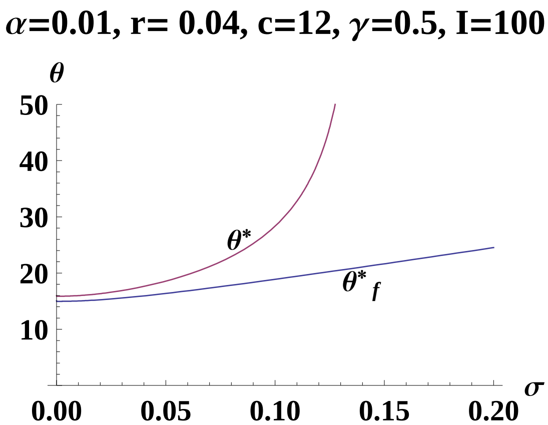

Figure 2: Thresholds for model with flexibility and without, θ_f^* and θ^* ($\alpha = 0.01$, $r = 0.04$, $c = 14$, $\gamma = 0.5$, and $I = 100$)

with $\gamma > 0$ and θ_t following the geometric Brownian motion of (1). We are concentrating on output price uncertainty by setting the input price that is the variable cost of operation, c, as known and constant. We determine the optimal q by maximizing the profit flow $\pi(q) = (\theta - \gamma q)q - cq$, implying that at each point in time q satisfies

$$q = \frac{\theta}{2\gamma} - \frac{c}{2\gamma}. \tag{12}$$

The fact that the optimal quantity can become negative by means of the maximization causes the main difference of this model compared to the first one. The firm will produce as long as $\theta > c$ but in case of $\theta < c$ a production stop would be the optimal strategy. Therefore we make the additional assumption, that operation can be temporarily and costlessly suspended when θ falls below c, and costlessly resumed later if θ rises above c. Considering $q^* = max\left(0, \frac{\theta}{2\gamma} - \frac{c}{2\gamma}\right)$, the profit flow is given by

$$\pi(\theta) = \left\{ \begin{array}{rl} \frac{\theta^2 - 2c\theta + c^2}{4\gamma} & \text{if } \theta > c \\ 0 & \text{if } \theta < c \end{array} \right\},$$

where the firm can set its optimal q at each point in time. This function is a convex w.r.t. the argument θ. In the following we will go through the steps of obtaining the value of the project, $V(P)$. In order to find the value of the project we consider the value of the whole sequence of operating options. Applying the dynamic programming approach, the value function must satisfy the following Bellman equation

$$V(\theta) = \pi(\theta)dt + E\left[V(\theta + d\theta)e^{-rdt}\right], \tag{13}$$

where r is the discount rate. Applying Ito's Lemma, substituting and rewriting leads to the following differential equation

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \alpha\theta\frac{\partial V}{\partial\theta} - rV(\theta) + \left[\left\{ \begin{array}{rl} \frac{\theta^2 - 2c\theta + c^2}{4\gamma} & \text{if } \theta < c \\ 0 & \text{if } \theta > c \end{array} \right.\right] = 0. \tag{14}$$

Equation (30) solved by familiar methods (see Appendix A), leading to the following project value

$$V(\theta) = \left\{ \begin{array}{rl} M_1\theta^{\beta_1} + M_2\theta^{\beta_2} & \text{for } \theta < c, \\ N_1\theta^{\beta_1} + N_2\theta^{\beta_2} + \frac{1}{4\gamma}\left[\frac{1}{r - 2\alpha - \sigma^2}\theta^2 - \frac{2c}{r-\alpha}\theta + \frac{c^2}{r}\right] & \text{for } \theta > c. \end{array} \right. \tag{15}$$

This solution has an intuitive economic interpretation: In the region $\theta < c$, operation is suspended and the project yields no current profit flow. But there is a positive probability that the price process will at some future time move into the region $\theta > c$, when operation will resume and profits will accrue. The value

$V(\theta)$, when $\theta < c$, is just the expected present value of such future flows. In the case of region $\theta > c$ we suppose for a moment that the firm is forced to continue operation of the project forever, even during those times when the risky revenues fall below c. What is the net worth of such a project?

It is just the expected present value of a profit stream $\frac{\theta^2-2c\theta+c^2}{4\gamma}$ calculated using the appropriate risk-adjusted discount rates. The profit flow is consisting of three terms. The last term c^2 is discounted simply by the original discount rate r. For the second one $-2c\theta$ we have to take into account the fact that θ has the growth rate α. We consider, that $E[\theta_t] = \theta e^{\alpha t}$, where θ is the initial level of the process θ_t. Multiplying by e^{-rt} and integrating we get $\frac{\theta}{r-\alpha}$. Therefore, we have to assume that $r > \alpha$. For the derivation of the discount rate of θ^2 we calculate

$$d(\theta^2) = \left(2\alpha + \sigma^2\right)\theta_t^2 dt + 2\sigma\theta_t^2 dW_t. \tag{16}$$

This leads to an expected present value of

$$\int_0^\infty \frac{1}{4\gamma} \exp\left[(2\alpha+\sigma^2)t\right] e^{-rt} dt = \frac{1}{4\gamma}\theta^2 \frac{1}{r-2\alpha-\sigma^2}.$$

For all this to make economic sense we assume that $r > 2\alpha + \sigma^2$. In case that $r < 2\alpha + \sigma^2$ our discount rate would be smaller than the expected growth rate of our process (as we can see in equation (16)). Thus waiting longer would always be a better policy, and the optimum would not exist.

Remark 1 *The networth* $N(\theta) = \frac{1}{4\gamma}\left[\frac{1}{r-2\alpha-\sigma^2}\theta^2 - \frac{2c}{r-\alpha}\theta + \frac{c^2}{r}\right]$ *is always positive for the whole range of* $\theta \geq c$.

The net worth constitutes the last three terms in the solution above. This net worth is always positive even for the region where $\theta < c$ and it raises very high for small θ. Therefore, we need a correction factor, for the possiblity to suspend operations in the future should θ fall below c, of the form $N_1\theta^{\beta_1} + N_2\theta^{\beta_2}$. This would normally be interpreted as an option value (see Dixit and Pindyck [10]). But in our case this factor has to be negative in contrast to the always positive net worth.

The constants in the solution (15) are determined using considerations that apply at the boundaries of the regions. We begin with $\theta < c$: As θ becomes very small, the resumption option c is unlikely to be invoked except perhaps in the very remote future. The expected present value of future profits should then go to zero, and so should the value of the project. But since β_2 is negative, θ^{β_2} goes to ∞ as θ goes to 0. Therefore the constant multiplying this term, M_2, should be zero.

Turning to $\theta > c$ we have to consider when θ becomes very large, the event of its falling below c becomes very unlikely. Therefore value of the correction factor, which isn't part of the revenue scheme, should then go to zero. But $N_1\theta^{\beta_1}$ would diverge. For this we have to rule out the positive power of θ, by setting $N_1 = 0$. This leads to the following project value:

$$V(\theta) = \begin{cases} M_1\theta^{\beta_1} & \text{for } \theta < c, \\ N_2\theta^{\beta_2} + \frac{1}{4\gamma}\left[\frac{1}{r-2\alpha-\sigma^2}\theta^2 - \frac{2c}{r-\alpha}\theta + \frac{c^2}{r}\right] & \text{for } \theta > c. \end{cases}$$

The analytical derivations of the two parameters M_1 and N_2 are shown in Appendix A.

Remark 2 $M_1 > 0$ *and* $N_2 < 0$ *for all the possible parameter ranges.*

Since the term in M_1 captures the expected profit from the option to resume operations in the future it should be positive. The constant N_2 is negative. The second term in (17) is always positive and raises very high for small θ. Therefore, the first term has to be negative in order to act as a correctionvalue because we have to consider the possibility of future suspension in case θ falls below c.

In the following we are illustrating this solution with a numerical example. Unless otherwise noted, we set $c = 10, I = 100, r = 0.04, \alpha = 0.01$ and $\sigma = 0.1$. Figure 3 shows $V(\theta)$ as a function of θ for $\sigma = 0$, 0.05, 0.075, and 0.1. When $\sigma = 0$, there is no possibility that θ will rise in the future. In this case the value of the project will just be the discounted profit function, $V(\theta) = \pi(\theta)/r$. For $\sigma > 0$, the upside potential for future profit is unlimited while the downside is limited to zero. Therefore, the greater σ, the greater is the

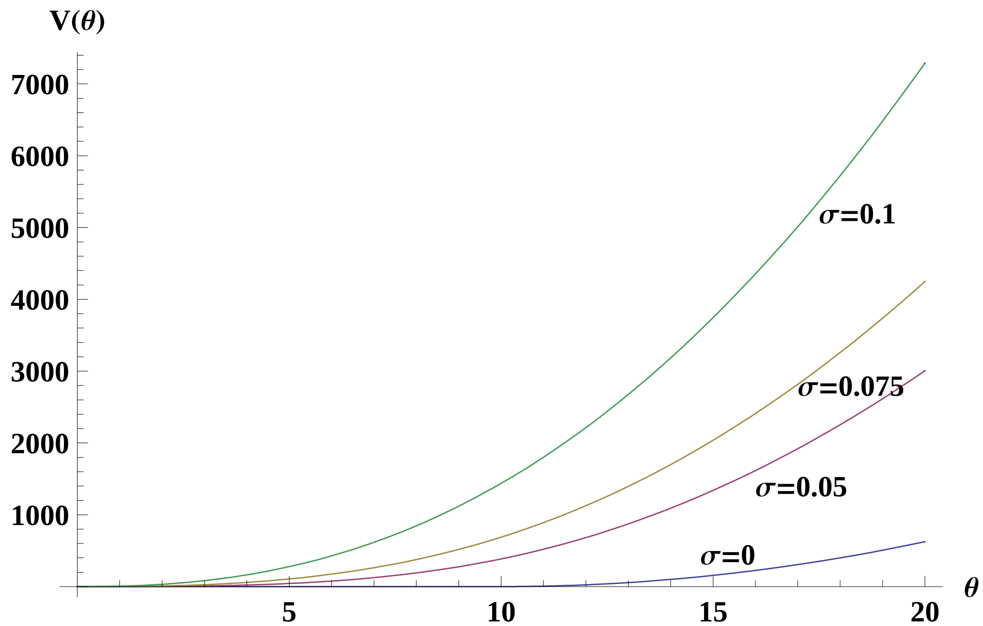

Figure 3: Project Value $V(\theta)$, for $\sigma = 0.05$, 0.075 and 0.1

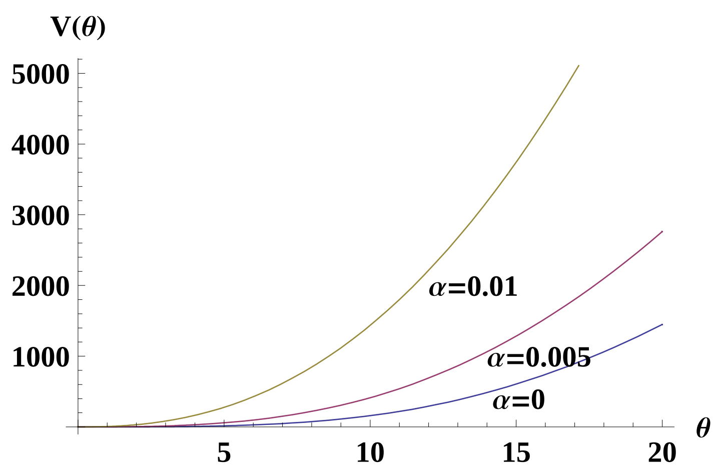

Figure 4: Project Value $V(\theta)$, for $\alpha = 0$, 0.005 and 0.01

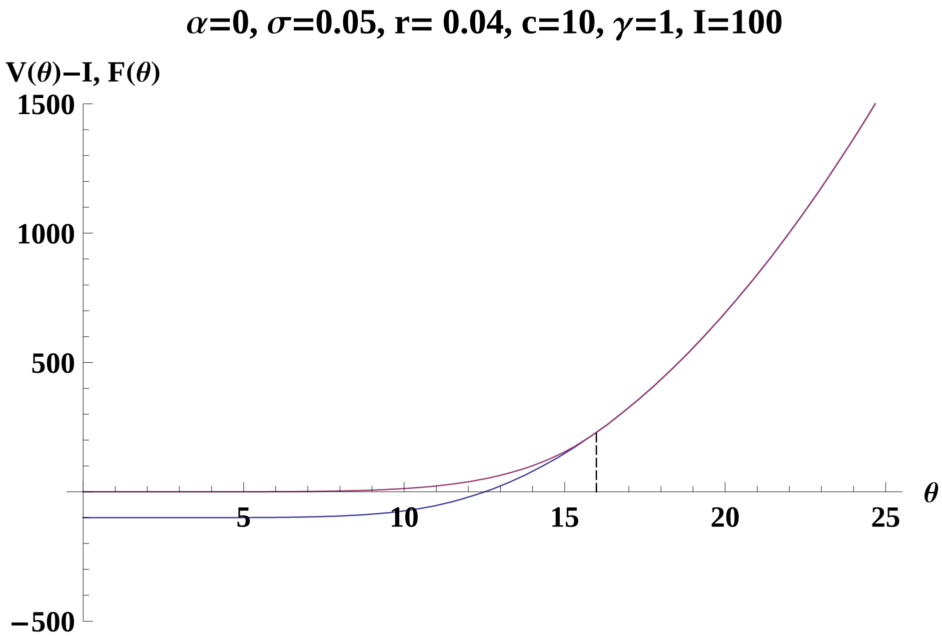

Figure 5: Value of Investment Opportunity, $F(\theta)$, and $V(\theta)-I$ ($\alpha=0$, $r=0.04$, $c=10$, $\gamma=1$, and $I=100$)

expected future flow of profit, and the higher is $V(\theta)$. Figure **??** clearly shows this expected result in the real options literature.

Figure 4 shows the value of the project for different values of α. In case we keep the discount rate fixed, a higher value of α means a higher expected rate of price appreciation, and hence a higher value for the project.

Knowing the value of the project, $V(\theta)$, we can find the value of the option to invest in the project, $F(\theta)$. Considering Dixit and Pindyck [10] we know that the value of the option to invest takes the form

$$F(\theta) = A_1\theta^{\beta_1} + A_2\theta^{\beta_2}.$$

Since $\theta = 0$ is an absorbing barrier, so that $F(0) = 0$, we know that $A_2 = 0$. At the optimal exercise point θ^* we have the usual conditions of value matching and smooth pasting. The option will not be exercised when $\theta < c$ because there is no reason to incur the investment cost I only to keep the project idle for some time. Formally this means that $A_1\theta^{\beta_1}$ cannot satisfy the value matching and smooth pasting with $M_1\theta^{\beta_1} - I$, therefore we use the solution for $V(\theta)$ in the operating region. Applying the value-matching and smooth-pasting conditions we can solve for A_1 and θ^*:

$$N_2(\theta^*)^{\beta_2}[\beta_1-\beta_2] + \frac{1}{4\gamma}\left[\frac{1}{r-2\alpha-\sigma^2}(\theta^*)^2(\beta_1-2) - \frac{2c}{r-\alpha}\theta^*(\beta_1-1) + \frac{c^2}{r}\beta_1\right] \quad = \quad I\beta_1 \tag{17}$$

Proposition 1 *Equation* (17) *has a unique threshold* $\theta^* > c$.

Equation (17), which gives the optimal investment rule, can easily be solved numerically. Proposition 1 states that (17) has a unique positive solution for θ^* that is larger than the variable cost of operation c.

Taking on our numerical example will help to illustrate this solution. The solution for $F(\theta)$ and θ^* is shown graphically in Figure 5. Unless otherwise noted, we set $\alpha = 0$, $\sigma = 0.05$, $r = 0.04$, $c = 10$, $\gamma = 1$, $I = 100$. The figure shows $F(\theta)$ and $V(\theta) - I$, where θ^* is the investment threshold satisfying the value-matching and smooth-pasting condition. Furthermore, we examine how these curves shift when σ or α are changed. We will show this in order to approve the general prediction of the real options literature, stating that a higher level of uncertainty increases the value of waiting and thus has a negative effect on investment.

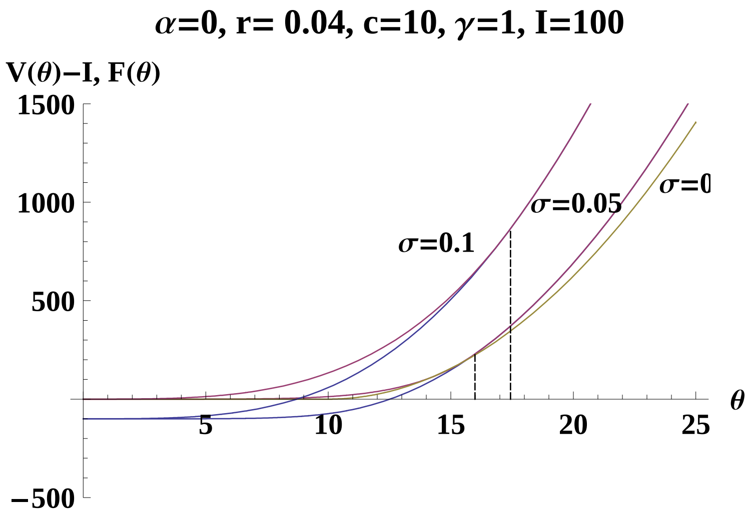

Figure 6: Value of Investment Opportunity, $F(\theta)$, and $V(\theta) - I$, for $\sigma = 0$, 0.05, and 0.1 ($\alpha = 0$, $r = 0.04$, $c = 10$, $\epsilon = 1.9$, and $I = 100$)

Figure 6 shows $F(\theta)$ and $V(\theta) - I$ for $\sigma = 0$, 0.05, and 0.1. When $\sigma = 0$ the critical price is 14. This value is equal to the Marshallian full cost (operating cost plus interest on the capital cost of investment), $(c + rI)$. At $\theta^* = 14$ the value of the project is equal to its cost of 100. As σ is increasing, both $V(\theta)$ and $F(\theta)$ increase. θ^* is 15.9886 for $\sigma = 0.05$, and 17.4341 for $\sigma = 0.1$. We already observed that an increase in σ results in an increase in $V(\theta)$ for any θ. But even though an increase in σ raises the value of the project, it also increases the critical price at which it is optimal to invest, as we can see in Figure 6. That arises from the fact, that for any θ, the value of the investment option (and thus the opportunity cost of investing), $F(\theta)$, increases even more than $V(\theta)$. Hence, increased uncertainty raises the value of waiting and thus decelerates investment.

Furthermore, an increase in α decreases the investment threshold θ^*. If the rate of increase in θ is greater, which means that the options on future production are worth more, it means that $V(\theta)$ increases (see Figure 4). The second effect influences to the opposite direction. A greater α means a greater expected rate of growth of $F(\theta)$, i.e. the opportunity cost of waiting to invest is smaller. This results in more incentive to exercise the investment option, rather than keep it alive. The first effect dominates, therefore a higher α results in a higher θ^*. This is illustrated in Figure 7, which shows $F(\theta)$ and $V(\theta) - I$ for $\alpha = 0$, 0.005, and 0.01.

3.2 Inflexible Model

Also for the model with the linear inverse demand function

$$p_t = \theta_t - \gamma q \quad with \quad \gamma > 0,$$

where

$$d\theta_t = \alpha\theta_t dt + \sigma\theta_t dW_t.$$

we will now relieve the possibility of quantity adaption after investment and derive the result for the optimal investment threshold of this inflexible model.

The profit of this firm at time t is denoted by $\Pi(t)$ and is equal to

$$\Pi(t) = (p_t - c)q_t = (\theta_r - \gamma q - c)q$$

The investment problem the firm is facing is formalized as follows

$$V(\theta_t, q) = \max_{T \geq t, q \geq 0} E\left[exp(-rT)\left(\int_{\tau=0}^{\infty} q(\theta_\tau - \gamma q - c)exp(-r\tau)d\tau - I\right)\right])$$

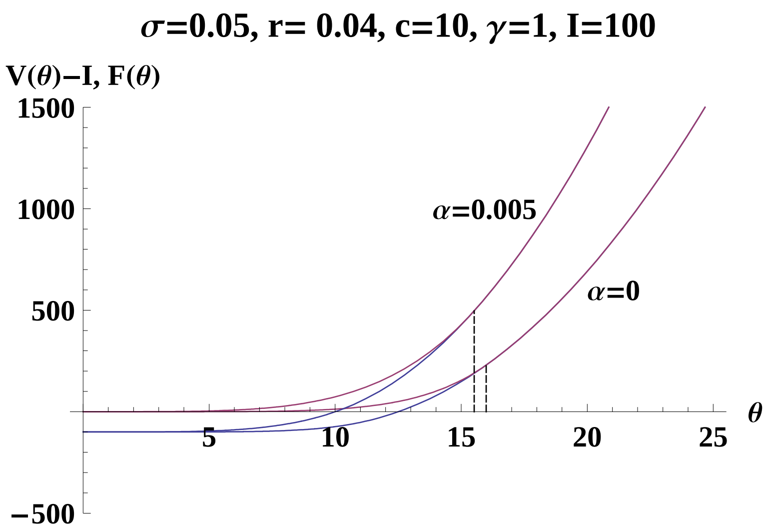

Figure 7: Value of Investment Opportunity, $F(\theta)$, and $V(\theta) - I$, for $\alpha = 0$, and 0.005 ($\alpha = 0$, $r = 0.04$, $c = 10$, $I = 100$)

where we denote V as the value of the firm. Here T is the stopping time at which the investment is made and q the quantity that the firm chooses. The expectation is conditional on the information that is available at time t. Applying the dynamic programming approach, we get the following expected value of the firm at the moment of investment given that the current level of θ_t is θ and the firm produces q units of quantity.

$$V(\theta, q) = \frac{q\theta_t}{r-\alpha} - \frac{(\gamma q + c)q}{r} - I \tag{18}$$

Maximizing with respect to q gives the optimal capacity size q^* for every given level of θ:

$$q^*(\theta) = \frac{1}{2\gamma}\left[\frac{\theta r}{r-\alpha} - c\right] \tag{19}$$

The value of the option to invest, denoted by F, is equal to

$$F(\theta) = A\theta^{\beta_1} \tag{20}$$

with β_1 the positive root of (4). To determine the investment threshold θ^* we employ the value matching and smooth pasting conditions:

$$F(\theta^*) = V(\theta^*, q), \tag{21}$$

$$\frac{\partial F(\theta)}{\partial \theta}|_{\theta=\theta^*} = \frac{\partial V(\theta, q)}{\partial \theta}|_{\theta=\theta^*} \tag{22}$$

Substitution of (18) and (20) into (21) and (22) and solving for θ^* gives

$$\theta^*(q) = \frac{r-\alpha}{q}\left(\frac{\beta}{\beta-1}\right)\left[\frac{(\gamma q + c)q}{r} + I\right] \tag{23}$$

Equation (19) and (23) are two equations in two variables which can be solved analytically.

3.3 Value of Flexibility

In order to get to know about the Value of Flexibility we have to compare the two values of the inflexible and flexible model as demonstrated in Section 2.3. We derived this value for some numerical examples without gaining a complete overview of the result. We considered it necessary to extend our model regarding capacity choice first. This extensions are the content of Chapter 5.

4 Simulation

In this section we use simulations in order to point out the difference between the two models, focusing on the optimal quantity function. Figure 8 shows a simulation of the process θ_t with parameter values $\alpha = 0.01$ and $\sigma = 0.1$. The process is simulated for a duration of 20 years divided into 10000 time steps. Furthermore, we set $c = 14$ and $I = 100$. Figure 8 shows that the process falls under the boundary of variable cost, $c = 14$, shortly after the beginning of the period under consideration and reaches first the lower investment threshold for the isoelastic model, $\theta^* = 19.62$, and then that one for the linear model, $\theta^* = 21.17$, in year 4. Between year 4 and 18 the process moves above the boundary of variable cost. In year 18 the process falls under the boundary of variable cost the first time after investment has been made. In the following two sections we look at the optimal quantity q^* as a function of process θ_t.

4.1 Optimal Output Simulation - Isoelastic Model

The isoelastic model gives the following optimal quantity.

$$q^* = \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}.$$

We set $\gamma = 0.1$ and get an optimal investment threshold of $\theta^* = 19.62$. Figure **??** shows the optimal quantity for the realization of the process θ_t at every instant. After the investment has been taken at point $q^* = 21.8$ the quantity will be attapted to the realisation of the process θ at every instant. We see, that optimal quantity is always positive even at the point where the process θ falls beyond the variable cost c in year 18 and production is not profitable anymore.

4.2 Optimal Output Simulation - Linear Model

The model with the linear inverse demand function gives the following optimal quantity q^*

$$q^* = max\left(0, \frac{\theta}{2\gamma} - \frac{c}{2\gamma}\right).$$

Figure 8 shows that the optimal point to invest is, when realisation of process θ_t reaches $\theta^* = 21.17$. Setting $\gamma = 1$ we get an optimal investment threshold of $\theta^* = 21.17$. Figure 10 shows the corresponding quantity to be produced at every instant depending on the realization of process θ_t. At the point where process θ_t falls below variable cost c in year 18, the crucial difference to the former model is clear - optimal quantity produces falls to 0 and operation is suspended temporarily. It is resumed later when θ_t rises above c.

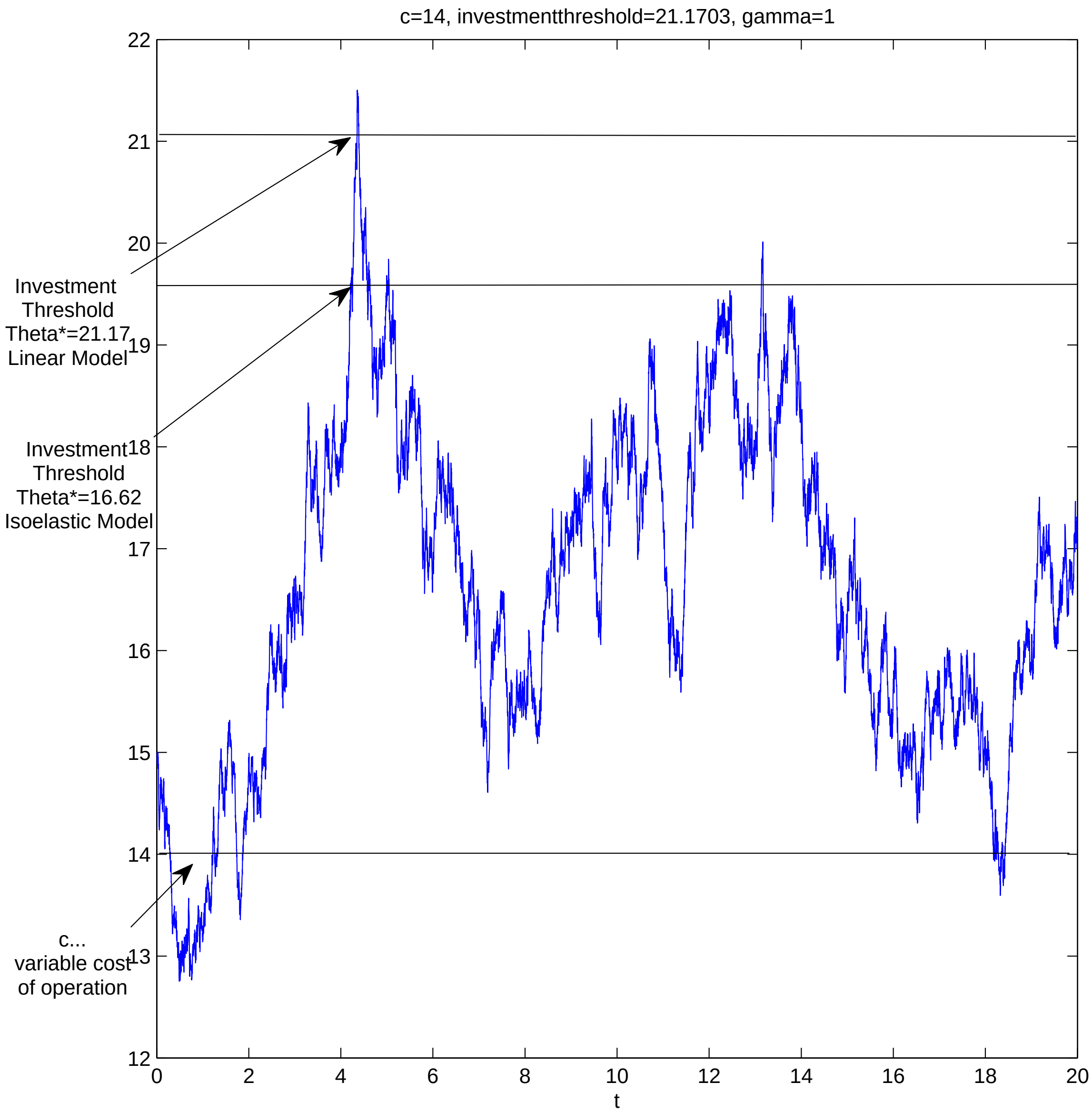

Figure 8: Simulation of Process θ^* ($\alpha = 0.01$, $\sigma = 0.1$, $r = 0.04$, $c = 14$, $I = 100$)

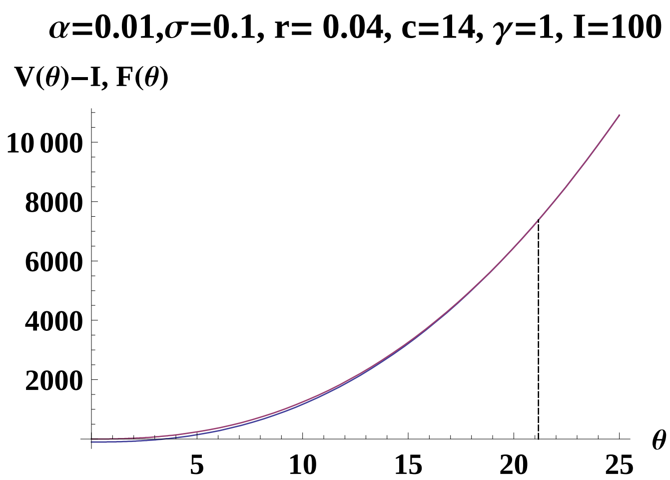

Figure 9: Value of Investment Opportunity, $F(\theta)$, and $V(\theta) - I$ and Investment Threshold θ^* ($\alpha = 0.01$, $\sigma = 0.1$, $r = 0.04$, $c = 10$, $I = 100$)

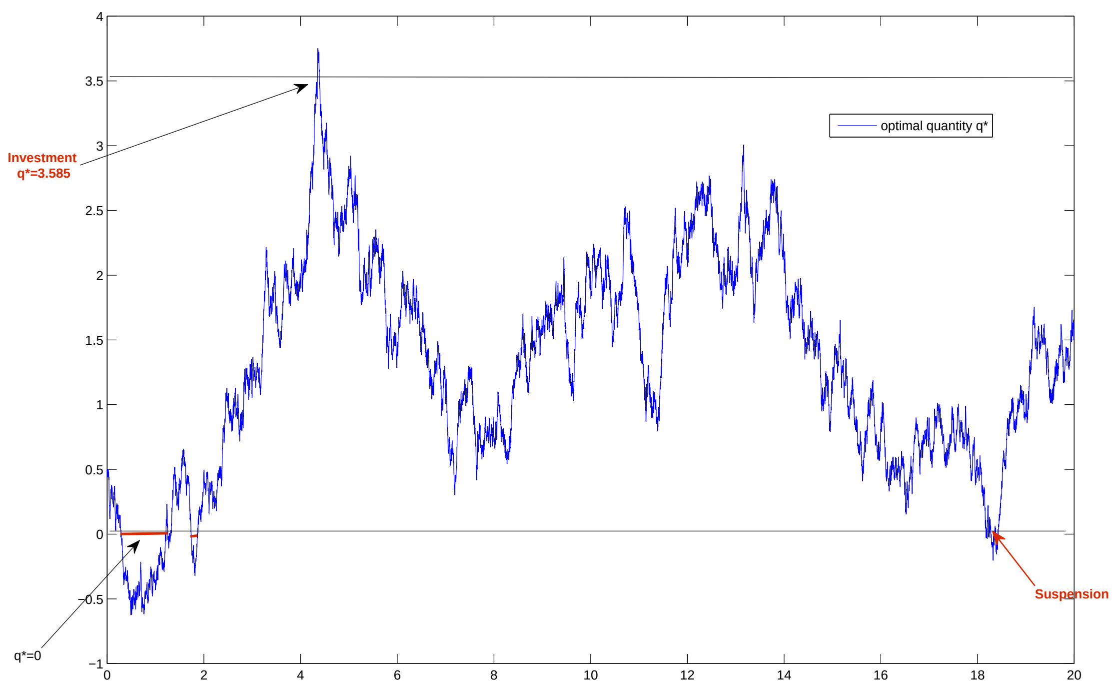

Figure 10: optimal quantity q^* for the range of 20 years ($\alpha = 0.01$, $\sigma = 0.1$, $r = 0.04$, $c = 14$, $I = 100$)

5 Variable Production and Capacity Choice

This section extends the idea of variable production considering limited capacity, where the firm chooses capacity at the moment it invests. The model is based on the previous one, but now the firm has to take into account capacity when it produces and choose for an upper capacity boundary when making the investment decision.

Capacity is denoted by K. After the investment it either holds that

$$q = K,$$

or

$$q < K,$$

where the latter holds in case of a low realization of the stochastic process θ_t. At the moment it invests the firms must also choose in how much capacity to invest. The investment costs are depending on the capacity level K. These costs are sunk and assumed to have the general form of $I(K) = \delta K^\lambda$ with $\lambda \leq 1$. This means that the marginal investment costs are decreasing with increasing installed capacity. Variable cost of operation are as before known and constant.

5.1 Linear Inverse Demand Function

First, we choose the linear inverse demand function

$$p = \theta - \gamma q. \tag{24}$$

As in Chapter 3 we determine the optimal q^* by maximizing the profit flow

$$q^* = \max_q \left(0, \frac{\theta}{2\gamma} - \frac{c}{2\gamma}\right). \tag{25}$$

Considering that we have limited capacity K we get the following function for the profit flow

$$\pi(\theta) = \begin{cases} (\theta - \gamma K - c)K & \text{for } \theta \geq 2\gamma K + c \\ \frac{(\theta-c)^2}{4\gamma} & \text{for } c \leq \theta < 2\gamma K + c \\ 0 & \text{for } 0 \leq \theta < c, \end{cases}$$

where we denote $\theta_1 = c$ and $\theta_2 = 2\gamma K + c$.

In the following we study the market entry of the firm. V denotes the value of the firm. The investment problem that the firm is facing is formalized as follows

$$V(\theta_t, K(t)) = \max_{T \geq t, K \geq 0} E\left[exp(-rT)\left(\int \pi(\tau)exp(-r\tau)d\tau - \delta K\right)\right],$$

where T is the stopping time at which investment is made and K the quantity that the firm chooses. The expectation is conditional on the information that is available at time t. Familiar steps lead to the following differential equation

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \alpha\theta\frac{\partial V}{\partial\theta} - rV(\theta) + \pi(\theta) = 0,$$

Solving this equation for $V(\theta, K)$ considering that we have three different regions and ruling out bubble solutions we get the following value of the project

$$V(\theta, K) = \begin{cases} N_2\theta^{\beta_2} + \frac{K}{r-\alpha}\theta - \frac{K(K\gamma+c)}{r} & \text{for } \theta \geq \theta_2 \\ M_1\theta^{\beta_1} + M_2\theta^{\beta_2} + \frac{1}{4\gamma}\left[\frac{\theta^2}{r-2\alpha-\sigma^2} - \frac{2c\theta}{r-\alpha} + \frac{c^2}{r}\right] & \text{for } \theta_1 \leq \theta < \theta_2 \\ L_1\theta^{\beta_1} & \text{for } 0 \leq \theta < \theta_1 \end{cases}$$

The derivations of this value are shown in Appendix B.

Subsequently we will go through the steps of obtaining the value of the option to invest and the optimal investment strategy. In order to get the optimal capacity choice, we ignore the investment timing decision for a moment which, means that the firm just decides either to invest or not. The firm has to find the capacity level K^* in which it is optimal to invest. Therefore, we calculate the derivative of the value of the project, $V(\theta, K)$, w.r.t. K and check the second derivative of the value function at the optimal K^*. Since we have three different regions, we have to calculate this optimal boundary for each of them separately and compare the value of the project at each of them. For matters of clearness we denote

$$V(\theta, K) = V_i \text{ for region i},$$

where

$$\begin{aligned} \text{Region I:} \quad & 0 \leq \theta < \theta_1 \\ \text{Region II:} \quad & \theta_1 \leq \theta < \theta_2 \\ \text{Region III:} \quad & \theta \geq \theta_2 \end{aligned}$$

We calculate the derivative of the value of the project w.r.t K and set it equal to zero, i.e.

$$\frac{\partial(V_3(\theta, K) - I(K))}{\partial K} = \frac{dN_2}{dK}\theta^{\beta_2} + \frac{\theta}{r-\alpha} - \frac{2K\gamma + c}{r} - \delta\lambda K^{\lambda-1} = 0 \tag{26}$$

$$\frac{\partial(V_2(\theta, K) - I(K))}{\partial K} = \frac{dM_1}{dK}\theta^{\beta_1} = 0 \tag{27}$$

$$\frac{\partial(V_1(\theta, K) - I(K))}{\partial K} = \frac{dL_1}{dK}\theta^{\beta_1} = 0 \tag{28}$$

The analytical derivation of these values is shown in Appendix B.

At this point we want to address the paper of Dangl [7]. He derives his results for a similar model using comparative statics. Since we want to elaborate on his work regarding the economic results, our choice of parameter values for the following figures corresponds to that one of Dangl.

Continuing with our derivations, we first look at region III ($\theta > \theta_2$): We are able to find solutions of equation (26) numerically for different values of θ. To make sure that this value is a maximum, we check the second order derivative. All this calculations can be looked up in Appendix B. Figure 11 shows the optimal choice K^* for region III.

Since for the special parameter values used for figure 11 the second derivative of $V_3 - I$ is zero, we have to consider the third derivative, which shows that K^* is a local maximum. Since K^* for V_3 is outside the considered boundary of region III (demonstrated in figure 11), the optimal capacity choice (for this region) is the boundary value $K_2(\theta) = \frac{\theta - c}{2\gamma}$.

Now we continue with deriving the optimal capacity choice, K^*, for (27) and (28). Since

$$\frac{dL_1}{dK}\theta^{\beta_1} = \frac{dM_1}{dK}\theta^{\beta_1},$$

the equation that has to be solved for K^* is the same in both regions. Figure 12 shows the optimal K^* for region $\theta < \theta_2$.

Finally, we have to decide in which region to install the capacity level. Since the firm does not start producing for realizations of the process θ_t in region I, where $\theta < c$, it does not make sense to invest there. In that case the firm would invest in capacity which it would not use for production yet and therefore paying the investment costs without gaining any profit. Thus, the firm prefers to wait for more information. The two remaining possibilities are: Either the firm invests at a realization of the stochastic process in region II. This means that it invests in capacity it does not fully use right at the moment of investment. The second possibility is to invest (in region III) in a capacity level that is fully used from the start. For our current example this means that the optimal choice of capacity K^* is either the boundary for region III or the FOC in region II. Therefore, we calculate the value of the firm for both solutions and compare them. Figure 13 shows this values for uncertainty level $\sigma = 0.1$. Since, for this numerical example the value of the project is higher for the optimal capacity level of region II, the optimal strategy is to choose the capacity level in that

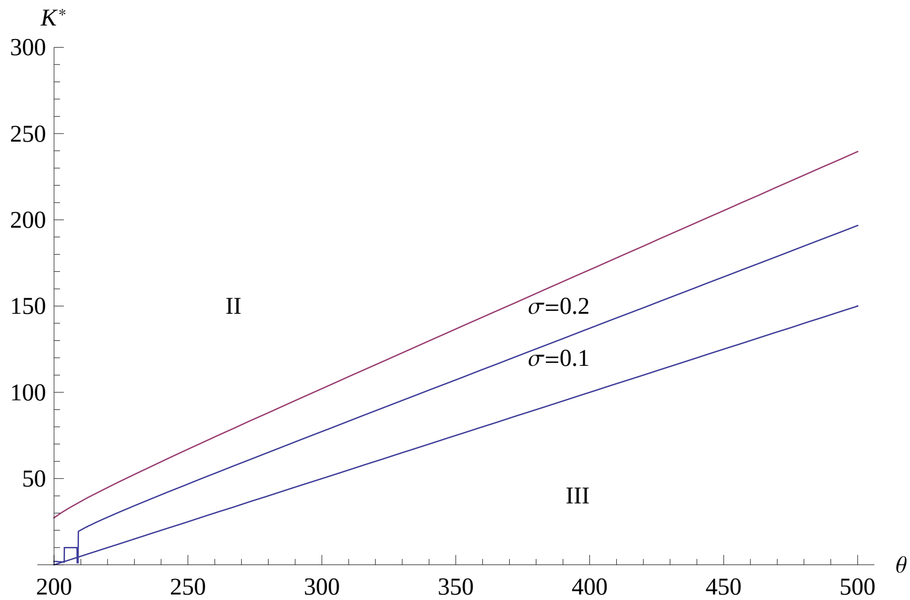

Figure 11: Optimal K, $K^*(\theta)$, for project value V_3 in region $\theta > \theta_1$ for $\sigma = 0.1,\ 0.2$ ($\alpha = 0.02$, $\sigma = 0.1$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$ and $\lambda = 0.7$)

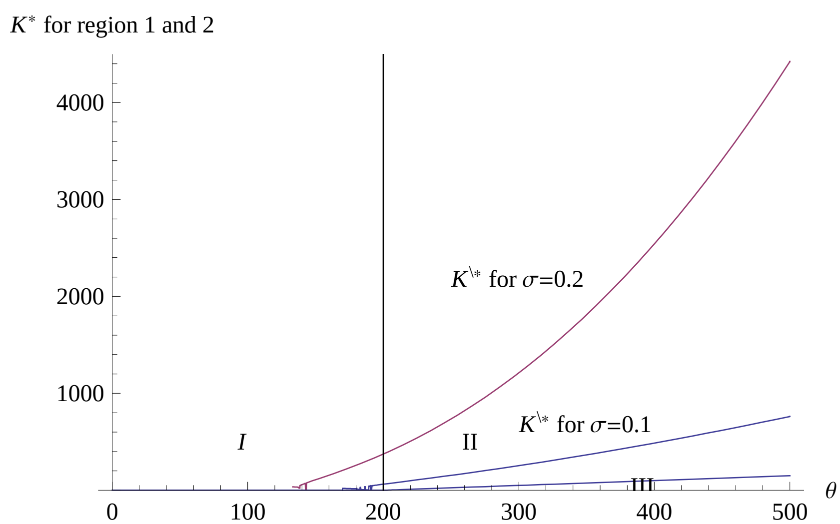

Figure 12: optimal K for the region of $\theta \leq \theta_2$, $K^*(\theta)$ ($\alpha = 0.02$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$ and $\lambda = 0.7$)

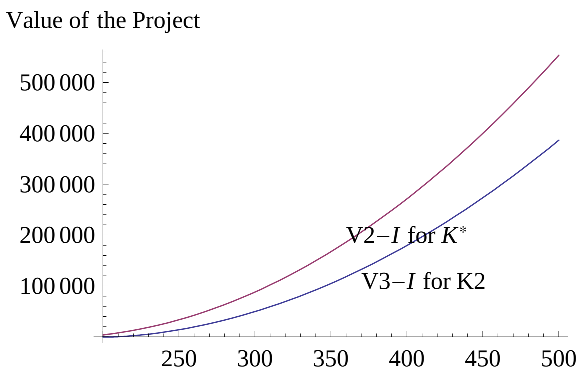

Figure 13: Project value for the optimal capacity choice in region II and III for $\sigma = 0.1$ ($\alpha = 0.02$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$ and $\lambda = 0.7$)

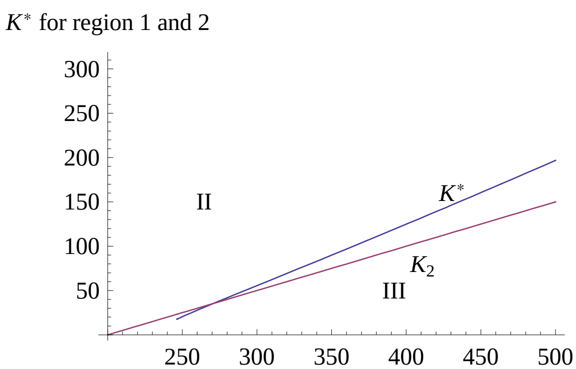

Figure 14: Project value for the optimal capacity choice in region II and III ($\sigma = 0.05$, $\alpha = 0.002$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$ and $\lambda = 0.7$)

region and leave some capacity idle for the moment. We get the same result of investing in region II also for uncertainty level of $\sigma = 0.2$.

While this numerical example coincides with that one of Dangl [7], he does not take into account the possibility of investing in region III. Dangl's derivation of the optimal capacity choice misses the differentiation between the three regions.

Since, our specific capacity choice for the latter example is affected by the particular choice of parameters, we want to investigate in a next step if and for which parameter values it is optimal to invest in region III.

Therefore, we look at another numerical example. This shows that there are cases for which it is optimal to produce up to full capacity right after investment is made. Changing the parameters to $\sigma = 0.05$ and $\alpha = 0.02$ shows that in region III it is again optimal to invest at the boundary K_2. The optimal capacity choice for region II is shown in figure 14. Figure 15 shows that the value of the project is this time higher for the boundary of region III, which means that the firm should invest at a capacity level which is fully used from the start. This affirms our expectation that for certain parameters it is optimal to invest in region III.

The next step is to find the optimal timing strategy. So we are looking for the optimal investment threshold θ^* which separates the 'waiting' from the 'investment region'. Therefore, we have to calculate the value of this option. Using the value matching and smooth pasting condition as in Chapter 3 will give the optimal investment strategy.

6 Conclusions

Our paper shows that a generalization of the inverse demand function can lead to the fact that one has to consider the possibility of suspension of production. In particular, we show that with a linear inverse

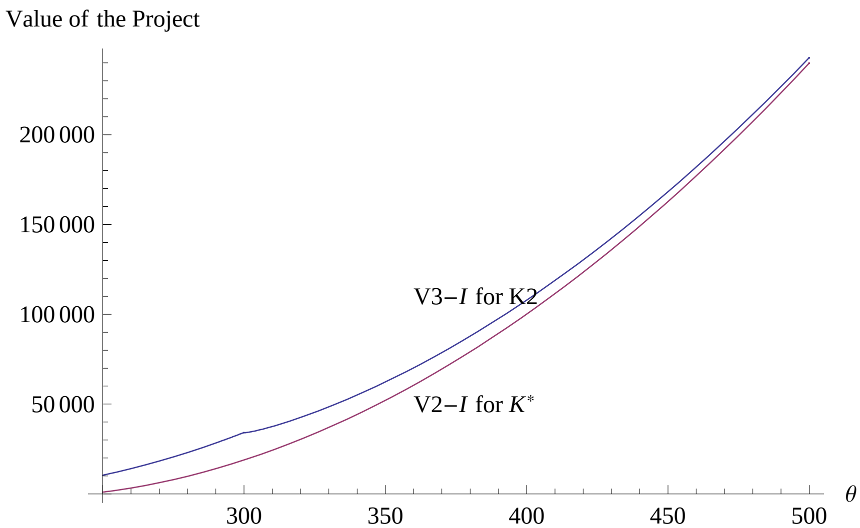

Figure 15: Project value for the optimal capacity choice in region II and III ($\sigma = 0.05$, $\alpha = 0.002$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$ and $\lambda = 0.7$)

demand function the profit maximization can lead to an optimal quantity of zero, while with a Iso-elastic inverse demand function once the investment decision is made it will be produced forever.

Furthermore, we analyze the value of flexibility by comparing these models with their inflexible matches and work on comparative statics considering the different investment thresholds.

As a second step we consider limited capacity where the firm can choose capacity at the moment it invests. Our results show that there are differences in the investment strategy for different parameter values. There is either the possibility to invest in capacity which is fully used right after investment or the firm leaves some capacity idle for the moment. In a next step we want to investigate the boundary which marks the point where the strategy changes from leaving some capacity idle after investment to producing up to full capacity at the first moment.

Considering recent literature a relevant extension will be the comparison of our results to that one in a duopoly setting. As already mentioned in the introduction, many papers consider operational flexibility in a competitive environment but in two- or three-stage decision settings. We are looking at the results for a continuous time setting model.

A Appendix

A.1 Proofs and Derivations of Chapter 2

The particular solution of equation (2) is the expected present value of a profit stream $K\theta^{\epsilon}$ calculated using the appropriate risk-adjusted discount rate. Using Ito's Lemma we can derive the mean and variance of the lognormal distributed θ_t^{ϵ}.

$$d(log\theta_t^{\epsilon}) = \left[\epsilon\alpha - \frac{1}{2}\epsilon\sigma^2\right] dt + \epsilon\sigma dW_t.$$

Therefore

$$log\theta_t^{\epsilon} \sim N\left[log\theta^{\epsilon} + \left(\epsilon\alpha - \frac{1}{2}\epsilon\sigma^2\right) T, \epsilon^2\sigma^2 T\right],$$

which means for θ_t^ϵ that it has mean

$$\exp(log\theta^\epsilon + (\epsilon\alpha - \frac{1}{2}\epsilon\sigma^2)t) = \theta^\epsilon \exp\left[(\left(\epsilon\alpha - \frac{1}{2}\epsilon\sigma^2\right) t)\right].$$

Then for any future t

$$E'\left[K\theta_t^\epsilon\right] = K\theta^\epsilon \exp\left[\left(\epsilon\alpha + \frac{1}{2}\epsilon(\epsilon-1)\sigma^2\right) t\right],$$

where E' denotes the expectation calculated relative to this new process. Multiplying by $\exp(-rt)$ and integrating we get the present value of this expectation

$$\int_0^\infty K\theta^\epsilon \exp\left[\left(\epsilon\alpha + \frac{1}{2}\epsilon(\epsilon-1)\sigma^2\right) t\right] \exp\left(-rt\right)dt = K\theta^\epsilon \frac{1}{r - \epsilon\alpha - \frac{1}{2}\epsilon(\epsilon-1)\sigma^2}.$$

By ruling out bubble solutions we can set B_1 in (3) equal zero. Therefore, we calculate the expected revenue of the project. This should be equal to the value of the project:

$$V(\theta(t)) = E\left[\int_{\tau=t}^\infty K\theta(\tau)^\epsilon exp\left[-r(\tau-t)\right]d\tau\right] - I.$$

Let $Z(\tau) = \theta(t+\tau)$ then we have that

$$\begin{aligned}
V(\theta(t)) &= V(Z(0)) \\
&= E\left[\int_{\tau=t}^\infty KZ^\epsilon(\tau-t)exp\left[-r(\tau-t)\right]d\tau\right] - I \\
&= E\left[\int_{\tau=0}^\infty KZ^\epsilon(\tau)exp\left[-r\tau\right]d\tau\right] - I.
\end{aligned}$$

We have that

$$Z(\tau) = Z(0)exp\left[(\alpha - \frac{1}{2}\sigma^2)\tau + \sigma W(\tau)\right],$$

which leads to

$$V(\theta(t)) = E\left[\int_{\tau=0}^\infty KZ^\epsilon(0)exp\left[\epsilon(\alpha - \frac{1}{2}\sigma^2)\tau + \epsilon\sigma W(\tau)\right]exp\left[-r\tau\right]d\tau\right] - I,$$

and with $Z(0) = \theta(t)$:

$$\begin{aligned}
V(\theta(t)) &= E\left[\int_{\tau=0}^\infty K\theta^\epsilon(t)exp\left[\epsilon(\alpha - \frac{1}{2}\sigma^2)\tau + \epsilon\sigma W(\tau)\right]exp\left[-r\tau\right]d\tau\right] - I \\
&= K\theta^\epsilon(t)\int_{\tau=0}^\infty\int_{\omega=-\infty}^\infty \frac{1}{\sqrt{\tau}\sqrt{2\pi}}exp\left[-\frac{\omega^2}{2\tau} + \epsilon(\alpha - \frac{1}{2}\sigma^2)\tau - r\tau + \epsilon\sigma\omega\right]d\omega d\tau - I \\
&= K\theta^\epsilon(t)\int_{\tau=0}^\infty exp\left[(-r + \epsilon\alpha - \frac{1}{2}\epsilon\sigma^2)\tau\right]\int_{\omega=-\infty}^\infty \frac{1}{\sqrt{\tau}\sqrt{2\pi}}exp\left[-\frac{(\omega-\epsilon\sigma\tau)^2}{2\tau} + \frac{\epsilon^2\sigma^2\tau}{2}\right]d\omega d\tau - I \\
&= \theta^\epsilon(t)\int_{\tau=0}^\infty exp\left[(-r +_\epsilon \alpha - \frac{1}{2}\epsilon\sigma^2)\tau\right]exp\left[\frac{1}{2}\epsilon^2\sigma^2\tau\right] - I \\
&= K\theta^\epsilon(t)\frac{1}{r - \epsilon\alpha - \frac{1}{2}\epsilon(\epsilon-1)\sigma^2} - I.
\end{aligned}$$

Taking notice of the fact, that in the expected revenue of the project there is no factor like $B_1\theta^{\beta_1}$, we can also set the second unkown in the project value B_1 equal to zero.

Explanation of $\beta_1 > \epsilon$: For the model to make economic sense we need that the risk-adjusted discount rate $\delta' = r - \epsilon\alpha - \frac{1}{2}\epsilon(\epsilon-1)\sigma^2$ is strictly bigger than zero, $\delta' > 0$. The expression of δ' is just the negative of our fundamental quadratic (4) evaluated at ϵ. Therefore, by requiring that $\delta' > 0$ also $Q(\epsilon) < 0$ has to be fulfilled. This means that ϵ must lie between the two roots of the quadratic, specifically $\epsilon < \beta_1$.

Proof of Lemma 1 $\theta_f^* < \theta^*$

$$
\begin{aligned}
\left[\left(\frac{\beta}{\beta-\epsilon}\right)\frac{I}{K}(r-\epsilon\alpha-\frac{1}{2}\epsilon(\epsilon-1)\sigma^2)\right]^\gamma &< \left[I(r-\alpha)\left(\frac{\beta}{\beta\gamma-1}\right)\right]^\gamma\left[\frac{(1-\gamma)r}{c(r-\alpha)}\right]^{\gamma-1} \\
(r-\epsilon\alpha-\frac{1}{2}\epsilon(\epsilon-1)\sigma^2)^\gamma &< (r-\alpha)r^{\gamma-1} \\
0 &< (r-\alpha)r^{\gamma-1} - (r-\epsilon\alpha-\frac{1}{2}\epsilon(\epsilon-1)\sigma^2)^\gamma
\end{aligned}
$$

We denote $f(\sigma) = (r-\alpha)r^{\gamma-1} - (r-\epsilon\alpha-\frac{1}{2}\epsilon(\epsilon-1)\sigma^2)^\gamma$. Since the following two points are fulfilled

1. $f(\sigma=0) = (r-\alpha)r^{\gamma-1} - (r-\epsilon\alpha)^\gamma$,
2. $\frac{df}{d\sigma} > 0$,

the inequality is approved.

A.2 Proofs and Derivations of Chapter 3

The function for the value of the project must satisfy the following Bellman equation

$$
V(\theta) = \pi(\theta)dt + E\left[V(\theta+d\theta)e^{-rdt}\right], \tag{29}
$$

where r is the discount rate. We apply Ito's Lemma to the expectation in the right-hand side of equation (29). Substituting this result into equation (29) and rewriting leads to the following differential equation

$$
\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \alpha\theta\frac{\partial V}{\partial\theta} - rV(\theta) + \pi(\theta) = 0,
$$

$$
\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \alpha\theta\frac{\partial V}{\partial\theta} - rV(\theta) + \left[\begin{cases} \frac{\theta^2-2c\theta+c^2}{4\gamma} & \text{if } \theta < c \\ 0 & \text{if } \theta > c \end{cases}\right] = 0. \tag{30}
$$

This is solved by familiar methods. The homogeneous part has two independent solutions θ^{β_1} and θ^{β_2} as usual, where β_1 (β_2) is the positive (negative) root of equation (4), where $\beta_1 > 1$ and $\beta_2 < 0$. The nonhomogeneous part is defined differently when $\theta < c$ and when $\theta > c$. Here we apply the method, Dixit and Pindyck [10, Chapter 6.2] use to solve a model with operating costs considering temporary suspension.

We solve the equation separately for $\theta < c$ and $\theta > c$ and then stitch together the two solutions at the point ($\theta = c$). In the region $\theta < c$, where we have $\pi(\theta) = 0$, only the homogeneous part of the equation remains. Therefore, the general solution is just a linear combination of the two power solutions corresponding to the two roots:

$$
V(\theta) = M_1\theta^{\beta_1} + M_2\theta^{\beta_2}, \tag{31}
$$

where the constants M_1 and M_2 remain to be determined. In the region $\theta > c$, we take another linear combination of the power solutions of the homogeneous part, and add a particular solution of the full equation. In order to get a particular solution we try the solution of the form $K_1\theta^2 + K_2\theta + K_3$,

$$
V(\theta) = N_1\theta^{\beta_1} + N_2\theta^{\beta_2} + K_1\theta^2 + K_2\theta + K_3.
$$

Substituting and solving for K_1, K_2 and K_3, we find the solution

$$
\frac{1}{4\gamma}\left[\frac{1}{r-2\alpha-\sigma^2}\theta^2 - \frac{2c}{r-\alpha}\theta + \frac{c^2}{r}\right]. \tag{32}
$$

Proof of Remark 1 $N(\theta) = \frac{1}{4\gamma}\left[\frac{1}{r-2\alpha-\sigma^2}\theta^2 - \frac{2c}{r-\alpha}\theta + \frac{c^2}{r}\right]$. The first the derivativ of the net worth is

$$\frac{dN(\theta)}{d\theta} = \frac{1}{4\gamma}\left[\frac{2\theta}{r-2\alpha-\sigma^2} - \frac{2c}{r-\alpha}\right],$$

which has it's minimum at $\theta = c\frac{r-2\alpha-\sigma^2}{r-\alpha} > 0$, is decreasing for $\theta < c\frac{r-2\alpha-\sigma^2}{r-\alpha}$ and increasing for$\theta > c\frac{r-2\alpha-\sigma^2}{r-\alpha}$. Therefore, for the considered range of parameters the net worth function is always positive.

In order to decide for the two constants M_1 and N_2 we consider the point $(\theta = c)$ where the two regions meet. The solution $V(\theta)$ must be continuously differentiable across c. We are equating the values and derivatives of the two component solutions at c, and get

$$\begin{aligned}
M_1 c^{\beta_1} &= N_2 c^{\beta_2} + \frac{1}{4\gamma}\left[\frac{1}{r-2\alpha-\sigma^2}c^2 - \frac{2c^2}{r-\alpha} + \frac{c^2}{r}\right], \\
\beta_1 M_1 c^{(\beta_1-1)} &= \beta_2 N_2 c^{(\beta_2-1)} + \frac{1}{4\gamma}\left[\frac{2c}{r-2\alpha-\sigma^2} - \frac{2c}{r-\alpha}\right].
\end{aligned}$$

These are two linear equations in the unknowns M_1 and K_2 which we are able to solve. We get the following values

$$M_1 = \frac{1}{4\gamma}c^{2-\beta_1}\frac{1}{\beta_1-\beta_2}\left[\frac{1}{r-2\alpha-\sigma^2}(2-\beta_2) + \frac{2}{r-\alpha}(\beta_2-1) - \frac{\beta_2}{r}\right], \tag{33}$$

$$N_2 = \frac{1}{4\gamma}c^{2-\beta_2}\frac{1}{\beta_1-\beta_2}\left[\frac{1}{r-2\alpha-\sigma^2}(2-\beta_1) + \frac{2}{r-\alpha}(\beta_1-1) - \frac{\beta_1}{r}\right]. \tag{34}$$

To analyze this result we need to argue first that $\beta_1 > 2$. For the model to make economic sense we need that the risk-adjusted discount rate $\delta' = r - 2\alpha - \sigma^2$ is strictly bigger than zero, $\delta' > 0$. The expression of δ' is just the negative of our fundamental quadratic (**??**) evaluated at 2. Therefore, by requiring that $\delta' > 0$ also $Q(2) < 0$ has to be fulfilled. This means that 2 must lie between the two roots of the quadratic, specifically $2 < \beta_1$.

Proof of Remark 2 M_1 can be rewritten as

$$M_1 = \frac{1}{4\gamma}c^{2-\beta_1}\frac{1}{\beta_1-\beta_2}L_1,$$

where we denote

$$L_1 := \frac{1}{r-2\alpha-\sigma^2}(2-\beta_2) + \frac{2}{r-\alpha}(\beta_2-1) - \frac{\beta_2}{r}.$$

Since the first three terms in (52) are always positive, the fourth term L has to be positive as well, in order to guarantee a positive M_1.

$$\begin{aligned}
\frac{1}{r-2\alpha-\sigma^2}(2-\beta_2) + \frac{2}{r-\alpha}(\beta_2-1) - \frac{\beta_2}{r} &\geq 0 \\
r(r-\alpha)(2-\beta_2) + r(r-2\alpha-\sigma^2)(\beta_2-1) - \beta_2(r-\alpha)(r-2\alpha-\sigma^2) &\geq 0 \\
r^2 + r\sigma^2 - \beta_2(r^2 + \alpha\sigma^2 + 2\alpha^2 - 2r\alpha) &\geq 0
\end{aligned}$$

Considering the fact that $\beta_2 < 0$ and $r^2 + \alpha\sigma^2 + 2\alpha^2 > r(2\alpha + \sigma^2) + \alpha\sigma^2 + 2\alpha^2 \geq 2r\alpha$ using the assumption $(r > 2\alpha + \sigma^2)$ we have shown that this inequality is always valid.

N_2 can be rewritten as

$$N_2 = \frac{1}{4\gamma}c^{2-\beta_2}\frac{1}{\beta_1-\beta_2}L_2,$$

where L_2 is denoted by

$$L_2 := \frac{1}{r-2\alpha-\sigma^2}(2-\beta_1) + \frac{2}{r-\alpha}(\beta_1-1) - \frac{\beta_1}{r}.$$

The first three terms of N_2 are always positive. In order to verify that N_2 is negative it has to be shown that L_2 is negative. Therefore we undertake the following calculations

$$\begin{aligned}
\frac{1}{r-2\alpha-\sigma^2}(2-\beta_1)+\frac{2}{r-\alpha}(\beta_1-1)-\frac{\beta_1}{r} &\leq 0\\
r(r-\alpha)(2-\beta_1)+r(r-2\alpha-\sigma^2)(\beta_1-1)-\beta_1(r-\alpha)(r-2\alpha-\sigma^2) &\leq 0\\
r^2+r\sigma^2-\beta_1(r^2+\alpha\sigma^2+2\alpha^2-2r\alpha) &\leq 0
\end{aligned}$$

Since $\beta_1 > 2$ the following inequality holds

$$r^2+r\sigma^2-\beta_1(r^2+\alpha\sigma^2+2\alpha^2-2r\alpha) \leq r^2+r\sigma^2-2(r^2+\alpha\sigma^2+2\alpha^2-2r\alpha)$$

So this Remark is shown if we can approve that the right part of the previous inequality is smaller or equal to zero. In order to show this, we rearrange the inequation

$$\begin{aligned}
r^2+r\sigma^2-2(r^2+\alpha\sigma^2+2\alpha^2-2r\alpha) &\leq 0\\
0 &\leq r(r-\sigma^2)-4r\alpha+2\alpha(\sigma^2+2\alpha).
\end{aligned} \tag{35}$$

The right-side we denote as $f(\sigma)$ which is a function of σ and show that it is positive for the relevant interval $0 \leq \sigma \leq \sqrt{r-2\alpha}$

$$f(\sigma) := r(r-\sigma^2)-4r\alpha+2\alpha(\sigma^2+2\alpha). \tag{36}$$

Therefore we calculate the values of $f(\sigma)$ for the two boundary points of the interval

$$\begin{aligned}
f(0) &= r^2-4r\alpha+4\alpha^2=(r-2\alpha)^2>0,\\
f(\sqrt{r-2\alpha}) &= 0.
\end{aligned}$$

Within the interval the function $f(\sigma)$ is decreasing, which we prove in the following

$$f'(\sigma)=-2r\sigma+4\alpha\sigma=2\sigma(2\alpha-r)\leq -2\sigma^2\leq 0,$$

where the inequatility holds because $(r > 2\alpha+\sigma^2)$. Therefore, the inequality in (35) holds.

Proof of Proposition 1 In order to be able to make a statemant about existence and number of thresholds of equation (17), we are looking for the zeropoints of the function $T(\theta)$ which we denote by

$$T(\theta) := N_2(\theta^*)^{\beta_2}\frac{(\beta_1-\beta_2)}{\beta_1}+\frac{1}{4\gamma}\left[\frac{1}{r-2\alpha-\sigma^2}(\theta^*)^2\frac{(\beta_1-2)}{\beta_1}-\frac{2c}{r-\alpha}\theta^*\frac{(\beta_1-1)}{\beta_1}+\frac{c^2}{r}\right]-I \tag{37}$$

We will show that at the point $(\theta = c)$ there is a saddle point with the function value $T(\theta = c) = -I$. In this saddlepoint our function is changing from concave to convex behaviour. Since the function $T(\theta)$ is going to $-\infty$ for $\theta \to 0$ and to $+\infty$ for $\theta \to \infty$ and we have a negative function value at the saddle point, we know that the function can cross the x-axis just once within the region $\theta > c$. In order to prove this behavior we will show the following five properties:

1. $T(\theta = c) = -I$
2. $\frac{dT(\theta)}{d\theta}|_{\theta=c} = 0$
3. $\frac{d^2T(\theta)}{d\theta^2}|_{\theta=c} = 0$
4. $\frac{d^3T(\theta)}{d\theta^3}|_{\theta=c} \neq 0$
5. $\frac{d^2T(\theta)}{d\theta^2} \geq 0$ for $\theta > c$

The first property shows that the function value is negative at $(\theta = c)$. Property two to four show the existence of a saddle point and property five the convex behavior in the region $\theta > c$.

We illustrate this behavior in figure 16.

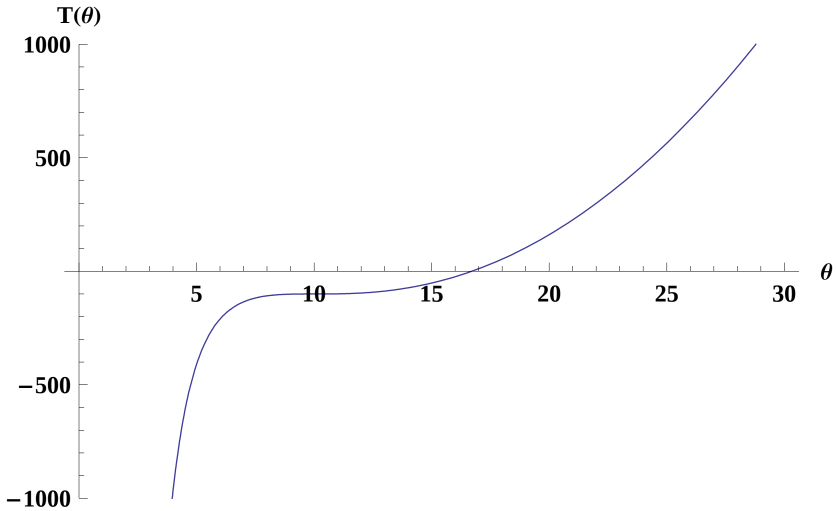

Figure 16:

1. First we show that $T(c) = -I$:

$$T(c) = N_2 c^{\beta_2} \frac{(\beta_1 - \beta_2)}{\beta_1} + \frac{1}{4\gamma}\left[\frac{1}{r-2\alpha-\sigma^2} c^2 \frac{(\beta_1 - 2)}{\beta_1} - \frac{2c^2}{r-\alpha}\frac{(\beta_1-1)}{\beta_1} + \frac{c^2}{r}\right] - I$$

Substituting N_2 by its value (see (34)) gives

$$\begin{aligned} T(c) &= \frac{1}{4\gamma} c^2 \frac{1}{\beta_1}\left[\frac{1}{r-2\alpha-\sigma^2}(2-\beta_1) + \frac{2}{r-\alpha}(\beta_1-1) - \frac{\beta_1}{r}\right] \\ &\quad + \frac{1}{4\gamma} c^2 \frac{1}{\beta_1}\left[\frac{1}{r-2\alpha-\sigma^2}(\beta_1-2) - \frac{2}{r-\alpha}(\beta_1-1) + \frac{\beta_1}{r}\right] - I \\ &= -I \end{aligned}$$

2.

$$\begin{aligned} \frac{dT(\theta)}{d\theta} &= N_2\beta_2(\theta^*)^{(\beta_2-1)}\frac{(\beta_1-\beta_2)}{\beta_1} + \frac{1}{4\gamma}\left[\frac{1}{r-2\alpha-\sigma^2}2\theta^*\frac{(\beta_1-2)}{\beta_1} - \frac{2c}{r-\alpha}\frac{(\beta_1-1)}{\beta_1}\right] \\ \frac{dT(\theta)}{d\theta}|_{\theta=c} &= N_2\beta_2 c^{(\beta_2-1)}\frac{(\beta_1-\beta_2)}{\beta_1} + \frac{1}{4\gamma}\left[\frac{2c}{r-2\alpha-\sigma^2}\frac{(\beta_1-2)}{\beta_1} - \frac{2c}{r-\alpha}\frac{(\beta_1-1)}{\beta_1}\right] \end{aligned}$$

for which substituting the value of N_2 gives

$$\begin{aligned} \frac{dT(\theta)}{d\theta}|_{\theta=c} &= \frac{1}{4\gamma}c\frac{\beta_2}{\beta_1}\left[\frac{(2-\beta_1)}{r-2\alpha-\sigma^2} + \frac{2(\beta_1-1)}{r-\alpha} - \frac{\beta_1}{r}\right] + \frac{1}{4\gamma}\left[\frac{2c}{r-2\alpha-\sigma^2}\frac{(\beta_1-2)}{\beta_1} - \frac{2c}{r-\alpha}\frac{(\beta_1-1)}{\beta_1}\right] \\ &= \frac{1}{4\gamma}\frac{c}{\beta_1}\left[\frac{\beta_2(2-\beta_1)}{r-2\alpha-\sigma^2} + \frac{2\beta_2(\beta_1-1)}{r-\alpha} - \frac{\beta_1\beta_2}{r}\right] + \frac{1}{4\gamma}\frac{c}{\beta_1}\left[\frac{2(\beta_1-2)}{r-2\alpha-\sigma^2} - \frac{2(\beta_1-1)}{r-\alpha}\right] \\ &= \frac{1}{4\gamma}c\frac{1}{\beta_1}\left[\frac{\beta_2(2-\beta_1)}{r-2\alpha-\sigma^2} + \frac{2\beta_2(\beta_1-1)}{r-\alpha} - \frac{\beta_1\beta_2}{r} + \frac{2(\beta_1-2)}{r-2\alpha-\sigma^2} - \frac{2(\beta_1-1)}{r-\alpha}\right] \\ &= \frac{1}{4\gamma}c\frac{1}{\beta_1}L_3 \end{aligned} \tag{38}$$

We denote the fourth term of expression (38) by L_3 and show that it is equal to zero. This makes the first derivative of $T(\theta)$ w.r.t. θ, equal to zero at $\theta = c$, which means that we have a stationary point in $\theta = c$.

$$\begin{aligned}
L_3 \quad &:= \quad \frac{\beta_2(2-\beta_1)}{r-2\alpha-\sigma^2} + \frac{2\beta_2(\beta_1-1)}{r-\alpha} - \frac{\beta_1\beta_2}{r} + \frac{2(\beta_1-2)}{r-2\alpha-\sigma^2} - \frac{2(\beta_1-1)}{r-\alpha} \\
&= \quad \frac{1}{r-2\alpha-\sigma^2}(\beta_1-2)(2-\beta_2) - \frac{2}{r-\alpha}(\beta_1-1))(1-\beta_2) - \frac{\beta_1\beta_2}{r} \\
&= \quad \frac{1}{r}\frac{1}{(r-\alpha)}\frac{1}{(r-2\alpha-\sigma^2)}L_4
\end{aligned}$$

where L_4 is denoted as follows and equal to zero. We use the following substitutions:

- $$\beta_1 + \beta_2 = 1 - \frac{2\alpha}{\sigma^2} \tag{39}$$
- $$\beta_1\beta_2 = -\frac{2r}{\sigma^2} \tag{40}$$

$$\begin{aligned}
L_4 \quad &:= \quad r(r-\alpha)(\beta_1-2)(2-\beta_2) - 2r(r-2\alpha-\sigma^2)(\beta_1-1)(1-\beta_2) - (r-\alpha)(r-2\alpha-\sigma^2)\beta_1\beta_2 \\
&= \quad (-4r^2+4r\alpha) + \beta_1\left[2r^2-2r\alpha\right] + \beta_2\left[2r^2-2r\alpha\right] + \beta_1\beta_2\left[-r^2+r\alpha\right] \\
&\qquad +(2r^2-4r\alpha-2r\sigma^2) + \beta_1\left[-2r^2+4r\alpha+2r\sigma^2\right] + \beta_2\left[-2r^2+4r\alpha+2r\sigma^2\right] \\
&\qquad +\beta_1\beta_2\left[2r^2-4r\alpha-2r\sigma^2\right] + \beta_1\beta_2\left[-r^2+3r\alpha-2\alpha^2+r\sigma^2-\alpha\sigma^2\right] \\
&= \quad -2r^2-2r\sigma^2+(\beta_1+\beta_2)(2r\alpha+2r\sigma^2) - \beta_1\beta_2(2\alpha^2+r\sigma^2+\alpha\sigma^2) \\
&=^{(39),(40)} \quad -2r^2-2r\sigma^2+(1-\frac{2\alpha}{\sigma^2})(2r\alpha+2r\sigma^2) + \frac{2r}{\sigma^2}(2\alpha^2+r\sigma^2+\alpha\sigma^2) \\
&= \quad 0
\end{aligned}$$

Therefore we get the result

$$\frac{dT(\theta)}{d\theta} = \frac{1}{4\gamma}c\frac{1}{\beta_1}\frac{1}{r}\frac{1}{(r-\alpha)}\frac{1}{(r-2\alpha-\sigma^2)}L_4 = 0$$

3. Now we want to show that at point $(\theta = c)$ the second derivative of the function T equals zero.

$$\begin{aligned}
\frac{d^2T(\theta)}{d\theta^2} &= N_2\beta_2(\beta_2-1)(\theta^*)^{(\beta_2-2)}\frac{(\beta_1-\beta_2)}{\beta_1} + \frac{1}{4\gamma}\frac{2}{r-2\alpha-\sigma^2}\frac{(\beta_1-2)}{\beta_1} \\
\frac{d^2T(\theta)}{d\theta^2}|_{\theta=c} &= N_2\beta_2(\beta_2-1)c^{(\beta_2-2)}\frac{(\beta_1-\beta_2)}{\beta_1} + \frac{1}{4\gamma}\frac{2}{r-2\alpha-\sigma^2}\frac{(\beta_1-2)}{\beta_1}
\end{aligned} \tag{41}$$

Substituting for N_2 (see (34))leads to

$$\begin{aligned}
\frac{d^2T(\theta)}{d\theta^2}|_{\theta=c} &= \frac{1}{4\gamma}\frac{\beta_2(\beta_2-1)}{\beta_1}\left[\frac{(2-\beta_1)}{r-2\alpha-\sigma^2} + \frac{2(\beta_1-1)}{r-\alpha} - \frac{\beta_1}{r}\right] + \frac{1}{4\gamma}\frac{2}{r-2\alpha-\sigma^2}\frac{(\beta_1-2)}{\beta_1} \\
= \frac{1}{4\gamma}\frac{1}{\beta_1}&\left[\frac{1}{r-2\alpha-\sigma^2}(\beta_1-2)(2-\beta_2(\beta_2-1))\frac{1}{r-\alpha}2(\beta_1-1)\beta_2(\beta_2-1) - \frac{1}{r}\beta_1\beta_2(\beta_2-1)\right] \\
&= \frac{1}{4\gamma}\frac{1}{r}\frac{1}{(r-\alpha)}\frac{1}{(r-2\alpha-\sigma^2)}L_5
\end{aligned}$$

where L_5 is denoted by

$$\begin{aligned}
L_5 \quad &:= \quad \left[r(r-\alpha)(\beta_1-2)(2-\beta_2(\beta_2-1)) + 2r(r-2\alpha-\sigma^2)(\beta_1-1)\beta_2(\beta_2-1) + \frac{1}{r}\beta_1\beta_2(\beta_2-1)\right] \\
&= \quad \beta_2(\beta_2-1)\left[-r(r-\alpha)(\beta_1-2) + 2r(r-2\alpha-\sigma^2)(\beta_1-1) - (r-\alpha)(r-2\alpha-\sigma^2)\beta_1\right] \\
&\qquad +2r(r-\alpha)(\beta_1-2) \\
&= \quad \beta_2(\beta_2-1)\left[-\beta_1(r\sigma^2+2\alpha^2+\alpha\sigma^2) + 2r(\alpha+\sigma^2)\right] + 2r(r-\alpha)(\beta_1-2)
\end{aligned} \tag{42}$$

substituting $\left[\beta_2(\beta_2-1)=\frac{2}{\sigma^2}(r-\alpha\beta_2)\right]$(which we get considering (**??**)) into (42) yields

$$
\begin{aligned}
L_5 &= \frac{2}{\sigma^2}(r-\alpha\beta_2)\left[-\beta_1(r\sigma^2+2\alpha^2+\alpha\sigma^2)+2r(\alpha+\sigma^2)\right]+2r(r-\alpha)(\beta_1-2)\\
&= -4r\alpha\beta_1-\frac{4r\alpha^2}{\sigma^2}\beta_1+4r\alpha(1+\frac{r}{\sigma^2})-\frac{4r\alpha}{\sigma^4}(\sigma^2(r+\alpha)+2\alpha^2)-\frac{4r\alpha}{\sigma^2}\beta_2(\alpha+\sigma^2)\\
&= -\frac{4r\alpha}{\sigma^2}(\beta_1+\beta_2)(\alpha+\sigma^2)+4r\alpha-\frac{4r\alpha^2}{\sigma^2}-\frac{8r\alpha^3}{\sigma^4}\\
&= 0 \qquad (43)
\end{aligned}
$$

Therefore, the second derivate of $T(\theta)$ at $(\theta=c)$, $\left[\frac{d^2T(\theta)}{d\theta^2}|_{\theta=c}\right]$, is equal to zero. This is a necessary condition for a saddlepoint.

4. For this point being a saddle point one also needs the lowest-order non-zero derivative to be of odd order (third, fifth, etc.). We show that the third order derivative is strictly positive for $c>0$ and therefore unequal zero.

$$
\begin{aligned}
\frac{d^3T(\theta)}{d\theta^3} &= N_2\beta_2(\beta_2-1)(\beta_2-2)\theta^{(\beta_2-3)}\frac{\beta_1-\beta_2}{\beta_1}\\
\frac{d^3T(\theta)}{d\theta^3}|_{\theta=c} &= N_2\beta_2(\beta_2-1)(\beta_2-2)c^{(\beta_2-3)}\frac{\beta_1-\beta_2}{\beta_1}>0
\end{aligned}
$$

$$2r(r-2\alpha-\sigma^2)-r(r-\alpha)+(r-\alpha)(r-2\alpha-\sigma^2)=-(r\sigma^2+2\alpha^2+\alpha\sigma^2) \qquad (44)$$

$$\beta_2(\beta_2-1)=\frac{2}{\sigma^2}(r-\alpha\beta_2) \qquad (45)$$

5. Concluding, we show that T is convex for $(\theta>c)$

$$
\begin{aligned}
\frac{d^3T(\theta)}{d\theta^3} &= N_2\beta_2(\beta_2-1)(\theta^*)^{(\beta_2-2)}\frac{(\beta_1-\beta_2)}{\beta_1}+\frac{1}{4\gamma}\frac{2}{r-2\alpha-\sigma^2}\frac{(\beta_1-2)}{\beta_1}\\
&= \frac{1}{4\gamma}c^{2-\beta_2}(\theta^*)^{(\beta_2-2)}\frac{\beta_2(\beta_2-1)}{\beta_1}\left[\frac{1}{r-2\alpha-\sigma^2}(2-\beta_1)+\frac{2}{r-\alpha}(\beta_1-1)-\frac{\beta_1}{r}\right]\\
&\quad +\frac{1}{4\gamma}\frac{2}{r-2\alpha-\sigma^2}\frac{(\beta_1-2)}{\beta_1}\\
&> \frac{1}{4\gamma}(\theta^*)^{2-\beta_2}(\theta^*)^{(\beta_2-2)}\frac{\beta_2(\beta_2-1)}{\beta_1}\left[\frac{1}{r-2\alpha-\sigma^2}(2-\beta_1)+\frac{2}{r-\alpha}(\beta_1-1)-\frac{\beta_1}{r}\right]\\
&\quad +\frac{1}{4\gamma}\frac{2}{r-2\alpha-\sigma^2}\frac{(\beta_1-2)}{\beta_1}\\
&= \frac{1}{4\gamma}\frac{\beta_2(\beta_2-1)}{\beta_1}\left[\frac{1}{r-2\alpha-\sigma^2}(2-\beta_1)+\frac{2}{r-\alpha}(\beta_1-1)-\frac{\beta_1}{r}\right]+\frac{1}{4\gamma}\frac{2}{r-2\alpha-\sigma^2}\frac{(\beta_1-2)}{\beta_1}\\
&= \frac{d^2T(\theta)}{d\theta^2}|_{\theta=c}\\
&= 0
\end{aligned}
$$

B Appendix

B.1 Value Function of the Project

We have to solve the following differential equation

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial^2V}{\partial\theta^2}+\alpha\theta\frac{\partial V}{\partial\theta}-rV(\theta)+\pi(\theta)=0,$$

for V. The nonhomogeneous part is defined differently for three regions. Using the method of Dixit and Pindyck [10, Section 6.2] we solve the equation separately for three regions and then stitch together the two solutions at the points ($\theta = \theta_1$) and ($\theta = \theta_2$).

Solving the differential equation for the three regions we get

$$V(\theta, K) = \begin{cases} N_1\theta^{\beta_1} + N_2\theta^{\beta_2} + \frac{K}{r-\alpha}\theta - \frac{K(K\gamma+c)}{r} & \text{for } \theta \geq \theta_2 \\ M_1\theta^{\beta_1} + M_2\theta^{\beta_2} + \frac{1}{4\gamma}\left[\frac{\theta^2}{r-2\alpha-\sigma^2} - \frac{2c\theta}{r-\alpha} + \frac{c^2}{r}\right] & \text{for } \theta_1 \leq \theta < \theta_2 \\ L_1\theta^{\beta_1} + L_2\theta^{\beta_2} & \text{for } 0 \leq \theta < \theta_1 \end{cases}$$

Considering the following two boundary conditions

$$V(0, K) = 0 \tag{46}$$

$$\lim_{\theta\to\infty} V_{\theta\geq\theta_2} = \frac{K}{r-\alpha}\theta - \frac{K(K\gamma+c)}{r} \tag{47}$$

we can set $L_2 = 0$ and $N_1 = 0$.

In order to get the values for the remaining four parameters we stitch together the three solutions at the points ($\theta = \theta_1$) and ($\theta = \theta_2$).

$$N_2\theta_2^{\beta_2} + \frac{K}{r-\alpha}\theta_2 - \frac{K(K\gamma+c)}{r} = M_1\theta_2^{\beta_1} + M_2\theta_2^{\beta_2} + \frac{1}{4\gamma}\left[\frac{\theta_2^2}{r-2\alpha-\sigma^2} - \frac{2c\theta_2}{r-\alpha} + \frac{c^2}{r}\right] \tag{48}$$

$$\beta_2N_2\theta_2^{\beta_2-1} + \frac{K}{r-\alpha} = \beta_1M_1\theta_2^{\beta_1-1} + \beta_2M_2\theta_2^{\beta_2-1} + \frac{1}{4\gamma}\left[\frac{2\theta_2}{r-2\alpha-\sigma^2} - \frac{2c}{r-\alpha}\right] \tag{49}$$

$$M_1\theta^{\beta_1} + M_2\theta^{\beta_2} + \frac{1}{4\gamma}\left[\frac{\theta^2}{r-2\alpha-\sigma^2} - \frac{2c\theta}{r-\alpha} + \frac{c^2}{r}\right] = L_1\theta^{\beta_1} \tag{50}$$

$$\beta_1M_1\theta^{\beta_1-1} + \beta_2M_2\theta^{\beta_2-1} + \frac{1}{4\gamma}\left[\frac{2\theta}{r-2\alpha-\sigma^2} - \frac{2c}{r-\alpha}\right] = \beta_1L_1\theta^{\beta_1-1} \tag{51}$$

Dividing (49) by β_2 and multiplying with θ_2 and than substracting that from (48) gives

$$M_1 = \theta_2^{-\beta_1}\frac{1}{\beta_1-\beta_2}\left[\frac{1}{4\gamma}\left[\frac{\theta_2^2}{r-2\alpha-\sigma^2}(\beta_2-2) - \frac{2c\theta_2}{r-\alpha}(\beta_2-1) + \beta_2\frac{c^2}{r}\right] + \beta_2\frac{K(\gamma K+c)}{r} - \frac{K\theta_2}{r-\alpha}\theta_2(\beta_2-1)\right] \tag{52}$$

Dividing (51) by β_1 and multiplying with θ_1 and than substracting that from (50) gives

$$M_2 = \theta_1^{-\beta_2}\frac{1}{\beta_1-\beta_2}\frac{1}{4\gamma}\left[\frac{\theta_1^2}{r-2\alpha-\sigma^2}(2-\beta_1) - \frac{2c\theta_1}{r-\alpha}(1-\beta_1) - \beta_1\frac{c^2}{r}\right]$$

Using the values of M_1 and M_2 we can derive the remaing parameter values

$$\begin{aligned} N_2 = \theta_2^{-\beta_2}\left[M_1\theta_2^{\beta_1} + M_2\theta_2^{\beta_2} + \frac{1}{4\gamma}\left[\frac{\theta_2^2}{r-2\alpha-\sigma^2} - \frac{2c\theta_2}{r-\alpha} + \frac{c^2}{r}\right] - \frac{K\theta_2}{r-\alpha} + \frac{K(\gamma K+c)}{r}\right] \\ L_1 = \theta_1^{-\beta_1}\left[M_1\theta_1^{\beta_1} + M_2\theta_1^{\beta_2} + \frac{1}{4\gamma}\left[\frac{\theta_1^2}{r-2\alpha-\sigma^2} - \frac{2c\theta_1}{r-\alpha} + \frac{c^2}{r}\right]\right] \end{aligned} \tag{53}$$

B.2 Derivatives

We calculate the derivative of the value of the project w.r.t K for each region, i.e.

$$\begin{aligned} \frac{\partial(V_3(\theta,K) - I(K))}{\partial K} &= \frac{\partial V_3(\theta,K)}{\partial K} - \frac{I(K)}{\partial K} && \text{for } \theta \geq \theta_2 \\ \frac{\partial(V_2(\theta,K) - I(K))}{\partial K} &= \frac{\partial V_2(\theta,K)}{\partial K} - \frac{I(K)}{\partial K} && \text{for } \theta_1 \leq \theta < \theta_2 \\ \frac{\partial(V_1(\theta,K) - I(K))}{\partial K} &= \frac{\partial V_1(\theta,K)}{\partial K} - \frac{I(K)}{\partial K} && \text{for } 0 \leq \theta < \theta_1 \end{aligned}$$

B.2.1 Region III

First we derive $\frac{\partial V_3(\theta,K)}{\partial K}$:

$$\frac{dV_3}{dK} = \frac{dN_2}{dK}\theta^{\beta_2} + \frac{\theta}{r-\alpha} - \frac{2K\gamma + c}{r}$$

Furthermore, we get the derivative of $N_2(M_1(\theta_2(K), K), \theta_2(K), K)$ w.r.t. K. Therefore, we need $\frac{dM_2}{dK}$ and $\frac{dM_1}{dK}$ which are

$$\begin{aligned}
\frac{dM_1}{dK} &= \frac{1}{\beta_1-\beta_2}\theta_2^{-\beta_1}[\frac{1}{2}\left[\frac{\theta_2}{r-2\alpha-\sigma^2}(\beta_2-2)(2-\beta_1) - \frac{2c}{r-\alpha}(\beta_2-1)(1-\beta_1) - \beta_1\beta_2\frac{c^2\theta_2^{-1}}{r}\right] \\
&+ 2\gamma\left[-\beta_1\beta_2\theta_2^{-1}\frac{\gamma K^2+Kc}{r} - \frac{K}{r-\alpha}(\beta_2-1)(1-\beta_1)\right] \\
&+ \beta_2\frac{2\gamma K+c}{r} - \frac{\theta_2}{r-\alpha}(\beta_2-1)] \qquad (54) \\
\frac{dM_2}{dK} &= 0
\end{aligned}$$

$$\frac{dN_2}{dK} = \frac{\partial N_2}{\partial M_1}\frac{dM_1}{dK} + \frac{\partial N_2}{\partial \theta_2}\frac{d\theta_2}{dK} + \frac{dN_2}{dK}$$

$$\begin{aligned}
\frac{dN_2}{dK} = \theta_2^{-\beta_2}[\theta_2^{\beta_1}\frac{dM_1}{dK} + 2\gamma(M_1(\beta_1-\beta_2)\theta_2^{\beta_1-1} + \frac{1}{4\gamma}\left[\frac{\theta_2}{r-2\alpha-\sigma^2}(2-\beta_2) - \frac{2c}{r-\alpha}(1-\beta_2) - \beta_2\frac{\theta_2^{-1}c^2}{r}\right] \\
-\frac{K}{r-\alpha}(1-\beta_2) - \beta_2\frac{K(\gamma K+c)}{r}\theta_2^{-1}) \\
-\frac{\theta_2}{r-\alpha} + \frac{2\gamma K+c}{r}]
\end{aligned}$$

After substituting the values for M_1 (52) and $\frac{dM_1}{dK}$ (54) and simplifying we get:

$$\begin{aligned}
\frac{dN_2}{dK} &= \theta_2^{-\beta_2}[\frac{1}{2}\left[\frac{\theta_2}{r-2\alpha-\sigma^2}\frac{(\beta_2-2)(2-\beta_1)}{\beta_1-\beta_2} - \frac{2c}{r-\alpha}\frac{(\beta_2-1)(1-\beta_1)}{\beta_1-\beta_2} - \frac{c^2\theta_2^{-1}}{r}\frac{\beta_1\beta_2}{\beta_1-\beta_2}\right] + \\
&2\gamma\left[-\theta_2^{-1}\frac{\gamma K^2+Kc}{r}\frac{\beta_1\beta_2}{\beta_1-\beta_2} - \frac{K}{r-\alpha}\frac{(\beta_2-1)(1-\beta_1)}{\beta_1\beta_2}\right] + \frac{2\gamma K+c}{r}\frac{\beta_2}{\beta_1-\beta_2} - \frac{\theta_2}{r-\alpha}\frac{\beta_2-1}{\beta_1-\beta_2} \\
&-\frac{\theta_2}{r-\alpha} + \frac{2K\gamma+c}{r}] \\
&= \theta_2^{-\beta_2}[\frac{1}{2}\left[\frac{\theta_2}{r-2\alpha-\sigma^2}\frac{(\beta_2-2)(2-\beta_1)}{\beta_1-\beta_2} - \frac{2c}{r-\alpha}\frac{(\beta_2-1)(1-\beta_1)}{\beta_1-\beta_2} - \frac{c^2\theta_2^{-1}}{r}\frac{\beta_1\beta_2}{\beta_1-\beta_2}\right] + \\
&2\gamma\left[-\theta_2^{-1}\frac{\gamma K^2+Kc}{r}\frac{\beta_1\beta_2}{\beta_1-\beta_2} - \frac{K}{r-\alpha}\frac{(\beta_2-1)(1-\beta_1)}{\beta_1-\beta_2}\right] \\
&+\frac{2\gamma K+c}{r}\frac{\beta_1}{\beta_1-\beta_2} - \frac{\theta_2}{r-\alpha}\frac{\beta_1-1}{\beta_1-\beta_2}] \qquad (55)
\end{aligned}$$

Furthermore we have to check the second order derivative to see if this K^* is a maximum or minimum:

$$\frac{\partial^2 V_3(\theta,K)}{\partial K^2} = \frac{d\left(\frac{dN_2}{dK}\right)}{dK}\theta^{\beta_2} - \frac{2\gamma}{r} \qquad (56)$$

where

$$
\begin{aligned}
\frac{d\left(\frac{dN_2}{dK}\right)}{dK} &= \frac{\partial\left(\frac{dN_2}{dK}\right)}{\partial\theta_2}\frac{d\theta_2}{dK} + \frac{\partial\left(\frac{dN_2}{dK}\right)}{\partial K} \\
&= \frac{1}{\beta_1-\beta_2}\theta_2^{-\beta_2}[\gamma\left[\frac{1}{r-2\alpha-\sigma^2}(\beta_2-2)(2-\beta_1)(1-\beta_2) + \frac{2c\theta_2^{-1}}{r-\alpha}\beta_2(\beta_2-1)(1-\beta_1) + \frac{c^2}{r\theta_2^2}\beta_1\beta_2(1+\beta_2)\right] \\
&+ 4\gamma^2\left[\beta_1\beta_2(1+\beta_2)\frac{\gamma K^2+Kc}{r\theta_2^2} + \beta_2(\beta_2-1)(1-\beta_1)\theta_2^{-1}\frac{K}{r-\alpha}\right] \\
&- 4\gamma\left[\beta_1\beta_2\frac{2\gamma K+c}{r\theta_2} + (\beta_1-1)(1-\beta_2)\frac{1}{r-\alpha}\right] \\
&+ \frac{2\gamma}{r}\beta_1]
\end{aligned}
$$

B.2.2 Region I and II

Now we look at region I and II. We derive the optimal capacity choice (K^*) for (27) and (28)

$$
\begin{aligned}
\frac{dV_2}{dK} = \frac{dM_1}{dK}\theta^{\beta_1} \\
= \theta^{\beta_1}\frac{1}{\beta_1-\beta_2}\theta_2^{-\beta_1}[\frac{1}{2}\left[\frac{\theta_2}{r-2\alpha-\sigma^2}(\beta_2-2)(2-\beta_1) - \frac{2c}{r-\alpha}(\beta_2-1)(1-\beta_1) - \beta_1\beta_2\frac{c^2\theta_2^{-1}}{r}\right] \\
+2\gamma\left[-\beta_1\beta_2\theta_2^{-1}\frac{\gamma K^2+Kc}{r} - \frac{K}{r-\alpha}(\beta_2-1)(1-\beta_1)\right] \\
+\beta_2\frac{2\gamma K+c}{r} - \frac{\theta_2}{r-\alpha}(\beta_2-1)]
\end{aligned}
\tag{57}
$$

$$
\frac{dV_1}{dK} = \frac{dL_1}{dK}\theta^{\beta_1} = \frac{dM_1}{dK}\theta^{\beta_1}
\tag{58}
$$

Since

$$
\frac{\partial V_1(\theta,K)}{\partial K} = \frac{\partial V_2(\theta,K)}{\partial K}
\tag{59}
$$

we get the same equation for K^* considering V_1 and V_2, respectively.

$$
\begin{aligned}
\frac{\partial V_2(\theta,K)}{\partial K} - \frac{I(K)}{\partial K} &= 0 \\
\frac{\partial V_2(\theta,K)}{\partial K} &= \frac{\partial I(K)}{\partial K} \\
\frac{\partial V_2(\theta,K)}{\partial K} &= \delta\lambda K^{\lambda-1}
\end{aligned}
$$

Now we look at the second order derivative to see if this K^* (for $\theta < \theta_2$ and $0 \le \theta < \theta_1$) is a maximum. Therefore we calculate

$$
\frac{\partial^2 V_2(\theta,K)}{\partial K^2} = \left(\frac{\partial\left(\frac{dM_1}{dK}\right)}{\partial\theta_2}\frac{d\theta_2}{dK} + \frac{\partial\left(\frac{dM_1}{dK}\right)}{\partial K}\right)\theta^{\beta_1}
\tag{60}
$$

$$
\begin{aligned}
\frac{\partial\left(\frac{dM_1}{d\theta_2}\right)}{\partial K} &= \frac{1}{\beta_1-\beta_2}[\frac{1}{2}\left[\frac{\theta_2^{-\beta_1}}{r-2\alpha-\sigma^2}(\beta_2-2)(2-\beta_1)(1-\beta_1)+\frac{2c\theta_2^{-\beta_1-1}}{r-\alpha}(\beta_2-1)(1-\beta_1)\beta_1+\beta_1\beta_2(\beta_1+1)\frac{c^2\theta_2^{-2-\beta_1}}{r}\right] \\
&+ 2\gamma\left[\beta_1\beta_2(\beta_1+1)\theta_2^{-2-\beta_1}\frac{\gamma K^2+Kc}{r}+\frac{K\theta_2^{-1-\beta_1}}{r-\alpha}(\beta_2-1)(1-\beta_1)\beta_1\right] \\
&- \beta_1\beta_2\theta_2^{-1-\beta_1}\frac{2\gamma K+c}{r}-\frac{\theta_2^{-\beta_1}}{r-\alpha}(\beta_2-1)(1-\beta_1)]
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
\frac{\partial^2 V_2(\theta,K)}{\partial K^2} &= (\frac{1}{\beta_1-\beta_2}[\gamma\left[\frac{\theta_2^{-\beta_1}}{r-2\alpha-\sigma^2}(\beta_2-2)(2-\beta_1)(1-\beta_1)+\frac{2c\theta_2^{-\beta_1-1}}{r-\alpha}(\beta_2-1)(1-\beta_1)\beta_1+\beta_1\beta_2(\beta_1+1)\frac{c^2\theta_2^{-2-\beta_1}}{r}\right] \\
&+ 4\gamma^2\left[\beta_1\beta_2(\beta_1+1)\theta_2^{-2-\beta_1}\frac{\gamma K^2+Kc}{r}+\frac{K\theta_2^{-1-\beta_1}}{r-\alpha}(\beta_2-1)(1-\beta_1)\beta_1\right] \\
&- 2\gamma\left[\beta_1\beta_2\theta_2^{-1-\beta_1}\frac{2\gamma K+c}{r}+\frac{\theta_2^{-\beta_1}}{r-\alpha}(\beta_2-1)(1-\beta_1)\right] \\
&+ 2\gamma\left[-\beta_1\beta_2\theta_2^{-1-\beta_1}\frac{2\gamma K+c}{r}-\frac{\theta_2^{-\beta_1}}{r-\alpha}(\beta_2-1)(1-\beta_1)\right]+\beta_2\theta_2^{-\beta_1}\frac{2\gamma}{r}])\theta^{-\beta_1} \\
&= (\frac{1}{\beta_1-\beta_2}\theta_2^{-\beta_1}[\gamma\left[\frac{1}{r-2\alpha-\sigma^2}(\beta_2-2)(2-\beta_1)(1-\beta_1)+\frac{2c\theta_2^{-1}}{r-\alpha}(\beta_2-1)(1-\beta_1)\beta_1+\beta_1\beta_2(\beta_1+1)\frac{c^2\theta_2^{-2}}{r}\right] \\
&+ 4\gamma^2\left[\beta_1\beta_2(\beta_1+1)\theta_2^{-2}\frac{\gamma K^2+Kc}{r}+\frac{K\theta_2^{-1}}{r-\alpha}(\beta_2-1)(1-\beta_1)\beta_1\right] \\
&- 4\gamma\left[\beta_1\beta_2\theta_2^{-1}\frac{2\gamma K+c}{r}-\frac{1}{r-\alpha}(\beta_2-1)(1-\beta_1)\right]+\beta_2\frac{2\gamma}{r}])\theta^{-\beta_1}
\end{aligned}
$$

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