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(Real-)Options, Uncertainty and Comparative Statics:

Are Black and Scholes mistaken?

Summary:

The purpose of this paper is to analyze the influence of uncertainty on the value of real options while allowing for a possible change in the value of the underlying asset. We show that the proposition of a strictly positive influence of uncertainty does not hold, if the value of the underlying asset changes due to a variation of the standard deviation. Only if the underlying risk is unsystematic or the binding relation between risk and return is neglected, the strictly positive effect of uncertainty can be retained. In all other cases, the influence becomes ambiguous. In addition, we discuss the consequences of our results on a more economic level to convey an understanding of when the procedure dealt with would be indicated.

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Keywords: Real Options; Uncertainty; Investment/Uncertainty-Relationship; Risk; Underlying

JEL: G13, G31, O32

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1. Introduction

Based on the work of *Myers* (1977), the real option-approach has broken new ground in the most diverse areas of application since the end of the 1980s. Within this approach, a nearly unanimous consensus has crystallized in the literature[1] on the positive effects of increasing uncertainty on the value of real options[2], directly opposing the fundamental principles of neo-classical investment and financing theory.[3] The work of *Black* and *Scholes* (1973) provides the theoretical foundation of this assertion, showing that the value of an option increases as the volatility of the underlying stock grows, provided all other parameters remain unchanged. However, it is particularly difficult to conceive using real options of how the value of the underlying (regularly calculated as the present value of future cash flows (DCF-value)) should not react to an increase in volatility. In this case, one rather has to assume a reduction of value. Thus, two countervailing effects influence the option value, whose net resulting tendency can vary. The article at hand seeks to examine these different net effects more closely.

This problem relates directly to the critique – dealt with under the term "investment/uncertainty-relationship"[4] – concerning the importance of uncertainty within the real option-approach. The discussion within this critique concentrates on the probability of exercising the option, thereby refuting the prevailing perception that growing uncertainty leads to a deceleration in investment. Indeed, the question as to how a change in the underlying influences the corresponding option valuation remains neglected, an omission which leaves a further and perhaps all the more significant aspect largely unstudied up until now. The single exception, *Davis* (2002), does take this factor into consideration and likewise comes to a contradictory result: The value of the growth option tends to fall with an increase in uncertainty.

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1 Cf. for previous practical-oriented literature *Smith/Triantis* (1995), p. 148; *Teisberg* (1995), p. 43; *Amram/Kulatilaka* (1999); *Copeland/Antikarov* (2001); *Damodaran* (2003), p. 141; *Damodaran* (o.J.), p. 7 ff. Cf. for purely scientific treatments *McDonald/Siegel* (1986); *Pindyck* (1988); *Paddock/Siegel/Smith* (1988), p. 504; *Williams* (1993); *Teisberg* (1994); *Trigeorgis* (1996); *Ott/Thompson* (1996) and *Schwartz/Moon* (2000). For an empirical study see *Haushalter et al.* (2002).

2 Very few exceptions and likewise references of other interpretations can be found in *Emery et al.* (1978); *Jagannathan* (1984); *McDonald/Siegel* (1985) and (1986), p. 722; *Kulatilaka/Perotti* (1998); *Trigeorgis* (1999) and *Dixit/Pindyck* (1994).

3 „This recommendation [to invest in projects with higher variance] is in direct contrast to the prescriptions derived from the traditional Markowitz (1952)-Tobin (1958) financial portfolio and Hillier (1963) real investment models." Cf. *Emery et al.* (1978), p. 364.

4 Cf. *Sarkar* (2000); *Cappuccio/Moretto* (2001); *Davis* (2002) and *Lund* (2005).

This article seeks to conceptually generalize *Davis*'s (2002) findings in consideration of the previously discussed literature and to expand from there into other option types. This combination will succeed in deriving closed-form solutions in the case of European entry and exit options, solutions on the basis of which clear separations concerning the value effects are possible. The principle conclusions will be as follow: In accord with *Davis* (2002), the analysis of the entry option reveals a value reduction in this option type vis-à-vis increasing uncertainty. For the exit option, however, one does not observe the reverse impact. Rather, the case of the exit option comes to an intensification of the traditional value effect. From this, the sensitivity of the real option value declines considerably relative to the somehow opaque “uncertainty“ parameter. Also, the analysis of uncertainty in *Merton*'s (1974) model always leads to the negative influence of this parameter on equity value as long as the rating of the firm is better or equal to the “BB“-rating class. Finally, one can subsequently resolve the “learning paradox“ of the real option-approach, i.e. the question as to why a firm with flexibilities would be interested in learning about the future given that its option value rises with the degree of uncertainty. These conclusions should enrich subsequent discussion with further aspects and enhance the acceptance of the real option-approach via a better theoretical foundation.

This article is organized as follows: In Chapter 2, the setup of the model will be elucidated, whereby the effect of volatility on the present value of cash flows as well as on the option component will be integrated. Building upon that, Chapter 3 will deal with the comparative statics for puts and calls of both European and American types on the basis of this modified approach. Chapter 4 will contain three example applications which should clarify the altered economic implications within the new framework. The article will end with a conclusion and an outlook for future areas of application.

2. Derivation of the Modified View

2.1 Mathematical Derivation

In the following, it will be shown by example how a change in the risk structure of the underlying affects the value of the classical entry and exit option.[5] To this end, the cash flow process associated with the potential entry and exit will be first derived into a present value representing the underlying of the respective option type. The derivation thus depends on prevailing procedures within the framework of real option-theory.[6]

The cash flow associated with the exercise of the option CF_t is assumed to be uncertain and follows a geometric Brownian motion of the form:[7]

$$dCF_t = \alpha CF_t dt + \sigma CF_t dB_t^1, \quad CF_0 = c_0 = known,$$

with drift α and volatility σ as well as the common assumptions in relation to probability space and filtration. In accordance to the objective, the process is now completed by a further risk component, which can be understood as an incremental accrual of uncertainty:[8]

$$dCF_t = \alpha CF_t dt + \sigma CF_t dB_t^1 + \varepsilon CF_t dB_t^2, \quad CF_0 = c_0. \tag{1}$$

Given the assumption of a complete capital market, the existence of tradable securities X^i can be ascertained, which are capable of hedging the risk B_t^i of the process,

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5 Cf. for a similar derivation Willershausen et al. (2007), p. 316 ff. The reasoning can also fundamentally be carried over to the case of financial options. In the framework of the real option-approach the c.p.-assumption is, however, less problematic since the capital market parameters as well as the equilibrium upon the capital market is not affected by parameter variations in general. An analogical transition could also be done relative to the riskfree interest rate.

6 Cf. *Dixit/Pindyck* (1994). Likewise, *Sarkar* (2000) approaches the problem referring to cash flows; however, his modification of the theory appears in *Dixit/Pindyck* (1994) as well. As an alternative to the valuation based on the Theory of Martingales the Dynamic Programming-approach is often chosen.

7 The inclusion of so-called „jump processes“ would also be possible as a matter of principle. This aspect was however already discussed briefly in *Dixit/Pindyck* (1994) as well as in *Mölls/Willershausen/Krag* (2005) and can thus be carried out in accordance with these thoughts. At this point, we will forgo further complicating the analysis with such jump processes. The insinuated geometric Brownian motion has no qualitative effect on the subsequent results. It was chosen strictly for providing an expedient comparison between the results disclosed here and the classical results. Note that the cash flow describes a cash flow rate that is – according to the standard – written with capital letters.

8 Without an essential limitation of generality, it is assumed that B_t^1 and B_t^2 are completely uncorrelated. The argument changes marginally, when this assumption is left out. In this case, a revision of the “basis-transformation” would need to be carried out. Cf. *Nielsen* (1999), p. 137 ff.

$i = 1,2$.[9] It is assumed that the X^i follow a stochastic differential equation of the form[10]:

$$dX_t^i = \mu_i X_t^i dt + \sigma_i X_t^i dB_t^i .$$

Provided this, the proposition from *Harrison* and *Pliska*[11] yields the existence of a definite martingale Q, to which the normalized price processes of each tradable stock are martingales. With the value of Q known, the present value V_t of the cash flows can be determined when they are generated starting in time t (i.e. when entering the project):[12]

$$V_t = e^{rt} E^Q\left(\int_t^\infty \frac{CF_s}{e^{rs}} ds \mid CF_t \right) = \frac{CF_t}{\delta}, \tag{2}$$

whereby[13]

$$\delta = r + \sigma \frac{\mu_1 - r}{\sigma_1} + \varepsilon \frac{\mu_2 - r}{\sigma_2} - \alpha = \overbrace{r + \sigma\theta_1 + \varepsilon\theta_2}^{Required\ Return} - \alpha , \tag{3}$$

which corresponds to the starting point of *Sarkar* (2000) and *Davis* (2002). The required rate of return can be interpreted in the sense of an instantaneous and stationary CAPM.[14] In so doing, *r* stands for the risk free interest rate and θ_i denotes the market price of risk of the respective risk class (B_t^i), which is presumed to be independent of the amount of risk of the cash flows. When interpreting the preceding equations, three different cases can be

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9 It is possible to forgo this assumption and choose an alternative approach. Cf. *Davis* (2002), p. 4 ff. as well as *Lund* (2005), p. 6 f.

10 It is not necessarily the case that $\sigma = \sigma_1$ or $\varepsilon = \sigma_2$. If it is additionally assumed that the volatilities of the processes correspond to one another, this would affect the rate of return shortfall (cf. the explanations further below). However, this assumption, which is found in much literature on the subject, is only seldom tenable. *Copeland/Antikarov* (2001), p. 94, give an especially demonstrative example: The value of a gold mine will feature a high correlation with the price of gold, but the volatilities connected with the objects are in general completely distinct. In principle, it is possible to produce the prefactor σ by redistributing the weighting in the duplicating portfolio. Since the risk class cannot be abandoned in doing so, the expected return $\tilde{\mu}_1$ of the rearranged portfolio should be determined according to the CAPM via $\tilde{\mu}_1 = r + \sigma \frac{\mu_1 - r}{\sigma_1}$ (cf. also *Øksendal* (2000), p. 254 ff). At first glance, the rate of return shortfall would then be reduced to $\tilde{\mu}_1 - \alpha$, but the substitution of $\tilde{\mu}_1$ in correspondence to the provided equation will subsequently result in the value $\delta = r + \sigma \frac{\mu - r}{\sigma_1} - \alpha$, as this article will show below.

11 Cf. *Harrison/Pliska* (1981) and (1983) and *Delbean/Schachermayer* (1994).

12 Cf. Section 6.1.

13 The reasoning shows that $\delta + \alpha$ depicts exactly the required costs of capital. Since CF follows a geometric Brownian movement, V_t is well defined.

14 The terminology chosen in this passage follows *Dixit/Pindyck* (1994), p. 148 f.

distinguished concerning the additional risk component B_t^2:

(1) Purely unsystematic risk (θ_2 = 0):

An addition of purely unsystematic risk does not change the required rate of return as well as the present value V_t. In this case, all the conclusions of classical real option-theory remain in effect.

(2) Systematic risk and an increase of cash flow (θ_2 > 0, CF_t ↑):

With an addition of systematic risk and a corresponding simultaneous increase of the cash flow CF_t, the present value V_t remains unchanged. This situation can, as a general rule, be denoted as a “new investment“. If a firm has the choice between several projects that all require the same present value (i.e. amount of investment), then it is to be assumed that with the choice of the riskier project the generated cash flows would be higher. In this case, the classical view of real option-theory also remains in effect.

(3) Systematic risk without an increase of cash flow (θ_2 > 0, CF_t→):

With an addition of systematic risk without a simultaneous rise in cash flow CF_t, the present value V_t declines. This situation circumscribes a “project already in force“. If an investment is already active and the volatility of the cash flows changes later on, a depreciation of value follows. In such a setting, *Davis* (2002) highlights that in real situations it could neither be assumed that the price of risk would be negative or zero, nor could the real rate of growth correspondingly be corrected by raising the variance.[15] Expressed in another way, the value of δ rises with an uptake of further risk when the development of the median cash flow remains constant and assuming a positive market price of risk in that risk class[16] ($\theta_2 > 0$).[17,18] The relation (2) at point $t = 0$ further implies

15 *Davis* (2002), p. 6 ff. gives numerous real examples of such a circumstance.

16 This is the realistic case. In most real projects, a positive beta value is assumed, thus implying that the associated market price of risk in the corresponding risk class is positive.

17 It is implied that through a change in the value of σ the value α does not change. The cash flow of the project alone does not therefore emerge from a complete market. It should not be assumed that higher uncertainty automatically results in a higher rate of return in the development of the cash flow. Thus, a firm can anticipate the median growth of cash flows and the appraisal of such growth does not change merely according to an increase in volatility (cf. *Davis* (2002)). Higher rates of return, which a project with higher volatility must exhibit, come about through a reduction of the present value. Alternatively, i.e. correcting for drift, two projects with completely distinct cash flow profiles would be compared. It should come as no surprise that the subsequent conclusions of such a case would stand in contradiction to the fundamental tenets of investment theory.

that an increase in δ c.p., i.e. with a hold constant (because it is observable) CF_0, results in a lower value of the underlying. For this reason, higher uncertainty works to *decrease the value* of the option's underlying. This accords with the fundamental tenets of investment theory. Using the *Ito* Formula, the value process V_t results in the following:

$$\begin{aligned} dV_t &= (r-\delta)V_t dt + \sigma V_t dB_t^{1,Q} + \varepsilon V_t dB_t^{2,Q} \\ &= (rV_t - CF_t)dt + \sigma V_t dB_t^{1,Q} + \varepsilon V_t dB_t^{2,Q}, \quad V_0 = CF_0 / \delta. \end{aligned}$$

As the expression shows, the volatility structure of the present value process results from the volatility of the development of the cash flow (cf. (1)). A completely uncoupled observation of both uncertainty structures should therefore also not be effected for more general cash flow processes. The alteration of the presumed volatility (σ) in the underlying in favor of a further risk component (ε) has – via the cash flow process – an effect upon the present value of the cash flows (V_0). Consequently, the present value V_0 of the underlying itself fundamentally stands in functional coherence with the parameter σ as well as ε.

In summary, it can be asserted that due to holding the present value constant at *t = 0* the classical view implicitly implies further assumptions when carrying out comparative statics.[19] A fixation of the present value entails nothing other than an incorporation of purely unsystematic (valuation-irrelevant) risk ($\theta_2 = 0$), provided that the underlying cash flow profile ($\alpha \nearrow$) is not supposed to become completely supplanted. This corresponds – graphically speaking – to a horizontal transition of the most diverse risk classes (cf. *Figure 1*). Accordingly, the classical view can be retained, if the rise in risk emerges in the form of unsystematic risk. If, however, systematic components gain a foothold, in the end only the present value reduction or cash flow process exchanges would remain.[20] Thus, a conflict with classical investment theory does not exist.

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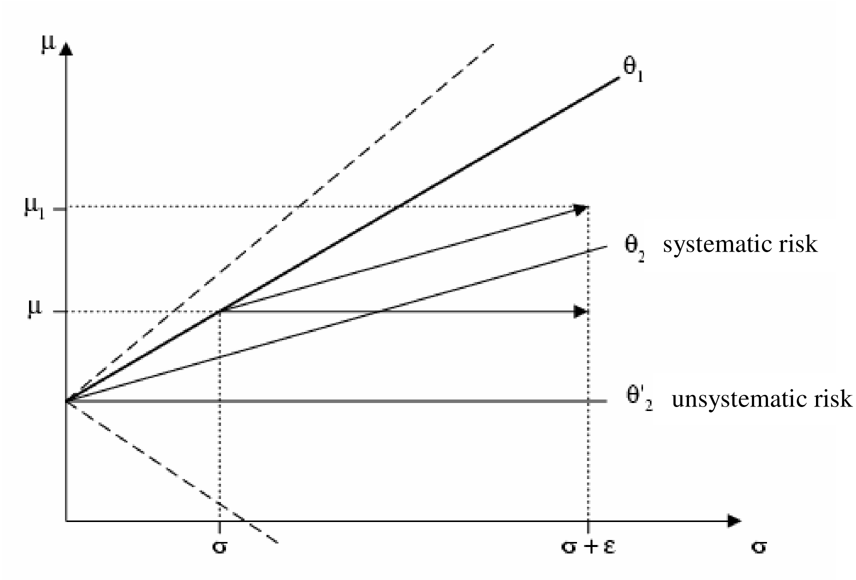
[18] *Sarkar* (2000) and (2003), *Cappuccio/Moretto* (2001), *Davis* (2002) and *Lund* (2005) all come to the same assertions.

[19] *Dixit/Pindyck* (1994), p. 334 warn of the uncritical use of comparative statics referencing those assumptions often held implicit.

[20] It is important to note that the distinction between risk aversion and risk neutrality is not essential for the purpose discussed here. Rather, the form of the project risk is more salient.

Figure 1: Classical vs. new view within the CAPM



2.2 Illustration of the Results

In this section, the above results will be illustrated for the classical entry and exit option. For the value of an American call option upon the underlying V_t with strike price I and infinite maturity, the following is valid:[21]

$$C(V_t) = \begin{cases} AV^{\beta}, & \textit{für } V < V^* \\ V - I, & \textit{für } V \geq V^* \end{cases}$$

with

$$\beta = \frac{1}{2} - (r-\delta)/\sigma^2 + \sqrt{\left[(r-\delta)/\sigma^2 - \frac{1}{2}\right]^2 + 2r/\sigma^2} > 1$$

$$A = (V^* - I)/(V^*)^{\beta} = \left(\frac{\beta-1}{I}\right)^{\beta-1}\left(\frac{1}{\beta}\right)^{\beta}.$$

The left illustration in *Figure 2* graphically plots this option for two distinct values of σ. It will become clear that an increase of uncertainty ($\sigma_1 < \sigma_2$) will have the option value rise.[22] With a fixed underlying (V_0), this leads directly to the assertion that uncertainly has a

[21] Cf. *Dixit/Pindyck* (1994), p. 136 ff.

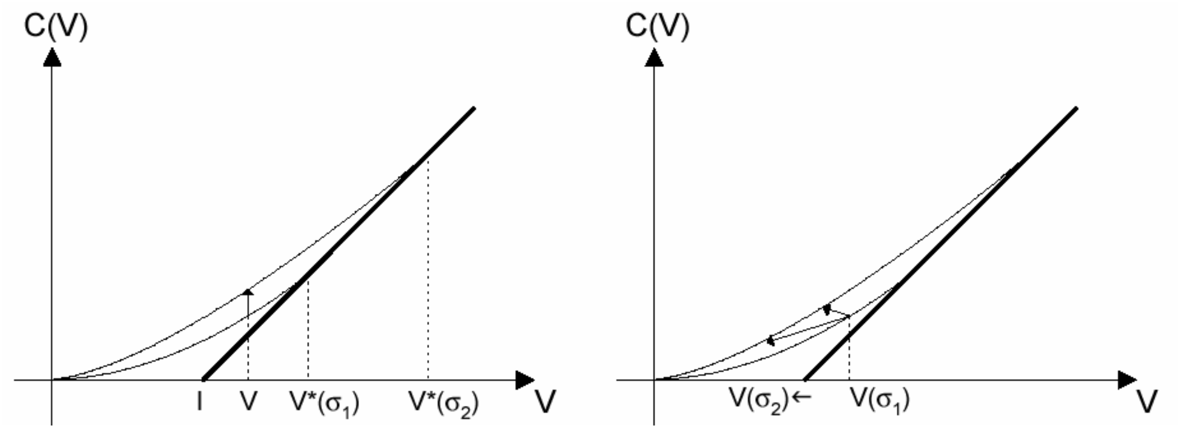
[22] In the above model this corresponds to an addition of $(\sigma_2 - \sigma_1)dB_t^1$, i.e. $\varepsilon = \sigma_2 - \sigma_1$.

strictly positive influence on the value (cf. the left illustration). Admittedly, such a conclusion depends decisively on the assumption that a rise in the volatility of cash flows does *not* influence the DCF-value. However, as it was shown in Section 2.1, this is only a reasonable assumption under certain premises, due to the relation

$$V_0 = \frac{CF_0}{\delta} = \frac{CF_0}{r + \sigma\theta_1 + \varepsilon\theta_2 - \alpha}.$$

Without these assumptions the value V_0 reacts to a change in the volatility structure in dependence with the risk class. A project risk ($\theta_2 = 0$) completely uncorrelated with the market results in the aforementioned constellation $V_0(\sigma_1) = V_0(\sigma_2)$. In the case $\theta_2 > 0$ it follows that $V_0(\sigma_1) > V_0(\sigma_2)$, and finally, when $\theta_2 < 0$, $V_0(\sigma_1) < V_0(\sigma_2)$ is valid. It is evident that, dependent on $V_0(\sigma_2)$, the cumulative value of the option can just as well rise as fall (cf. the illustration to the right in *Figure 2*, where only the (realistic) case $\theta_2 > 0$ is diagrammed).[23]

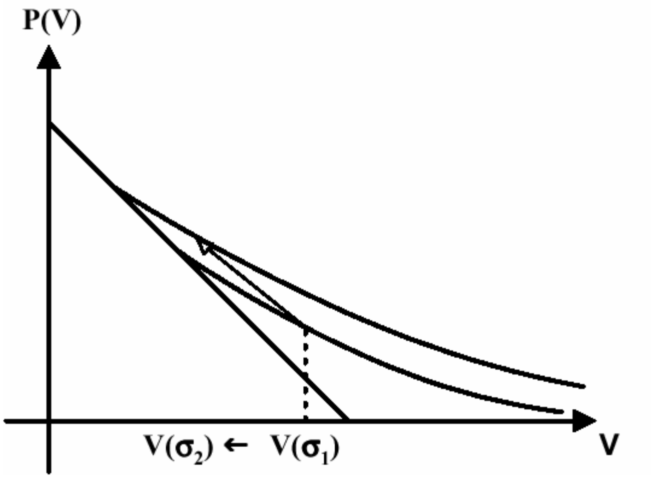
Figure 2: Old vs. new perspective relative to the American call



[23] According to the new approach an increase in the parameter σ no longer necessarily leads to a later entry (V^*), since an increase in δ c.p. causes a decline in the entry threshold. Cf. *Sarkar* (2000) for a more elaborate discussion on a comparable question. *Sarkar* (2000) also chooses the above depiction of δ, but a discussion of how a change of the parameter σ within the comparativ statics affects this assumption remains absent.

The universal validity of the statement suggested by the illustration at the left in *Figure 2* is in no case tenable, however. Rather, a fundamental conflict between the two described effects persists: Growing uncertainty entails both a fall in the DCF-value as well as a relative rise in the value of flexibility. An essentially homogenous picture results in the case of an American put option with infinite maturity (cf. *Figure 3*). For a project risk that is positively correlated to the market ($\theta_2 > 0$), both effects act in the same, value-appreciating direction. This is not remarkable insofar as put options directly insure against distressing conditions.

Figure 3: Illustration of the new perspective relative to the American put



3. Comparative Statics

In this chapter, the critique treated in the previous sections will be analyzed quantitatively. Within the practical framework of comparative statics-analysis it is always assumed that an increase of the factor σ describes the growth in uncertainty. In light of the above deliberation, this corresponds to the addition of a second and identical Brownian motion, i.e. $dB_t^2 = dB_t^1$. Because of this, the problem of the form of the risk (systematic or unsystematic) from which the increase in uncertainty originates, reduces to the question of the type of risk this particular Brownian motion (B_t) exhibits.

Four classical option types will be examined[24]: First, the simple European call and put option will be dealt with, followed by the American entry and exit option with infinite maturity. The first two cases result in closed-form solutions, while in the latter two situations we revert to numerical results due to prevailing complexity and ambivalence concerning the resulting effects. The procedure itself differentiates from predominating approaches[25] in one decisive aspect: It does not see the value V_0 as fixed, but rather determines this parameter through the (implicit) equation

$$V_0 = \frac{CF_0}{\delta} = \frac{CF_0}{r - \alpha + \sigma\theta}$$

for each choice of σ and then calculates the option value using the modified underlying.[26] Only the value CF_0 is assumed to remain constant (because its observable) and therefore corresponds to an invariant dividend rate ($V_0\delta = CF_0$). Moreover, in the following the exercise of the call option will be interpreted as an investment and the exercise of the put option will be viewed as a disinvestment.

3.1 Analysis of European Call Options

This examination takes as its point of departure the relationship of the European call option value (*C*) with strike price *I* and maturity *T* according to the *Black/Scholes*-Formula on a dividend-paying stock:[27]

$$C = V_0 e^{-\delta T} N(d_1) - e^{-rT} I\, N(d_2),$$

whereby $N(\cdot)$ denotes the cumulative standardized normal distribution $d_1 = \frac{\ln(V_0 / I) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$. Assuming that V_0 is completely independent from the value of σ, the partial derivation of the call option on a dividend-paying stock with maturity *T* results in the following:[28]

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[24] Cf. for an analogical analysis Willershausen et al. (2007), p. 318 ff.

[25] For exceptions see *Sarkar* (2000) und *Davis* (2002) in particular.

[26] Thus, the so-called „option delta“ determines the change in the option value.

[27] Cf. *Hull* (2005).

[28] Cf. for the derivation *Hull* (2005). Note that V_0 as the current stock price is always greater than zero. N' denotes the derivation of the cumulative standardized normal distribution.

$$\frac{\partial C}{\partial \sigma} = V_0 e^{-\delta T} N'(d_1)\sqrt{T} > 0. \tag{4}$$

This well-established relation founds the assertion that growing uncertainty in the form of σ positively influences option values. The assumption, however, that V_0 itself is a function of σ slightly changes the picture:[29]

$$\frac{\partial C}{\partial \sigma} = V_0 e^{-\delta T}\left(-\theta(\frac{1}{\delta}+T)N(d_1) + N'(d_1)\sqrt{T}\right). \tag{5}$$

The leading component of the derivative is novel. It is induced by the reduction of the present value and displays the ambivalent value effects. The term $-\theta(\frac{1}{\delta}+T)N(d_1)$ is less than zero for positive values of θ and δ, and can exceed the value $N'(d_1)\sqrt{T}$ with a suitable choice of parameters. In this case, increasing uncertainty would have a negative influence. The above formula makes it furthermore clear that the impact of the present value completely separates itself from the increase in flexibility (as expressed by the second summand) in an additive manner. As a comparison with the equation in Formula 4 displays, the final term remains completely unaltered in its form.[30] These results reveal the need for caution when interpreting partial derivatives. Derivation (4) merely states that in two projects with the same DCF-value one must choose the one which exhibits a greater risk (σ), since the real options connected with this project are more valuable.[31] If one observes changes in risk within a project, then no general conclusion on the alteration of value in the real option can be obtained (cf. (5)). Instead, such a case requires a more precise examination of the corresponding circumstances.[32]

At this point it is appropriate to determine – as *Davis* (2002) aptly puts it – whether the real option is „in the money“ or „out of the money“.[33] In the first case, an increase in

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29 Cf. Section 6.2.

30 As shown in Section 6.2, this is independent of the special form of the functional dependency $\delta(\sigma)$.

31 Note again that the net present value-rule in this case would result in an indifference regarding the decision-making.

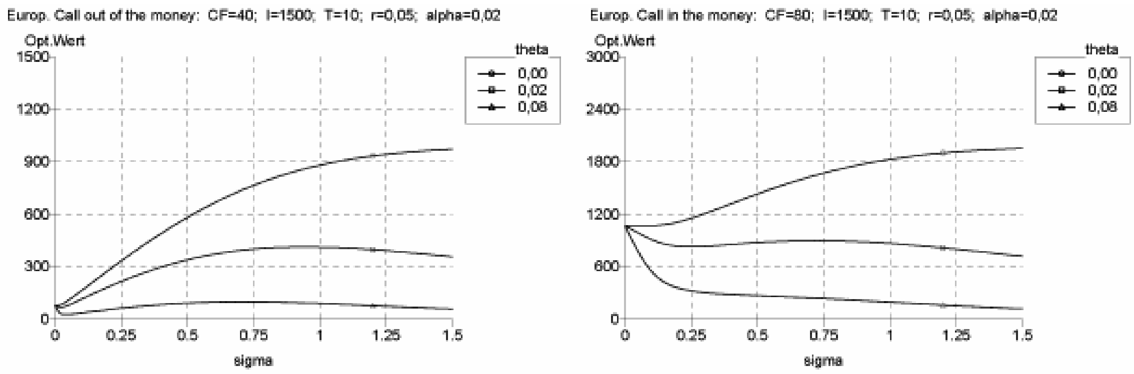
32 The unconditional assertion of a positive influence, as it has been raised in other works (cf. Footnote 1) in clear distinction from naive DCF-methods, should be viewed extremely critical in light of the above explanation. *Huchzermeier/Loch* (2001) present a single exception, breaking down and interpreting the influence of uncertainty and risk very precisely on the basis of a binomial structure. *Trigeorgis* (1999), p. 372 f., references this aspect in a figure by distinguishing the effect of increasing uncertainty into the influence on the „static NPV“ and the „option premium.“

33 Mathematically speaking, the case differentiation is reflected in the various values of the cumulative normalized distribution: Being „in the money“ results in a higher value for the cumulative normalized distribution, while being „out of the money“ lends itself to a lower value.

uncertainty works predominately toward a decrease in the present value, since the option character is not particularly pronounced in this range. From here, the option value tends to decrease in sum. However, since the option character gains in importance relatively with each reduction in the value of the underlying, the decline in value has a distinct absolute minimum. The opposite result appears in the second case. Here, the option is "out of the money" and the option value is essentially born upon the option character. An increase in uncertainty additionally raises the value as well as mostly compensates for the reduction in the present value.

Figure 4 displays this situation graphically. The graph to the left shows the behavior for the case of being „out of the money". In the classical situation (represented at $\theta = 0$, since δ remains constant at this point)[34], the value of the real option increases as expected in a strictly monotonic fashion. However, with an increase in θ, the effect on the value becomes ambivalent. A compensation of the present value effect no longer occurs in certain situations. The influence reverses itself. The graph to the right describes the behavior in the alternative range. The classical view ($\theta = 0$) anticipates a rise in value. The modified view, in contrast, tends to state a decrease in value, which turns out all the more clearly with an increasing market price of risk in risk class θ. This confirms the results of *Davis* (2002) in the European case as well.

Figure 4: European call dependent on σ („out of the money" vs. „in the money")



[34] Addition of purely unsystematic risk. There is no necessity for a reduction in the present value.

3.2. Analysis of American Call Options

The case of an American call option with infinite maturity only marginally differs from its European counterpart.[35] *Figure 5* shows the value of the call option dependent on σ with CF_0 held constant in both cases of being „in the money“ and „out of the money.“ It can be once again observed here that with an increase of the market price of risk in the given risk class (represented by $\rho_{Market,x} \cdot \theta_{Market}$, whereby ρ denotes the coefficient of correlation) the effect of a positive influence becomes mitigated to the point that this effect becomes reversed (cf. the graph to the right in particular). This reversal already occurs in completely realistic choices of parameters. Thus, the lowest curve in the graph to the right refers to a market price of risk of the risk class at a level of 0.08, which with a risk-free interest rate of $r = 5\%$, a market risk of $\sigma = 18\%$, and a 10% market rate of return yields a correlation of the project risk with the market portfolio of approximately 0.3 ($0.08 \approx \frac{10\% - 5\%}{18\%} \cdot 0.3$).[36] Accordingly, what has thus far been discussed is not a fanciful construct. As seen already in the European case, the respective medial run of the function shows the ambivalent behavior of value especially clearly. Smaller values of σ result in an anticipated negative influence on the value. If the degree of uncertainty increases further, however, this positively affects the value of the option at first. In this case, the increase in flexibility compensates for the reduction in the present value which accompanies the growth in risk. Here, this general conclusion concurs with the classical view. A further expansion of uncertainty finally leads to yet another reversal of the tendency of influence.

As expected, the conclusions of *Davis* (2002) are confirmed for the American entry and growth option. Additionally, the importance of the market price of risk should be highlighted, insofar as the above discussion shows how sensitively the influence of uncertainty reacts to the market price of risk of the underlying.

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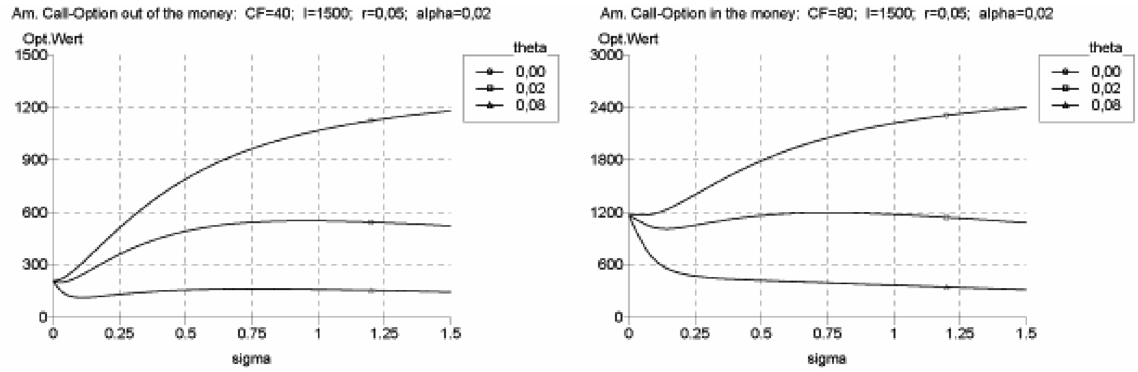
[35] In this case, the problem of the optimal investment threshold (i.e. the value of the underlying at which an investment is optimal) as well as its associated investment probability come furthermore into play. Among the given conditions, this applies for the threshold $V^* = [\beta/(\beta-1)]\, I$ (cf. *Dixit/Pindyck* (1994) for a depiction of β in particular). If one accounts for the fact that $\beta = \beta(\sigma, \delta(\sigma))$ is valid, with the determination of the investment probability one must allow for the change in not only the distribution but also – via the optimal threshold – the variable δ. This aspect will not be further discussed here since this situation has already been dealt with in the noted works and the examination here seeks to focus on the value effect.

[36] It could be claimed, for example, that Volkswagen exhibits a one-year correlation of approximately 0.7 with the German DAX. In such a case, the risk class would then reach approximately 20%, clearly higher than the 8% reference value.

Figure 5: American call dependent on σ („out of the money“ vs. „in the money“)



3.3 Analysis of European Put Options

As was the case with the European call option, the European put option (*P*) for a dividend-paying stock with maturity *T* and strike price *I* results in a closed-form solution in regard to the influence of increasing uncertainty on the value. The statement

$$\frac{\partial P}{\partial \sigma} = V_0 e^{-\delta T}\left[\underbrace{\left(1 - N(d_1)\right)}_{>0}\theta\left(\frac{1}{\delta}+T\right) + N'(d_1)\sqrt{T}\right] > 0$$

is valid as long as $\theta, \delta, V_0 > 0$.[37] This result differs from the typical depiction – as it already did in the case of the entry option – by the existence of a further summand. Due to the put call parity this summand resembles the above formula, but also now exhibits a positive sign that reflects an even stronger impact of an increased uncertainty on the value than assumed by the prevailing view. Restrictively it should be noted that the influence on the value could become ambivalent here as well if the growth in uncertainty arises from a risk negatively correlated with the market, i.e. for $\theta < 0$. However, this situation would have more of a theoretical rather than practical relevance.[38] One can once again observe that the value effects (additively) separate from each other, leaving the flexibility component completely intact despite the changing functional relation. For a graphical representation of this situation it is referenced to the results of the following section.

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[37] Cf. Section 6.2.

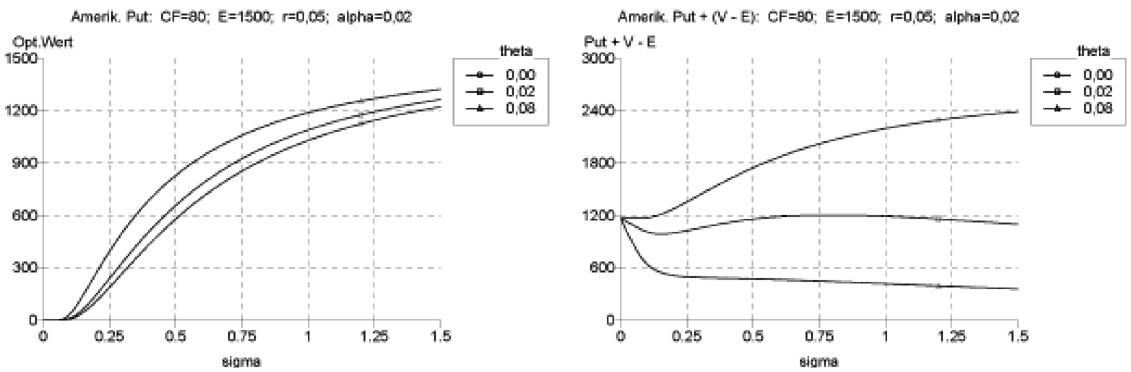
[38] The case $\theta = 0$ results, as expected, in the classical presentation of a strictly positive influence on value.

3.4 Analysis of American Put Options

Concerning the value effect the case of the American put option results in a rather unsurprisingly homogenous picture. Numerical analysis not only confirms the prevailing view, but also reveals an influence to be all the more incisive under the new perspective. The hedge in regard to poor market developments that display the economic kernel of the put option become increasingly meaningful under the modified view: Risk-averse investors „punish“ the higher level of uncertainty by lowering the market value. Accordingly, the underlying of the option becomes worse in regard to its net present value, what is to the final benefit of put option's holder. The graph to the left in *Figure 6* shows the difference between the two views ($\theta = 0$ vs. $\theta \neq 0$). In this particular configuration, the distinction between being „in the money“ vs. „out of the money“ bears only marginal differences. With that in mind, an explicit comparison between the two will not be carried out.

However, one should note that the holder of a real put option (exit option) also tends to own the underlying itself, restricting the significance of an isolated analysis of the option's value. For this reason, the graph to the right charts the more relevant value effect for the sum of the put option and the underlying reduced for the strike price.[39] This sum results however in nearly identical patterns compared to the case of the American call option noted above due to the put call parity.[40]

Figure 6: American put as well as sum of the put and reduced underlying dependent on σ



[39] The value was reduced by *E* in order to use the put call parity. The sum of the put plus the underlying is greater by amount *E*.

[40] It should be noted that the put call parity is only an initial approximation in the American case. For this reason, the behavior of the value is not completely identical.

4. Application of the Results

The consequences of the results in the previous sections will be discussed in this chapter on a more economic level in order to convey an understanding of when the procedure dealt with here would be indicated. Hence, the following sections will focus on three areas. Firstly, the question will be clarified as to how a false estimation of the parameter σ would affect the value of a real option. In light of the previous deliberation, it will thereby be shown that the high sensitivity of the calculus becomes mitigated. Going further, this chapter will once again challenge the valuation of equity for a leveraged firm within *Merton*'s (1974) model. The latter example will serve to clarify a fundamental inconsistency within the real option-approach, i.e. if „… [real] options are in general increasing functions of uncertainty whereas learning reduces uncertainty, why would we want to learn?" (*Martzoukos/Trigeorgis* (2001), p. 2). The classical view can only explain this paradox with great difficulty.

4.1 Uncertainty about the Degree of Uncertainty

In order to calculate a real option value one has to quantify the height of the parameters involved at the best. To this end, the estimation of the risk class ($\theta_i = \theta_{Market} \cdot \rho_{Market,i}$) as well as of the real expected growth rate of cash flows can be done more easily in comparison to an appropriate ascertainment of the particularly relevant parameter σ.[41] Since the classical view exhibits a high sensitivity in this respect, the application of the real option-reasoning often encounters difficulties in practice.[42] However, the diagrams in Chapter 3 show that the corridor in which the value of the real option fluctuates according to varying (realistic) values of σ observably contracts under the new approach. The graph on the left of *Figure 5* plots just such a change in the margin from approximately [200,1200] to [200,600] with a θ of 0.02 and to [200,100] with a θ of 0.08. This corresponds to a reduction of 60% and 90% respectively.[43] In this way, the high

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41 Cf. *Baecker/Hommel* (2004), p. 27 ff. for a general discussion of this problem. Cf. for a similar depiction Willershausen et al. (2007), p. 324 f.

42 Cf. *Baecker/Hommel* (2004), p. 27 ff.

43 The percentage values are based on the following calculation: One minus the new interval width divided by the classical view's interval width.

sensitivity of the calculus becomes mitigated as long as the given conditions are met.[44]

However, the above considerations only apply to call-sided types of real options. As shown in Section 3.3, the new approach accentuates the sensitivity of put-like option types. If it is brought into consideration that the owner of a real put option generally tends to own the underlying itself, the sensitivity of the call carries over to the sum of the put and the underlying, as already seen.[45] Consequently, a reduction in sensitivity can be deduced here as well. A practical application of the real option-approach looks to be more sensible in light of this.

4.2 Value of Equity in the Merton Model

The results from Chapter 3 can be drawn upon here to investigate the influence of uncertainty on the value of equity in limited liability corporations, thereby clarifying the issue as to whether equity should really always seek the maximal risk. In order to do this, the simple standard structural model of *Merton* (1974) will be called upon.[46] In this model, equity is understood as a call option on the firm's assets, since the assets only then transfer over into equity if the debt capital is amortized.[47] Under the classical view, this model subsequently asserts that equity holders prefer investment projects with the greatest possible risk because of its call-like value structure. Yet, in doing so, the dependency of asset values on the underlying risk and therefore the elementary risk/return-relationship become implicitly negated.

Given the above outcomes, the equity investor is faced with a complex investment decision, one which must account for the underlying as well as the option value in equal measure. The investment with the highest risk thus does not always prove itself to be the most advantageous. As the previous deliberations show, the decision tends to disfavor the riskier project so long as no additional cash flow effect registers (i.e. a larger α in the

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44 Corresponding to the new approach, the input parameter σ should for example be treated as an interval (e.g. between 10% and 30%) given the parameters α (the cash flows' real rate of growth) and θ (the risk class of the project), thereby leading to a reduction of the margin.

45 Cf. the graph to the right in *Figure 3.* The shifting around the constant *E* in this image does not affect the sensitivity.

46 *Black/Scholes* (1973) already applied the option pricing-theory to the valuation of debt, an idea later developed further in *Merton* (1974). Cf. *Perotti/Rossetto* (2007) for an analysis of equity carve outs as strategic real options.

47 Note that in the European case the views on equity as owning a call or a put are equivalent. In the American case equity has to be seen as a put option.

project with higher uncertainty).[48]

In order to underpin the previous assertions quantitatively, the results of a simulation will be presented in the following. For this purpose, a rating class $RAT \in \{Aa, A, Baa, Ba, B\}$ together with the cumulative probabilities $CumPD(RAT)$ have firstly been selected based on *Moody*'s (2006). Subsequently, a parameter vector $\varphi = (T, r_f, \delta, \theta, \sigma)$ and then L/V_t were implicitly determined in a way that the cumulative probability of failure associated with the rating class is precisely obtained, i.e.

$$P[\text{Default}] = P[V_T \leq L] = N(-\tilde{d}_1) \stackrel{!}{=} N\left(\frac{\ln(L/V_t) - (r_f + \theta\sigma - \delta + \sigma^2/2)}{\sigma\sqrt{T}}\right). \quad (6)$$

This was repeated for all rating classes from Aa to B and for all parameter combinations $\Phi=\{\varphi: 1 \leq T \leq 10, 0\% \leq r_f \leq 10\%, 1\% \leq \delta \leq 10\%, 5\% \leq \theta \leq 40\%, 5\% \leq \sigma \leq 100\%\}$. *Figure 7* depicts the *maximal* values of the Vega – that is, the derivative of the equity with respect to the volatility – per rating class and risk class (θ) as well as maturity (T). The left graph shows that the Vega for all rating classes Ba and better is always negative, i.e. an increase in volatility leads to a lower equity value. Only with very poor credit standings (rating class B) a rise in volatility can lead to a higher equity value. The graph to the right shows that this is the case only with longer maturities, however. Thus, for example, even for rating class B the Vega always becomes negative given short maturities of only a year. All of these observations confirm existing banking practices concerning credit risk: On the one hand, covenants as a rule stipulate limitations on new investments (CAPEX). As demonstrated in Section 2.1, new investments conform to the classical view of option pricing-theory, since the investment amount is of fixed size and the cash flows are regularly higher with high-risk investments. Additionally, firms with poorer credit standings typically only garner short-term credit. The danger of a conflict between equity holders and outside creditors becomes greater with longer-term credit, as the graph to the left shows. Such a danger does not persist at better credit ratings since these options depict „very deep in the money"-options that are dominated by the present value effect.

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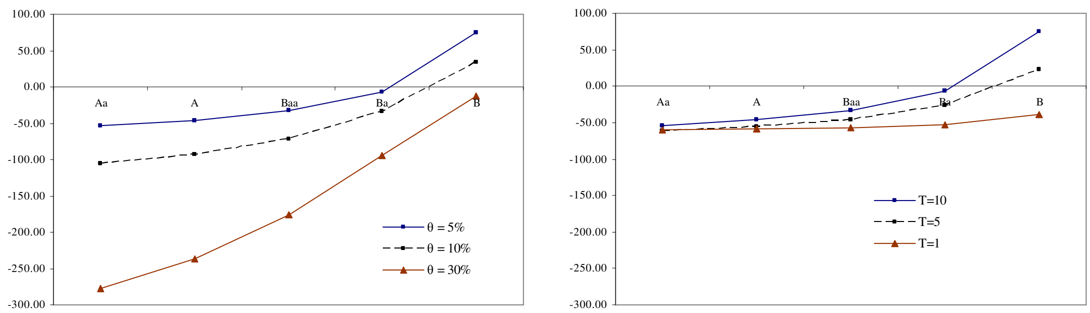
48 Cf. *Figure 4*. The homogeneity of interests between equity and debt in junk bonds – as postulated by *Leland/Toft* (1996) – must be critically examined in the same way. This article cannot, however, carry out an exhaustive discussion of this issue.

Figure 7: Maximal Vega according to the modified view, extending over all realisitic parameter combinations as a function of the rating, risk class (θ), and maturity (T) of debt in Merton's (1974) model



4.3 Influence of Learning

In light of the real option-approach, one must ask what sense it makes to collect information intent on a „reduction of uncertainty“ (in a value-based sense). It is questionable, for example, whether a firm that carries out a market study on the commercial success of a potential product would be doing itself a devaluating disservice by dispelling uncertainty in such a way. This conclusion would hold within the classical view in its pure form. A more sophisticated understanding of the problem that takes into account the above deliberations provides a different and much more sensible picture. A reduction of uncertainty in the cash flows from a potential marketing of a product would in fact depress the value of flexibility.[49] At the same time – in contradiction with the classical view – the dissolution of uncertainty would moreover change the underlying present value, thereby altering its level. The presumption of a static DCF-value independent of σ is not advisable in such a case. Risk-averse market actors would „reward“ the additional knowledge, i.e. the reduction of the margin, with a relatively higher present value. The acquisition of information would even intensify the higher valuation of the underlying, insofar as enhanced knowledge and the reduced uncertainty are often accompanied by a positive cash flow effect. Hence, the firm should then always take up the costs of reducing uncertainty if the gain in the underlying would compensate for the loss in the option value and if the costs

49 The dissolution of uncertainty finally makes the temporal progression of the DCF-value truly visible. In the extreme case, i.e. with complete certainty of the course of cash flow and thereby of the inter-temporal development of the DCF-value, the optimal alternative would be apparent and determined from the outset. Additional flexibility in comparison to the simple net present value-rule would be (economically speaking) worthless.

generated can be covered by the value difference.[50]

Thus, it is evident here as well that learning about future developments and investments to alleviate uncertainty always makes sense if the corresponding risk is systematic and if the generated information can reduce the margin of possible states.[51] However, in the case of purely unsystematic risk, like that conjectured by the classical view, such an investment becomes economically senseless, since a present value effect would remain absent.[52]

5. Conclusion

The previous discussion has shown that one cannot stick to the sweeping claim of a positive influence of increasing uncertainty on the value of real options, as is commonly argued in the pertinent literature. Rather, concerning the results of comparative statics-analysis one has to adopt a more nuanced perspective that accounts for the transection of value-relevant interrelations. This article developed this new view and presents a modified standpoint which can bring the real option-approach in accord with the fundamental tenets of investment theory. Thus, a conflict between these two does not emerge. Quite to the contrary: Consistent with classical theoretical approaches, it can be substantiated that the value of a real option depends vitally on the form of underlying risk. The correlation with the market portfolio must therefore also be seen as an essential measure for the determination of value within the framework of the real option-approach. The variance alone can only conditionally inform.

Along with this fundamental insight, the results further show that the high sensitivity of the real option-calculus vis-à-vis the somehow opaque parameter σ mitigates under the modified perspective. Practical application is supposed to find this circumstance especially beneficial since this parameter cannot generally be determined exactly due to incomplete information. In addition, this article's results weaken the conclusion of the classical *Merton* (1974) model pertaining to pronounced risk propensity among equity holders. The new perspective only recognizes this phenomenon for unsystematic risk.

[50] Once again one comes down to the question as to whether an investment in the dissolution of uncertainty exhibits a positive capital value.

[51] Respectively the cash flow process could be altered in a way that the investment would become profitable.

[52] Presumably the investment only entails a reduction of σ.

Rather, the adding of value-relevant risk tends to expose equity – despite its option character – to a dip in value if a positive cash flow effect does not accompany the rise in risk, thereby cancelling out the value-decreasing risk/return-relationship.

Future contributions to the real option-approach should incorporate the above results as well as the critical conclusions of the mentioned works concerning the investment probability in their treatments. In particular, the analyses dealing with uncertainty should show in detail in which form the elementary risk/return-relationship finds relevance, allowing the derivation of more conclusive and to some extent better founded conclusions.

6. Mathematical Appendix

6.1 Derivation of the Present Value Formula

Under the assumption of a complete capital market, the existence of traded stocks X^i capable of hedging the risk of the process B_t^i $(i = 1,2)$ can be ascertained. It is assumed that the X_t^i follow a stochastic differential equation of the form

$$dX_t^i = \mu_i X_t^i dt + \sigma_i X_t^i dB_t^i .$$

Given the assumption of a complete capital market and the proposition of *Harrison* and *Pilska*[53], one can deduce the existence of a unique martingale measure Q, in terms of which the standardized price processes of each tradable stock are martingales. The processes X_t^i can thus be written as:

$$\begin{aligned} dX_t^i &= rX_t^i dt - rX_t^i dt + \mu_i X_t^i dt + \sigma_i X_t^i dB_t^i \\ &= rX_t^i dt + \sigma_i X_t^i (\underbrace{\frac{\mu_i - r}{\sigma_i}}_{\theta_i} dt + dB_t^i) \\ &= rX_t^i dt + \sigma_i X_t^i dB_t^{i,Q} . \end{aligned}$$

A simple zero explement was carried out between the first and second line, while between the second and third line, the proposition from *Girsanov*[54] was used, whereby:

[53] Cf. *Harrison/Pliska* (1981) and (1983) and *Delbean/Schachermayer* (1994).
[54] Cf. *Øksendal* (2000).

$$dB_t^i = dB_t^{i,Q} - \frac{\mu_i - r}{\sigma_i} dt.$$

The process of the temporal fluctuation of the cash flow CF_t under Q results in:

$$\begin{aligned} dCF_t &= \alpha CF_t dt + \sigma CF_t dB_t^1 + \varepsilon CF_t dB_t^2 \\ &= \alpha CF_t dt + \sigma CF_t (dB_t^{1,Q} - \theta_1 dt) + \varepsilon CF_t (dB_t^{2,Q} - \theta_2 dt) \\ &= (\alpha - \sigma\theta_1 - \varepsilon\theta_2) CF_t dt + \sigma CF_t dB_t^{1,Q} + \varepsilon CF_t dB_t^{2,Q} \\ &= (r - \delta) CF_t dt + \sigma CF_t dB_t^{1,Q} + \varepsilon CF_t dB_t^{2,Q} \end{aligned},$$

whereby[55]

$$\delta := \overbrace{r + \sigma\theta_1 + \varepsilon\theta_2}^{Required\ \ Return} - \alpha.$$

The value $\theta_i := \frac{\mu_i - r}{\sigma_i}$ depicts – as already known from above – the market price of the corresponding risk class (B_t^i).[56] If one observes the temporal progress of cash flows as a contingent claim, the present value V_t of the discounted cash flows starting in time *t* (i.e. when entering the project) corresponds to the valuation formula for any contingent claim:[57]

$$\begin{aligned} V_t &= e^{rt} E^Q \left(\int_t^\infty \frac{CF_s}{e^{rs}} ds \mid CF_t\right) = e^{rt} E^Q \left(\int_t^\infty e^{-\delta s} \frac{CF_s}{e^{(r-\delta)s}} ds \mid CF_t\right) \\ &= e^{rt} \int_t^\infty e^{-\delta s} E^Q \left(\frac{CF_s}{e^{(r-\delta)s}} \mid CF_t\right) ds = e^{rt} \int_t^\infty e^{-\delta s} e^{-(r-\delta)t} CF_t ds \\ &= e^{\delta t} CF_t \int_t^\infty e^{-\delta s} ds = \frac{CF_t}{\delta}. \end{aligned}$$

55 δ represents the rate of return shortfall (cf. *McDonald/Siegel* (1984)).

56 θ can also be interpreted as $\theta = \rho_{Market,X} \frac{\mu_{Market} - r}{\sigma_{Market}}$, that is, the risk class correlated with the market price of risk in a certain manner.

57 It is assumed that cash flows are accumulated indefinitely and that $\delta > 0$ in order to assure true integrability. The transition from the first to the second line takes place through the generalization of the proposition of *Fubini* (cf. *Duffie* (2003), p. 333). In the second line it was utilized that CF_t suitably standardized with $e^{(r-\delta)s}$ is a martingale under Q. Cf. *Øksendal* (2000), p. 55.

6.2 Comparative statics

This section will verify the provided formula for the partial derivation. It is to derive $\frac{\partial C}{\partial \sigma}$, whereas it is assumed that

$$C = V_0(\sigma)e^{-\delta(\sigma)T} N(d_1(\sigma)) - Ie^{-rT} N(d_2(\sigma)).$$

Within the presentation – differently than in the text – the dependence of the parameters on σ are emphasized. Noted as ' the derivative with respect to each value of σ results in

$$\frac{\partial C}{\partial \sigma} = (V_0 e^{-\delta T})' N(d_1) + V_0 e^{-\delta T} N'(d_1) d_1^{'} - Ie^{-rT} N'(d_2) d_2^{'}.$$

Provided that $V_0 = CF_0 / \delta$ and $\delta = r + \sigma\theta - \alpha$, it follows that in the first part of the derivative

$$\begin{aligned}(V_0 e^{-\delta T})' N(d_1) &= \left(-\theta \frac{CF_0}{\delta^2} e^{-\delta T} + V_0(-\theta\, Te^{-\delta T}) \right) N(d_1) \\ &= -\theta\, V_0 e^{-\delta T} N(d_1) \left(\frac{1}{\delta} + T \right) < 0,\end{aligned}$$

given $\theta, \delta, V_0 > 0$. The second part of the above derivative differs from the classical derivation (cf. *Hull* (2005) among others) by the assumed functional dependence for $\delta(\sigma)$. Despite this change, the classical outcome is – independent of the concrete form of the functional dependence – preserved, as the following calculations show:

$$V_0 e^{-\delta T} N'(d_1) d_1^{'} - Ie^{-rT} N'(d_2) d_2^{'} \;=\; V_0 e^{-\delta T} N'(d_1) \left(d_1^{'} - \frac{I}{V_0} e^{-(r-\delta)T} \frac{N'(d_2)}{N'(d_1)} d_2^{'} \right).$$

It is further valid that:

$$\frac{N'(d_2)}{N'(d_1)} = \frac{N'(d_1 - \sigma\sqrt{T})}{N'(d_1)} = \frac{e^{-\frac{1}{2}(d_1 - \sigma\sqrt{T})^2}}{e^{-\frac{1}{2}d_1^2}} = e^{d_1\sigma\sqrt{T} - \frac{1}{2}\sigma^2 T},$$

whereby

$$e^{d_1\sigma\sqrt{T}} = e^{\ln(\frac{V_0}{I}) + (r - \delta + \frac{\sigma^2}{2})T}.$$

Substituting twice simplifies the derivative to:

$$V_0 e^{-\delta T} N'(d_1)(d_1' - d_2').$$

Finally, given $d_2' = (d_1 - \sigma\sqrt{T})' = d_1' - \sqrt{T}$, it follows for the derivative altogether:

$$\frac{\partial C}{\partial \sigma} = V_0 e^{-\delta T}\left[-\theta\, N(d_1)\left(\frac{1}{\delta} + T\right) + N'(d_1)\sqrt{T}\right].$$

As previously noted, the latter summand is exactly concordant with the classical formula. A novel, however, is the rest of the derivative that is caused by the reduction of the present value and that displays the ambivalent value influence.

The European put option (*P*) of a dividend-paying stock results via the put-call parity

$$P = C - V_0 e^{-\delta T} + I\, e^{-rT}$$

in the partial derivative:

$$\frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} - (V_0 e^{-\delta T})' + 0$$

and likewise with recourse to the above results:

$$\frac{\partial P}{\partial \sigma} = V_0 e^{-\delta T}\left[\underbrace{\left(1 - N(d_1)\right)}_{>0}\theta\left(\frac{1}{\delta} + T\right) + N'(d_1)\sqrt{T}\right] > 0,$$

as long as $\theta, \delta, V_0 > 0$.

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