Stochastic forest growth with harvesting and recovery decisions

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Abstract

A stochastic forest rotation model in the Faustmann tradition is presented and exemplified. The model combines harvesting decisions with options to recover or clean up to restore the land after very unfavorable evolutions of the stochastic growth process. Uncertainty is shown to have a generally ambiguous effect on the optimal choice of investment strategy. It is also shown how such models can be related to theory of optimal inventory control.

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1 Introduction

Rotation models in the Faustmann (1849) tradition address the question of when to cut and re-plant a growing forest on a certain piece of land. Decision problems of this kind have attracted increasing attention in recent years, as evidenced by the quantitative survey in Newman (2002). From a theory perspective, this is partly due to the development of real options theory, which has enabled more satisfying specifications of uncertainty than earlier techniques. From an application perspective, it is due to more focus on environmental economics and renewable resources in particular.

Faustmann-inspired models can be described and exemplified in various ways as some kind of harvesting decisions. For simplicity, this paper sticks to the original example with a forest stand that is to be cut and re-planted. The potential income is then a result of physical growth and possible quality changes, all characterized by a stochastic process, while costs are fixed. Clarke and Reed (1989) were among the first to assess this stochastic rotation problem in a real options context, but without presenting an explicit solution. Willassen (1998) used impulse control theory when providing an explicit solution under the assumption of a continuous and autonomous Ito forest growth process. Wilassen's results were confirmed by Sødal (2002), who used a less sophisticated methodology based on discount factors to derive a simplified closed-form rotation formula.

Other recent studies in the field include Chang (2005), who explores the sensitivity of Willassen's Faustmann formula for various parameters in the case with geometric Brownian forest growth, and Alvarez (2004) and Alvarez and Virtanen (2006), who present conditions under which various stochastic rotation problems are solvable. Alvarez and Koskela (2003) focus on interest rate uncertainty, while Insley and Rollins (2005) develop a two-factor model with linear growth and mean-reverting prices. Saphores (2003) includes risk of extinction. The model of this paper is similar to that of Saphores in including risk of extinction. However, where he associates extinction and reinvestment with a fixed level of the underlying stochastic variable, the related decision here will be an endogenous recovery investment based on marginal valuation which need not imply investment at an absorbing barrier. Recovery due to unfavorable growth could in principle be a viable option at many stages of a stochastic, unsuccessful growth path. The characteristics of the problem determines whether it might be better to wait for a natural recovery.

The optimal strategy in the standard Faustmann setting consists of cutting the forest stand as soon as it reaches a certain value. By cutting and re-planting, the value is brought back to the initial level. Hence, if the value of a new forest stand is V_0 and the cost of cutting and re-plantning is a constant C ($\geq V_0$), the optimal value at which to cut is some fixed value that exceeds C. In a deterministic environment, the optimal rotation period will be constant. In an environment with uncertain growth, the rotation period will be uncertain as some generations of trees grow faster than others. Such a decision strategy is optimal for many stochastic growth processes but it ignores one option that could be valuable under some assumptions: the option to recover or restore the land in cases with declining forest value. This implies a possible optimal strategy where forest growth should also be restarted when the value reaches some fixed value *below* V_0. There will be a net loss in the short run but the investment could still be preferred as it moves the forest out of a miserable state that could otherwise be long-lasting or permanent. Our objective is to describe such options, investigate the circumstances under which they are valuable, and relate the description of the decision problem to previous research. The characteristics of the various models to be discussed, are visualized in Figure 1.

< *Insert Figure* 1 *here* >

The first chart of Figure 1 shows the dynamics of the standard Faustmann model, for which harvesting takes place and the growth process is brought back to a fixed starting value as soon as the value of the forest hits a certain upper threshold. The process may possibly eventually stop in case a lower absorbing barrier exists. The next chart shows how forced recovery decisions may change the dynamics of the model, with rotation at either of two barriers. An absorbing barrier may still exist at or below the recovery barrier but it no longer implies a final stage of the model. The third chart in Figure 1 shows the similar dynamics of a standard barrier control model for which there is a combined fixed and variable benefit or cost of changing the current resource stock. Now the growth process is not brought back to a fixed starting value, but to an endogenous control point which depends on what barrier that was hit. Models of such optimal control of brownian motion are discussed and related to models in the Faustmann tradition towards the end of the paper.

The remaining part of the paper is structured as follows: Section 2 presents the general rotation model and the optimal rotation policy as visualized by the first two charts of Figure 1. Section 3 shows that, when growth is geometric Brownian, the optimal strategy coincides with the one derived by Willassen (1998) and simplified by Sødal (2002). In this case no lower threshold value for rotation exists. Sections 4 and 5 contain counterexamples. Section 4 focuses on how uncertainty affects decisions to re-invest in cases with a lower absorbing barrier as in Saphores (2003). Section 5 studies the opposite kind of a process, with an upper barrier for the size of the forest stand. The section applies a stochastic process which, despite its simplicity, does not seem to be in common use. The relationship between uncertainty, growth and the choice of optimal strategy in terms of whether recovery is ever optimal is shown to be non-trivial. For example, increased uncertainty could favor one or the other strategy depending on whether an absorbing barrier is far from or close to the initial value of the forest. The answer also depends on whether the expected growth rate is high or low. Section 6 shows how the generalized Faustmann problem can be related to optimal inventory control problems in the tradition of Scarf (1960), Clark and Scarf (1960) +++, as visualized by the third chart of Figure 1. Section 6 concludes.

2 The model

The (quality-adjusted) size or, equivalently, the potential sales price, of a stand of trees at time of planting equals a constant P_0, after which it starts to grow according to a continuous and autonomous Ito process

$$dP = f(P)dt + g(P)dz \tag{1}$$

Here $f(P)$ and $g(P)$ are drift and volatility terms and dz is a standard Wiener process, depending on the price. By incurring the fixed cost C when the price equals a general P, the owner receives a utility or income $V(P)$, where V is a measurable function of P.[1] Then the net revenue from one rotation is $V(P) - C$. Rotation also includes planting a new stand, so the value of the forest takes on the initial value, $V(P_0)$, immediately after the investment. The discount rate is constant and equal to ρ.

When is it optimal to start a new rotation in terms of a decision to cut and re-plant? Since the growth process is autonomous and costs are constant, the

[1] The standard assumption is $V(P) = P$, .i.e, no difference between growth of the forest and growth of the potential income or utility from cutting it. The distinction will prove to be useful. The function $V(P)$ could take on many shapes but it is typically assumed to be increasing within the interesting range for P. We usually refer to P as the price (or the forest size) and to V as the value (or utility), ignoring what the exact interpretations might be. The stochastic process for P is usually referred to as a growth process.

decision problem looks exactly the same at any two points in time as long as the price coincides at those instants. Then the optimal decision must also be exactly the same and there is just one decision to make: either start a new rotation or continue waiting. In case of a new rotation, the forest size is brought back to P_0. Since the process is continuous this proves that there are at most two relevant trigger prices for any decision policy: one on the upper side of P_0 and one on the lower side. In case both thresholds apply, denote the higher one by P_H and the lower one by P_L. An optimal decision policy is fully characterized by the two specific choices for P_H and P_L that together maximize the expected net present value over all future rotations; i.e., the value of the land including the stand of trees. Within the range (P_L, P_H), which includes the initial P_0 for a new stand, the optimal decision is to wait. As soon as P_H or P_L is hit, the net revenue is either $V(P_H) - C$ or $V(P_L) - C$ depending on what threshold is hit first. In both cases the growth process is restarted at P_0. What happens outside the closed interval $[P_L, P_H]$ is of no interest because such values are never reached.

The economic intuition behind the standard solution to this problem, for which rotation only takes place at an upper threshold, P_H, is familiar: optimal cutting and replanting is caused by decreasing returns from growth combined with discounting. The economic intuition behind an additional, lower switch, P_L, is that the forest stand will occupy the land even if its value declines. Therefore it may be better to try out again by cutting and planting once more instead of waiting very long and possibly forever for a natural recovery.

Starting at time zero and forest size P_0, the expected net present value of the stand, including all future rotations, can be written as a sum of two conditional expectations:

$$W_0 = E[e^{-\rho T_H}](V(P_H) - C + W_0) + E[e^{-\rho T_L}](V(P_L) - C + W_0) \qquad (2)$$

The variable T_H in this expression is the first hitting time when the underlying stochastic variable is to move from P_0 to P_H conditional on not hitting P_L first. As long as P_H is hit first, the net revenue $V(P_H) - C$ is achieved and the process is restarted at P_0. Then the net present value is back at W_0 because the process has no memory and costs are fixed. The obtained net gain, $V(P_H) - C + W_0$, is discounted by the expected discount factor, $E[e^{-\rho T_H}]$. If, on the contrary, the lower threshold is hit first, the similar revenue at time of rotation equals $V(P_L) - C + W_0$. This is discounted by the factor $E[e^{-\rho T_L}]$, where T_L is the first hitting time from P_0 to P_L conditional on not hitting P_H first.

The two expected discount factors, $E[e^{-\rho T_H}]$ and $E[e^{-\rho T_L}]$, can be written as functions of the variables P_0, P_L and P_H; see Dixit et al. (1999) and Sødal (2006) and the references therein. Denote the expected discount factor $E[e^{-\rho T_H}]$ by $Q(P_0, P_H, P_L)$ and the discount factor $E[e^{-\rho T_L}]$ by $Q(P_0, P_L, P_H)$. A simple rearrangement of (2) yields

$$W_0 = \frac{Q(P_0, P_H, P_L)(V(P_H) - C) + Q(P_0, P_L, P_H)(V(P_L) - C)}{1 - Q(P_0, P_H, P_L) - Q(P_0, P_L, P_H)} \qquad (3)$$

The optimal decision policy is found by maximizing this expression with respect to P_H and P_L. If the Q- and V- functions are sufficiently differentiable, the first-order conditions for a maximum are $\partial W_0/\partial P_H = 0$ and $\partial W_0/\partial P_L = 0$. These conditions are spelled out in *Appendix*. When P_L is fixed at zero, the formula corresponds to the value function with reinvestment in Saphores (2003) when ignoring minor differences in the formal set-up.[2]

For simplicity, the same rotation cost, C, has been assumed for both types of investment. More sophisticated versions can be developed where the rotation cost depends on the type of investment, or the cost could be made an endogenous variable as discussed later in this paper.

In general, the conditional discount factor function $Q(P, P_1, P_2)$ that applies to a motion from an arbitrary P to a fixed P_1 without hitting a fixed P_2 first, where P is located between P_1 and P_2, is found by solving the differential equation

$$\frac{1}{2}\sigma^2(P)\frac{\partial^2 Q(P, P_1, P_2)}{\partial P^2} + \mu(P)\frac{\partial Q(P, P_1, P_2)}{\partial P} - \rho Q(P, P_1, P_2) = 0 \qquad (4)$$

There are two boundary conditions: $Q(P_1, P_1, P_2) = 1$ and $Q(P_2, P_1, P_2) = 0$. This determines the two conditional discount factors in (3).

When only the upper threshold applies, we have $Q(P_0, P_H, P_L) \to D(P_0, P_H)$ in optimum, where $D(P_0, P_H)$ is the expected (unconditional) discount factor for a motion from P_0 to P_H, while $Q(P_0, P_L, P_H) \to 0$; see Dixit et al. (1999). Hence, eq. (3) simplifies to

$$W_0 = \frac{D(P_0, P_H)(V(P_H) - C)}{1 - D(P_0, P_H)} \qquad (5)$$

as in Sødal (2002). Then the first-order condition for a maximum with respect to P_H can be written

$$\frac{V(P_H) - C}{V(P_H)} = (1 - D(P_0, P_H))\frac{\gamma}{\epsilon} \qquad (6)$$

where $\epsilon = -D'(V)/(D(V)/V)$ and $\gamma = V'(P)/(V(P))/P$.

3 Geometric Brownian growth

The geometric Brownian process is characterized by $f(P) = \mu P$ and $g(P) = \sigma P$ in (1), where μ and σ are constants. We require that $\mu < \rho$.[3] Then the differential equation (4) with boundary conditions yields

$$Q(P_0, P_1, P_2) = \frac{(P_0/P_1)^\beta - (P_2/P_0)^\alpha (P_2/P_1)^\beta}{1 - (P_2/P_1)^{\alpha+\beta}} \qquad (7)$$

[2] Saphores (2003) expresses the discount factor that applies to a particular threshold as a product of the discount factor for a first-hit of either P_L or P_H, muliplied with the probability of hitting the particular threshold in question before the other one. This does not appear to be entirely correct, as it does not account for differences in the time it takes to reach either threshold.

[3] If $\mu \geq \rho$, growth is not sufficiently suppressed by discounting, and the expected net present value will growth beyond all limits by postponing cutting into infinity; see Willassen (1998).

where $\alpha > 0$ and $\beta > 1$. Moreover, $-\alpha$ and β are the two roots of the characteristic quadratic equation

$$\frac{1}{2}\sigma^2 x^2 + (\mu - \frac{1}{2}\sigma^2)x - \rho = 0 \tag{8}$$

Willassen (1998) finds that it can never be optimal to reinvest at a lower threshold $P_L(< P_0)$ under geometric Brownian growth, assuming that $V(P) = P$ (which implies $\gamma = 1$). One way to prove this is by showing that $\partial W_0/\partial P_L > 0$ for any P_L in the interval $(0, P_H)$ along the boundary $\partial W_0/\partial P_H = 0$. This is done in *Appendix*. Moreover, we have $Q(P_0, P_H, P_L) \to D(P_0, P_H) = (P_0/P_H)$ as $P_L \to 0$. Then $\epsilon \to \beta$, which inserted into (6) yields the stochastic rotation formula for this process as in Willassen (1998) and Sødal (2002).

It is a standard result that a geometric Brownian motion for which $\mu < \frac{1}{2}\sigma^2$, will go to zero with probability one in the long run. Then the forest eventually dies and is never re-planted. Sufficient expected growth relative to uncertainty is required to avoid such permanent extinction. The convergence requirement $\mu < \rho$ implies $\beta > 1$ and sets a limit, so the geometric Brownian motion may not be well suited to represent many Faustmann problems of practical interest. The requirement also explains why a recovery investment is never optimal. By combining (5) and (6), and using L'Hopital's rule it can be shown that $W_0 < P_0$ and that $W_0 \to P_0(< C)$ as $\beta \to 1$. Then the net present value of the land for any $P < P_0$ cannot exceed C, so the value of the land is rather low under these assumptions.

This section has demonstrated some limitations of the geometric Brownian model in expressing the growth pattern of a renewable resource like a forest stand. More realistic model versions should be able to represent at least the following two features:

1. Extinction combined with by a recovery investment as it is rarely observed that a single failure of growth prevents the owner of land from making further investments.

2. Expected growth which is not constant but which tends to decrease when approaching some maximum level determined by nature.

The subsequent two sections focus on each of these issues.

4 Extinction and recovery

Due to externalities, for example because individual species nourish each other, some renewable resources may need a critical size in order to survive. At the macro level, it has proven difficult to recover forests in some arctic regions after those that once existed were cut down. At the micro level, a new tree may have small chances of surviving the next winter or storm unless it reaches a certain size soon enough. The underlying forces behind such mechanisms could be complicated, but even a simplified treatment should allow for the main consequence, namely risk of extinction. This can be done most easily by including an absorbing barrier, P_L, some place below P_0 in the Faustmann model, although not necessarily at zero as assumed in Saphores (2003).

The only adjustment of the net present value (3) that is needed when P_L is an absorbing barrier, is to maximize the net expected present value (3) only with respect to P_H; then check whether this implies a higher net present value than without such forced recovery. If rotation is only to take place when P_H is hit before P_L, all costs and incomes stop as soon as the lower barrier is hit. The simplified version of (3) to be maximized becomes

$$W_0 = \frac{Q(P_0, P_H, P_L)(V(P_H) - C)}{1 - Q(P_0, P_H, P_L)} \tag{9}$$

Table 1 contains a set of results for both strategies in ten cases with absorbing barriers distributed evenly between zero and the initial value, $P_0 = 1$. Growth is geometric Brownian above the absorbing barrier. The results for the strategy with investment only at the upper threshold are denoted by $\widehat{W}$ and $\widehat{P}$, while $\widetilde{W}$ and $\widetilde{P}$ represent the strategy with investment also at P_L.

< *Insert Table 1 here* >

Continuity of the value function implies, in Table 1 as elsewhere, that

$$\widehat{W} \lesseqgtr \widetilde{W} \iff \widehat{P} \gtreqless \widetilde{P} \tag{10}$$

The optimal strategy (with the higher W and lower P_H) is visualized by printing the numbers for the optimal choice in bold characters in Table 1. Uncertainty is fairly high ($\sigma = 0.2$) and no growth is expected ($\mu = 0$). Together with the cost assumptions ($P_0 = 1.0$ and $C = 1.1$), this makes forced recovery unviable for all cases in Table 1.[4] Such a policy even leads to a negative net present value when P_L is high enough. One main reason is the assumption $\mu = 0$, which implies too small growth opportunities.

The results change by increasing the growth expectations from $\mu = 0$ to $\mu = 0.08$, close to the discount rate ($\rho = 0.10$); see Table 2. Both the net present value and the optimal investment threshold increase all over. More interestingly, the optimal policy also changes. Forced recovery is still not optimal if the absorbing barrier is close to zero, but for $P_L \geq 0.5$ the growth opportunities are too high to be missed.

< *Insert Table 2 here* >

Table 3 uses the same numbers as Table 2 except for volatility, which is increased to $\sigma = 0.3$. Now recovery is only optimal for intermediate P_L-values. This is due to a combination of effects: with very high uncertainty, rapid growth in expectation does not give enough protection against extinction if P_L is close enough to P_0. On the other hand, the net cost of enforced recovery, $P_L - C$, can

[4] If rotation only takes place at the upper threshold, non-negative profit can always be ruled as there is a net gain ($P_H - C$) at each rotation, and no other costs apply.

be too high if the extinction barrier is very far from P_0. Only in the intermediate range is a policy with a recovery investment optimal.

$<$ *Insert Table* 3 *here* $>$

Table 2 and Table 3 combined also clarify the impact of uncertainty on forest value including land (W_0). There are two effects: uncertainty increases the value of waiting but also the risk of extinction. Therefore increased uncertainty is positive if the extinction threshold is far from P_0 but negative if it is close to P_0.

The rotation cost also matters. Table 4 uses the same numbers as Table 3 except for higher rotation cost ($C = 1.5$). This increases the upper threshold P_H and lowers the value of land for both decision policies, and recovery after extinction is no longer optimal for any P_L. Note also that very high $\widetilde{P}$-values appear when P_L is close to P_0 as the risk of having to invest several times before a successful next growth path is established makes the owner less inclined to invest in the first place.

$<$ *Insert Table* 4 *here* $>$

To conclude, risk of extinction by hitting an absorbing lower barrier may influence investment timing and investment strategy greatly via effects that are not trivial and generally ambiguous in terms of uncertainty and other variables. For the geometric Brownian case studied here, strict assumptions seem needed if a strategy with forced recovery after extinction is to be optimal. Exclusive rotation at an upper threshold seems to be the normal case also when there exists an absorbing barrier above zero but below P_0. The next section sets up a scene where the optimal policy is more likely to include both types of rotation investments.

5 Limits to growth

Suppose there exists an natural maximum for forest size. This can be done by letting growth and uncertainty develop according to the following process,

$$dP = \mu(\overline{P} - P)dt + \sigma(\overline{P} - P)dz \tag{11}$$

where μ and σ are constants and, typically, $\mu \geq 0$ (with no upper constraint). This can be called an inverted geometric Brownian motion because the volatility rate as well as the expected growth rate is proportional to the distance from the upper barrier, $\overline{P}$. This makes economic sense as growth is often most uncertain for immature natural resources.

A process like (11) can take on negative values. That is not worrysome here as it may indeed require extraordinary efforts to replace and old, rotten forest stand. Similar arguments hold for harvesting of some other renewable resources. It will typically also not be of interest to start a new rotation for very low, negative P-values; therefore such outcomes can simply be interpreted as times when the forest is very far from having economic value that makes harvesting an option of interest. A lower extinction barrier can also easily be imposed. Since such barriers were studied in the previous section we abstain from introducing them here.

The growth process (11) is not affected by the location of the origin: drift and volatility parameters are only affected by the distance from the upper barrier; not by the distance from zero. This is not appealing, since natural resources like trees are real assets that are subject scale effects of various kinds. To encount for such effects, let the value of the stand be given by a cubic function:

$$V(P) = P^3 \tag{12}$$

This could reflect growth in three dimensions, but the exact type of power function (or alike) is less important. The main objective is describe growth by a process that is independent of explicit time, which is convenient from technical reasons, but where growth still is slow at first and then faster until natural decreasing returns set in. In the deterministic case ($\sigma = 0$), it follows that the V-process can be written

$$V(t) = \overline{P} - (\overline{P} - P_0)e^{-\mu t}$$

which is S-shaped in time Thus it resembles an ordinary one-factor production function except for the definition of the input factor; in this real options model, calendar time is a kind of variable input factor from which more output arises. By and large, this also holds for a stochastic growth path.

Setting $f(P) = \mu(\overline{P} - P)$ and $g(P) = \sigma(\overline{P} - P)$, and solving (4) with boundary conditions leads to the conditional discount factor

$$Q(P_0, P_1, P_2) = \frac{\left(\frac{\overline{P}-P_1}{\overline{P}-P_0}\right)^a - \left(\frac{\overline{P}-P_0}{\overline{P}-P_2}\right)^b \left(\frac{\overline{P}-P_1}{\overline{P}-P_2}\right)^a}{1 - \left(\frac{\overline{P}-P_1}{\overline{P}-P_2}\right)^{a+b}} \tag{13}$$

where $-a$ and b are solutions to the quadratic equation

$$\frac{1}{2}\sigma^2 x^2 - (\mu + \frac{1}{2}\sigma^2)x - \rho = 0 \tag{14}$$

implying $a, b > 0$. The optimal decision policy is determined as usual by maximizing (3) based on (12), (13) and (14). Table 5 reports the results for one data set with no expected growth ($\mu = 0$), with forest size starting from $P_0 = 0$ and constrained by $\overline{P} = 10$, and a unit-sized rotation cost ($C = 1$). Rotation on both sides of P_0 is now optimal as long as the volatility exceeds approximately five percent.

< *Insert Table* 5 *here* >

The value of the recovery option increases further by introducing a growth trend. This can be seen from Table 6, where $\mu = 0.02$.

< *Insert Table* 6 *here* >

Table 7 applies the same numbers as Table 6 except for twice as high rotation cost ($C = 2$). Increasing the rotation cost in this manner reduces the value of the recovery option.

< *Insert Table 7 here* >

The results above show that the upper threshold for investment (P_H) increases with uncertainty as in most investment models of this kind. The lower threshold (P_L) increases typically but not always with uncertainty. The convex shape of $V(P)$ in the interesting domain for investment ($P > C$) is probably a main reason for the ambiguity. All results in this section also indicate that main effect of the recovery option is to increase the value of the land including the forest (W_0). The option seems to have minor impact on the optimal policy at the upper threshold. This can be seen by comparing the columns $\widehat{P}_H$ and $\widetilde{P}_H$. It remains to be studied whether empirical data would produce similar results. In general, this should be expected in cases where the typical rotation period is long relative to the discount rate.

6 Relationship to inventory control models

There is a close relationship between the stochastic rotation model of this paper and inventory control models (Scarf (1960), Scarf and Clark (1960) etc. etc.... +++). Suppose a resource stock P which evolves stochastically produces a flow of utility (in monetary units) that is equal to $u(P)$, where u is a concave function that reaches a maximum for some P. The expected net present value of the flow can be written

$$U(P) = E\left[\int_0^\infty u(P)e^{-\rho t}dt\right] \tag{15}$$

Suppose that the process for P is continuous and autonomous as usual, and also that it is so regular that the slope characteristics of $u(P)$ carry over to $U(P)$. Regulation of the inventory requires only a fixed cost, I, and the optimal level is a fixed P_0. Then the optimal inventory control problem boils down to the question of when to reset the stock to P_0 when a level which is either too high ($P_H > P_0$) or too low ($P_L < P_0$) is reached. What are the optimal P_H and P_L at which to reset?

Following the logic of previous sections, the net present value to be maximized, starting from P_0, can be written

$$W_0 = U_0 + Q(P_0, P_H, P_L)(U_0 - U_H - I + W_0) + Q(P_0, P_L, P_H)(U_0 - U_L - I + W_0) \tag{16}$$

where $U_i = U(P_i)$ for $i = 0, P, H$. The net present value U_0 is obtained with no further regulation of the inventory. The value of the option to regulate down when the inventory gets to high, is given by the next term on the right-hand side of (16). The net utility gain when the option is exercised equals $U_0 - U_H$, and the cost is I, after which one is back to the initial situation with net present value W_0. By defining $C = I - U_0$ and $V(P) = -U(P)$, it is possible to rewrite (16) as follows:

$$W_0 = U_0 + \frac{Q(P_0, P_H, P_L)(V(P_H) - C) + Q(P_0, P_L, P_H)(V(P_L) - C)}{1 - Q(P_0, P_H, P_L) - Q(P_0, P_L, P_H)} \tag{17}$$

This is identical to (3) except for a constant (U_0). Thus, the presented stochastic rotation problem in the Faustmann tradition is equivalent to a standard inventory control problem.

There are more parallels. As shown by Sødal (2002) in a simpler context, the optimal size of the stock, P_0, can be determined endogenously by maximizing W_0 from (17) with respect to P_H, P_L and P_0.

Alternatively, a variable cost element could be introduced by which regulation becomes more costly the larger the shift. This yields a barrier control (S,s)-model (see Dixit (199x), xxx). In our context it corresponds to partial harvesting as, for example, in the model by Saphores (2003), but also including regulatory efforts during crises. For example, it may be possible to protect a renewable resource against extinction in this way without having an ambition to make profit from harvesting in the short run. A model with these extensions must reflect that the magnitude of regulation could depend on what barrier is hit, P_H or P_L. Therefore P_0 must be replaced by two endogenous variables, P_h and P_l, where $P_L < P_l \leq P_h < P_H$. Suppose that the current inventory size is a constant P_0 between P_l and P_h. (P_0 is now just a fixed number with no specific interpretation as it was in the previous settings.) Then the expected net present value at start, W_0, can be derived from the following:

$$\begin{aligned} W_0 &= U_0 + Q(P_0, P_H, P_L)(U_h - U_H - I - c_H(P_H - P_h) + W_h) + \\ &\quad Q(P_0, P_L, P_H)(U_l - U_L - I - c_L(P_L - P_l) + W_l) \end{aligned} \tag{18}$$

$$\begin{aligned} W_h &= Q(P_h, P_H, P_L)(U_h - U_H - I - c_H(P_H - P_h) + W_h) + \\ &\quad Q(P_h, P_L, P_H)(U_l - U_L - I - c_L(P_L - P_l) + W_l) \end{aligned} \tag{19}$$

$$\begin{aligned} W_l &= Q(P_l, P_H, P_L)(U_h - U_H - I - c_H(P_H - P_h) + W_h) + \\ &\quad Q(P_l, P_L, P_H)(U_l - U_L - I - c_L(P_L - P_l) + W_l) \end{aligned} \tag{20}$$

Equation (19) is similar to (16) except for two variable cost elements. The constant c_H is a variable cost per unit reduction of P when starting from P_H. The similar variable cost is c_L when starting in the lower end. These variable costs could result from selling or buying assets at a fixed price $c > 0$. Then we have $c_L = -c_H = c$. The solution to the problem, yielding the optimal P_L, P_l, P_h and P_H, is found by solving (18), (19) and (20) for W_0 and maximizing with respect to P_L, P_l, P_h and P_H. See Dixit (199x) +++ for a discussion of the charactertistics of this optimal regulation problem in a specific case.

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7 Final remarks

This paper has confirmed some well known relationships between uncertainty and investment but also pointed at some new ones. Uncertainty was shown to have a ambiguous effects with respect to valuation of investment options in the presence of extinction barriers. This was also discovered by Saphores (2003), but we have seen none of his examples of uncertainty having an ambiguous effect on when to invest in terms of the trigger price (P_H). Increasing uncertainty implied higher P_H in all numerical cases. On the other hand, uncertainty was shown to have an ambiguous effect on the choice of investment strategy in terms of whether to enforce recovery or not. The effect of uncertainty on the optimal trigger price for such recovery investments (P_L) was also ambiguous in the models with endogenous thresholds of this kind.

The final section pointed at a close relationship between stochastic models of forest growth and models of inventory control. Most such continuous-time models are concerned with optimal regulation of Brownian motion. The limited amount of cross references between the two strands of literature in this field indicate that researcher working within each tradition could have more to learn from each other.

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8 Appendix

This appendix states the first-order conditions for a maximum of the expected net present value function in Proposition 1, $\partial W_0/\partial P_H = 0$ and $\partial W_0/\partial P_L = 0$, where W_0 is given by eq. (3). Define the following functions and abbreviations, assuming that all derivates exist:

$$
\begin{gathered}
Q_H = Q(P_0, P_H, P_L), \quad Q_L = Q(P_0, P_L, P_H), \qquad \text{(A1)} \\
Q_H^H = \frac{\partial Q_H}{\partial P_H}, \quad Q_L^H = \frac{\partial Q_L}{\partial P_H}, \; \epsilon_H^H = -Q_H^H \frac{P_H}{Q_H}, \quad \epsilon_L^H = Q_L^H \frac{P_H}{Q_L}, \\
Q_H^L = \frac{\partial Q_H}{\partial P_L}, \quad Q_L^L = \frac{\partial Q_L}{\partial P_L}, \; \epsilon_H^L = -Q_H^L \frac{P_L}{Q_H}, \quad \epsilon_L^L = Q_L^L \frac{P_L}{Q_L}, \\
V_H = V(P_H), \quad V_L = V(P_L), \; V_H' = V'(P_H), \quad V_L' = V'(P_L), \\
\gamma_H = V'(P_H) \frac{P_H}{V_H}, \quad \gamma_L = V'(P_L) \frac{P_L}{V_L}
\end{gathered}
$$

$V(P)$ is assumed non-decreasing so all elasticities (ϵ_H^H, ϵ_L^H, ϵ_H^L, ϵ_L^L, γ_H, γ_L) are defined as non-negative numbers. Differentiating and summing up terms, the first-order condition $\partial W_0/\partial P_H = 0$ can be written as

$$\left(Q_H^H - Q_L Q_H^H + Q_L^H Q_H\right) V_H + \left(Q_H - Q_H Q_H - Q_L Q_H\right) V_H' + \left(Q_L^H - Q_H Q_L^H + Q_H^H Q_L\right) V_L - \left(Q_H^H + Q_L^H\right) C = 0 \tag{A2}$$

By use of the elasticity definitions, this becomes

$$\begin{gathered} Q_H \epsilon_H^H \left(Q_L (V_H - V_L) - (V_H - C)\right) - \\ Q_L \epsilon_H^L \left(Q_L (V_H - V_L) + C - V_L\right) + \\ (1 - Q_H - Q_L)\gamma_H Q_H V_H = 0 \end{gathered} \tag{A3}$$

Likewise, the first-order condition $\partial W_0/\partial P_L = 0$ can be written

$$\left(Q_L^L - Q_H Q_L^L + Q_H^L Q_L\right) V_L + \left(Q_L - Q_L Q_L - Q_H Q_L\right) V_L' + \left(Q_H^L - Q_L Q_H^L + Q_L^L Q_H\right) V_H - \left(Q_L^L + Q_H^L\right) C = 0 \tag{A4}$$

which by the elasticity definitions becomes

$$\begin{gathered} Q_L \epsilon_L^L (Q_H (V_H - V_L) + C - V_L) - \\ Q_H \epsilon_H^L (Q_L (V_H - V_L) + V_H - C) + \\ (1 - Q_H - Q_L)\gamma_L Q_L V_L = 0 \end{gathered} \tag{A5}$$

In the geometric Brownian case, the discount factor function (7) implies $Q_H^H = -\beta Q_H/P_H$, $Q_H^L = \beta Q_H/P_L$, $Q_L^H = -\alpha Q_L/P_H$ and $Q_L^L = \alpha Q_L/P_L$. The first-order condition (A2) can be rewritten as

$$\begin{gathered} (\beta Q_H + \alpha Q_L)\frac{C}{P_L} = (\beta Q_H - (\beta - \alpha) Q_H Q_L)\frac{P_H}{P_L} - \\ Q_H (1 - Q_H - Q_L)\frac{P_H}{P_L} + ((\beta - \alpha) Q_H Q_L + \alpha Q_L) \end{gathered} \tag{A6}$$

Then it follows that

$$\begin{aligned} \frac{\partial W_0}{\partial P_L} &= (\beta - \alpha) Q_H Q_L + \alpha Q_L + (1 - Q_H - Q_L) Q_L \\ &+ (\beta Q_H - (\beta - \alpha) Q_H Q_L)\frac{P_H}{P_L} - (\beta - \alpha) Q_H Q_L - \alpha Q_L + (\beta Q_H + \alpha Q_L)\frac{C}{P_L} \\ &= (1 - Q_H - Q_L) Q_L + (\beta (1 - Q_L) Q_H + \alpha Q_H Q_L)\frac{P_H}{P_L} > 0 \end{aligned} \tag{A7}$$

As long as $P_H \neq P_L$ the inequality sign is ensured by the properties of the discount factor function, since $0 < Q_H, Q_L < 1$, $0 < Q_H + Q_L < 1$ and $P_H, P_L, \alpha, \beta > 0$. This shows that the expected net present value, W_0, is strictly increasing for all values of P_L, so it is optimal to raise P_L all the way up to P_H, thereby making it irrelevant and establishing the corner solution.

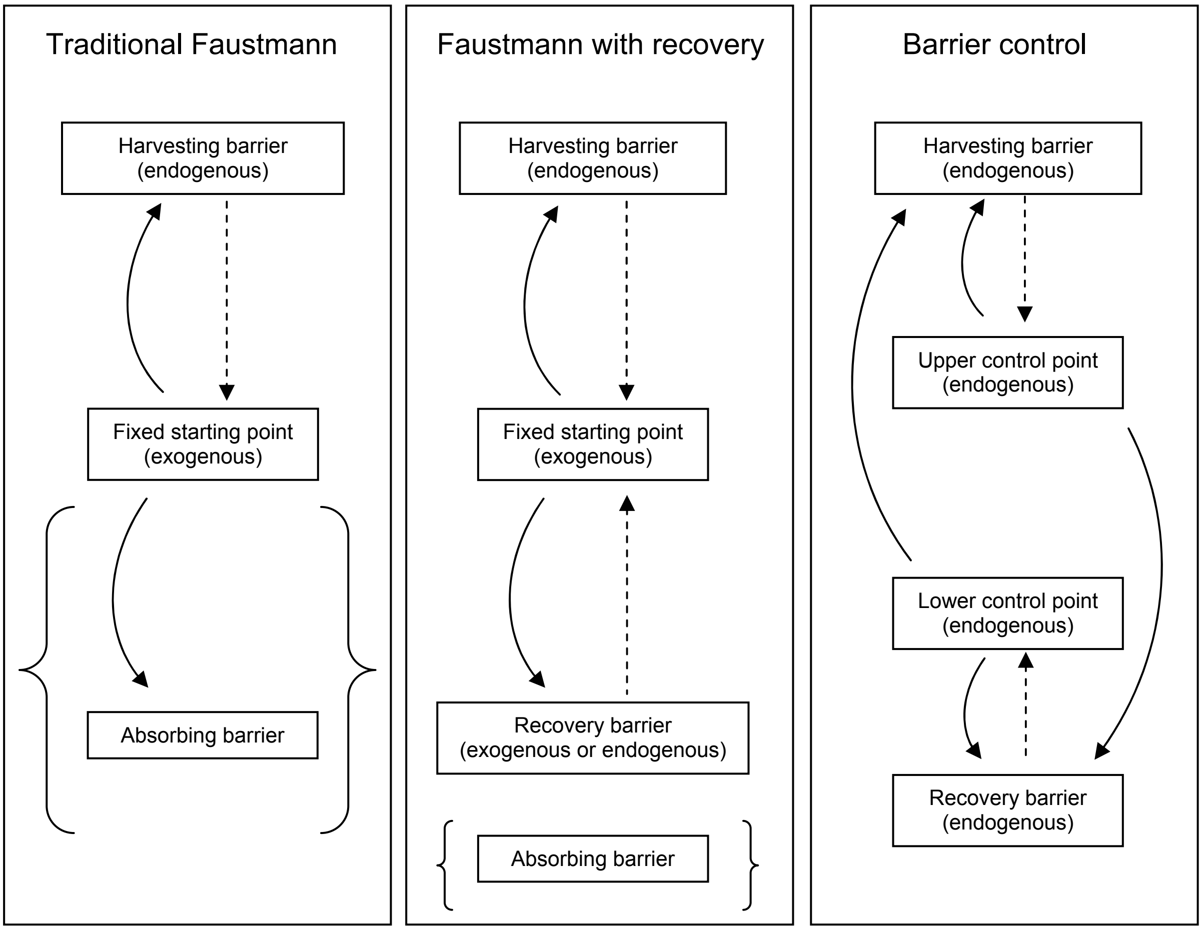

Figure 1

$\hat{P}_L$	$\tilde{W}$	$\tilde{P}_H$	$\hat{W}$	$\hat{P}_H$
0.0	**0.192644**	**1.413895**	**0.192644**	**1.413895**
0.1	0.175551	1.422379	**0.192637**	**1.413898**
0.2	0.138036	1.442220	**0.192489**	**1.413970**
0.3	0.086934	1.472161	**0.191652**	**1.414378**
0.4	0.024440	1.513817	**0.188949**	**1.415701**
0.5	-0.050052	1.571431	**0.182453**	**1.418913**
0.6	-0.140668	1.653918	**0.169520**	**1.425454**
0.7	-0.258773	1.781504	**0.147086**	**1.437275**
0.8	-0.438086	2.011260	**0.112294**	**1.456863**
0.9	-0.835106	2.604328	**0.063320**	**1.487221**

Table 1

(P_0=1.0, C=1.1, μ=0, ρ=0.1, σ=0.2)

$\hat{P}_L$	$\widetilde{W}$	$\widetilde{P}_H$	$\hat{W}$	$\hat{P}_H$
0.0	**0.706265**	**2.438238**	**0.706265**	**2.438238**
0.1	0.706232	2.438427	**0.706260**	**2.438263**
0.2	0.705735	2.441284	**0.706081**	**2.439297**
0.3	0.703699	2.453027	**0.704632**	**2.447639**
0.4	0.698588	2.482630	**0.698633**	**2.482367**
0.5	**0.688416**	**2.542097**	0.681372	2.583658
0.6	**0.670198**	**2.650131**	0.642458	2.817432
0.7	**0.638025**	**2.844400**	0.569252	3.267622
0.8	**0.575669**	**3.227878**	0.448292	4.019967
0.9	**0.408370**	**4.268622**	0.264587	5.163430

Table 2
(P_0=1.0, C=1.1, μ=0.08, ρ=0.1, σ=0.2)

$\hat{P}_L$	$\widetilde{W}$	$\widetilde{P}_H$	$\hat{W}$	$\hat{P}_H$
0.0	**0.747982**	**2.672294**	**0.747982**	**2.672294**
0.1	0.742875	2.700174	**0.747111**	**2.677019**
0.2	0.730094	2.771650	**0.740650**	**2.712445**
0.3	0.710897	2.883291	**0.722757**	**2.813728**
0.4	0.684933	3.041712	**0.688314**	**3.020642**
0.5	**0.650295**	**3.264379**	0.633694	3.374944
0.6	**0.602448**	**3.588560**	0.556801	3.910852
0.7	**0.530606**	**4.099954**	0.456349	4.647199
0.8	**0.403382**	**5.044261**	0.331097	5.591438
0.9	0.072460	7.569701	**0.179556**	**6.748561**

Table 3
(P_0=1.0, C=1.1, μ=0.08, ρ=0.1, σ=0.3)

$\hat{P}_L$	$\widetilde{W}$	$\widetilde{P}_H$	$\hat{W}$	$\hat{P}_H$
0.0	0.654922	6.415289	**0.654922**	**6.415289**
0.1	0.644277	6.490908	**0.654308**	**6.419641**
0.2	0.615229	6.698371	**0.649736**	**6.452104**
0.3	0.567664	7.041145	**0.636891**	**6.543514**
0.4	0.497740	7.550705	**0.611443**	**6.725514**
0.5	0.396488	8.297430	**0.569308**	**7.029241**
0.6	0.244671	9.430007	**0.506854**	**7.483957**
0.7	-0.003213	11.297454	**0.421023**	**8.115702**
0.8	-0.482700	14.934774	**0.309297**	**8.946422**
0.9	-1.860997	25.420068	**0.169572**	**9.994007**

Table 4
(P_0=1.0, C=1.5, μ=0.08, ρ=0.1, σ=0.3)

σ	$\widetilde{W}$	$\widetilde{P}_L$	$\widetilde{P}_H$	$\hat{W}$	$\hat{P}_H$
0.000				**0.000**	**-**
0.025				**0.159**	**1.715**
0.050				**1.409**	**2.573**
0.075	**4.402**	**-0.710**	**3.339**	4.374	3.341
0.100	**12.136**	**-0.654**	**3.881**	9.381	3.984
0.125	**24.102**	**-0.619**	**4.316**	16.603	4.519
0.150	**40.458**	**-0.596**	**4.670**	26.119	4.967
0.175	**61.253**	**-0.579**	**4.961**	37.957	5.344
0.200	**86.482**	**-0.566**	**5.202**	52.112	5.666
0.225	**116.111**	**-0.556**	**5.403**	68.566	5.942
0.250	**150.094**	**-0.549**	**5.573**	87.292	6.181
0.275	**188.383**	**-0.543**	**5.718**	108.258	6.389
0.300	**230.931**	**-0.538**	**5.841**	131.435	6.571

Table 5

(ρ=0.1, μ=0, C=1, P_0=0, $\bar{P}=10$)

σ	$\widetilde{W}$	$\widetilde{P}_L$	$\widetilde{P}_H$	$\hat{W}$	$\hat{P}_H$
0.000	-	-	-	**5.497**	**3.524**
0.025	**6.748**	**-0.502**	**3.693**	6.739	3.694
0.050	**10.753**	**-0.542**	**4.053**	10.323	4.071
0.075	**18.081**	**-0.547**	**4.427**	16.102	4.489
0.100	**29.119**	**-0.544**	**4.758**	24.050	4.884
0.125	**44.094**	**-0.539**	**5.039**	34.171	5.238
0.150	**63.129**	**-0.534**	**5.275**	46.464	5.550
0.175	**86.280**	**-0.530**	**5.474**	60.921	5.824
0.200	**113.567**	**-0.527**	**5.642**	77.527	6.065
0.225	**144.989**	**-0.524**	**5.785**	96.261	6.276
0.250	**180.534**	**-0.522**	**5.907**	117.102	6.464
0.275	**220.183**	**-0.520**	**6.011**	140.030	6.630
0.300	**263.918**	**-0.518**	**6.101**	165.025	6.777

Table 6

(ρ=0.1, μ=0.02, C=1, P_0=0, $\bar{P}=10$)

σ	$\widetilde{W}$	$\widetilde{P}_L$	$\widetilde{P}_H$	$\hat{W}$	$\hat{P}_H$
0.000				**5.372**	**3.586**
0.025				**6.606**	**3.750**
0.050	**10.202**	**-0.875**	**4.115**	10.170	4.116
0.075	**16.733**	**-0.807**	**4.501**	15.924	4.525
0.100	**26.782**	**-0.779**	**4.843**	23.844	4.913
0.125	**40.628**	**-0.762**	**5.134**	33.933	5.261
0.150	**58.410**	**-0.750**	**5.378**	46.192	5.570
0.175	**80.192**	**-0.741**	**5.583**	60.613	5.841
0.200	**105.992**	**-0.734**	**5.757**	77.179	6.080
0.225	**135.809**	**-0.729**	**5.904**	95.871	6.290
0.250	**169.628**	**-0.724**	**6.029**	116.668	6.476
0.275	**207.427**	**-0.721**	**6.137**	139.549	6.641
0.300	**249.186**	**-0.718**	**6.230**	164.495	6.788

Table 7

(ρ=0.1, μ=0.02, C=2, P_0=0, $\bar{P}=10$)