The Optimal Timing for the Construction of an International Airport: a Real Options Approach with Multiple Stochastic Factors and Shocks

Paulo Pereira*
Artur Rodrigues†
Manuel J. Rocha Armada‡

University of Minho, Portugal

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Abstract

In this paper we study the option to invest in a new international airport, considering that the *benefits* of the investment behave stochastically. In particular, the number of passengers, and the cash flow per passenger are both assumed to be random. Additionally, positive and negative shocks are also incorporated, which seems to be realistic for this type of projects.

Accordingly, we propose a new real options model which combines *two stochastic factors* with *positive* and *negative shocks*.

While the authors developed this model having as a reference the project for the new Lisbon airport, the model can be applied to other investments in airports and, eventually, with minimal adaptations, it can also be applied to projects in different areas.

*PhD candidate in Finance. School of Economics and Management - University of Minho. E-mail address: *pjmorp@gmail.com*.

†Corresponding author. Assistant Professor in Finance. Management Research Unit - School of Economics and Management - University of Minho - 4710-057 Braga - Portugal. Phone: +351 253 604 564, Fax: +351 253 289 724. E-mail address: *artur.rodrigues@eeg.uminho.pt*.

‡Professor of Finance. Management Research Unit - School of Economics and Management - University of Minho. E-mail address: *rarmada@eeg.uminho.pt*.

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1 Introduction

The decision to invest in an international airport demands, usually, a huge amount of money. This, combined with the fact that this type of investment is, in a large scale, irreversible, becomes a very important *timing problem*.

The real options theory, realizing that the uncertainty surrounding a project gives the firms a valuable option to defer the project implementation, tries to determine the optimal moment to invest[1].

Related to our work is that of Smit (2003), where he analyzes the investment in an airport infrastructure. Particularly focused on the airport expansion with strategic interactions, the author combines real option theory and game theory, through a discrete-time model.

Also related is Paxson and Pinto (2005). In this paper the authors develop a continuous-time model for the option to invest in a duopoly market under competition, disaggregating revenues into two variables: the profit per unit and the quantity sold. We derive the model for two stochastic variables differently, and, more important, we extend the model in order to allow jumps, which can be *positive* and *negative* shocks. These shocks are random events which modify, in a discrete way, the level of the state variable, and are particularly relevant in the airport business.

This model has been developed having as a reference the project for the new Lisbon international airport, but we think the model is sufficiently flexible to be applied to projects in different areas.

2 The Model

Assume an investor, which can be the Government or a public/private joint-venture, interested in investing in the construction of a new airport. This type of investment requires huge amounts of capital, which is, in a large scale, sunk once spent.

There are several sources of uncertainty in this type of project, since it depends upon the air transportation industry: the higher the number of flying passengers, the higher the revenues for the airport.

In this paper, instead of working with one stochastic factor (the net cash flow, or the gross project value) we disaggregate it into two components.

[1]See Dixit and Pindyck (1994) and Trigeorgis (1996) for a general overview on real options.

Note that the total cash flow for the airport can be expressed as being the total number of passengers multiplied by the average cash flow per passenger.

So, in our model, we have two basic random variables - the number of passengers arriving/departing from the airport, as well as the cash flow per passenger - governed by two different stochastic processes, which we allow to be correlated.

As we will see later, our model also accounts for other sources of uncertainty: we call them "shocks", which are basically random/unexpected events with a major (positive or negative) impact[2] on the airport revenues.

2.1 The Basic Case: One Stochastic Factor

Let us start with the simplest situation, where only the number of passengers is stochastic. Let x be the number of passenger per unit of time (e.g.: per year), and R be the deterministic cash flow per passenger. So, the total cash flow, in a given period of time is xR.

Let the future values for x be random, according to the following GBM[3]:

$$dx = \alpha_x x dt + \sigma_x x dZ_x \tag{1}$$

where α_x is the (expected) growth rate of the number of passengers, σ_x the related standard deviation, and dZ_x the increment of the Wiener process.

The net cash flow per passenger (R) is assumed to grow deterministically[4]:

$$dR = \alpha_R R dt \tag{2}$$

$F(x)$ represents the project value function which must satisfy the following ordinary differential equation (ODE), during the continuation period (when it is not yet optimal to start constructing the airport):

$$\frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 F(x)}{\partial x^2} + (r - \delta_x) x \frac{\partial F(x)}{\partial x} - rF(x) = 0 \tag{3}$$

subject to the boundary conditions:

$$F(0) = 0 \tag{4}$$

[2]As we will see, an example of a negative shock is the terrorist attack occurred on September 11, 2001. That unexpected event reduced significantly the number of passengers on the airports, almost all over the world.

[3]Marathe and Ryan (2005) analyzed the validity of the assumption of the geometric brownian motion process for the U.S. airline passenger emplanements and could not reject the hypothesis of normality and independence of the log ratios.

[4]This will be relaxed later on, where R will be modeled as a stochastic variable.

$$F(x^*) = \frac{x^* e^{\alpha_x n} R e^{\alpha_R n}}{k - \alpha_x - \alpha_R} e^{-kn} - K \tag{5}$$

$$F'(x^*) = \frac{e^{\alpha_x n} R e^{\alpha_R n}}{k - \alpha_x - \alpha_R} e^{-kn} \tag{6}$$

where $\delta_x = k - \alpha_x$, k is the equilibrium rate of return, and n represents the number of years for the construction.

The boundary 5, the so-called value matching condition, gives the NPV of the project for the moment when it is optimal to invest. The first part of the left-hand side of the equation represents the gross project cash flow, and K is the present value of all the expenditures required to proceed the construction of the airport.

The general solution for the equation 3 takes the form:

$$F(x) = Ax^{\beta_1} + Bx^{\beta_2} \tag{7}$$

where β_1 and β_2 are the two roots of the fundamental quadratic:

$$\frac{1}{2}\sigma_x^2 \beta(\beta - 1) + (r - \delta_x)\beta - r = 0 \tag{8}$$

and so:

$$\begin{aligned} \beta_1 &= \frac{1}{2} - \frac{r - \delta_x}{\sigma_x^2} + \sqrt{\left(-\frac{1}{2} + \frac{r - \delta_x}{\sigma_x^2}\right)^2 + \frac{2r}{\sigma_x^2}} > 1 \\ \beta_2 &= \frac{1}{2} - \frac{r - \delta_x}{\sigma_x^2} - \sqrt{\left(-\frac{1}{2} + \frac{r - \delta_x}{\sigma_x^2}\right)^2 + \frac{2r}{\sigma_x^2}} < 0 \end{aligned} \tag{9}$$

In order to respect the first boundary condition 4, and since Bx^{β_2} goes to infinity as x goes to zero, we must set $B = 0$. From hereafter let $\beta \equiv \beta_1$.

Solving the problem, taking into account the other two boundaries, we get the following solutions, respectively, for the airport value function and for the so-called trigger value, which represents the value of x for which is optimal to start the construction:

$$F(x) = \begin{cases} \frac{K}{\beta - 1}\left(\frac{x}{x^*}\right)^\beta & \text{for } x < x^* \\ \frac{x e^{\alpha_x n} R e^{\alpha_R n}}{k - \alpha_x - \alpha_R} e^{-kn} - K & \text{for } x \geq x^* \end{cases} \tag{10}$$

$$x^* = \frac{\beta}{\beta - 1} \frac{k - \alpha_x - \alpha_R}{e^{\alpha_x n} R e^{\alpha_R n} e^{-kn}} K \tag{11}$$

Finite concession

Until now we have assumed that the investor acquires a perpetual concession. If, alternatively, the concession is finite (for example, for m years, including the construction period), the solution would be:

$$F(x) = \begin{cases} \frac{K}{\beta-1}\left(\frac{x}{x^*}\right)^{\beta} & \text{for } x < x^* \\ \frac{xe^{\alpha_x n}Re^{\alpha_R n}}{k-\alpha_x-\alpha_R}e^{-kn}\left(1-e^{(\alpha_x+\alpha_R-k)(m-n)}\right)-K & \text{for } x \geq x^* \end{cases} \tag{12}$$

$$x^* = \frac{\beta}{\beta-1}\frac{k-\alpha_x-\alpha_R}{e^{\alpha_x n}Re^{\alpha_R n}e^{-kn}\left(1-e^{(\alpha_x+\alpha_R-k)(m-n)}\right)}K \tag{13}$$

Note that the NPV (lower part of equation 12) is the difference of two perpetuities with a time lag of $m-n$ years.

2.2 Two Stochastic Factors

Let us now incorporate more uncertainty in the process. In the previous section, only the number of passengers was considered to be random. Now we extend the randomness to the net cash flow per passenger.

Let x behave as in 1, and assume now that R (the net cash flow per passenger) follow a similar GBM process:

$$dR = \alpha_R R dt + \sigma_R R dZ_R \tag{14}$$

Since x and R are both stochastic, their product, which corresponds to the total profit for the airport, will also be stochastic.

Instead of working with the two stochastic factors, we can reduce them to a single one. Let $P(x,R) = xR$, and so:

$$\begin{aligned} dP &= \frac{\partial P(x,R)}{\partial x}dx + \frac{\partial P(x,R)}{\partial R}dR \\ &+\frac{1}{2}\left[\frac{\partial^2 P(x,R)}{\partial x^2}(dx)^2 + \frac{\partial^2 P(x,R)}{\partial R^2}(dR)^2\right] \\ &+\frac{\partial^2 P(x,R)}{\partial x \partial R}dxdR \end{aligned}$$

Noting that $\frac{\partial^2 P(x,R)}{\partial x^2} = \frac{\partial^2 P(x,R)}{\partial R^2} = 0$ and $\frac{\partial^2 P(x,R)}{\partial x \partial R} = 1$, the previous equation becomes:

$$dP = xdR + Rdx + dxdR \tag{15}$$

Substituting for dx and dR, and after rearranging:

$$dP = (\alpha_x + \alpha_R + \rho\sigma_x\sigma_R)\, Pdt + (\sigma_x dZ_x + \sigma_R dZ_R)\, P \tag{16}$$

where $E\left[dZ_x dZ_R\right] = \rho dt$.

Using the standard procedures[5], we find the ODE that must be followed by $F(P)$:

$$\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 F(P)}{\partial P^2} + (r - \delta_P)P\frac{\partial F(P)}{\partial P} - rF(P) = 0 \tag{17}$$

where $\sigma_P^2 = \sigma_x^2 + \sigma_R^2 + 2\rho\sigma_x\sigma_R$, $\delta_P = k - \alpha_P$ and $\alpha_P = \alpha_x + \alpha_R + \rho\sigma_x\sigma_R$, subject to the following boundary conditions:

$$F(0) = 0 \tag{18}$$

$$F(P^*) = \frac{P^* e^{\alpha_P n}}{k - \alpha_P} e^{-kn} - K \tag{19}$$

$$F'(P^*) = \frac{e^{\alpha_P n}}{k - \alpha_P} e^{-kn} \tag{20}$$

After taking into account the boundary 18, the general solution for the equation 17 takes the form:

$$F(P) = CP^\gamma \tag{21}$$

where γ is the positive root of the quadratic equation:

$$\frac{1}{2}\sigma_P^2\gamma(\gamma - 1) + (r - \delta_P)\gamma - r = 0 \tag{22}$$

which is:

$$\gamma = \frac{1}{2} - \frac{r - \delta_P}{\sigma_P^2} + \sqrt{\left(-\frac{1}{2} + \frac{r - \delta_P}{\sigma_P^2}\right)^2 + \frac{2r}{\sigma_P^2}}$$

Determining C and P^* using the boundaries 19 and 20, we get the following solutions for $F(P)$ and P^*:

$$F(P) = \begin{cases} \frac{K}{\gamma - 1}\left(\frac{P}{P^*}\right)^\gamma & \text{for } P < P^* \\ \frac{P e^{\alpha_P n}}{k - \alpha_P} e^{-kn} - K & \text{for } P \geq P^* \end{cases} \tag{23}$$

[5]See Appendix A.

$$P^* = \frac{\gamma}{\gamma - 1} \frac{k - \alpha_P}{e^{\alpha_P n} e^{-kn}} K \tag{24}$$

The solution for the finite concession case is:

$$F(P) = \begin{cases} \frac{K}{\gamma-1}\left(\frac{P}{P^*}\right)^{\gamma} & \text{for } P < P^* \\ \frac{Pe^{\alpha_P n}}{k-\alpha_P} e^{-kn}\left(1 - e^{(\alpha_P - k)(m-n)}\right) - K & \text{for } P \geq P^* \end{cases} \tag{25}$$

$$P^* = \frac{\gamma}{\gamma - 1} \frac{k - \alpha_P}{e^{\alpha_P n} e^{-kn}\left(1 - e^{(\alpha_P - k)(m-n)}\right)} K \tag{26}$$

2.3 Two Stochastic Factors with Positive and Negative Shocks

Assume now that there are some random events which, in a discrete way, changes the level of the state variable.

These events are shocks which seem to be particularly relevant in airports business. See, as an example, the *negative shock* coming from the terrorist attack on September 11, or, the less tragic negative shock coming from the negative impact of some new competing airport; a *positive shock*, could be the occurrence of an important international event, such as the Olympic games or the world football championship.

In order to incorporate these shocks, we change equation 16 adding an additional term:

$$dP = (\alpha_x + \alpha_R + \rho\sigma_x\sigma_R)\,Pdt + (\sigma_x dZ_x + \sigma_R dZ_R)\,P + dqP \tag{27}$$

where:

$$dq = \begin{cases} (1+u) & \text{with probability} \quad \lambda_u dt \\ (1-d) & \text{with probability} \quad \lambda_d dt \\ 0 & \text{with probability} \quad 1 - (\lambda_u + \lambda_d)dt \end{cases} \tag{28}$$

and $(1+u)$ represent the *positive shock*, and $(1-d)$ represent the *negative shock*; u and d are deterministic parameters.

So $F(P)$ must satisfy now the following differential equation:

$$\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 F(P)}{\partial P^2} + (r - \delta_P) P \frac{\partial F(P)}{\partial P} - rF(P) +$$
$$+ \lambda_u \left[F\left((1+u)P\right) - F(P) \right] + \lambda_d \left[F\left((1-d)P\right) - F(P) \right] = 0 \quad (29)$$

Rearranging, we get:

$$\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 F(P)}{\partial P^2} + (r - \delta_P) P \frac{\partial F(P)}{\partial P} - (r + \lambda_u + \lambda_d) F(P) +$$
$$+ \lambda_u F\left((1+u)P\right) + \lambda_d F\left((1-d)P\right) = 0 \quad (30)$$

Note that, assuming that the events are independent, the probability of the occurrence of a *positive shock* and a *negative shock* during the same period of time dt is: $\lambda_u \lambda_d (dt)^2$. Calling the standard arguments, $(dt)^2$ goes faster to zero as $dt \to 0$, and so this probability can be ignored.

The expected variation of P, is now $\alpha_s = \alpha_P + \lambda_u u - \lambda_d d$, to incorporate the effects of the shocks.

The solution must be found subject to the following boundaries:

$$F(0) = 0 \quad (31)$$

$$F(P^*) = \frac{P^* e^{\alpha_s n}}{k - \alpha_s} e^{-kn} - K \quad (32)$$

$$F'(P^*) = \frac{e^{\alpha_s n}}{k - \alpha_s} e^{-kn} \quad (33)$$

The general solution is of the form:

$$F(P) = D P^{\phi} \quad (34)$$

where ϕ is the positive root of the non-linear equation:

$$\frac{1}{2}\sigma_P^2 \phi(\phi - 1) + (r - \delta_P)\phi - (r + \lambda_u + \lambda_d) + \lambda_u (1+u)^{\phi} + \lambda_d (1-d)^{\phi} = 0 \quad (35)$$

which, on the contrary to what happens to β (equation 8) and γ (equation 22), must be solved numerically; D, as well as the trigger P^*, are to be determined using the boundaries 33 and 32.

Accordingly, the solutions are:

$$F(P)=\begin{cases}\frac{K}{\phi-1}\left(\frac{P}{P^*}\right)^{\phi} & \text{for } P<P^* \\ \frac{Pe^{\alpha_s n}}{k-\alpha_s}e^{-kn}-K & \text{for } P\geq P^*\end{cases} \tag{36}$$

$$P^*=\frac{\phi}{\phi-1}\frac{k-\alpha_s}{e^{\alpha_s n}e^{-kn}}K \tag{37}$$

For the case of a finite concession period, the solution is given by:

$$F(P)=\begin{cases}\frac{K}{\phi-1}\left(\frac{P}{P^*}\right)^{\phi} & \text{for } P<P^* \\ \frac{Pe^{\alpha_s n}}{k-\alpha_P}e^{-kn}\left(1-e^{(\alpha_s-k)(m-n)}\right)-K & \text{for } P\geq P^*\end{cases} \tag{38}$$

$$P^*=\frac{\phi}{\phi-1}\frac{k-\alpha_s}{e^{\alpha_s n}e^{-kn}\left(1-e^{(\alpha_s-k)(m-n)}\right)}K \tag{39}$$

3 The effect of uncertainty

It is commonly accepted that uncertainty increases option values. Nevertheless such relationship depends on the effect, less studied, of the uncertainty on the underlying asset value, the NPV, in our case[6]. Note that it depends on the equilibrium rate of return (k), which is related with the uncertainty of the project. According to the CAPM such relationship is expressed as:

$$k=r+\lambda\rho_{PM}\sigma_P \tag{40}$$

where $\lambda=\frac{r_M-r}{\sigma_M}$ is the market price of risk, ρ_{PM} the correlation of the total revenues with the market and σ_M the market volatility.

The net effect of an increase of the project volatility depends on the indirect effect it can have through the correlation, and if the increased volatility is unaccompanied by the market volatility. The sign of the effect can be null, negative or positive.

Assuming that the increased volatility leaves unchanged both the correlation and the market price of risk, it implies a higher cost of capital, reducing the NPV. In the case of a single stochastic factor model, an increase in the volatility to σ_x', would increase the equilibrium rate of return

[6]See Davis (2002) for a related discussion.

Parameter	Description	Value
x	Current number of passenger per year	11.5 million
R	Current mean cash flow per passenger (EUR)	8
$P \equiv xR$	Current total cash flow per year (EUR)	92 million
K	Present value of the investment cost (EUR)	3000 million
k	Equilibrium rate of return	0.090
r	Risk-free interest rate	0.030
α_x	Expected growth rate of x	0.04
σ_x	Standard deviation of x	0.15
α_R	Expected growth rate of R	0.020
σ_R	Standard deviation of R	0.01
ρ	Correlation between the two stochastic variables	0.00
n	Number of years for the construction	7
m	Number of years of the concession	30

Table 1: Base-case parameters for the project considering two stochastic factors.

to[7]:

$$k' = \frac{\sigma'_x}{\sigma_x}(k-r)+r \tag{41}$$

4 A Numerical Example

4.1 Considering Two Stochastic Factors

Assume a project for the construction of a new airport, considering that both the number of passengers and the cash-flow per passenger behave stochastically. The basic parameters are in Table 1.

The main objective is to determine the optimal timing to start the construction of the new airport (which is given by the critical total annual cash flow, P^*), and additionally, the value of the option to invest, and of the option to defer the project implementation.

Table 2 shows the output values. For the base-case parameters the construction should only start when the present value of the expected annual cash flow is 203.2 million euros for the perpetual case concession, whereas for the finite concession case, it must reach 407.7 million euros.

The option to defer construction adds 948.4 million euros to the tradi-

[7] Note that

$$\lambda \rho_{xM}\sigma_x = \frac{k-r}{\sigma_x} = \frac{k'-r}{\sigma'_x}$$

.

tional NPV, which is negative, making the investment opportunity worthwhile. The option almost doubles its value for the case of a finite concession. However the probability of investing in a finite concession is very small, as will also be shown in the the following Figures.

The second stochastic variable adds to the investment opportunity 246 million euros and 77.5 million euros for the perpetual and finite concessions, respectively.

		Two factors		**One factor**	
Output	**Description**	**Perp.**	**Finite**	**Perp.**	**Finite**
x^*	Critical number of passengers (million)			19.4	38.8
P^*	Critical total annual cash flow (EUR million)	203.2	407.7		
$F(x)$	Value of the investment opportunity (EUR million)	434.2	93.6	188.2	16.1
NPV	Value of the project if implemented today (EUR million)	-514.2	-1761.0	-514.2	-1761.0
OD	Value of the *option to defer* the construction (EUR million) $[OD = F(x) - NPV]$	948.4	1854.6	702.4	1777.1

Table 2: Output values for the base-case parameters.

Figures 1 to 5 show the impact of some parameters. The expected growth rates of either the number of passengers or the cash flow per passenger (Figure 1) and the equilibrium rate of return (Figure 2) are some of the key drivers of the value of the investment opportunity.

The difference between the perpetual and finite concessions, both in the NPV and investment opportunity values, increases significantly with the growth rates (Figure 1), and deacreases with the equilibrium rate of return (Figure 2). As the growth rates become closer to the equilibrium rate of return (i.e. $k - \alpha_x - \alpha_R$ approaches zero) the NPV goes to infinity.

The assumption, underlying the perpetual concession, that the number of passengers grows infinitly is, surely, unrealistic, while for the growth rate of the cash flow per passenger, such assumption is more realistic.

Higher growth rates, after a certain value, increases the critical value of the total cash flow in a finite concession, which is not the case for the perpetual concession. The same patern can be found for the equilibrium rate of return (Figure 2) and the volatility (Figure 5). The critical cash flow increases monotonically with the investment cost (Figure 3) and the time needed to construct the airport (Figure 4) both for the perpetual and finite concessions.

As discussed previously in section 3, a higher uncertainty may decrease the NPV. Figure 5 presents the results for the model with a single factor,

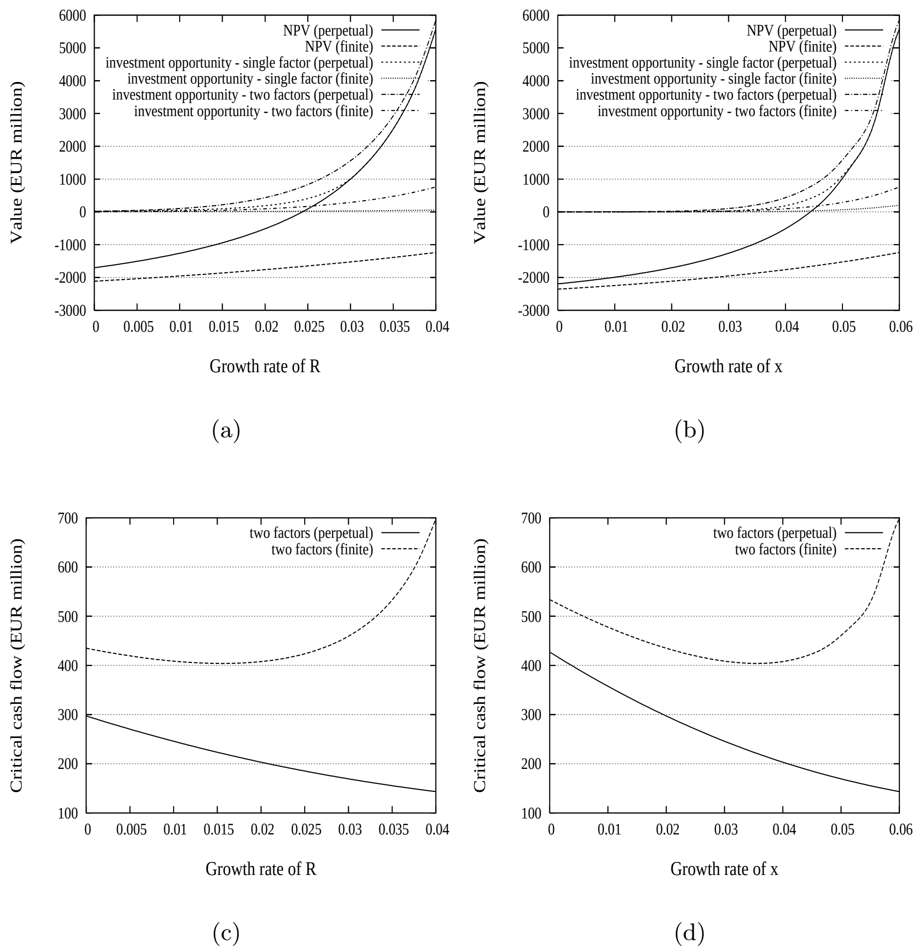
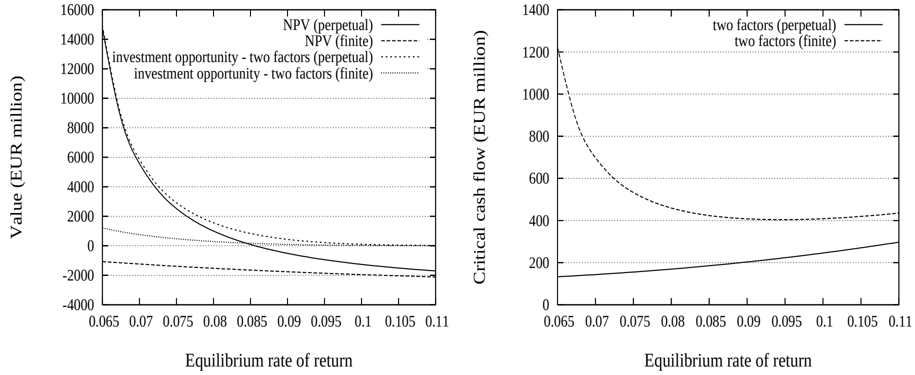
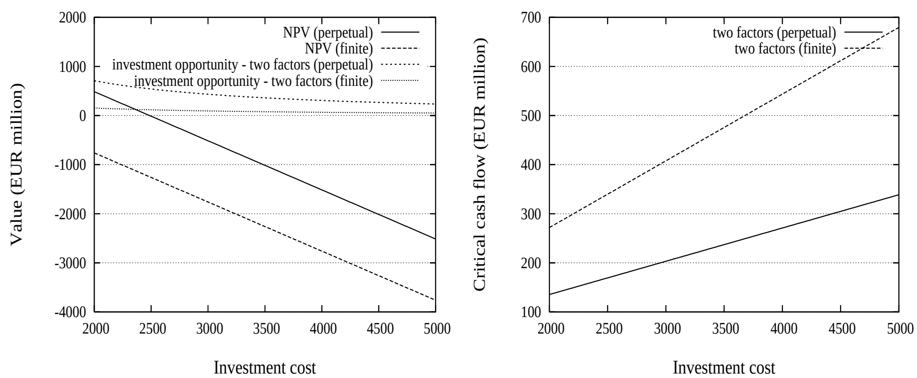


Figure 1: The impact of the growth rates



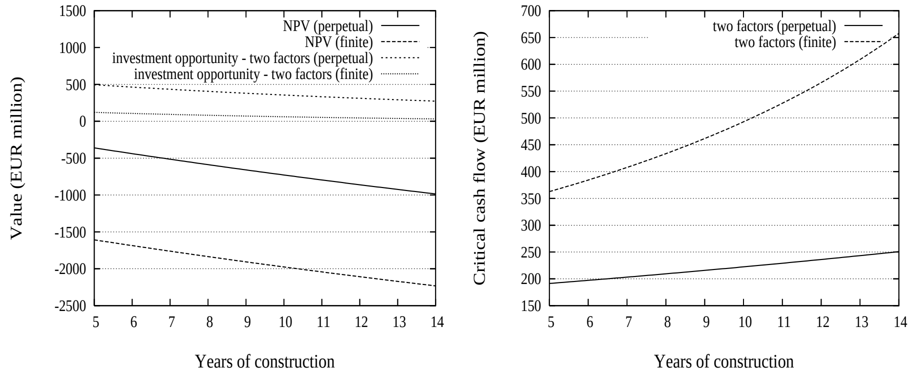
(a) (b)

Figure 2: The impact of the equilibrium rate of return



(a) (b)

Figure 3: The impact of the investment cost



(a) (b)

Figure 4: The impact of the number of years of construction

and shows that, if we assume that the covariance with the market remains unchanged, a higher uncertainty reduces the NPV through an increased required rate of return, reducing the probability of investing. Even reducing both the NPV and the investment opportunity values, a higher volatility increases the value of the option do defer.

4.2 Considering Two Stochastic Factors with Shocks

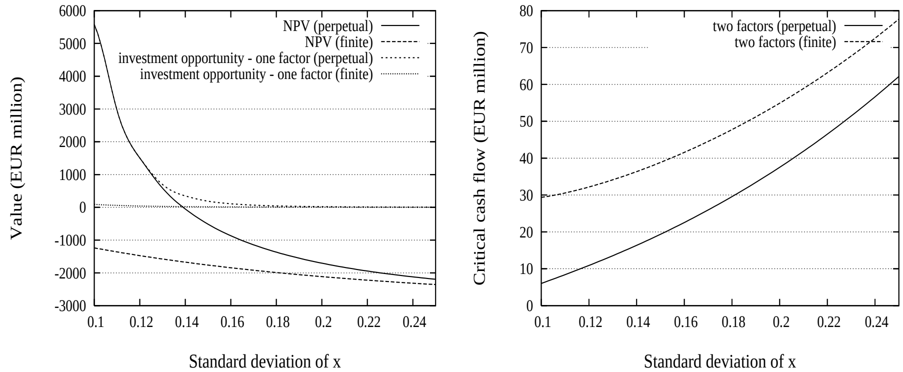
Let us consider now the existence of some unexpected *shocks*, which, as we saw, can have a positive or a negative impact on the level of the state variable (total cash flow), increasing or decreasing it in a discrete way.

Let the parameters be as presented in Table 1, and additionally consider the *shocks* and the respective probabilities presented in Table 3).

Parameter	**Description**	**Value**
u	Magnitude of the *positive shock*	0.1
d	Magnitude of the *negative shock*	0.15
λ_u	Probability of occurrence of the *positive shock*	0.1
λ_d	Probability of occurrence of the *positive shock*	0.1

Table 3: Parameters for *shocks* and probabilities.

The critical P (total annual cash flow), as well as the value of the investment opportunity are presented in Table 4. The effect of adding shocks



(a) (b)

Figure 5: The impact of the volatility of the number of passengers

depends on the relative magnitudes and probabilities of the positive and negative shocks (Figure 6). If the magnitude of the positive shock is lower than the magnitude of the negative shock, the effect is negative. A higher(lower) magnitude of the positive(negative) shocks increases(decreases) the value of the project.

		Without shocks		**With shocks**	
Output	**Description**	**Perp.**	**Finite**	**Perp.**	**Finite**
P^*	Critical total annual cash flow (EUR million)	203.2	407.7	232.2	419.9
$F(x)$	Value of the investment opportunity (EUR million)	434.2	93.6	244.7	60.1
NPV	Value of the project if implemented today (EUR million)	-514.2	-1761.0	-942.6	-1862.4
OD	Value of the *option to defer* the construction (EUR million) $[OD = F(x) - NPV]$	948.4	1854.6	1187.3	1922.6

Table 4: Output values for the base-case parameters, considering the *shocks*.

5 Conclusions and Future Research

This paper presents a real options model to value the option to build a new international airport, considering that the number of passenger and the

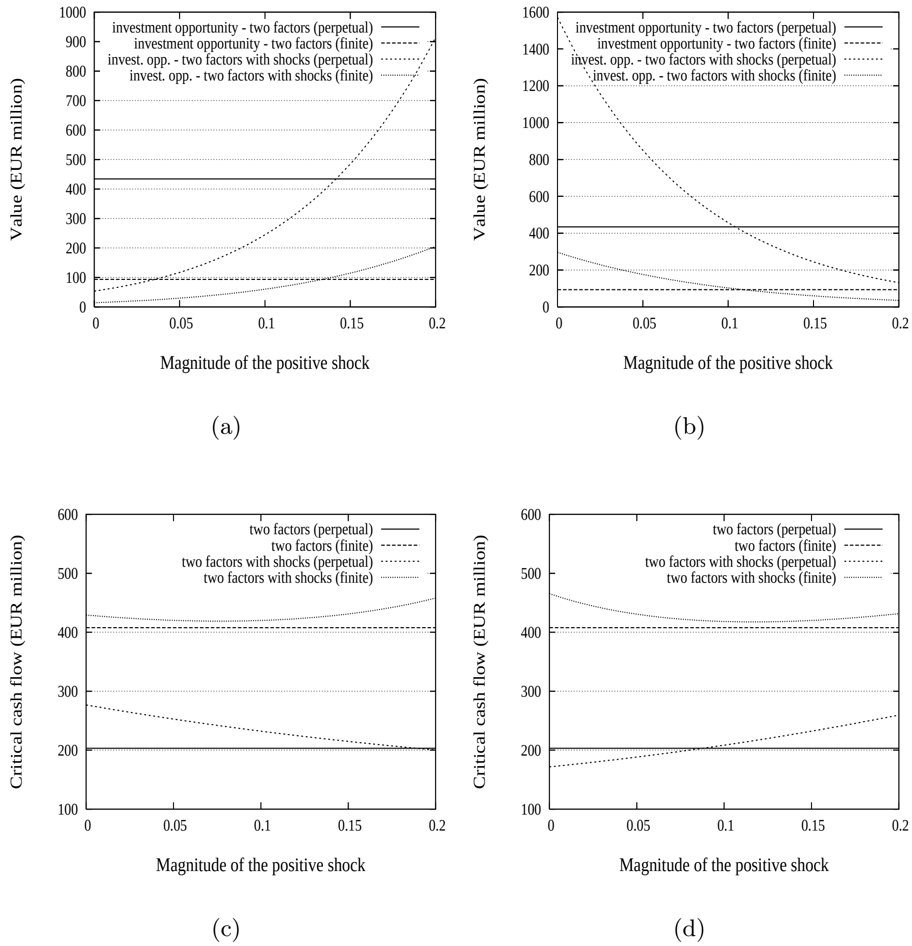


Figure 6: The impact of the shocks

cash flow per passenger behave stochastically. Additionally, positive and negative shocks are also incorporated, which seems to be realistic for this type of projects.

The option to defer this type of heavy investment projects is highly dependent on the assumptions about the expected growth rates, time needed for construction and the equilibrium rate of return.

We have shown that higher uncertainty of the stochastic variables lowers the value of the project and increases the option to delay it.

One of the most interesting conclusions is related to concession terms of the project. If the contract allows the builder to delay the investment, making it optional and not compulsory, he would available to pay more for the concession or, alternatively would require less public expenditure. However, a finite concession is very unlikely to induce investment, requiring more financial support of the government, if the desired outcome is the construction of the infrastructure.

Calibration of this model may imply the adjustment of some assumptions about the behavior of the stochastic variables.

Several extensions to the model, including, for example, the valuation of other options, the possibility of investing continuously and the effect of competition, may be added, at the cost of more complexity and the use of numerical valuation methods.

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A Appendix

From equations 1 and 14 we saw that:

$$dx = \alpha_x x dt + \sigma_x x dZ_x \tag{42}$$
$$dR = \alpha_R R dt + \sigma_R R dZ_R \tag{43}$$

and according to equation 15:

$$dP = xdR + Rdx + dxdR \tag{44}$$

Substituting (42) and (43) in (44), we get:

$$\begin{aligned} dP &= x\left(\alpha_R R dt + \sigma_R R dZ_R\right) \\ &+ R\left(\alpha_x x dt + \sigma_x x dZ_x\right) \\ &+ \left(\alpha_x x dt + \sigma_x x dZ_x\right)\left(\alpha_R R dt + \sigma_R R dZ_R\right) \end{aligned} \tag{45}$$

Noting that $xR = Rx = P$, (45) comes

$$
\begin{aligned}
dP &= \alpha_R P dt + \sigma_R P dZ_R \\
&+ \alpha_x P dt + \sigma_x P dZ_x \\
&+ (\alpha_x x dt + \sigma_x x dZ_x)(\alpha_R R dt + \sigma_R R dZ_R)
\end{aligned} \tag{46}
$$

The term $(\alpha_x x dt + \sigma_x x dZ_x)(\alpha_R R dt + \sigma_R R dZ_R)$ is equal to $\rho \sigma_x \sigma_R P dt$. Note that we ignore the terms dt of order $3/2$[8] or higher, since they go faster to zero as dt goes to zero. Additionally, $dZ_x dZ_R = \rho dt$. According to this (5) comes, after rearranging:

$$dP = (\alpha_R + \alpha_x + \rho \sigma_x \sigma_R) P dt + (\sigma_R dZ_R + \sigma_x dZ_x) P \tag{47}$$

In order to simplify, let $\alpha_P = \alpha_R + \alpha_x + \rho \sigma_x \sigma_R$, so (47) reads now:

$$dP = \alpha_P P dt + (\sigma_R dZ_R + \sigma_x dZ_x) P \tag{48}$$

Consider now a portfolio consisting on the option to invest ($F(P)$) and on a short position on $\frac{\partial F(P)}{\partial P}$ units of the project (or of a perfect correlated asset or portfolio). The value of this portfolio is:

$$\Pi = F(P) - \frac{\partial F(P)}{\partial P} P \tag{49}$$

Considering that the short position demands the payment $\delta_P = k - \alpha_P$ (see, for example, Dixit and Pindyck (1994) for details), the total return for the portfolio Π during the period dt is:

$$dF(P) - \frac{\partial F(P)}{\partial P} dP - \delta_P P \frac{\partial F(P)}{\partial P} dt \tag{50}$$

Applying the Ito's Lemma, $F(P)$ must follow the ODE:

$$dF(P) = \frac{1}{2} \frac{\partial^2 F(P)}{\partial P^2} (dP)^2 + \frac{\partial F(P)}{\partial P} dP \tag{51}$$

Substituting, we can write the total return for the portfolio as follows:

[8] $dZ_R \times dt = \epsilon_t \sqrt{dt} \times dt = \epsilon_t dt^{3/2}$, where ϵ_t is a random variable that follows a normal distribution with the first two moments being $(0, 1)$.

$$\frac{1}{2}\frac{\partial^2 F(P)}{\partial P^2}(dP)^2 - \delta_P P \frac{\partial F(P)}{\partial P} dt \tag{52}$$

Let us now calculate $(dP)^2$:

$$\begin{aligned}(dP)^2 &= [\alpha_P P dt + (\sigma_R dZ_R + \sigma_x dZ_x)P]^2 \quad (53)\\ &= (\alpha_P P dt)^2 + 2\alpha_P P dt(\sigma_R dZ_R + \sigma_x dZ_x)P \\ &+ (\sigma_x dZ_x + \sigma_R dZ_R)^2 P^2 \quad (54)\end{aligned}$$

Ignoring all terms dt of order higher than 1 we get:

$$(dP)^2 = [(\sigma_x dZ_x)^2 + (\sigma_R dZ_R)^2 + 2\sigma_x dZ_x \sigma_R dZ_R)]P^2 \tag{55}$$

Since $(dZ_x)^2 = dt$, $(dZ_R)^2 = dt$, and $dZ_x dZ_R = \rho dt$, $(dP)^2$ is simply

$$(dP)^2 = (\sigma_x^2 + \sigma_R^2 + 2\rho\sigma_x\sigma_R)dtP^2 \tag{56}$$

and so, substituting:

$$\frac{1}{2}(\sigma_x^2 + \sigma_R^2 + 2\rho\sigma_x\sigma_R)dtP^2\frac{\partial^2 F(P)}{\partial P^2} - \delta_P P \frac{\partial F(P)}{\partial P} dt \tag{57}$$

Note that this is risk-free, and so the adequate return is the risk-free rate. Which means that, during short period dt, the return for the portfolio must be:

$$r\Pi dt = r\left(F(P) - \frac{\partial F(P)}{\partial P}P\right)dt \tag{58}$$

and so:

$$\begin{aligned}\frac{1}{2}(\sigma_x^2 + \sigma_R^2 + 2\rho\sigma_x\sigma_R)dtP^2\frac{\partial^2 F(P)}{\partial P^2} &- \delta_P P \frac{\partial F(P)}{\partial P} dt \\ &= r\left(F(P) - \frac{\partial F(P)}{\partial P}P\right)dt \quad (59)\end{aligned}$$

After dividing both sides of the equation by dt and after rearranging, we get:

$$\frac{1}{2}(\sigma_x^2 + \sigma_R^2 + 2\rho\sigma_x\sigma_R)P^2\frac{\partial^2 F(P)}{\partial P^2} + (r - \delta_P)P\frac{\partial F(P)}{\partial P} - rF(P) = 0 \quad (60)$$

or, as it appears in equation 17:

$$\frac{1}{2}\sigma_P^2 P^2\frac{\partial^2 F(P)}{\partial P^2} + (r - \delta_P)P\frac{\partial F(P)}{\partial P} - rF(P) = 0 \quad (61)$$

where $\sigma_P^2 = \sigma_x^2 + \sigma_R^2 + 2\rho\sigma_x\sigma_R$.