Dynamic acquisition of investment information under uncertainty: a learning-options approach*

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Abstract

This paper studies the optimal acquisition of incremental information on an irreversible investment opportunity that is subject to economic uncertainty through a random payoff shock. Imperfect information is introduced by assuming that the payoff and the investment cost are affected by distinct multiplicative signals that at the outset are unobservable to the investor. The problem of the investor then is to decide when, if at all, to acquire the signals given that the alternative is to invest with the prior estimates. The incurred costs being completely sunk, going ahead with an acquisition translates to exercising an irreversible real option, a learning option. We show that the optimal acquisition (learning) policies are represented by two simple stopping times for the payoff shock. The policies balance the trade-off between the acquisition costs and the fact that postponing an acquisition increases the risk of learning information that would have been more beneficial when incorporated into decision making earlier. In particular, postponing increases the risk of forfeiting the optimal perfect-information investment.

Keywords: information acquisition, irreversible investment, real option

JEL subject classification: D80, G31

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1 Introduction

Besides economic uncertainty observed to resolve ex ante, investment opportunities are commonly subject to a type of uncertainty that resolves only ex post: imperfect information on the investment payoff and on the investment cost. For an investor this poses a problem as the ex-ante investment policy then tends to deviate from the perfect-information policy, leading either to over-investment by suboptimally rushing or to under-investment by suboptimally prolonging. The problem is mitigated by operational flexibility for information acquisition. The economic motivation for an acquisition stems from the expected contribution of information to the value of the optimal investment policy. The contribution represents the *value of information.* The incurred costs being completely sunk, going ahead with an acquisition translates to exercising an option on an irreversible investment, a *learning option*.

In this paper we contribute to the literature on dynamic investment and information acquisition by studying optimal *active acquisition* through exercise of learning options. We most significantly break away from *passive acquisition* and learning-by-doing (at least implicitly) allowed for by a wide range of investment models. For example, the models proposed by Roberts and Weitzman [1981] and Weitzman et al. [1981] allow for passive acquisition in the context of a multistage development project where the firm updates its beliefs about the final payoff at the completion of each stage. The seminal time-to-build model proposed by Majd and Pindyck [1987] allows for passive acquisition in the same fashion. Pennings and Lint [1997] construct a jump-diffusion model in the context of an R&D project, allowing for the fact that upon the arrival of strategic information (on competing standards, for example) a firm updates its projection on the payoff (market revenues) generated by a new product. Grenadier [1999] allows for passive acquisition in the context of an investment game where investors start out with imperfect and heterogeneous information, showing that in equilibrium there may be strategic incentives to delay in order to infer information from the investment decisions of the opponents (see Essay II of this thesis for discussion). Decamps et al. [2002] and Roche [2005] allow for passive acquisition in technically elaborate models where the stochastic process governing the investment value involves an unobservable growth rate, and the firms Bayesian update their beliefs about this rate while observing the process.

Now, the immediate conclusion is that since passive acquisition bases on delaying rather than on an autonomous policy, it has no bearing on the important question whether and when investors opt to acquire incremental information. This brings us to active acquisition. Along the lines of Murto [2004], we study active acquisition from the viewpoint of an investor holding a monopoly right to an irreversible investment. Extending from the Murto model where one signal affects jointly the net investment payoff, we assume that the investment payoff and the investment cost are affected by distinct multiplicative signals that at the outset are unobservable to the investor. The optimal overall acquisition policy is then governed by a sequence of two learning options. The learning process is extreme in that by exercising a learning option the investor learns perfectly the true value of the desired signal. From the viewpoint of Bayesian learning this means that the posterior distribution for the signal is perfectly concentrated on the true value. The optimal policy balances the trade-off between the acquisition costs and the fact that postponing an acquisition increases the risk of learning information that would have been more useful when incorporated into decision making earlier. In particular, postponing increases the risk of forfeiting the optimal perfect-information investment. The policy also involves a trade-off between the acquisition costs and the accuracy of the prior signal estimates. If a prior estimate is accurate and the cost is large then it is intuitive that the investor scraps the learning option altogether.

For analytic tractability and clarity of economic argumentation, we make two additional assumptions. Firstly, economic uncertainty is represented by a multiplicative payoff shock that fluctuates according to a geometric Brownian motion. Secondly, the acquisition sequence is exogenous to the degree that the option to learn the cost information (signal) is created by the acquisition of the payoff information (signal). This in the popular real-options terminology means that the flexibility for acquisition is rendered more complex by *intraproject compoundness* (see e.g. Schwartz and Trigeorgis [2001], pp. 86-93).

Summing up, the principal questions of the paper are:

1. Given economic uncertainty and the exogenous order of acquisitions, what is the value of the investment-payoff information and the investment-cost information?
2. Given the values, what are the optimal acquisition policies?
3. What are the comparative static properties of the optimal acquisition policies?

The main findings are as follows. From a particular convexity property exhibited by the investment option, we first confirm the contribution of information to the investment policy. Working backwards in the acquisition sequence and making use of the standard dynamic-programming procedure presented in Dixit and Pindyck [1994], we then introduce a computationally efficient two-step method to show that the optimal acquisition policies are represented by simple stopping times for the shock. The policy for the cost information incorporates the payoff information learned previously. The policy for the payoff information in turn incorporates the option to learn the cost information. The aggregate policy follows by combining the two policies. As a special case, the acquisitions occur simultaneously. We confirm that a learning option assumes a negative value when the prior signal estimate is accurate and the acquisition cost is large. As argued above, the investor then scraps the learning option. Finally, we expand on the policies by inspecting the comparative-statics properties of the shock triggers (learning triggers) implicit in the stopping times.

The paper proceeds as follows. Section 2 sets up the investment environment and introduces the perfect-information case as the starting point. Section 3 solves for the optimal acquisition policies and carries out the comparative statics. Section 4 concludes and suggests directions for further research.

2 Basic setup

We consider an investor holding a monopoly right to an irreversible investment opportunity under economic uncertainty. Extending from the problem studied by Pindyck [1991], the investor starts out with imperfect information on the true value of the opportunity. The investment payoff is given by θX where θ is an unobservable signal and X is an observable random shock. The investment cost is given by λI where λ is an unobservable signal distinct from θ and I is an observable constant.

The payoff signal is selected by nature from the uniform distribution on a closed support Θ,

$$f(s) = \begin{cases} \frac{s}{|\Theta|}, & s \in \Theta \\ 0, & s \notin \Theta, \end{cases}$$

where denotes the Lebesgue measure of Θ. The cost signal is selected from the uniform distribution on a closed support Λ,

$$g(u) = \begin{cases} \frac{u}{|\Lambda|}, & u \in \Lambda \\ 0, & u \notin \Lambda. \end{cases}$$

Letting $[a, b]^+ = b$ and $[a, b]^- = a$, the prior signal estimates are represented by the associated expectations

$$\mu_\theta = \frac{\Theta^+ + \Theta^-}{2},$$

and

$$\mu_\lambda = \frac{\Lambda^+ + \Lambda^-}{2},$$

The accuracies of the estimates are proxied by $|\Theta|$ and $|\Lambda|$.

The shock fluctuates according to the geometric Brownian motion

$$\frac{dX}{X} = \mu dt + \sigma dz,$$

where μ is a growth rate parameter, σ is a proportional volatility parameter and dz is the increment of a standard Brownian motion. We assume that the fluctuation is spanned by the financial markets, which gives rise to equivalent risk-neutral valuation in the way shown by Dixit and Pindyck [1994, pp. 121-124]. The differential operator associated with the risk-neutral version of the shock is given by

$$\mathcal{A} = \frac{1}{2}\sigma^2 X^2 \frac{\partial^2}{\partial X^2} + (r - \delta) X \frac{\partial}{\partial X},$$

where r is the risk-free interest rate and $\delta < r$ is the equilibrium rate of return shortfall.

The investor holds operational flexibility to acquire information on the signals at a fixed cost prior to going ahead with the investment. The optimal acquisition-investment policy is then the one of a sequential capital-budgeting problem with the possibility of two acquisition stages and one investment stage. Following the reference model proposed by Murto [2004], the learning process is extreme in that upon acquisition the investor learns perfectly the true value of the desired signal. From the viewpoint of Bayesian updating this means that the posterior distribution for the signal is perfectly concentrated on the true value. Henceforth, we reserve the term *ex post* for the time immediately after an acquisition and the term *ex ante* for the time immediately before an acquisition.

The economic motivation for a signal acquisition follows from the expected contribution of incremental information to the value of the optimal investment policy, as governed by the investment option. For elaborating this further, let us consider the perfect-information case where both θ and λ are observed. The investment problem then reads as

$$W^*(X; \theta, \lambda) = \sup_{\tau} \mathbf{E}_X [e^{-r\tau} V^*(X(\tau); \theta, \lambda)], \tag{1}$$

where

$$V^*(X; \theta, \lambda) = \theta X - \lambda I,$$

and $\mathbf{E}_X$ denotes an expectation operator associated with the shock starting at $X_0 = X$. The supremum is taken over the set of all stopping times adapted to the shock process.

We can rewrite W^* as

$$W^*(X; \theta, \lambda) = \begin{cases} F^*(X; \theta, \lambda), & X < X^*(\theta, \lambda) \\ V^*(X; \theta, \lambda), & X \geq X^*(\theta, \lambda), \end{cases}$$

where F^* is interpreted as the value of the investment option and X^* denotes the yet unknown optimal investment trigger for the shock.

The optimal investment policy and the solution to (1) is represented by the stopping time

$$\tau^*(X;\theta,\lambda) = \inf\{t \geq 0 : X(t) \geq X^*(\theta,\lambda), X(0) = X\}.$$

An application of Itô's lemma shows that F^* is governed by the stochastic Bellman equation

$$\mathcal{A}F^* - rF^* = 0,$$

subject to the boundary condition $F^*(0;\theta,\lambda) = 0$. It follows that

$$F^*(X;\theta,\lambda) = A^*(\theta,\lambda)X^{\beta},$$

where

$$\beta = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + 2\frac{r}{\sigma^2}} > 1,$$

and $A^* > 0$ is an unknown constant.

The standard dynamic-programming (smooth-fitting) procedure of selecting the trigger X^* so as to make W^* continuously differentiable gives

$$X^*(\theta,\lambda) = \frac{\beta}{\beta-1}\frac{\lambda I}{\theta},$$

and

$$F^*(X;\theta,\lambda) = \left(\frac{X}{X^*(\theta,\lambda)}\right)^{\beta} V^*(X^*;\theta,\lambda) = \frac{1}{\beta-1}\left(\frac{X}{X^*(\theta,\lambda)}\right)^{\beta}\lambda I.$$

Taking the differentials establishes that F^* is convex in both θ and λ:

$$\frac{\partial^2 F^*}{\partial\theta^2} > 0, \frac{\partial^2 F^*}{\partial\lambda^2} > 0.$$

From Jensen's inequality it then follows that the expected ex-post value of the investment option is larger than the ex-ante value. For example, if both signals are unobservable then the investor computes

$$\frac{1}{|\Theta|}\int_{\Theta} F^*(X; s, \mu_\lambda)\mathbf{1}(X < X^*(s, \mu_\lambda))f(s)ds > F^*(X; \mu_\theta, \mu_\lambda),$$

and

$$\frac{1}{|\Lambda|}\int_{\Lambda} F^*(X; \mu_\theta, u)\mathbf{1}(X < X^*(\mu_\theta, u))g(u)du > F^*(X; \mu_\theta, \mu_\lambda),$$

where the indicator functions allow for the technicality that the domain of definition for an investment option is bounded from above by the investment trigger.

For simplifying the optimal acquisition policies, we finally add the following assumptions:

(A1) The acquisition of the payoff signal θ (θ-*information*) incurs a sunk cost $A_\theta = a_\theta I > 0$. The acquisition of the cost signal λ (λ-*information*) incurs a sunk cost $A_\lambda = a_\lambda I > 0$.

(A2) The sequence of the signal acquisitions is exogenous to the degree that λ cannot be acquired before θ.

3 Optimal acquisition policies

We find the optimal acquisition policies by working backwards in the acquisition sequence, from the acquisition of the λ-information to that of the θ-information. As established above, the motivation for the acquisitions follows from the convexity in θ and λ of the investment-option value F^*. The economic reasoning is that incremental information contributes in expected terms to the value of the investment policy through better investment timing. This contribution, the difference between the expected ex-post value and the ex-ante value, is interpreted as the *value of information*. The value fluctuates unpredictably as a function of the payoff shock. The incurred costs being completely sunk, going ahead with acquisition then translates to exercising an irreversible real option, a *learning option*. The overall acquisition policy involves intraproject option compoundness in that the option to learn the λ-information is created by the exercise of the option to learn the θ-information. To put it differently, the investment into the θ-information creates the opportunity to invest into the λ-information.

For brevity, we refer to the learning options as the λ-*option* and the θ-*option*. The respective acquisition triggers, or learning triggers at which the options are exercised in exchange for the value of information, are referred to as the λ-*trigger* and the θ-*trigger*.

The policies are found in two stages as follows:

1. Find the value of information (λ-information and θ-information);
2. Employ standard dynamic programming to find the optimal acquisition policy (λ-trigger and θ-trigger).

Policy for investment-cost information. Since λ is unobserved at this stage, the ex-ante value of the investment opportunity is given by

$$W^{-\lambda}(X;\theta) = \begin{cases} F^{-\lambda}(X;\theta), & X < X^*(\theta,\mu_\lambda) \\ V^{-\lambda}(X;\theta), & X \geq X^*(\theta,\mu_\lambda), \end{cases}$$

where

$$F^{-\lambda}(X;\theta) = F^*(X;\theta,\mu_\lambda) = \frac{1}{\beta-1}\left(\frac{X}{X^*(\theta,\mu_\lambda)}\right)^\beta \mu_\lambda I,$$

and

$$V^{-\lambda}(X;\theta) = V^*(X;\theta,\mu_\lambda) = \theta X - \mu_\lambda I.$$

Value of information. The value of the λ-information is of the form

$$I^\lambda(X;\theta) = G^{-\lambda}(X;\theta) - W^{-\lambda}(X;\theta),$$

where $G^{-\lambda}$ denotes the expected ex-post value of the investment opportunity. The acquisition problem thus reads as

$$W^\lambda(X;\theta) = \sup_\tau \mathbf{E}_X[e^{-r\tau}V^\lambda(X(\tau);\theta)], \tag{2}$$

where

$$V^\lambda(X;\theta) = I^\lambda(X;\theta) - A_\lambda,$$

and the supremum is taken over the set of all stopping times adapted to the shock process. For $W^\lambda < 0$ there does not exist a feasible acquisition policy for the λ-information at all. The investor then scraps the acquisition flexibility and sticks to the prior estimate μ_λ in making the investment decision. This occurs for example when A_λ is too large or when μ_λ is sufficiently accurate, as will be seen shortly.

Since the investor ex post observes both θ and λ, $G^{-\lambda}$ derives from the investment policy implied by the perfect-information trigger

$$X^*(\theta,\lambda) = \frac{\beta}{\beta-1}\frac{\lambda I}{\theta}.$$

From the fact that $X^*(\theta,\lambda)$ decreases with λ, we assert that for all X there exists a unique threshold $\lambda^*(X)$ such that upon learning $\lambda < \lambda^*(X)$ the investor goes simultaneously ahead with the investment, while upon learning $\lambda > \lambda^*(X)$ he delays by keeping the investment option. Since going ahead with the investment is now optimal if

$$X \geq \frac{\beta}{\beta-1}\frac{\lambda I}{\theta},$$

we get on inverting that

$$\lambda^*(X) = \lambda^*(X;\theta) = \frac{\beta-1}{\beta}\frac{\theta X}{I}. \tag{3}$$

For $X < X^*(\theta,\mu_\lambda)$ it thus holds that

$$V^\lambda(X;\theta) = G^{-\lambda}(X;\theta) - F^{-\lambda}(X;\theta) - A_\lambda,$$

and for $X \geq X^*(\theta,\mu_\lambda)$ that

$$V^\lambda(X;\theta) = G^{-\lambda}(X;\theta) - V^{-\lambda}(X;\theta) - A_\lambda,$$

where

$$G^{-\lambda}(X;\theta) = \frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*(X;\theta)} V^*(X;\theta,u)g(u)du + \tag{4}$$

$$\frac{1}{|\Lambda|}\int_{\lambda^*(X;\theta)}^{\Lambda^+} F^*(X;\theta,u)g(u)du. \tag{5}$$

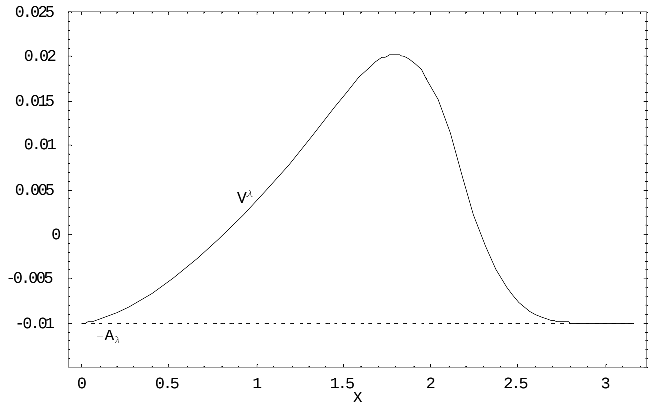

Figure 1: net value of λ-information

Figure 1 illustrates V^λ when $\Lambda = [1,2]$ and the other parameters are

$$(\sigma, \delta, \theta, a_\lambda, I) = (0.1, 0.02, 2, 0.01, 1).$$

We see that

$$V^\lambda(X;\theta) \to -A_\lambda,$$

as $X \to 0$ or $X \to \infty$. The former limit reflects the fact that the λ-information becomes worthless when the value of the underlying investment opportunity vanishes with the shock: since we have for all $(\theta, \lambda) \in \Theta \times \Lambda$ that $V^*(X;\theta,\lambda) \to 0$ as $X \to 0$, we must also have for all θ that $I^\lambda(X;\theta) \to 0$ as $X \to 0$. The latter limit reflects the fact that the λ-information becomes worthless when the value of the investment opportunity grows unboundedly with the shock: since we have for all $(\theta, \lambda) \in \Theta \times \Lambda$ that $V^*(X;\theta,\lambda) \to \infty$ as $X \to \infty$, we must also have for all θ that $I^\lambda(X;\theta) \to 0$ as $X \to \infty$.

Acquisition trigger. We can rewrite W^λ as

$$W^\lambda(X;\theta) = \begin{cases} F^\lambda(X;\theta), \ X < X_\lambda(\theta) \\ V^\lambda(X;\theta), \ X_\lambda(\theta) \le X \le X_*(\theta), \end{cases}$$

where F^λ is interpreted as the value of the λ-option, $X_\lambda(\theta)$ denotes the yet unknown λ-trigger for a given θ and

$$X_*(\theta) = \arg\max_X V^\lambda(X;\theta).$$

Now let $X < X_*(\theta)$, so that X lies in the increasing part of V^λ. The optimal acquisition policy and the solution to the problem (2) is then represented by the stopping time

$$\tau^\lambda(X;\theta) = \inf\{t \geq 0 : X_\lambda(\theta) \leq X \leq X_*(\theta), X(0) = X\}.$$

The option value is governed by the stochastic Bellman equation

$$\mathcal{A}F^\lambda - rF^\lambda = 0,$$

subject to the boundary condition $F^\lambda(0;\theta) = 0$. Thus,

$$F^\lambda(X;\theta) = B_\lambda(\theta)X^\beta,$$

where $B_\lambda(\theta)$ is an unknown constant. The λ-trigger is by construction related to $X^*(\theta,\lambda)$ as

$$X_\lambda(\theta) = X^*(\theta,\lambda^*) = \frac{\beta}{\beta-1}\frac{\lambda^* I}{\theta}, \tag{6}$$

where

$$\lambda^* = \lambda^*(X_\lambda)$$

denotes the λ-threshold upon the optimal acquisition for which the investor simultaneously goes ahead with the investment.

The above leaves us with the unknowns X_λ and B_λ. For transparency and ease of argumentation, we introduce a two-step approach to find them in a closed form. The first step treats λ^* as exogenous and employs the standard dynamic-programming procedure to determine X_λ as a function of λ^*, $X_\lambda(\lambda^*;\theta)$. The second step takes the X_λ from equation (6) and sets the two X_λ at hand equal to solve for λ^*. Having done this, it remains to substitute λ^* back into (6). The two-step approach will prove particularly instrumental for finding the θ-trigger.

Step 1. Fix $\lambda^* > 0$ as an exogenous constant and let W_0^λ denote an auxiliary value function obtained from W^λ by restricting $\lambda^*(X) = \lambda^*$ for all X,

$$W_0^\lambda(X) = W^\lambda(X)|_{\lambda(X)=\lambda^*}.$$

Then $X_\lambda = X_\lambda(\lambda^*; \theta)$ follows from the standard dynamic-programming procedure of making W_0^λ continuously differentiable: both a value-matching and a high-contact condition must hold at X_λ. Keeping the notation for the λ-option, the value-matching condition reads as

$$F^\lambda(X_\lambda; \theta) = V_0^\lambda(X_\lambda; \theta), \tag{7}$$

where V_0^λ is the restricted net value of the λ-information, given by

$$V_0^\lambda(X) = \frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} V^*(X; \theta, u)g(u)du + \frac{1}{|\Lambda|}\int_{\lambda^*}^{\Lambda^+} F^*(X; \theta, u)g(u)du -$$
$$W^{-\lambda}(X; \theta) - A_\lambda.$$

The high-contact condition reads as

$$\frac{\partial F^\lambda}{\partial X}(X_\lambda; \theta) = \frac{\partial V_0^\lambda}{\partial X}(X_\lambda; \theta). \tag{8}$$

Since we have from Appendix A.1 that V^λ slopes downwards at $X^*(\theta, \mu_\lambda)$, it must also be that

$$X_\lambda < X^*(\theta, \mu_\lambda).$$

The conditions are thus solved for X_λ with V^λ involving

$$W^{-\lambda}(X; \theta) = F^{-\lambda}(X; \theta).$$

Appendix A.2 shows that

$$X_\lambda = X_\lambda(\lambda^*; \theta) = \frac{\beta}{\beta - 1}\frac{(\lambda^*)^2 - (\Lambda^-)^2 + 2|\Lambda|a_\lambda}{2\theta(\lambda^* - \Lambda^-)}I, \tag{9}$$

where it remains to find λ^*.

Step 2. The obtained $X_\lambda(\lambda^*; \theta)$ is λ-trigger conditional on that simultaneous investment is ex post optimal for $\lambda > \lambda^*$. For consistency it must therefore be that

$$X_\lambda(\lambda^*;\theta) = X^*(\theta,\lambda^*),$$

or

$$\frac{\beta}{\beta-1}\frac{(\lambda^*)^2-(\Lambda^-)^2+2|\Lambda|a_\lambda}{2\theta(\lambda^*-\Lambda^-)}I = \frac{\beta}{\beta-1}\frac{\lambda^* I}{\theta},$$

which collapses to the fixed-point expression

$$\lambda^* = \frac{(\lambda^*)^2-(\Lambda^-)^2+2|\Lambda|a_\lambda}{2(\lambda^*-\Lambda^-)}. \tag{10}$$

Rearranging this results in a quadratic equation for λ^* as

$$(\lambda^*)^2 - 2(\Lambda^-)\lambda^* + ((\Lambda^-)^2 - 2|\Lambda|a_\lambda) = 0. \tag{11}$$

The equation has the roots $\Lambda^- \pm \sqrt{2|\Lambda|a_\lambda}$. Now, since economic uncertainty induces an option value for delaying, the investor will never go ahead with a costly acquisition (exercise the λ-option) until there is a possibility that the learnt λ is immediately useful for the optimal investment policy, as implied by $X^*(\theta,\lambda)$. For $A_\lambda > 0$ there must in other words be a positive probability that simultaneous investment is optimal ex post. This imposes

$$X_\lambda(\theta) = \frac{\beta}{\beta-1}\frac{\lambda^* I}{\theta} > \frac{\beta}{\beta-1}\frac{\Lambda^- I}{\theta} = X^*(\theta,\Lambda^-),$$

so that we select

$$\lambda^* = \Lambda^- + \sqrt{2|\Lambda|a_\lambda}.$$

Substituting this back into (6) now gives the λ-trigger as

$$X_\lambda = X_\lambda(\theta) = \frac{\beta}{\beta-1}\frac{\Lambda^- + \sqrt{2|\Lambda|a_\lambda}}{\theta}I.$$

The somewhat less transparent equivalent approach is to determine X_λ in one step by employing the dynamic-programming procedure directly with W^λ instead of indirectly with W_0^λ. Appendix A.3 provides the technical details. Figure 2 illustrates the determination of X_λ for the above parameters. Since then

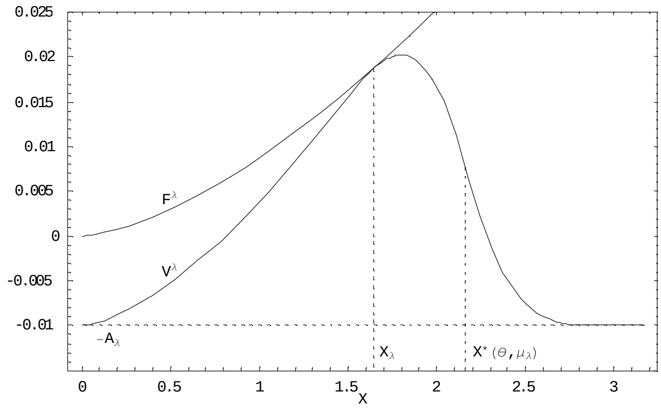

Figure 2: optimal acquisition trigger for λ

$X_\lambda = 1.696 < X^*(\theta, \mu_\lambda) = 2.228$ and $\lambda^* = \lambda^*(X_\lambda) = 1.141$, the investor opts to acquire the λ-information the moment the shock hits 1.696 (at time τ^λ), and goes ahead with the investment simultaneously should he then learn $\lambda < 1.141$. Should the investor to the contrary learn $\lambda > 1.141$, he opts to delay by keeping the investment option until the shock has climbed further to $X^*(\theta, \lambda) > X_\lambda$.

Comparative statics. We now proceed to the comparative statics of X_λ to gain further insight into the optimal acquisition policy for the λ-information. Since $\lambda^* = \Lambda^-$ for $a_\lambda = 0$,

$$X_\lambda(\theta) = X^*(\theta, \Lambda^-) = \frac{\beta}{\beta - 1} \frac{\Lambda^-}{\theta} I.$$

The interpretation is that in the absence of acquisition costs the optimal acquisition policy represented by τ^λ involves no risk of learning λ-information that would have been more beneficial when incorporated into the investment decision earlier. The optimal acquisition-investment policy in other words becomes indistinguishable from the irreversible-investment problem studied by Pindyck [1991] in that there is no risk of ex post forfeiting the optimal perfect-information investment. For $a_\lambda > 0$ the investor delays the acquisition beyond $X^*(\theta, \Lambda^-)$ in reflection of the fact that the return on the acquisition cost must be sufficiently large to compensate for the value of the λ-option as an opportunity cost. The return is given by

$$\frac{V^\lambda(X_\lambda; \theta)}{A_\lambda} = \frac{1}{\beta - 1} \frac{C_\lambda}{a_\lambda}.$$

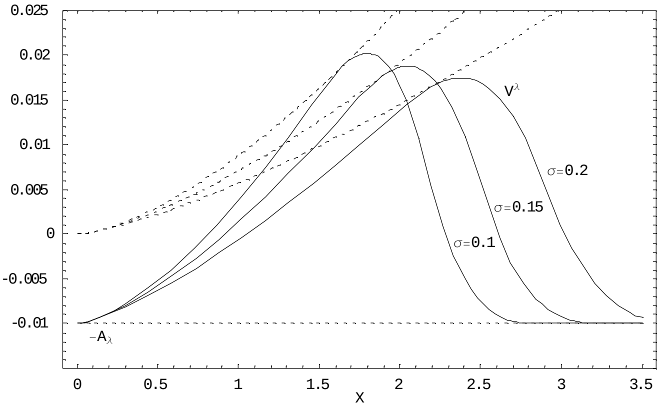

Figure 3: the effect of the payoff-shock volatility

The effect of economic uncertainty on the optimal acquisition policy is conveyed by the option multiple $\beta/(\beta-1) > 1$. Since the multiple increases in the shock volatility σ, so does X_λ. This reflects on one hand (i) the delaying-option function of the λ-option under irreversibility, and on the other hand (ii) a positive feedback from the ex-post investment decision. First, as a larger σ increases the uncertainty over the value of the λ-information, it makes delaying the acquisition more attractive due to an asymmetry between upside and downside risk. Second, as a larger σ also delays the ex-post investment for all λ, it makes the preceding acquisition less urgent. Figure 3 illustrates the relationship between X_λ and σ. From the points where the F^λ-graphs (dashed curves) join smoothly onto the V^λ-graphs we confirm that X_λ increases in σ. The payoff signal θ has the opposite effect due to a negative feedback from the ex-post investment decision. The support Λ affects the optimal acquisition policy through (i) the expected investment cost $\mu_\lambda I$ and (ii) the accuracy of the prior estimate μ_λ on λ. First, increasing μ_λ such that $|\Lambda|$ is controlled for (shifting Λ^- and Λ^+ to the right by an equal amount) results in a strictly larger X_λ: keeping the accuracy fixed, a larger μ_λ delays the acquisition since in expected terms there is less urgency for the ex-post investment. Second, increasing $|\Lambda|$ such that μ_λ is controlled for (taking a mean-preserving spread in Λ) results in a strictly lower X_λ: keeping μ_λ fixed, a larger $|\Lambda|$ advances the acquisition since the return on acquisition cost increases as μ_λ gets less accurate. Technically this follows from the convexity in λ of the investment-option value F^*. Since a mean-preserving spread in Λ by Jensen's inequality increases the expected ex-post value of F^* for all X, also V^λ/A_λ increases. Figure 4 illustrates the relationship between the accuracy and V^λ by taking three mean-preserving spreads in Λ.

As X_λ decreases with decreasing $|\Lambda|a_\lambda$, the comparative statics appears to suggest counterintuitively that the acquisition is urgent when the estimate μ_λ is accurate and the acquisition cost is relatively large. This is but a technical particularity as the dynamic-programming procedure results in a positive X_λ also when the value of λ-information is everywhere negative. We indeed have

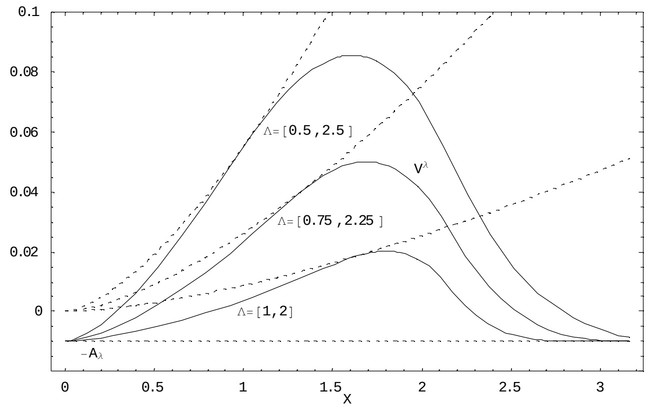

Figure 4: the effect of the prior-estimate accuracy

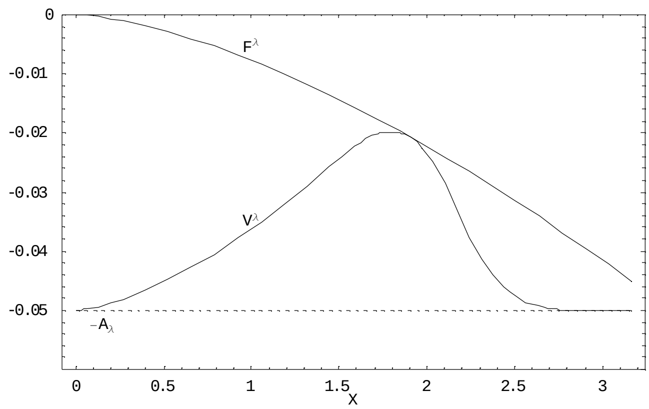

Figure 5: λ-option scrapped

$V^\lambda(X_\lambda;\theta) < 0$ and thus $F^\lambda(X_\lambda;\theta) < 0$ when $|\Lambda|/a_\lambda$ is sufficiently small. This is illustrated by Figure 5 when $a_\lambda = 0.05$ and the other parameters are as above. The investor then scraps the λ-option and sticks to μ_λ in making the investment decision.

Value functions. Appendix A.4 employs λ^* and X_λ to show that

$$V^\lambda(X_\lambda(\theta);\theta) = \frac{1}{\beta - 1} C_\lambda I,$$

and

$$F^\lambda(X;\theta) = \left(\frac{X}{X_\lambda(\theta)}\right)^\beta V^\lambda(X_\lambda(\theta);\theta) = \frac{1}{\beta-1}\left(\frac{X}{X_\lambda(\theta)}\right)^\beta C_\lambda I,$$

where

$$C_\lambda = 2a_\lambda + (\Lambda^-)\sqrt{\frac{2a_\lambda}{|\Lambda|}} + \frac{1}{2-\beta}\frac{(\lambda^*/\Lambda^+)^\beta(\Lambda^+)^2 - (\lambda^*)^2}{|\Lambda|} - \qquad (12)$$

$$\left(\frac{\lambda^*}{\mu_\lambda}\right)^\beta \mu_\lambda.$$

The value of the opportunity to acquire the λ-information prior to investing thus becomes

$$W^\lambda(X;\theta) = \begin{cases} \frac{1}{\beta-1}\left(\frac{X}{X_\lambda(\theta)}\right)^\beta C_\lambda I, & X < X_\lambda(\theta) \\ \frac{1}{\beta-1}C_\lambda I, & X = X_\lambda(\theta) \\ V^\lambda(X), & X_\lambda(\theta) < X \leq X_*(\theta), \end{cases}$$

where the exact form of V^λ can be located in Appendix A.1.

Acquisition policy. For parameter configurations for which $C_\lambda < 0$, the investor scraps the λ-option (the opportunity to acquire the λ-information) and sticks to the estimate μ_λ since then $F^\lambda(X;\theta) < 0$ ($W^\lambda(X_\lambda;\theta) < 0$). This is the scenario illustrated previously by Figure 5. Scrapping most notably does not take place when μ_λ is sufficiently inaccurate and/or when the cost factor a_λ is sufficiently low. Then $C_\lambda \geq 0$ which is equivalent to $X_\lambda \leq X_*$.

Appendix B goes on to establish the instrumental result that the relationship between X_λ and X_* remains fixed as θ varies: for all θ,

$$X_*(\theta) = qX_\lambda(\theta),$$

where q solves

$$\left(\mu_\lambda^{1-\beta}|\Lambda| - \frac{1}{2-\beta}(\Lambda^+)^{2-\beta}\right)(\lambda^*)^{\beta-1}q^{\beta-1} + \frac{\beta-1}{2-\beta}\lambda^* q + \Lambda^- = 0. \qquad (13)$$

This enables us to expand on scrapping as

Lemma 1 *No-scrapping condition. A sufficient (and also necessary) condition for the λ-option not be scrapped by the investor is that*

$$\left(\mu_\lambda^{1-\beta}|\Lambda| - \frac{1}{2-\beta}(\Lambda^+)^{2-\beta}\right)(\lambda^*)^{\beta-1} + \frac{\beta-1}{2-\beta}\lambda^* + \Lambda^- \leq 0,$$

for $|\Lambda| > 0$

Proof. Since

$$C_\lambda \geq 0 \Leftrightarrow X_\lambda(\theta) \leq X_*(\theta),$$

and $X_* = qX_\lambda$, the inequality follows from first setting $q = 1$ in equation (13) and then requiring that $q \geq 1$. ■

Summing up the above, we are now ready to state the optimal acquisition policy for the λ-information.

Proposition 1 *Suppose that the no-scrapping condition holds. In the examined investment environment the optimal acquisition policy for the λ-information is then represented by the stopping time*

$$\tau^\lambda(X;\theta) = \inf\{t \geq 0 : X_\lambda(\theta) \leq X(t) \leq X_*(\theta), X(0) = X\},$$

where

$$X_\lambda(\theta) = \frac{\beta}{\beta-1}\frac{\lambda^*}{\theta}I = \frac{\beta}{\beta-1}\frac{\Lambda^- + \sqrt{2|\Lambda|a_\lambda}}{\theta}I,$$

If the condition does not hold then the investor scraps the λ-option and sticks to the prior estimate μ_λ in making the investment decision ($\tau^\lambda = \infty$).

The table below finally computes X_λ when σ, a_λ and θ vary and the other parameters are as above.

σ	a_λ	θ	X_λ
0.1	0	1	2.883
0.2	0.01	1.5	2.930
0.3	0.02	2	3.166
0.4	0.03	2.5	3.561
0.5	0.04	3	4.057

Acquisition of investment-payoff information. Taking Lemma 1 as given, we now move backwards in the acquisition sequence to establish the optimal acquisition policy for the θ-information. Since also λ is unobserved, the ex-ante value of the investment opportunity is given by

$$W^{-(\theta,\lambda)}(X) = \begin{cases} F^{-(\theta,\lambda)}(X), & X < X^*(\mu_\theta, \mu_\lambda) \\ V^{-(\theta,\lambda)}(X), & X \geq X^*(\mu_\theta, \mu_\lambda), \end{cases}$$

where

$$F^{-(\theta,\lambda)}(X) = \frac{1}{\beta - 1} \left(\frac{X}{X^*(\mu_\theta, \mu_\lambda)} \right)^\beta \mu_\lambda I,$$

and

$$V^{-(\theta,\lambda)}(X) = \mu_\theta X - \mu_\lambda I.$$

The optimal ex-post policy generally has four alternatives:

1. Postpone the acquisition of the λ-information prior to investing (keep both the λ-option and the investment option);
2. Acquire the λ-information simultaneously and postpone the investment (exercise the λ-option, and given the learnt λ keep the investment option);
3. Acquire the λ-information and invest simultaneously (exercise the λ-option, and given the learnt λ exercise also the investment option);
4. Go ahead with the investment without acquiring the λ-information (stick to the prior estimate μ_λ).

The first three cases can be labeled in heuristic terms as "learn low θ", "learn first high θ and then high λ" and "learn first high θ and then low λ". The fourth case describes situations where the investor either scraps the λ-option straight out or finds it optimal to leave the λ-option unexercised in response to the resolution of uncertainty.

Auxiliary assumptions. Supposing that the no-scrapping condition holds for the λ-option, the θ-option involves multi-level compoundness in that the investor upon learning θ holds a sequence of two options, with the λ-option involving compoundness through the perfect-information investment option. Since the investor is not forced to acquire the λ-information prior to investing, the sequence represents intraproject compoundness with the embedded flexibility to stick to the prior estimate μ_λ in making the investment decision.

For making the analysis more interesting, we assume that the λ-option is not scrapped straight out:

(A3) The no scrapping-condition of Lemma 1 holds for the λ-option, so that

$$C_\lambda \geq 0 \Leftrightarrow X_\lambda(\theta) \leq X_*(\theta),$$

where C_λ is given as in equation (12).

For tractability and ease of argumentation, we also restrict attention to a special case of the support Λ:

(A4) The support Λ has $\Lambda^- = 0$ and thus $|\Lambda| = \Lambda^+$.

It follows that

$$X_\lambda(\theta) = \frac{\beta}{\beta - 1} \frac{\sqrt{2|\Lambda| a_\lambda}}{\theta} I,$$

and from equation (13) that

$$q = \left(\frac{1-\beta}{2^{\beta-1}(2-\beta)-1} \right)^{\frac{1}{\beta-2}} \frac{\sqrt{2}}{2} \sqrt{\frac{|\Lambda|}{a_\lambda}}. \tag{14}$$

The maximiser X_* collapses to

$$X_*(\theta) = \left(\frac{1-\beta}{2^{\beta-1}(2-\beta)-1} \right)^{\frac{1}{\beta-2}} \frac{\beta}{\beta - 1} \frac{|\Lambda|}{\theta} I,$$

and the no-scrapping condition assumed by (A3) to

$$\frac{|\Lambda|}{a_\lambda} \geq 2 \left(\frac{1-\beta}{2^{\beta-1}(2-\beta)-1} \right)^{\frac{2}{2-\beta}},$$

where we recall that $|\Lambda|$ proxies the accuracy of the estimate μ_λ.

From the fact that X_λ is inversely related to θ, we assert that for all X there exists a θ-threshold $\theta^*(X)$ such that upon learning $\theta < \theta^*(X)$ the investor opts to delay the acquisition of the λ-information by time τ^{λ}. Since the investor is indifferent about the acquisition if

$$X = X_{\lambda}(\theta) = \frac{\beta}{\beta - 1} \frac{\lambda^* I}{\theta},$$

we get on inverting that

$$\theta^*(X) = \frac{\beta}{\beta - 1} \frac{\lambda^* I}{X} = \frac{\beta}{\beta - 1} \frac{\sqrt{2|\Lambda| a_{\lambda}}}{X} I.$$

Moreover, the θ-trigger is related to X_{λ} as

$$X_{\theta} = X_{\lambda}(\theta^*) = \frac{\beta}{\beta - 1} \frac{\lambda^*}{\theta^*} I,$$

where

$$\theta^* = \theta^*(X_{\theta})$$

is the yet unknown θ-threshold applicable upon the optimal acquisition. Note the analogy to $X_{\lambda}(\theta) = X^*(\theta, \lambda^*)$.

Now suppose that the investor has forfeited the optimal acquisition of the λ-information through learning $\theta > \theta^*(X)$ ($X > X_{\lambda}(\theta)$). The optimal ex-post policy then depends on whether $X < X_*(\theta)$ or $X > X_*(\theta)$. Since for $X < X_*(\theta)$ the value of the λ-information slopes upwards, the optimal ex-post policy is to go ahead with the acquisition of the λ-information simultaneously: $\tau^{\lambda}(X; \theta) = 0$. The return on the acquisition cost then satisfies

$$\frac{V^{\lambda}(X; \theta)}{A_{\lambda}} > \frac{V^{\lambda}(X_{\lambda}; \theta)}{A_{\lambda}} = \frac{1}{\beta - 1} \frac{C_{\lambda}}{a_{\lambda}}.$$

For $X > X_*(\theta)$ the value of the λ-information slopes downwards and $\tau^{\lambda}(X; \theta)$ is rendered inapplicable. This reflects the fact that the optimal acquisition policy for the λ-information conditional on the shock starting at $X(0) < X_*(\theta)$ is fundamentally different from the policy conditional on the shock starting at $X(0) > X_*(\theta)$. For $X(0) > X_*(\theta)$ the possible acquisition of the λ-information occurs either simultaneously or when the shock hits the applicable λ-trigger from above, instead of from below[1]. From this we go on to argue that upon learning $\theta > \theta^*(X)$ for $X > X_*(\theta)$ the ex-post policy has three possibilities. Firstly, if X lies above $X^*(\theta, \mu_\lambda)$ then the investor goes ahead with the investment simultaneously, sticking to the prior estimate μ_λ. Secondly, if X lies below the applicable λ-trigger then the investor goes ahead with the acquisition of the λ-information simultaneously. Finally, if X lies between the applicable λ-trigger and $X^*(\theta, \mu_\lambda)$ then the investor opts to delay by keeping both the λ-option and the investment option. It follows that along the trajectories where the shock drops to the applicable λ-trigger before climbing to $X^*(\theta, \mu_\lambda)$, the investor is observed to acquire the λ-information prior to going ahead with the investment. Along the trajectories where the shock climbs to $X^*(\theta, \mu_\lambda)$ first, the investor is observed to stick to the prior estimate.

For further tractability, we rule out the above three ex-post scenarios by assuming that there is no possibility of V^λ sloping downwards upon the optimal acquisition of the θ-information at X_θ. Since τ^λ is then applicable, the optimal ex-post policy restricts to two alternatives: the investor either goes ahead with the acquisition of the λ-information simultaneously ("high θ") or delays the acquisition by keeping the λ-option for time $\tau^\lambda > 0$ ("low θ"). Note that investment prior to the acquisition of the λ-information is ruled out by construction, as $X_\lambda(\theta) < X^*(\theta, \mu_\lambda)$ for all θ.

Since X_* is inversely related to θ, the assumption becomes

$$X_\theta = \frac{\beta}{\beta - 1}\frac{\lambda^*}{\theta^*} I \leq q \frac{\beta}{\beta - 1}\frac{\lambda^*}{\Theta^+} I = X_*(\Theta^+),$$

or $\theta^* \geq \Theta^+/q$. From equation (14) then:

(A5) The θ-threshold θ^* satisfies

$$\theta^* \geq \left(\frac{1-\beta}{2^{\beta-1}(2-\beta) - 1}\right)^{\frac{1}{2-\beta}} (\Theta^+) \sqrt{\frac{2a_\lambda}{|\Lambda|}},$$

where $|\Lambda| = \Lambda^+$.

[1]For $X(0) > X_*(\theta)$, $X_\lambda(\theta)$ follows from employing the dynamic-programming procedure with

$$F_\lambda(X; \theta) = B_\lambda(\theta) X^\alpha,$$

where

$$\alpha = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + 2\frac{r}{\sigma^2}} < 0.$$

Although the only modification is to replace X^β with X^α, the resulting $X_\lambda(\theta)$ is significantly different from the one obtained above.

The lower bound for X_θ follows directly from economic considerations. Since the intraproject compound function of the θ-option under (A3) is to create the λ-option and uncertainty induces an option value for delaying, the investor will never go ahead with a costly acquisition (exercise the θ-option) before there is a positive probability that the discovered θ calls for simultaneous acquisition of the λ-information (exercise of the λ-option). This imposes

$$X_\theta = \frac{\beta}{\beta - 1}\frac{\lambda^*}{\theta^*} I \geq \frac{\beta}{\beta - 1}\frac{\lambda^*}{\Theta^+} I = X_\lambda(\Theta^+),$$

and thus in conjunction with assumption (A5),

$$\left(\frac{1-\beta}{2^{\beta-1}(2-\beta)-1}\right)^{\frac{1}{2-\beta}} (\Theta^+)\sqrt{\frac{2a_\lambda}{|\Lambda|}} \leq \theta^* \leq \Theta^+.$$

The upper bound holds with equality for $a_\theta = 0$ since in the absence of acquisition costs the optimal acquisition policy must involve no risk of ex post forfeiting the optimal acquisition of the λ-information: $X_\theta|_{a_\theta=0} = X_\lambda(\Theta^+)$. We return to this point in more detail below.

The following remark is now obvious.

Remark 1 *Since $\theta^* = \Theta^+$ for $a_\theta = 0$ and θ^* must be inversely related to a_θ to make X_θ increase in a_θ, assumption (A5) can always be enforced by selecting a sufficiently small a_θ.*

Figure 6 illustrates the bounds for $\Lambda = [0, 2]$ and

$$(\Theta^+, \sigma, r, \delta, a_\lambda, I) = (3, 0.1, 0.05, 0.02, 0.01, 1).$$

The dashed curves plot V^λ for two ex-post scenarios $\theta < \Theta^+$. The bounds are as in the table below.

q	3.449
$X_\lambda(\Theta^+)$	0.192
$X_*(\Theta^+)$	0.663
Θ^+/q	0.870

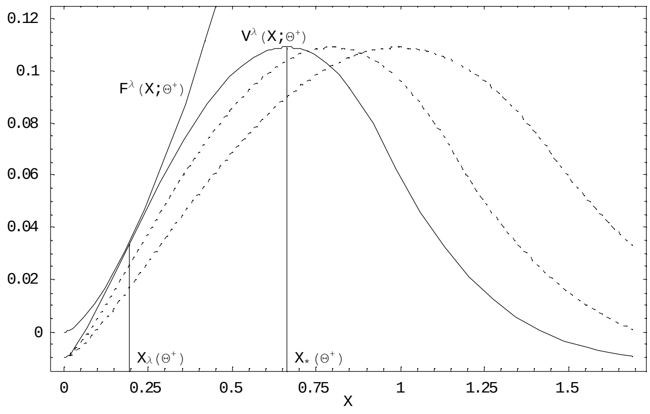

Figure 6: lower and upper bounds for X_θ

Value of information. Equipped with the auxiliary assumptions (A3)-(A5), we now proceed to find the value of the θ-information. As with the λ-information, the value is found by moving backwards from the optimal ex-post policy.

The value is of the form

$$I^\theta(X) = G^{-(\theta,\lambda)}(X) - W^{-(\theta,\lambda)}(X),$$

where $G^{-(\theta,\lambda)}$ denotes the expected ex-post value of the compound opportunity to acquire the λ-information and invest. The acquisition problem thus reads as

$$W^\theta(X) = \sup_\tau \mathbf{E}_X(e^{-r\tau} V^\theta(X(\tau))), \tag{15}$$

where

$$V^\theta(X) = I^\theta(X) - A_\theta,$$

and the supremum is taken over the set of all stopping times adapted to the shock process.

Suppose first that the investor has learnt $\theta < \theta^*(X)$. Since the optimal ex-post policy then is to delay the acquisition of the λ-information by keeping the λ-option for time τ^λ, the ex-post value of the compound opportunity becomes

$$H(X;\theta) = \mathbf{E}_X(e^{-r\tau^\lambda(X;\theta)}(G^{-\lambda}(X_\lambda(\theta);\theta) - A_\lambda)),$$

where $G^{-\lambda}(X;\theta)$ is given as in equation (4). Appendix C shows that

$$H(X;\theta) = \frac{1}{\beta-1} E_\theta \left(\frac{X}{X_\lambda(\theta)} \right)^\beta I,$$

where

$$\begin{aligned} E_\theta &= (\Lambda^-)\sqrt{\frac{2a_\lambda}{|\Lambda|}} + \frac{1}{2-\beta} \frac{(\lambda^*/\Lambda^+)^\beta (\Lambda^+)^2 - (\lambda^*)^2}{|\Lambda|} + 2a_\lambda \qquad (16) \\ &= \frac{2}{2-\beta} \left((\lambda^*/\Lambda^+)^{\beta-2} + 1 - \beta \right) a_\lambda, \end{aligned}$$

where the latter equation follows from $\Lambda^- = 0$ and $(\lambda^*)^2 = 2|\Lambda| a_\lambda$.

Weighting by the probability mass of $\theta < \theta^*(X)$ now gives the conditional expected ex-post value

$$G_1^{-(\theta,\lambda)}(X) = \frac{1}{|\Theta|} \int_{\Theta^-}^{\theta^*(X)} H(X;s) f(s) ds.$$

Suppose next that the investor has learnt $\theta > \theta^*(X)$. Since the optimal ex-post policy then is to simultaneously acquire the λ-information, the investor ends up acquiring perfect information on the investment opportunity at total cost $A_\theta + A_\lambda = (a_\theta + a_\lambda)I$. The investor simultaneously acts on the information and goes ahead with the investment (exercises the investment option) should the shock be above the discovered perfect-information investment trigger. This is $X > X^*(\theta,\lambda)$ or equivalently $\lambda < \lambda^*(X;\theta)$, where we recall that

$$\lambda^*(X;\theta) = \frac{\beta-1}{\beta} \frac{\theta X}{I}.$$

The rectangle

$$\{(s,u) : s > \theta^*(X),\ u\} \cap (\Theta \times \Lambda)$$

thus separates to two distinct regions representing the optimal investment decisions upon simultaneous acquisitions. As this result is critical to the analysis, we write

Lemma 2 *Should the investor simultaneously learn*

$$(\theta, \lambda) \in \mathcal{I}(X) = \{(s, u) : s > \theta^*(X),\, u < \lambda^*(X; s)\} \cap (\Theta \times \Lambda),$$

the payoff

$$V^*(X; \theta, \lambda) = \theta X - \lambda I$$

is sufficiently large for also going ahead with the investment simultaneously. Should the investor instead learn

$$(\theta, \lambda) \in \mathcal{D}(X) = \{(s, u) : s > \theta^*(X),\, u > \lambda^*(X; s)\} \cap (\Theta \times \Lambda),$$

the payoff falls short of the sufficient level and the investor delays by keeping the investment option for time $\tau^*(X; \theta, \lambda)$.

Figure 7 illustrates the two regions.

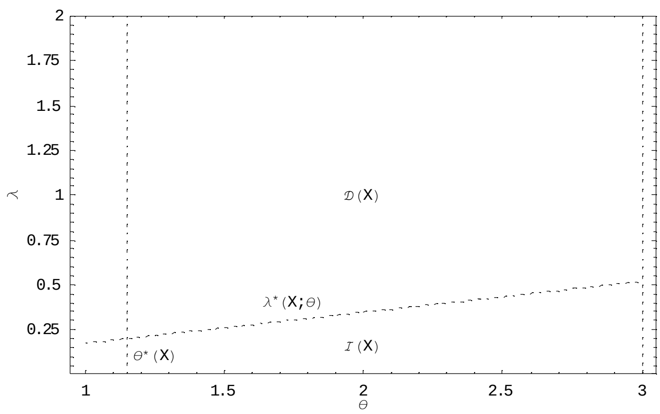

Figure 7: investment and delay regions

Weighting by the probability masses of the regions and recalling that the value of the investment option upon simultaneous acquisitions is

$$F^*(X; \theta, \lambda) = \frac{1}{\beta - 1} \left(\frac{X}{X^*(\theta, \lambda)} \right)^{\beta} \lambda I,$$

now gives the conditional expected ex-post value

$$G_2^{-(\theta,\lambda)}(X) = \frac{1}{|\Theta||\Lambda|}\int\int_{\mathcal{I}(X)} V^*(X;s,u)f(s)g(u)dsdu+$$

$$\frac{1}{|\Theta||\Lambda|}\int\int_{\mathcal{D}(X)} F^*(X;s,u)f(s)g(u)dsdu - \frac{1}{|\Theta|}\int_{\theta^*(X)}^{\Theta^+} A_\lambda f(s)ds.$$

The expected ex-post value of the compound opportunity follows as

$$G^{-(\theta,\lambda)}(X) = \sum_{j=1,2} G_j^{-(\theta,\lambda)},$$

and the value of the θ-information as

$$I^\theta(X) = \sum_{j=1,2} G_j^{-(\theta,\lambda)} - V^{-(\theta,\lambda)}(X),$$

for $X < X(\mu_\theta, \mu_\lambda)$ and

$$I^\theta(X) = \sum_{j=1,2} G_j^{-(\theta,\lambda)} - F^{-(\theta,\lambda)}(X),$$

for $X \geq X(\mu_\theta, \mu_\lambda)$.

Acquisition trigger. We can rewrite W^θ as

$$W^\theta(X) = \begin{cases} F^\theta(X), & X < X_\theta \\ V^\theta(X), & X = X_\theta, \end{cases}$$

where

$$F^\theta(X) = B_\theta X^\beta$$

is interpreted as the value of the θ-option. The optimal acquisition policy and the solution to the problem (15) is then represented by the stopping time

$$\tau^{\theta}(X) = \inf\{t \geq 0 : X(t) = X_{\theta}, X(0) = X\}.$$

We find X_{θ} by a two-step approach analogous to that applied for finding X_{λ}. The first step treats the threshold $\theta^* = \theta^*(X_{\theta})$ as exogenous and employs the standard dynamic-programming procedure to determine $X_{\theta} = X_{\theta}(\theta^*)$.

The second step employs

$$X_{\theta} = \frac{\beta}{\beta - 1}\frac{\lambda^* I}{\theta^*} \tag{17}$$

to solve for θ^*.

Step 1. Fix $\theta^* > 0$ as an exogenous constant. Since the perfect-information investment trigger X^* is related to X_{θ} as

$$X^*(\theta^*, \lambda^*) = X_{\lambda}(\theta^*) = X_{\theta},$$

the λ-threshold for investment upon optimal simultaneous acquisitions is then given by

$$\lambda^*(X^*(\theta^*, \lambda^*); \theta) = \lambda^{**}(\theta, \theta^*) = \frac{\lambda^*}{\theta^*}\theta.$$

From this the regions $\mathcal{I}$ and $\mathcal{D}$ follow as

$$\mathcal{I}^* = \{(s, u) : s > \theta^*,\ u < \lambda^{**}(s, \theta^*)\} \cap (\Theta \times \Lambda),$$

and

$$\mathcal{D}^* = \{(s, u) : s > \theta^*,\ u > \lambda^{**}(s, \theta^*)\} \cap (\Theta \times \Lambda).$$

Now let W_0^{θ} denote an auxiliary function obtained from W^{θ} by restricting

$$\theta^*(X) = \theta^*, \mathcal{I}(X) = \mathcal{I}^*, \mathcal{D}(X) = \mathcal{D}^*,$$

for all X. Keeping the notation for the θ-option, the corresponding value-matching condition reads as

$$F^\theta(X_\theta) = V_0^\theta(X_\theta), \tag{18}$$

where V_0^θ is the restricted net value of the θ-information, given by

$$V_0^\theta(X) = \frac{1}{|\Theta|}\int_{\Theta^-}^{\theta^*} H(X;s)f(s)ds + \frac{1}{|\Theta||\Lambda|}\int\int_{\mathcal{I}^*} V^*(X;s,u)f(s)g(u)dsdu +$$
$$\frac{1}{|\Theta||\Lambda|}\int\int_{\mathcal{D}^*} F^*(X;s,u)f(s)g(u)dsdu - \frac{1}{|\Theta|}\int_{\theta^*}^{\Theta^+} A_\lambda f(s)ds - A_\theta.$$

The high-contact condition reads as

$$\frac{\partial F^\theta}{\partial X}(X_\theta) = \frac{\partial V_0^\theta}{\partial X}(X_\theta). \tag{19}$$

Omitting here the tedious computations, we can show that the conditions solve for $X_\theta(\theta^*)$ as

$$X_\theta(\theta^*) = \frac{\beta}{\beta-1}\frac{2(\theta^*)^3 - (3/2)\phi(\theta^*)^2 - (1/2)(\Theta^+)^3}{(\theta^*)^3 - (\Theta^+)^3}\frac{\lambda^*}{\theta^*}I,$$

where

$$\phi = \Theta^+ + \frac{a_\theta}{a_\lambda}|\Theta| = \left(1 + \frac{a_\theta}{a_\lambda}\right)\Theta^+ - \frac{a_\theta}{a_\lambda}\Theta^-.$$

It now remains to find θ^*.

Step 2. The obtained $X_\theta(\theta^*)$ is the θ-trigger conditional on that (*i*) simultaneous acquisition of the λ-information is ex post optimal for $\theta > \theta^*$ and that (*ii*) simultaneous investment is then optimal for $\lambda < \lambda^{**}(\theta, \theta^*)$, as implied by X_λ. Since from equation (17) now

$$X_\theta(\theta^*) = \frac{2(\theta^*)^3 - (3/2)\phi(\theta^*)^2 - (1/2)(\Theta^+)^3}{(\theta^*)^3 - (\Theta^+)^3}X_\lambda(\theta^*),$$

it must thus be for consistency that

$$\frac{2(\theta^*)^3 - (3/2)\phi(\theta^*)^2 - (1/2)(\Theta^+)^3}{(\theta^*)^3 - (\Theta^+)^3} X_\lambda(\theta^*) = X_\lambda(\theta^*).$$

Dividing throughout by $(1/2)X^*(\theta^*, \lambda^*)$ and rearranging results in a third-order equation for θ^* as

$$2(\theta^*)^3 - 3\phi(\theta^*)^2 + (\Theta^+)^3 = 0. \tag{20}$$

Finally substituting $\lambda^* = \sqrt{2|\Lambda| a_\lambda}$ into equation (17), we get the θ-trigger for $\Lambda^- = 0$ as

$$X_\theta = \frac{\beta}{\beta - 1} \frac{\sqrt{2|\Lambda| a_\lambda}}{\theta^*} I.$$

Figure 8 illustrates the determination of X_θ together with the upper bound $X_*(\Theta^+)$ when $(a_\theta, \Theta^-) = (0.02, 1)$ and the other parameters are as above. Since then $X_\theta = 0.478 < X_*(\Theta^+) = 0.663$, the investor opts to acquire the θ-information the moment the shock hits 0.478 from below. Should the investor then learn $\theta < \theta^* = 1.205$, he delays the acquisition of the λ-information by keeping the λ-option for time $\tau^\lambda(X_\theta; \theta)$, until the shock has climbed further to $X_\lambda(\theta) > X_\theta$. Should the investor to the contrary learn $\theta > \theta^* = 1.205$, he simultaneously goes ahead with the acquisition of the λ-information. If it turns out that $\lambda < \lambda^{**}(\theta, \theta^*) = 0.166\theta$, so that $(\theta, \lambda) \in \mathcal{I}^*$, then the investor simultaneously acts on the obtained perfect information and goes ahead with the investment also. If it to the contrary turns out that $\lambda > \lambda^{**}(\theta, \theta^*)$, so that $(\theta, \lambda) \in \mathcal{D}^*$, then the investor delays by keeping the investment option for time $\tau^*(X_\theta; \theta, \lambda)$, until the shock has climbed further to $X^*(\theta, \lambda) > X_\theta$.

Remark 2 *In the general case $\Lambda^- \geq 0$ the solution for θ^* is more complex due to a number of inconvenient extra terms in the third order equation. From Appendix D we namely have that then*

$$\left(2\lambda^* - 3\Lambda^-\right)\lambda^*(\theta^*)^3 + 3((-\lambda^* + 2\Lambda^-)(\Theta^+)\lambda^* - 2|\Theta||\Lambda| a_\theta)(\theta^*)^2 -$$

$$3(\Theta^+)^2(\Lambda^-)\lambda^*\theta^* + (\Theta^+)^3(\lambda^*)^2 = 0,$$

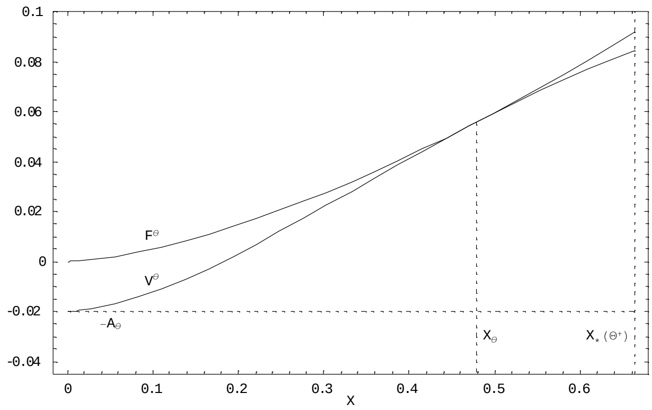

Figure 8: optimal acquisition trigger for θ

where

$$\lambda^* = \Lambda^- + \sqrt{2|\Lambda| a_\lambda}$$

For $\Lambda^- = 0$ *the equation collapses to (20).*

Comparative statics. Roughly, the comparative statics of X_θ separates to:

1. The effect of the θ-information parameters;
2. The effect of the shock parameters;
3. The effect of compoundness.

The comparative statics of X_θ with respect to the θ-information parameters replicates that of X_λ with respect to the corresponding λ-information parameters. For example, taking a mean-preserving spread in Θ will result in a strictly lower X_θ: keeping the estimate μ_θ fixed, a larger $|\Theta|$ advances the acquisition of the θ-information since the return on the acquisition cost increases as μ_θ gets less accurate. Similarly, the comparative statics of X_θ with respect to the shock parameters replicates that of X_λ. Most importantly, X_θ increases in the shock volatility σ through the option multiple $\beta/(\beta-1)$. This reflects on one hand (i) the delaying-option function of the θ-option under irreversibility, and on the other hand (ii) a positive feedback from the ex-post policy τ^λ to τ^θ. Since uncertainty makes the acquisition of the λ-information less urgent due to the delaying-option effect and a positive feedback from the perfect-information investment policy τ^*, it also makes the preceding acquisition of the θ-information less urgent.

Generally, letting "$\rightarrow$" stand for one-directional feedback, we have

$$\tau^* \ (X^*) \rightarrow \tau^\lambda \ (X_\lambda) \rightarrow \tau^\theta \ (X_\theta).$$

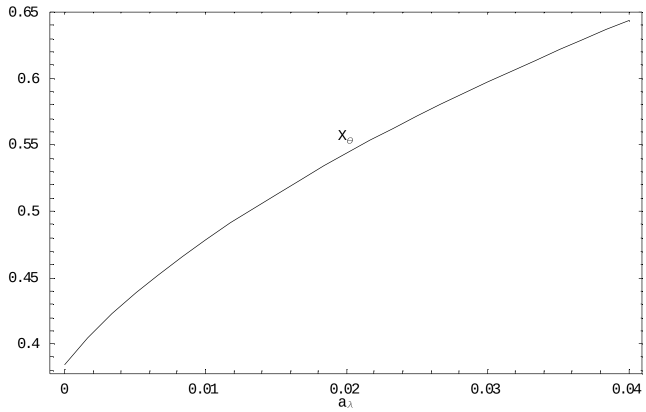

Figure 9: optimal trigger and a_λ (compoundness)

Note that τ^* has an indirect feedback effect on τ^θ through τ^λ.

The feedback from τ^λ to τ^θ is a manifestation of strict intraproject option compoundness as the λ-option is created by the exercise of the θ-option. The effect of this compoundness is most visible from the fact that X_θ depends on the λ-information parameters. As these parameters are changed, X_λ and X_θ move in parallel. For example, a larger a_λ results in a strictly higher X_θ. This is illustrated by Figure 9.

For $a_\theta = 0$ equation (20) collapses to

$$2(\theta^*)^3 - 3(\Theta^+)(\theta^*)^2 + (\Theta^+)^3 = 0,$$

which implies $\theta^* = \Theta^+$ and thus

$$X_\theta = X_\lambda(\Theta^+) = \frac{\beta}{\beta-1}\frac{\lambda^*}{\Theta^+}I.$$

The interpretation is that in the absence of acquisition costs the optimal acquisition policy represented by τ^θ involves no risk of learning θ-information that would have been more beneficial when incorporated into the acquisition policy τ^λ for the λ-information earlier: for $a_\theta = 0$ there is no risk of ex post forfeiting the optimal acquisition of the λ-information. For $a_\theta > 0$ the investor delays the acquisition beyond $X_\lambda(\Theta^+)$ in reflection of the fact that the return on the acquisition cost must be sufficiently large to compensate for the value of the θ-option as an opportunity cost.

Letting $a_\lambda \to 0$, we can finally show that

$$X_\theta \to \frac{\beta}{\beta-1}\frac{4}{3}\frac{\sqrt{|\Lambda| a_\theta}}{|\Theta|}I = \frac{\beta}{\beta-1}\frac{4}{3}\frac{\sqrt{\Lambda^+ a_\theta}}{|\Theta|}.$$

The limit gives the acquisition trigger for perfect information when the total acquisition cost amounts to $A_\theta = a_\theta I$ and $\Lambda^- = 0$.

Value functions. Omitting here the tedious computations, we can show that

$$F^\theta(X) = B_\theta X^\beta = \frac{1}{\beta - 1}\left(\frac{X}{X_\theta}\right)^\beta C_\theta I,$$

and

$$V^\theta(X_\theta) = F^\theta(X_\theta) = \frac{1}{\beta - 1} C_\theta I,$$

where

$$C_\theta = \frac{1-\beta}{2-\beta}\frac{2a_\lambda}{(\theta^*)^2}\frac{1}{|\Theta|}\int_{\theta^*}^{\Theta^+} s^2 f(s)ds + \frac{1}{\beta+1}\frac{1}{|\Theta|}\left(\theta^* - (\Theta^-)^{\beta+1}(\theta^*)^{-\beta}\right)E_\theta +$$
$$\frac{1}{(\beta+1)(2-\beta)}\frac{1}{|\Theta|}\left(\frac{\lambda^*}{\theta^*}\right)^\beta \left(\Lambda^+\right)^{-\beta+1}\left((\Theta^+)^{\beta+1} - (\theta^*)^{\beta+1}\right) - \left(\frac{\lambda^*\mu_\theta}{\theta^*\mu_\lambda}\right)^\beta \mu_\lambda,$$

and E_θ is given as in equation (16). The value of the opportunity to acquire the θ-information prior to acquiring the λ-information and investing thus becomes

$$W^\theta(X) = \begin{cases} \frac{1}{\beta-1} C_\theta \left(\frac{X}{X_\theta}\right)^\beta I, & X < X_\theta \\ \frac{1}{\beta-1} C_\theta I, & X = X_\theta. \end{cases}$$

The no scrapping-condition for the θ-option is $C_\theta \geq 0$. The condition is violated for example when the estimate μ_θ is sufficiently accurate and/or the cost factor a_θ is too large.

Acquisition policy. Summing up the above, we are ready to state the optimal acquisition policy for the θ-information.

Proposition 2 *Suppose that the acquisition of the θ-information induces the flexibility to acquire the λ-information and that the no-scrapping conditions for the λ-option holds and the θ-option hold. In the examined investment environment the optimal acquisition policy for the θ-information is then represented by the stopping time*

$$\tau^{\theta}(X) = \inf\{t \geq 0 : X(t) = X_{\theta}, X(0) = X\},$$

where

$$X_{\theta} = \frac{\beta}{\beta - 1} \frac{\sqrt{2|\Lambda| a_{\lambda}}}{\theta^*} I,$$

and θ^* *solves*

$$2(\theta^*)^3 - 3\phi(\theta^*)^2 + (\Theta^+)^3 = 0,$$

where $\phi = \Theta^+ + \frac{a_\theta}{a_\lambda}|\Theta|$.

The table below computes X_θ when σ and a_θ vary.

σ	a_θ	X_θ
0.1	0	0.192
0.2	0.01	0.513
0.3	0.02	0.876
0.4	0.03	1.374
0.5	0.04	2.037

Optimal acquisition-investment policy. From Proposition 1 and Proposition 2, we finally state the optimal compound acquisition-investment policy as follows.

Proposition 3 *Suppose that the no-scrapping conditions for the* λ*-option and the* θ*-option hold. In the examined investment environment the acquisition of the* θ*-information then occurs at time* $\tau^{\theta}(X(0))$. *Should the investor learn*

$$\theta \geq \theta^*,$$

he simultaneously goes ahead with the acquisition of the λ*-information. In the event that*

$$\lambda \leq \lambda^{**}(\theta) = \frac{\lambda^*}{\theta^*}\theta,$$

the investor simultaneously goes ahead with the investment ("high θ*, low* λ*"). In the event that* $\lambda > \lambda^{**}$ *("high* θ*, high* λ*"), the investor delays the investment until time*

$$\tau^{\theta}(X_0) + \tau^*(X_\theta; \theta, \lambda)$$

Should the investor at time $\tau^{\theta}(X_0)$ learn

$$\theta < \theta^*,$$

he delays the acquisition of the λ-information until time

$$\tau^{\theta}(X_0) + \tau^{\lambda}(X_\theta; \theta).$$

In the event that then

$$\lambda \leq \lambda^*(\theta),$$

the investor simultaneously goes ahead with the investment ("low θ, low λ"). In the event that $\lambda > \lambda^*(\theta)$ ("low θ, high λ"), he delays the investment until time

$$\tau^{\theta}(X_0) + \tau^{\lambda}(X_\theta; \theta) + \tau^*(X_\lambda(\theta); \theta, \lambda).$$

Figure 10 illustrates an optimal acquisition-investment policy when the shock starts at $X_0 = 0.45$ and the other parameters are as above. Along the plotted trajectory the investor goes ahead with the acquisition of the θ-information when the shock hits $X_\theta = 0.478$ at time $\tau^{\theta}(X_0) = 0.349$. Having then learned $\theta = 1.15 < \theta^* = 1.205$, the investor delays the acquisition of the λ-information until the shock hits $X_\lambda(\theta) = 0.501$ at time $\tau^{\theta}(X_0) + \tau^{\lambda}(X_\theta; \theta) = 0.405$. Having then obtained perfect information through learning $\lambda = 0.21 > \lambda^* = 0.20$, the investor goes ahead with the investment when the shock hits $X^*(\theta, \lambda) = 0.526$ at time $\tau^{\theta}(X_0) + \tau^{\lambda}(X_\theta; \theta) + \tau^*(X_\lambda(\theta); \theta, \lambda) = 0.635$.

4 Concluding remarks

This paper provides a real-options framework to expand on the valuation and optimal acquisition of investment-related information under economic uncertainty. The framework bases on the fact that incremental information leads to a more rational investment policy (investment timing) and thereby contributes in expected terms to the value of the investment opportunity. The expected contribution is the real-options interpretation for the value of information. Since the value fluctuates randomly with the value of the investment opportunity, the flexibility to acquire information prior to going ahead with the investment is in itself a specific real option, an option to learn at a sunk cost at any point of time.

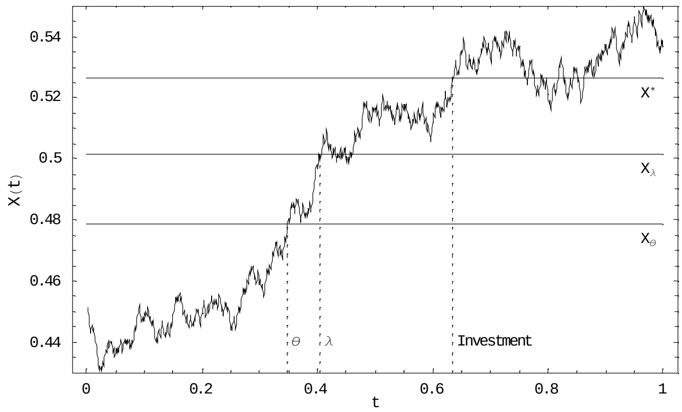

Figure 10: optimal acquisition-investment policy

Assuming that both the investment payoff and the investment cost involve an unobservable multiplicative signal, we introduce a two-step method to solve for the optimal acquisition policies in terms of two stopping times. The first stopping time represents the optimal acquisition policy for the payoff signal (payoff information), while the second stopping time represents the optimal acquisition policy for the cost signal (cost information). The first policy is rendered more elaborate by intraproject compoundness as it is also assumed that the cost signal becomes available only after the acquisition of the payoff signal. Simultaneous acquisitions along the lines of the Murto model (Murto [2004]) take place as a special case. The stopping times have the typical real-options form as the associated acquisition triggers involve the option multiple $\beta/(\beta-1)$ to account for the acquisition option as an opportunity costs. The comparative statics of the triggers is informative of the fundamentals underlying optimal acquisition. For example, we show that increasing economic uncertainty or the accuracy of the prior signal estimate delay the acquisitions while increasing the acquisition costs unsurprisingly work to the opposite direction.

The model can be extended in various ways. First, endogenising the order of the acquisitions is important as incremental cost information may well be more useful ex ante than incremental payoff information. For example, it is intuitive that with an accurate prior estimate on the cost signal and an inaccurate estimate on the payoff signal accurate the investor opts to acquire the payoff signal first. Second, the state space can be extended to two dimensions by replacing I with a random shock. Finally, more elaborate signal distributions may be introduced to allow for different information scenarios.

Appendix A.1. Acquisition of the investment-cost information: a trigger inequality

We here show that

$$X_\lambda(\theta) < X^*(\theta, \mu_\lambda) = \frac{\beta}{\beta-1}\frac{\mu_\lambda}{\theta}I$$

is necessary for the optimality of X_λ as V^λ is already decreasing at $X^*(\theta, \mu_\lambda)$. Since the λ-threshold is given by

$$\lambda^*(X) = \frac{\beta-1}{\beta}\theta\frac{X}{I},$$

we have

$$\lambda^*(X^*(\theta, \mu_\lambda)) = \mu_\lambda \in \Lambda,$$

so $V^\lambda(X;\theta)$ can be evaluated conditional on $\lambda^*(X) \in \Lambda$. This gives

$$V^\lambda(X;\theta) = \frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*(X)} V^*(X;\theta,u)g(u)du + \frac{1}{|\Lambda|}\int_{\lambda^*(X)}^{\Lambda^+} F^*(X;\theta,u)g(u)du -$$
$$F^{-\lambda}(X;\theta) - A_\lambda,$$

and on expanding

$$V^\lambda(X;\theta) = -\frac{1}{2}\frac{(\beta-1)^2}{\beta(2-\beta)}\frac{(\theta X/I)^2}{|\Lambda|}I + \frac{1}{(\beta-1)(2-\beta)}\left(\frac{X}{X^*(\theta,\Lambda^+)}\right)^\beta \frac{(\Lambda^+)^2}{|\Lambda|}I -$$
$$\frac{\Lambda^-}{|\Lambda|}\theta X - \frac{1}{\beta-1}\left(\frac{X}{X^*(\theta,\mu_\lambda)}\right)^\beta \mu_\lambda I - A_\lambda.$$

Differentiating with respect to X therefore gives

$$\frac{dV^\lambda}{dX} = -\frac{(\beta-1)^2}{\beta(2-\beta)}\frac{\theta^2}{|\Lambda|}\frac{X}{I} + \frac{\beta}{(\beta-1)(2-\beta)}\frac{1}{X}\left(\frac{X}{X^*(\theta,\Lambda^+)}\right)^\beta \frac{(\Lambda^+)^2}{|\Lambda|}I -$$

$$\frac{\Lambda^-}{|\Lambda|}\theta - \frac{\beta}{\beta-1}\frac{1}{X}\left(\frac{X}{X^*(\theta,\mu_\lambda)}\right)^\beta \mu_\lambda I,$$

and on substituting $X^*(\theta, \mu_\lambda)$

$$\begin{aligned}\frac{dV^\lambda}{dX}(X^*(\theta,\mu_\lambda)) &= \frac{1-\beta}{2-\beta}\frac{\theta\mu_\lambda}{|\Lambda|} + \frac{1}{2-\beta}\left(\frac{\mu_\lambda}{\Lambda^+}\right)^\beta \frac{(\Lambda^+)^2\theta}{|\Lambda|\mu_\lambda} - \frac{\Lambda^-}{|\Lambda|}\theta - \theta \\ &= \frac{1-\beta}{2-\beta}\frac{\theta\mu_\lambda}{|\Lambda|} + \frac{1}{2-\beta}\left(\frac{\mu_\lambda}{\Lambda^+}\right)^{\beta-2}\frac{\mu_\lambda}{|\Lambda|}\theta - \frac{\Lambda^+}{|\Lambda|}\theta.\end{aligned}$$

We now assert that

$$\frac{dV^\lambda}{dX}(X^*(\theta,\mu_\lambda)) < 0.$$

For $\beta > 2$ this holds since then

$$\frac{1-\beta}{2-\beta}\frac{\mu_\lambda}{|\Lambda|}\theta + \frac{1}{2-\beta}\left(\frac{\mu_\lambda}{\Lambda^+}\right)^{\beta-2}\frac{\mu_\lambda}{|\Lambda|}\theta < 0.$$

For $1 < \beta < 2$ this holds since then

$$\begin{aligned}\frac{dV^\lambda}{dX}(X^*(\theta,\mu_\lambda)) &< \frac{1-\beta}{2-\beta}\frac{\mu_\lambda}{|\Lambda|}\theta + \frac{1}{2-\beta}\frac{\mu_\lambda}{|\Lambda|}\theta \\ &< \frac{1-\beta}{2-\beta}\frac{\mu_\lambda}{|\Lambda|}\theta - \frac{1}{2-\beta}\frac{\mu_\lambda}{|\Lambda|}\theta < 0.\end{aligned}$$

Now fix $X'_\lambda \geq X^*(\theta,\mu_\lambda)$ and $X''_\lambda < X'_\lambda$ such that $V^\lambda(X''_\lambda;\theta) > V^\lambda(X'_\lambda;\theta)$. Since the first-passage time of $\{X_t\}$ to X''_λ is strictly smaller than to X'_λ, almost surely, the policy of acquiring the λ-information (exercising the λ-option) at X''_λ generates a strictly larger present expected value than the policy of acquiring the λ-information at X'_λ. But then X'_λ is suboptimal: it is necessary for optimality that $X_\lambda < X^*(\theta,\mu_\lambda)$. Graphically the suboptimality of the triggers $X_\lambda \geq X^*(\theta,\mu_\lambda)$ is shown by the fact there is no way to fit the λ-option value $F^\lambda(X;\theta) = B_\lambda X^\beta \geq 0$ smoothly onto V^λ at X-values for which V^λ is decreasing.

Appendix A.2. Acquisition trigger for the investment-cost information

We here invoke the standard dynamic-programming procedure together with the restricted λ-information value $V_0^\lambda(X;\theta)$ to determine the acquisition trigger $X_\lambda = X_\lambda(\theta)$ for the λ-information as a function of the λ-threshold λ^*. Since $X_\lambda < X^*(\theta, \mu_\lambda)$ by Appendix A.1, we let $W^{-\lambda}(X;\theta) = F^{-\lambda}(X;\theta)$ and thus

$$V_0^\lambda(X;\theta) = \frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} V^*(X;\theta,u)g(u)du + \frac{1}{|\Lambda|}\int_{\lambda^*}^{\Lambda^+} F^*(X;\theta,u)g(u)du - F^{-\lambda}(X;\theta) - A_\lambda.$$

Expanding this gives

$$V_0^\lambda(X;\theta) = \frac{1}{|\Lambda|}\theta X\int_{\Lambda^-}^{\lambda^*} g(u)du - \frac{1}{|\Lambda|}I\int_{\Lambda^-}^{\lambda^*} ug(u)du + \frac{1}{(\beta-1)(-\beta+2)}\left(\frac{\beta-1}{\beta}\theta\frac{X_\lambda}{I}\right)^\beta \frac{(\Lambda^+)^{-\beta+2} - (\lambda^*)^{-\beta+2}}{|\Lambda|}I - \frac{1}{\beta-1}\left(\frac{X}{X^*(\theta,\mu_\lambda)}\right)^\beta \mu_\lambda I - A_\lambda.$$

The value-matching condition

$$F^\lambda(X_\lambda;\theta) = V_0^\lambda(X_\lambda;\theta)$$

therefore reads as

$$B_\lambda(X_\lambda)^\beta = \theta X_\lambda\frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} g(u)du - I\frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} ug(u)du + \frac{1}{(\beta-1)(-\beta+2)}\left(\frac{\beta-1}{\beta}\theta\frac{X_\lambda}{I}\right)^\beta \frac{(\Lambda^+)^{-\beta+2} - (\lambda^*)^{-\beta+2}}{|\Lambda|}I - \frac{1}{\beta-1}\left(\frac{X_\lambda}{X^*(\theta,\mu_\lambda)}\right)^\beta \mu_\lambda I - A_\lambda.$$

The high-contact condition

$$\frac{\partial F^{\lambda}}{\partial X}(X_{\lambda};\theta) = \frac{\partial V_0^{\lambda}}{\partial X}(X_{\lambda};\theta)$$

reads as

$$\beta B_{\lambda}(X_{\lambda})^{\beta-1} = \theta\frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} g(u)du +$$
$$\frac{\beta}{(\beta-1)(-\beta+2)}\left(\frac{\beta-1}{\beta}\frac{\theta}{I}\right)^{\beta}\frac{(\Lambda^+)^{-\beta+2}-(\lambda^*)^{-\beta+2}}{|\Lambda|}\frac{1}{X_{\lambda}^{1-\beta}}I -$$
$$\frac{\beta}{\beta-1}\left(\frac{1}{X^*(\theta,\mu_{\lambda})}\right)^{\beta}\mu_{\lambda}\frac{1}{X_{\lambda}^{1-\beta}}I.$$

Multiplying the high-contact condition by X_{λ}/β and taking the result together with the value-matching condition implies

$$\frac{\beta-1}{\beta}\theta X_{\lambda}\frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} g(u)du = A_{\lambda} + I\frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} ug(u)du.$$

Expanding the integrals and substituting $A_{\lambda} = a_{\lambda}I$ finally gives

$$X_{\lambda}(\theta) = X_{\lambda}(\lambda^*;\theta) = \frac{\beta}{\beta-1}\frac{(\lambda^*)^2-(\Lambda^-)^2+2|\Lambda|a_{\lambda}}{2\theta(\lambda^*-\Lambda^-)}I,$$

where it remains to solve for λ^*.

Appendix A.3. Acquisition trigger for the investment-cost information: an equivalent derivation

We here determine $X_{\lambda}(\theta)$ in one step by invoking the dynamic programming procedure with

$$\lambda^*(X) = \frac{\beta-1}{\beta}\theta\frac{X}{I}.$$

Expanding

$$G^{-\lambda}(X;\theta) = \frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*(X)} V^*(X;\theta,u)g(u)du + \frac{1}{|\Lambda|}\int_{\lambda^*(X)}^{\Lambda^+} F^*(X;\theta,u)g(u)du$$

then gives the value of the λ-information as

$$V^{\lambda}(X;\theta) = -\frac{1}{2}\frac{(\beta-1)^2}{\beta(2-\beta)}\frac{(\theta X/I)^2}{|\Lambda|}I + \frac{1}{(\beta-1)(2-\beta)}\left(\frac{X}{X^*(\theta,\Lambda^+)}\right)^{\beta}\frac{(\Lambda^+)^2}{|\Lambda|}I -$$

$$\frac{\Lambda^-}{|\Lambda|}\theta X - \frac{1}{\beta-1}\left(\frac{X}{X^*(\theta,\mu_\lambda)}\right)^{\beta}\mu_\lambda I + \frac{1}{2}\frac{(\Lambda^-)^2}{|\Lambda|}I - A_\lambda.$$

The value-matching condition therefore reads as

$$B_\lambda(X_\lambda)^{\beta} = -\frac{1}{2}\frac{(\beta-1)^2}{\beta(2-\beta)}\frac{(\theta X_\lambda/I)^2}{|\Lambda|}I + \frac{1}{(\beta-1)(2-\beta)}\left(\frac{X_\lambda}{X^*(\theta,\Lambda^+)}\right)^{\beta}\frac{(\max)\Lambda^2}{|\Lambda|}I -$$

$$\frac{\Lambda^-}{|\Lambda|}\theta X_\lambda - \frac{1}{\beta-1}\left(\frac{X_\lambda}{X^*(\theta,\mu_\lambda)}\right)^{\beta}\mu_\lambda I + \frac{1}{2}\frac{(\Lambda^-)^2}{|\Lambda|}I - A_\lambda.$$

The high-contact condition reads as

$$\beta B_\lambda(X_\lambda)^{\beta-1} = -\frac{(\beta-1)^2}{\beta(2-\beta)}\frac{\theta^2}{|\Lambda|}\frac{X_\lambda}{I} + \frac{\beta}{(\beta-1)(2-\beta)}\left(\frac{1}{X^*(\theta,\Lambda^+)}\right)^{\beta}\frac{(\Lambda^+)^2}{|\Lambda|}\frac{1}{X_\lambda^{1-\beta}}I -$$

$$\frac{\Lambda^-}{|\Lambda|}\theta - \frac{\beta}{\beta-1}\left(\frac{1}{X^*(\theta,\mu_\lambda)}\right)^{\beta}\mu_\lambda\frac{1}{X_\lambda^{1-\beta}}I.$$

Multiplying the high-contact condition by X_λ/β and taking the result together with the value-matching condition gives on rearranging the quadratic equation

$$z^2 - 2(\Lambda^-)z + ((\Lambda^-)^2 - 2|\Lambda|a_\lambda) = 0,$$

where

$$z = z(X_\lambda) = \frac{\beta - 1}{\beta}\theta\frac{X_\lambda}{I}.$$

Solving the quadratic and then inverting the solution gives

$$X_\lambda = X_\lambda(\theta) = \frac{\beta}{\beta - 1}\frac{\Lambda^- + \sqrt{2|\Lambda| a_\lambda}}{\theta}I,$$

and we are done.

Appendix A.4. Value functions: investment-cost information

We here invoke the high-contact condition in Appendix A.2 to determine explicitly the value of the λ-option,

$$F^\lambda(X;\theta) = B_\lambda X^\beta.$$

Then $V^\lambda(X_\lambda;\theta)$ follows from the facts that $V^\lambda(X_\lambda;\theta) = V_0^\lambda(X_\lambda;\theta)$ and

$$V_0^\lambda(X_\lambda;\theta) = F^\lambda(X_\lambda;\theta).$$

Multiplying throughout by X_λ/β the high-contact condition becomes

$$B_\lambda X_\lambda^\beta = \frac{1}{\beta}\theta X_\lambda \frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} g(u)du + \frac{1}{(\beta-1)(2-\beta)}(\lambda^*)^\beta \frac{(\Lambda^+)^{-\beta+2} - (\lambda^*)^{-\beta+2}}{|\Lambda|}I -$$
$$\frac{1}{\beta-1}\left(\frac{X_\lambda}{X^*(\theta,\mu_\lambda)}\right)^\beta \mu_\lambda I$$
$$= \frac{1}{\beta-1}\lambda^* I \frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} g(u)du + \frac{1}{(\beta-1)(2-\beta)}(\lambda^*)^\beta \frac{(\Lambda^+)^{-\beta+2} - (\lambda^*)^{-\beta+2}}{|\Lambda|}I -$$
$$\frac{1}{\beta-1}\left(\frac{\lambda^*}{\mu_\lambda}\right)^\beta \mu_\lambda I,$$

where we have employed

$$X_\lambda = X_\lambda(\theta) = \frac{\beta}{\beta-1}\frac{\lambda^*}{\theta}I.$$

Now letting

$$C_\lambda = \lambda^* \frac{1}{|\Lambda|}\int_{\Lambda^-}^{\lambda^*} g(u)du + \frac{1}{2-\beta}(\lambda^*)^\beta \frac{(\Lambda^+)^{-\beta+2} - (\lambda^*)^{-\beta+2}}{|\Lambda|} - \left(\frac{\lambda^*}{\mu_\lambda}\right)^\beta \mu_\lambda$$

$$= 2a_\lambda + (\Lambda^-)\sqrt{\frac{2a_\lambda}{|\Lambda|}} + \frac{1}{2-\beta}\frac{(\lambda^*/\Lambda^+)^\beta(\Lambda^+)^2 - (\lambda^*)^2}{|\Lambda|} - \left(\frac{\lambda^*}{\mu_\lambda}\right)^\beta \mu_\lambda I.$$

and noting that then

$$B_\lambda = \frac{1}{\beta-1}\frac{C_\lambda}{(X_\lambda(\theta))^\beta},$$

we get

$$F^\lambda(X;\theta) = \frac{1}{\beta-1}\left(\frac{X}{X_\lambda(\theta)}\right)^\beta C_\lambda I,$$

and

$$V^\lambda(X_\lambda;\theta) = \frac{1}{\beta-1}C_\lambda I.$$

We are now done.

Appendix B. Maximiser for the payoff-information value function

We here show that the maximiser of $V^\lambda(X;\theta)$ is of the form $X_* = qX_\lambda(\theta)$ where $q > 1$. From Appendix A.1 we directly have that

$$\frac{dV_\lambda}{dX}(X_*) = -\frac{(\beta-1)^2}{\beta(2-\beta)}\frac{\theta^2}{|\Lambda|}\frac{X_*}{I} + \frac{\beta}{(\beta-1)(2-\beta)}\frac{1}{X_*}\left(\frac{X_*}{X^*(\theta,\Lambda^+)}\right)^\beta \frac{(\Lambda^+)^2}{|\Lambda|}I -$$

$$\frac{\Lambda^-}{|\Lambda|}\theta - \frac{\beta}{\beta-1}\frac{1}{X_*}\left(\frac{X_*}{X^*(\theta,\mu_\lambda)}\right)^\beta \mu_\lambda I = 0,$$

or

$$-\frac{(\beta-1)^2}{\beta(2-\beta)}\frac{(\theta X_*)^2/I}{|\Lambda|}+\frac{\beta}{(\beta-1)(2-\beta)}\left(\frac{X_*}{X^*(\theta,\Lambda^+)}\right)^\beta\frac{(\Lambda^+)^2}{|\Lambda|}I-$$

$$\frac{\Lambda^-}{|\Lambda|}\theta X_*-\frac{\beta}{\beta-1}\left(\frac{X_*}{X^*(\theta,\mu_\lambda)}\right)^\beta\mu_\lambda I=0.$$

Now letting

$$X_*=qX_\lambda(\theta)=q\frac{\beta}{\beta-1}\frac{\lambda^*}{\theta}I$$

gives

$$-\frac{\beta}{2-\beta}\frac{(\lambda^* q)^2}{|\Lambda|}+\frac{\beta}{(\beta-1)(2-\beta)}\left(\frac{\lambda^* q}{\Lambda^+}\right)^\beta\frac{(\Lambda^+)^2}{|\Lambda|}-$$
$$\frac{\beta}{\beta-1}\frac{\Lambda^-}{|\Lambda|}\lambda^* q-\frac{\beta}{\beta-1}\left(\frac{\lambda^* q}{\mu_\lambda}\right)^\beta\mu_\lambda=0,$$

which on multiplying throughout by $-(\beta-1)|\Lambda|/\beta\lambda^* q$ results in

$$\left(\mu_\lambda^{1-\beta}|\Lambda|-\frac{1}{2-\beta}(\Lambda^+)^{2-\beta}\right)(\lambda^*)^{\beta-1}q^{\beta-1}+\frac{\beta-1}{2-\beta}\lambda^* q+\Lambda^-=0.$$

We are now done.

Appendix C. Value function $H(X;\theta)$

We here determine explicitly the value function

$$H(X;\theta)=\mathbf{E}_X(e^{-r\tau^\lambda(X;\theta)}(G^{-\lambda}(X_\lambda(\theta);\theta)-A_\lambda)).$$

The definition of V^λ immediately gives

$$G^{-\lambda}(X_\lambda(\theta);\theta) = V^\lambda(X_\lambda(\theta);\theta) + F^{-\lambda}(X_\lambda(\theta);\theta) + A_\lambda,$$

where

$$F^{-\lambda}(X_\lambda(\theta);\theta) = \frac{1}{\beta-1}\left(\frac{\lambda^*}{\mu_\lambda}\right)^\beta \mu_\lambda I,$$

and from Appendix A.4

$$V^\lambda(X_\lambda(\theta);\theta) = V_0^\lambda(X_\lambda(\theta);\theta) = \frac{1}{\beta-1}C_\lambda I,$$

with

$$C_\lambda = 2a_\lambda + (\Lambda^-)\sqrt{\frac{2a_\lambda}{|\Lambda|}} + \frac{1}{2-\beta}\frac{(\lambda^*/\Lambda^+)^\beta(\Lambda^+)^2 - (\lambda^*)^2}{|\Lambda|} - \left(\frac{\lambda^*}{\mu_\lambda}\right)^\beta \mu_\lambda I.$$

We thus get

$$H(X_\lambda(\theta);\theta) = \frac{1}{\beta-1}E_\theta I,$$

where

$$E_\theta = \Lambda^-\sqrt{\frac{2a_\lambda}{|\Lambda|}} + \frac{1}{2-\beta}\frac{(\lambda^*/\Lambda^+)^\beta(\Lambda^+)^2 - (\lambda^*)^2}{|\Lambda|} + 2a_\lambda.$$

From this it is standard to show by an application of Itô's lemma and the boundary condition $H(0;\theta) = 0$ that

$$H(X;\theta) = \left(\frac{X}{X_\lambda(\theta)}\right)^\beta H(X_\lambda(\theta);\theta) = \frac{1}{\beta-1}E_\theta\left(\frac{X}{X_\lambda(\theta)}\right)^\beta I.$$

We are now done.

Appendix D. General threshold equation

We here determine the solution equation for the θ-threshold θ^* in the general case $\Lambda^- \geq 0$. From the smooth-pasting conditions (18) and (19) and

$$\lambda^{**}(\theta, \theta^*) = \frac{\lambda^*}{\theta^*}\theta,$$

it follows that

$$\left(2((\Theta^+)^3 - (\theta^*)^3)\frac{\lambda^*}{\theta^*} + 3((\theta^*)^2 - (\Theta^+)^2)\Lambda^-\right)\frac{\beta - 1}{\beta}\frac{X_\theta}{I}$$

$$= 3(\theta^* - \Theta^+)(\Lambda^-)^2 + \left(\frac{\lambda^*}{\theta^*}\right)^2((\Theta^+)^3 - (\theta^*)^3) + 6|\Theta||\Lambda|a_\theta + 6(\Theta^+ - \theta^*)|\Lambda|a_\lambda.$$

Solving this for X_θ we get

$$X_\theta = \frac{\beta}{\beta - 1}\frac{\phi_1(\theta^*)}{\phi_2(\theta^*)}I,$$

where $\phi_1(\theta^*)$ is a third-order polynomial given by

$$\phi_1(\theta^*) = 2\left(-2\lambda^* + 3\Lambda^-\right)\lambda^*(\theta^*)^3 +$$
$$3((\lambda^* - 2\Lambda^-)(\Theta^+)\lambda^* + 2|\Lambda||\Theta|a_\theta)(\theta^*)^2 + (\Theta^+)^3(\lambda^*)^2,$$

and $\phi_2(\theta^*)$ is a fourth-order polynomial given by

$$\phi_2(\theta^*) = (-2\lambda^* + 3\Lambda^-)(\theta^*)^4 - 3(\Theta^+)^2(\Lambda^-)(\theta^*)^2 + 2(\Theta^+)^3\lambda^*\theta^*.$$

Since on the other hand

$$X_\theta = X_\lambda(\theta^*) = \frac{\beta}{\beta - 1}\frac{\lambda^*}{\theta^*}I,$$

we have

$$\frac{\beta}{\beta-1}\frac{\phi_1(\theta^*)}{\phi_2(\theta^*)}I=\frac{\beta}{\beta-1}\frac{\lambda^*}{\theta^*}I,$$

which on substituting the $\phi_j(\theta^*)$ and rearranging implies that

$$\left(2\lambda^*-3\Lambda^-\right)\lambda^*(\theta^*)^3-3((\lambda^*-2\Lambda^-)(\Theta^+)\lambda^*+2|\Theta||\Lambda|a_\theta)(\theta^*)^2-$$
$$3(\Theta^+)^2(\Lambda^-)\lambda^*\theta^*+(\Theta^+)^3(\lambda^*)^2=0.$$

We are now done.

References

[1] Decamps, J. P. and Mariotti, T. and Villeneuve, S. [2002]: Investment Timing Under Incomplete Information. Unpublished manuscript.

[2] Dixit, A. K., and Pindyck, R. S. [1994]: *Investment under Uncertainty.* Princeton University Press.

[3] Grenadier, S. R. [1999]: Information Revelation Through Option Exercise. *The Review of Financial Studies, Vol. 12, No. 1, pp. 95-129.*

[4] Majd, S., and Pindyck, R. S. [1987]: Time to Build, Option Value, and Investment Decisions. *Journal of Financial Economics, Vol. 18, No. 1, pp. 7-27*

[5] Murto, P. [2004]: Valuing Options to Learn: Optimal Timing of Information Acquisition. Unpublished draft. *Real Options: Theory Meets Practise, 8th Annual International Conference.*

[6] Paddock. J. L., and Siegel, D. R., and Smith, J. L. [1988]: Option Valuation of Claims on Real Assets: The Case of Offshore Petroleum Leases. *Quarterly Journal of Economics, Vol. 103, pp. 479-508.*

[7] Pennings, E., and Lint, O. [1997]: The option value of advanced R&D. *European Journal of Operational Research, Vol. 103, pp. 83-94.*

[8] Pindyck, R. S. [1991]: Irreversibility, Uncertainty, and Investment. *Journal of Economic Literature. Vol. 29, No. 3, pp. 1110-1148.*

[9] Roberts, K., and Weitzman, M. L. [1981]: Funding Criteria for Research, Development, and Exploration Projects. *Econometrica, Vol. 49, No. 5, pp. 1261-1288.*

[10] Roche, H. [2005]: The Value of Waiting to Learn. Unpublished manuscript, Departmento de Administración, Instituto Tecnológico Autónomo de México.

[11] Schwartz, E. S. (ed.), and Trigeorgis, L. (ed.) [2001]: *Real Options and Investment under Uncertainty. Classical Readings and Recent Contributions.* The MIT Press.

[12] Weitzman, M. L., and Newey, W., and Rabin, M. [1981]: Sequential R&D Strategy for Synfuels. *The Bell Journal of Economics, Vol. 12, No. 2, pp. 574-590.*