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**“MARKET RISK AND PROCESS UNCERTAINTY IN PRODUCTION OPERATIONS”**

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## **“MARKET RISK AND PROCESS UNCERTAINTY IN PRODUCTION OPERATIONS”**

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### **ABSTRACT**

By adopting a real options framework we develop a production control model that jointly incorporates process and market uncertainties. In this model, process uncertainty is defined by random fluctuations in the outputs' yield and market risk through demand uncertainty for the output. In our approach, production outputs represent commodities or items for which financial contracts do not trade. Outputs are also functionally linked to the level of input inventories. To extend the model's applicability to a wide range of production industries, inputs are modeled to reflect either renewable, or partially renewable or non-renewable resources.

Given this setting, techniques of stochastic control theory are employed to obtain value maximizing production policies in a constrained capacity environment. The rate of production is modeled as an adapted positive real-valued process and analogously evaluated as a sequence of complex real options. Since optimal adjustments to the rate of production also functionally depend on the outputs' yield, we optimally establish “trigger boundaries” justifying controlled variations to the rate of production over time. In this context, we provide closed form analytic results and demonstrate their robustness with respect to the stochastic (including mean reverting) processes considered. Using these results, we also demonstrate that the value (net of holding costs) accrued to the producer from having an inventory of the output is equivalent to the producer's reservation price to operationally curb its process yield. These generalizations extend the scope of model applicability and provide a basis for applying the real options methodology in the operations arena. The model is explored numerically using a stylized example that allows for both output and demand uncertainty and achieves greater realism by incorporating an element of smoothing into the sequence of production decisions.

## (I) INTRODUCTION

The integration of uncertainty and management of risk has become central to research in the operations arena. Research on this front addresses two fundamental questions, albeit in different contexts. First, how do operating (and only quite recently, financial) uncertainty and its consequent risk(s) affect operating policies? Second, what are the ensuing economic ramifications and insights when uncertainty is explicitly accounted for and related risk(s) are truncated? In this paper we consider optimal operating policies for the firm in the presence of uncertainty, within an option theoretic context. The use of options theory enables us to examine operating policies dynamically, rather than to settle for the static solutions seen in most analyses of inventory problems. To that end, we exclusively focus on “quantity” uncertainty, its risk implications, and impact on production policies.<sup>1</sup> “Quantity” risk is due to either *process* uncertainty, or *market* uncertainty, or *both*. In this paper, we assume the latter. Specifically, we model process uncertainty by random fluctuations in the outputs’ yield and represent market uncertainty through variations in demand for the output. We extend our model’s applicability by considering situations where outputs reflect products for which financial contracts (e.g. futures contracts) do not trade such as the manufacturing or petrochemical industries. This is in contrast to commodity-based production/extraction applications of the real options methodology where the output commodity (copper, oil, etc.) is typically a traded asset. We model the required inputs for production as either renewable or partially renewable or non-renewable resources and use the state of input inventories as a mechanism to characterize the output levels. This generality expands the range and the nature of applications possible.

Several factors motivate the objectives of this paper and the approach adopted for reaching them. We recognize that the simultaneous introduction of both demand and process (or yield) uncertainty is not an entirely new concept.<sup>2</sup> For instance, in manufacturing industries firms may face demand uncertainty and be supplied with defective input materials. Yield uncertainty in manufacturing tends to affect technology choice, supplier alternatives, product design, and inventory options. In the operations literature demand and yield uncertainty are customarily treated as random variables. This modeling alternative suggests a static approach to the analysis in that the influence of time on the assumed distributions and the resulting policies are effectively ignored. However, as we demonstrate, the concurrent treatment of demand and yield variables as (generalized) stochastic processes within a real options context avoids this problem. This approach also provides a dynamic framework to value the resulting operating policies.

As remarked earlier, the majority of real options applications involve valuing investment opportunities in complete markets. This is typified by production efforts in mining/extraction based industries with

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<sup>1</sup> For most operational environments, commonly encountered sources of uncertainty may be *generically* classified into “Time”, “Quantity”, and “Price” categories. Time uncertainty is primarily due to variations in lead-times, arrival times, or processing times. Quantity uncertainty can be either external or internal. The former is typified by forecast errors of future sales or by random demand fluctuations. In the latter case, quality, design, or technology factors and complexities in processing often result in random variations in the actual yields observed. Price uncertainty for any input/output in competitive markets results from shifts in supply and demand for the item. In situations where the inputs/outputs reflect standard commodities for which futures contracts trade, their downside risk can be blocked by taking a hedged position. In almost all other situations the risk of price uncertainty is borne by the producer. The only exception reflects cases where prices are contractually fixed so that, in effect, a forward contract is at hand.

<sup>2</sup> See, for instance, Moynzadeh and Lee (1987).

outputs that are traded commodities and with output prices that reflect the major source of uncertainty. Market completeness implies that a portfolio of existing traded securities can be constructed that perfectly replicates the project's payoffs. Thus, absent arbitrage opportunities, the market value of this replicating portfolio must be the same as the value of the investment project. In other words, the value of a non-traded investment project is "the price the investment opportunity would have if it were traded".<sup>3</sup> Market completeness is an undeniably powerful concept in the valuation of contingent claims in that all risks can be *perfectly hedged* by constructing a replicating portfolio from existing traded assets.<sup>4</sup> As pointed out by Brennan and Schwartz (1985), there are several advantages to the valuation problem when a replicating portfolio can be constructed.<sup>5</sup> First, the need for estimates of the expected growth rate in the underlying source of uncertainty is eliminated. Second, discount rates need not be derived since in the absence of (market priced) risks, the risk free rate is the appropriate discount rate.

Other measures of market risk including exchange rate or demand uncertainty also have, albeit to a lesser extent, found their way into the current (real options) literature. Brennan and Schwartz (1985) introduce the notion of production flexibility in mining projects with multiple options to open, close and to subsequently abandon the project; the notion of market risk is captured through output prices, which are assumed random in nature. Furthermore, the output is also taken to be non-renewable, homogenous in its composition and therefore, not subject to variability. By extending Brennan and Schwartz's elegant model to account for output variability, Kamrad and Ernst (2001) consider situations where the extracted resource is non-homogenous. The general solution to the classical "duration" problem of a long-term renewable resource is provided by Morck, Schwartz and Stangeland (1990). In their production control model, price and the level of inventories reflect the sources of uncertainty. Andreou (1990) provides a basic model for valuing flexible plant capacity when demand conditions are uncertain whereas Chung (1990) develops a contingent claims model of demand uncertainty with a stochastic production function. He and Pindyck (1992) consider an investment model of flexible production capacity. Kamrad and Ritchken (1994) use a multi-jump lattice approach to establish optimal production and inventory policies in the face of multiple input price uncertainty. Kouvelis and Sinha (1994) examine investment mode options and exchange rate uncertainty. Exchange rate related work includes Dasu and Li (1997) who develop optimal operating policies; Huchzermeier and Cohen (1996) address operational flexibility concerns for the purposes of strategic global manufacturing; and Kogut and Kulatilaka (1994) who consider production shifts among plants in a network of manufacturing centers. Li and Kouvelis (1999) provide an in depth analysis of risk sharing in supply contracts. Kamrad and Siddique (2004) also consider supply contract valuation problems in the presence of multiple exchange rates, profit sharing strategies and supplier reaction options.

The application of real options methodology to a broader set of production-based problems, such as manufacturing, where the input(s) or the output(s) do not reflect traded commodities has received only

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<sup>3</sup> See Mason and Merton (1985) for a more detailed overview.

<sup>4</sup> See Harrison and Kreps (1979).

<sup>5</sup> When risks cannot be perfectly hedged by a replicating portfolio (i.e. incomplete markets), preferences and aversions toward risk enter the analysis. This, however, requires specifying the utility function of the representative investor.

limited attention.<sup>6</sup> In particular, there has been little discussion of “process” risk which is typified by reliability issues, lead-time uncertainty, system breakdowns, or output yield variability. The risk induced by process uncertainty is idiosyncratic and not compensated for in the financial markets. This is not to imply that process risk has no downside ramifications or related costs. Frequently, managers protect against the downside effects of process risk by producing larger lots and through maintaining inventories from which their contractual obligations is met. Of course, this comes at the expense of increased production and holding costs.

In this paper, process uncertainty is characterized by output yield factor defined as the usable proportion of the total amount produced.<sup>7</sup> By incorporating the yield factor into the analysis, we can explicitly account for the inherent heterogeneity problem existing in various forms in essentially all production processes. Though vastly different from the more traditional models of yield variability encountered in the manufacturing literature, our approach maintains the parallel that the yield factor is modeled as an independent multiplier to the output quantity. In light of this setup, we assume risk aversion and pursue a real options approach within Merton’s (1973) Intertemporal Capital Asset Pricing Model. Our model is derived in a general equilibrium context that parallels the findings of Constantinides (1978) and McDonald and Siegel (1985). Using techniques of stochastic control theory, we optimally adjust the rate of production in a manner consistent with a value-maximizing criterion.

We assume that the risk associated with the output’s yield is idiosyncratic and does not induce an additional premium. We also assume a JIT production setting in that there are no finished product inventories. This assumption may seem counterintuitive or restrictive in the presence of a yield variability problem. It may reasonably be argued that stockpiling finished goods is in effect an operational hedge to the yield uncertainty problem especially in light of demand uncertainty and constrained production capacities. However, we can justify this supposition in order to demonstrate that the value (net of holding costs) accrued to the producer from having an inventory of the output is equivalent to the producer’s reservation price to operationally hedge its process yield. In other words, the maximum dollar value that the management should be willing to expend in order to alleviate its yield risk. In that vein, we provide an upper bound premium for the inventory as an option. Alternatively, this value may be interpreted as the present value of a sequence of forward contracts with

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<sup>6</sup> Kamrad and Lele (1998) consider price uncertainty and “system failure” risk and develop an optimal production and maintenance expenditure policy in light of a warranty on shared failure repair costs. Hodder and Triantis (1993) consider production flexibility and use real options to value switching between alternative operating states. Nembhard, Shi and Park (2000) and Nembhard, Shi and Aktan (2002, 2003) use a real options framework in the context of quality control and to evaluate the potential benefits of product outsourcing.

<sup>7</sup> The literature on yield problems in production enjoys a rich heritage. Initially, and motivated by yield variations in agricultural crops, Karlin (1958) considered the inventory implications of uncertain outputs. By considering a multi-period ordering environment and assuming that production yields are binomially distributed, Mazzola, McCoy, and Wagner (1987) derive an EOQ model that allows for backlogging. Given demand uncertainty and defective shipment quantities, Moinzadeh and Lee (1987) provide an analysis of a continuous review model. The inventory implications of a finite horizon, single product production model in the presence of demand and yield uncertainty is due to Gerchak, Vickson and Parlar (1988). By assuming a single period, single product, multistage production setting with random production yields at each stage, Lee and Yano (1988) develop a model indicating that stage dependent costs and a convex function of each stage’s production. Bitran and Dasu (1992) develop a model for establishing ordering policies in the presence of stochastic yields and demand substitutability. An extensive and rich review of the random yield literature can be found in Yano and Lee (1995).

different delivery times in order to balance out the spot output shortages that arise due to yield variability. Nonetheless, we assume that the producer maintains an inventory of the needed raw materials through which the finished goods are produced. We model this inventory system in such a way to allow for the full characterization of the resource to be exploited. That is, the resource in question may be renewable, partially renewable or non-renewable. This is important since within our setup we can easily distinguish the type of industry within which the production effort lies. For instance, if the raw material inventory is a renewable resource, the production effort represents a manufacturing operation. On the other hand, if the reserves reflect a nonrenewable resource, the case at hand typifies mining or other extraction based operations.

Although our results are intended for the manufacturing industry in particular, our model lends its use in a number of related contexts. For instance, with appropriate modifications the model can also be employed to address yield and demand variability issues in service operations including but not limited to the commercial airline industry. In that case, we would define production as the number of flights operated at a given time, whereas the yield would be the percentage occupancy on those flights.

The paper is organized as follows. In the next section we define the notation, state the necessary assumptions and develop an options based production control model resulting in a Bellman valuation equation subject to appropriate boundary conditions. In section III, we present closed form solutions for the optimal production policies and provide economic insights to them. In general, the Bellman equation does not lend itself to an analytic solution, so it must be solved numerically to obtain results. In section IV, by invoking the Feynman-Kac results, we show how to derive the solution numerically along with the resulting optimal production policy to be followed. To that end, an efficient multinomial lattice approach provides the basis for approximating the stochastic evolution of the state variables. The numerical solutions are obtained recursively by superimposing a dynamic programming algorithm on the lattice. This procedure is then illustrated through a stylized example where we also include a penalty function to smooth the sequence of production decisions. Section V provides concluding remarks.

## ( II ) Assumptions and Model Development

Let  $Z_D(t) \in \mathbb{R}$  and  $Z_Y(t) \in \mathbb{R}$  define standard Brownian motions that are martingales with respect to the probability space,  $(\Omega, \mathcal{F}, \mathcal{T}, \mathcal{P})$ . The filtered probability space,  $(\Omega, \mathcal{F}, \mathcal{T}, \mathcal{P})$  is defined over the pre-established time interval  $[0, \tau]$  where the augmented filtration,  $\mathcal{F} = \{\mathcal{F}_t : t \in [0, \tau]\}$ , is right-continuous and increasing. In general, let the process depicting uncertainty be defined by  $\{X(t) : t \geq 0\}$ , where its sample path is posited by an Ito differential equation of the form:

$$dX(t) = M(X, t)dt + S(X, t)dZ_X(t) \quad (1)$$

The drift function,  $M(X, t)$  denotes the instantaneous change in  $X(t)$ . The volatility function,  $S(X, t)$  denotes the standard deviation of the growth rate, and  $dZ_X(t)$  is an instantaneous increment to the Brownian motion;  $Z_X(t) \in \mathbb{R}$  defined above. To develop the valuation model, let  $V(D, Y, I, t; q)$  represent the production value at time  $t$  given that the demand is  $D(t)$ , the yield factor is  $Y(t)$ , the level of input inventory (or remaining untapped resource level) is  $I(t)$ , and where the production rate is set at  $q(t)$ .

$D(t)$  and  $Y(t)$  are stochastic processes of the general type defined by equation (1) and the function  $V(D, Y, I, t; q)$  is taken to be Ito differentiable. The producer's actions are depicted by the rate of production,  $q(D, I, t) \equiv q(t), t \in [0, \tau]$  with  $q(t) \in (0, Q)$  where  $Q$  defines the current available production capacity. In our set up,  $q = \{q(t) : t \in [0, \tau]\}$  is an adapted positive real-valued process. The flexibility afforded by having the option to revise operating policies in reaction to both market (i.e. demand) and process (i.e. yield) uncertainty is value additive and as such is viewed as a sequence of (real) nested options. We also assume that the production cost function,  $K(q(t))$  is non-decreasing in  $q(t)$  and that  $K''(q(t)) \geq 0$ , depicting increasing marginal cost of producing an additional unit of the output. Finally, we assume that the market price for the output is deterministic and is denoted by  $\pi(t)$  so that the yield-affected revenue resulting from producing at rate  $q(t)$  at time  $t$ , is  $q(t)Y(t)\pi(t)$ . We now specify the components of the model.

### **Demand process**

The output's demand process  $\{D(t) : t \geq 0\}$  is depicted by one of the following alternative forms of equation (1):

$$dD(t) = D(t)\{\alpha_D dt + \sigma_D dZ_D(t)\} \quad D(0) = D_0 > 0 \quad (2a)$$

$$dD(t) = \kappa[\mu_D - D(t)]dt + \sigma_D[D(t)]^\eta dZ_D(t) \quad D(0) = D_0 > 0 \quad (2b)$$

In expression (2a), the constant drift parameter  $\alpha_D$  represents the instantaneous expected growth rate in demand and the constant per unit variance of the growth rate is  $\sigma_D^2$ . The demand process captured by (2a) implies that the conditional distribution of  $D(t)$  given  $D(s), t > s \in [0, \tau]$ , is lognormal and that  $D(t) > 0$  for all  $t \in [0, \tau]$ , if  $D_0 > 0$ . Expression (2b) reflects a generalized "mean reverting" process. The mean reversion constant,  $\kappa \geq 0$  can be thought of an elastic force that pulls  $D(t)$  to its long run mean  $\mu_D$  which is assumed to be positive. As  $\kappa \geq 0$  becomes smaller, the excursions around the mean become longer. In general,  $\eta$  is arbitrary. For  $\eta = 1$ , the process is known as an Ornstein-Uhlenbeck process or an inhomogeneous geometric mean reverting process. When  $\eta = 1/2$ ,  $D(t)$  follows a non-central  $\chi^2$  (chi-squared) distribution with finite mean and variance. The above specifications accommodate a wide range of probability distributions for the demand process depending on the parametric choices. Equation (2b) formally reduces to (2a) only when  $\eta = \mu_D = 0$  and  $\kappa = -\alpha_D$ , a set of conditions precluded by our specification.

We assume demand substitutability is not an alternative and that backlogging is not allowed. Furthermore, the producer does not stockpile finished products and hence there are no finished goods inventory concerns. This implies that given the available production capacity, the producer aims to meet as much of the demand as possible. To reduce the potential for overage costs, we implicitly impose a penalty constraint to that effect. We also assume that the producer's actions do not affect the market demand for the output.

### Inventory level

Given that there are no finished good inventories in meeting the demand for the output, the producer simply produces at rate  $q(t)$  in a manner that maximizes the operating profits. The producer, however, maintains an inventory of needed raw materials from which finished goods are produced. Let  $I(t), t \in [0, \tau]$  define time  $t$  level of input inventory. Supposing  $I(0) = I_0 > 0$  defines the initial known level of the resource, we have,

$$dI(t) = [\phi(q(t)) - q(t)]dt \quad (3)$$

The function  $\phi(\cdot)$  is used to determine whether or not the resource in question is renewable. Although, many forms for this function could be examined, in this paper we consider only the simple functional form,  $\phi(q(t)) = \xi q(t)$  with  $0 \leq \xi \leq 1.0$ .<sup>8</sup> Thus equation (3) becomes  $dI(t) = (\xi - 1)q(t)dt$ . When  $\xi = 0$ , the resource considered is non-renewable and the RHS of equation (3) simply reduces to  $-q(t)$ . When  $\xi = 1.0$ , the resource is instantaneously renewable: in other words, the supply is effectively infinite. In a limiting sense, a JIT inventory policy illustrates the notion of  $\xi = 1.0$ . All other cases (i.e.  $0 < \xi < 1$ ) present a partially renewable resource with the RHS of equation (3) becoming  $(\xi - 1)q(t)$ . In this case, the rate of depletion or extraction is faster than the rate of replenishment. Note that when  $\xi > 1.0$ , implies a raw material reserve reordering policy related to production. In an extraction-based industry this may be interpreted as an incentive for acquiring new reserves when the rate of production is increasing.

### Net output

The net usable output resulting from production at rate  $q(t)$  is defined as  $q(t)Y(t)$ . The yield variable  $\{Y(t), t \geq 0\}$  is conceptualized as an independent multiplier to the output rate and is constrained such that  $0 \leq Y(t) \leq 1.0$  for all  $t \geq 0$ . The yield process  $\{Y(t), t \geq 0\}$ , denoted simply by  $Y$  when no ambiguity arises, is assumed to follow the generalized stochastic process:

$$dY(t) = \mu(Y, t)dt + \sigma_Y(Y, t)dZ_Y(t) \quad Y(0) = Y_0 > 0 ; 0 \leq Y(t) \leq 1.0 \quad (4)$$

Thus  $Y(t)$  describes the state of the production process at time  $t$  and its realization is known at the instant that the incremental production decision is made. We defer specifying functional forms for  $\mu(\cdot)$  and  $\sigma_Y(\cdot)$  and address this concern in our results' section and in light of a contextually meaningful distribution for  $Y(t)$  to follow. We assume the Brownian increments defined earlier are orthogonal. That is, the demand and yield processes are instantaneously uncorrelated:

$$\rho_{DY} dt = E(dZ_D(t) \cdot dZ_Y(t)) = 0 \quad (5)$$

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<sup>8</sup> It is not within the scope of this paper to optimally establish  $\xi$ . However, the solution procedure shown in section IV can provide a numerical optimization approach.



### Model formulation

As a preliminary to developing our valuation model, we assume the existence of a (hypothetical) financial security that has the same covariance with market return as does demand.<sup>9</sup> Suppose  $W(t)$  depicts the price of this traded security at time  $t \in [0, \tau]$  and that the equilibrium growth rate on this financial asset is  $\alpha_w$ . We assume that the instantaneous change in the price level of this security is characterized by the following stochastic differential equation,

$$dW(t) = W(t)\alpha_w dt + W(t)\sigma_w dZ_w(t) \quad (6)$$

Expression (6) defines a geometric Brownian motion, with  $\alpha_w$  and  $\sigma_w$  reflecting the constant drift (expected rate of return) and volatility (standard deviation of rate of return) parameters. Here,  $\alpha_w$  represents the equilibrium rate of return on a financial security having the same covariance with the market return as does the demand. Let  $\alpha_M$  and  $\sigma_M$  define the instantaneous expected rate of return (drift) and the standard deviation of the rate of return on the market, respectively. The unexpected rate of return component defined by,  $\sigma_M dZ_M(t)$ , with  $Z_M(t) \in \mathbb{R}$  as a standard Brownian motion that is also a martingale with respect to the probability space,  $(\Omega, \mathcal{F}, \mathfrak{I}, \mathcal{P})$ . The constant and riskless rate of return is depicted by  $r$ . Employing Merton's (1973) Intertemporal Capital Asset Pricing Model, the market premium on this financial security is given by  $\lambda \rho_{WM} \sigma_w$ , which for valuation purposes is equivalent to  $\lambda \rho_{DM} \sigma_D$ . Here,  $\rho_{WM}$  and  $\rho_{DM}$  define the instantaneous correlation on returns between the financial security and the market and that of the demand and the market, respectively.<sup>10</sup> By definition,

$$\lambda = \frac{\alpha_M - r}{\sigma_M} \quad (7)$$

Furthermore, let the rate of return shortfall be defined by  $\psi = \alpha_w - \alpha_D$ , with  $\psi$  unrestricted in sign. By employing an intertemporal CAPM approach, the equilibrium rate of return on the financial security must reflect an adjustment for systematic risk. In this context, we have:

$$\alpha_w = r + \lambda \rho_{DM} \sigma_D \quad (8)$$

Recall, by definition that  $\lambda \rho_{WM} \sigma_w = \lambda \rho_{DM} \sigma_D$ .<sup>11</sup> Given this setup, we can obtain  $V(t) \equiv V(D, Y, I, t; q)$  using a replicating portfolio approach. To that end, consider portfolio  $G(t)$  consisting of a long position in  $V(\cdot)$  together with a short position of  $\delta$  units in security,  $W(t)$ . The instantaneous change in the value of this portfolio, in light of the necessary cash flow adjustment is,

$$dG(t) = dV(t) - \delta dW(t) + \{q(t)Y(t)\pi(t) - K(q(t))\} dt \quad (9)$$

<sup>9</sup> We will use this condition to arrive at equations (8 and 10).

<sup>10</sup> By definition  $\rho_{DM} \sigma_M \sigma_D = \rho_{WM} \sigma_M \sigma_w$ : the security has the same 'return' covariance with the market as does the demand. Hence  $\rho_{DM} \sigma_D = \rho_{WM} \sigma_w$ . That is,  $E(dZ_M(t) \cdot dZ_D(t)) \sigma_D = E(dZ_M(t) \cdot dZ_w(t)) \sigma_w$  which is (8) using a CAPM framework.

<sup>11</sup> See also Constantinides (1979) and, McDonald and Siegel (1985).

The expression in curly brackets denotes the reward function  $R(q(t))$ . To ensure the existence of only diversifiable risks, set  $\delta = (D/W) \cdot (\partial V(\cdot) / \partial D(t))$ . Absent arbitrage opportunities, this implies that the expected return on this portfolio,  $E(dG(t))$  should be the riskless rate so that,

$$E(dG(t)) = rV(t)dt \quad (10)$$

The producer seeks to maximize the net present value of the project. To that end, by applying Ito's lemma to the right hand side of equation (9), taking the resulting expectations, and equating it to expression (10) we obtain the desired Bellman valuation equation. Using equation (2a), it follows without loss of generality that,<sup>12</sup>

$$\text{Max}_{q \in [0, Q]} \left\{ \frac{\partial V}{\partial D} D(r - \psi) + \frac{\partial V}{\partial Y} \mu(Y, t) + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial D^2} D^2 \sigma_D^2 + \frac{1}{2} \frac{\partial^2 V}{\partial Y^2} \sigma_Y^2(Y, t) + q[Y\pi - \frac{\partial V}{\partial I} (1 - \xi)] - K(q) - rV \right\} = 0 \quad (11)$$

s.t.

$$\lim_{Y \rightarrow 0} V(D, Y, I, t; q) = 0 \quad (12a)$$

$$\lim_{D \rightarrow 0} V(D, Y, I, t; q) = 0 \quad (12b)$$

$$\lim_{D \rightarrow \infty} \frac{V(D, Y, I, t; q)}{D} < \infty \quad (12c)$$

$$q(t) \in [0, Q] \quad (12d)$$

$$V(D, Y, 0, t; q) = 0 \quad (12e)$$

$$V(D, Y, I, \tau; q) = C(I, \tau) \quad (12f)$$

$$V(D, Y, I, t; q) + P(I, t) \geq 0 \quad (12g)$$

$$H = \omega_1 [D(t) - Y(t)q(t)]^{\omega_2} \quad (12h)$$

Furthermore, in some situations, a mean reverting process is a more appropriate description for the demand variable. Our model can easily accommodate this adjustment. In particular, if the demand process were defined according to expression (2b), then the resulting Bellman valuation equation is:

$$\text{Max}_{q \in [0, Q]} \left\{ \frac{\partial V}{\partial D} [D(r - \tilde{\psi})] + \frac{\partial V}{\partial Y} \mu(Y, t) + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial D^2} D^2 \sigma_D^2 + \frac{1}{2} \frac{\partial^2 V}{\partial Y^2} \sigma_Y^2(Y, t) + q[Y\pi - \frac{\partial V}{\partial I} (1 - \xi)] - K(q) - rV \right\} = 0 \quad (13)$$

subject to the same constraints as before: that is (12a-g). The only difference between the Bellman equations (11) and (13) is the first term inside the brackets involving the partial derivative with respect to  $D(t)$  with;

$$\tilde{\psi} = \tilde{\alpha}_w - \tilde{\alpha}_D \quad \text{with} \quad \tilde{\alpha}_D = \left(\frac{\kappa}{D}\right) \mu_D \quad \text{and} \quad \tilde{\alpha}_w = \alpha_w + \kappa \quad (14)$$

where, as in equation (8),  $\alpha_w = r + \lambda \rho_{DM} \sigma_D$

<sup>12</sup> Note that changes in the yield variable are assumed independent of the overall economy and therefore not priced.

### Assumptions

Equations (12a-h) characterize the constraints to the above Bellman equation (11). First, we assume that the value goes to zero as the yield goes to zero. That is,  $Y(t) = 0$  is an absorbing state, since the production effort becomes valueless (12a). Likewise,  $D(t) = 0$  is an implied absorbing state given equation (2): i.e. equation (12b). We further assume that the value function is bounded above by  $cD(t)$  where  $c < \infty$  is a constant (12c). This regularity condition is stated for completeness; it is trivially satisfied since the production capacity is bounded above by  $Q$ , as indicated by equation (12d). We also assume that the value drops to zero when the input level inventory falls to zero. The terminal value at the close of the project is defined by function  $C(I, \tau)$  via equation (12f). To account for shutdown as a flexibility option, equation (12g) ensures that the operating value of the production effort exceeds the corresponding shutdown cost, as reflected by  $P(I, t)$ . To discourage production overages and shortages, we impose a penalty function characterized by constraint (12h), where the constants  $\omega_1, \omega_2 \geq 0$ .

Several issues regarding equation (11) are noteworthy. In particular, consider the first term in equation (11) reflecting the quantity  $(r - \psi)$ . This quantity has effectively replaced the original drift term of the demand process as a result of a replicating strategy barring arbitrage opportunities (see equations (9) and (10)). In the current context, the portion  $(r - \psi)$  is the revised drift representing the average growth rate for the demand process. Specifically,  $(r - \psi)$  characterizes the “market” adjusted instantaneous growth rate of demand with  $\psi$  unrestricted in its sign and where  $\psi = \alpha_w - \alpha_D$ . As stated earlier,  $\alpha_w$  is the expected rate of return on a financial asset having the same (financial) risk as the demand variable. When  $\psi < 0$ , it implies that the expected growth rate of demand exceeds the equilibrium rate of return on a security having the same financial risk in the market as does the demand. In contrast, when  $\psi > 0$ , the expected growth rate of demand is less than the aforementioned equilibrium rate of return.<sup>13</sup> These findings are consistent with Constantinides (1978) and McDonald and Siegel (1985); the later includes a more detailed and intuitive discussion. In light of the above results, for valuation purposes the demand process initially defined by equation (2a) may be depicted as:

$$dD(t) = D(t)\{(r - \psi)dt + \sigma_D dZ_D(t)\} \quad D(0) = D_0 > 0 \quad (15)$$

In what follows, closed form results regarding the optimal policies are provided. However, the value of the project, as characterized by equation (11), must be solved for numerically since the Bellman valuation equation does not yield closed form results. An alternative approach would be to formulate the valuation problem as a stochastic dynamic program, subject to appropriate constraints. Indeed, in section IV we adopt such an approach to obtain results numerically. However, we believe that the CCA formulation provides additional and complementary insights into the decision-making process.

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<sup>13</sup> If instead of demand uncertainty we were to represent market uncertainty through stochastic prices for the output and had it been the case where the output were a commodity for which futures contracts traded, then  $\psi$  would have analogously represent the commodity’s convenience yield instead. See Brennan and Schwartz (1985).

### (III) Optimal Production Policies

This section provides closed form results for the optimal operating policies. In order to obtain explicit results, we assume that the production cost function  $K(q(t))$  is quadratic and non-decreasing (monotone) in the rate of production  $q(t)$  :

$$K(q(t)) = k_0 + k_1 q(t) + k_2 q^2(t) \quad (16)$$

where the monotonicity conditions imply that  $k_1, k_2 \geq 0$ . We further assume that the drift and volatility function functions for the yield process are constant, so that  $\mu(Y, t) = \mu$  and  $\sigma_Y(Y, t) = \sigma_Y$ . That is:

$$dY(t) = \mu dt + \sigma_Y dZ_Y(t) \quad Y(0) = Y_0 > 0 \quad (17)$$

thereby implying that the distribution of the yield variable  $Y(t)$ , given  $Y(s)$  with  $t > s$ , and  $t, s \in [0, \tau]$  follows a doubly truncated normal distribution with zero as an absorbing barrier and with one as a reflecting barrier for  $Y(t)$ . If the input inventory level at time  $t \in [0, \tau]$  drops to zero, so will the value function as constraint (12e) indicates. To ensure production remains a viable option, we assume that (input) inventory is strictly positive during the planning horizon. If this were not the case, we would simply devise an optimal policy over time until the inventory is fully exhausted, and that would complete the analysis. Thus, we assume that the input inventory level remains positive over the period  $[0, \tau]$ . A sufficient condition for this assumption is that  $I(0) > \tau Q(1 - \xi)$ .

#### Theorem 1:

Let  $I(0) > \tau Q(1 - \xi)$  with  $0 \leq \xi < 1$ . Suppose that the demand process is defined by either expression (2a) or (2b), and that the production cost function,  $K(q(t))$  is an increasing convex quadratic function in the production rate,  $q(t)$ . The optimal production policy  $\{q^*(t), q^* \in [0, Q]\}$ ,  $t \in [0, \tau]$  is given by:

$$q^*(t) = \text{Min}\left[q_c(t), \frac{D(t)}{Y(t)}\right] \quad (18)$$

where

$$q_c(t) = \begin{cases} Q & \text{if } Y(t) \geq \bar{Y}(t) \\ \tilde{q}(t) & \text{if } Y^*(t) < Y(t) < \bar{Y}(t) \\ 0 & \text{if } Y(t) \leq Y^*(t) \end{cases}$$

with

$$\tilde{q}(t) = \frac{\pi(t)Y(t) - (k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1 - \xi))}{2k_2} \quad (19)$$

$$\bar{Y}(t) = \frac{2k_2 Q + (k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1 - \xi))}{\pi(t)} \quad (20)$$

$$Y^*(t) = \frac{(k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1 - \xi)) + 2(k_0 k_2)^{1/2}}{\pi(t)} \quad (21)$$

**Proof:** See Appendix A

Equation (19) has an economic interpretation in that the optimal production rate, given the bounds on the yield variable, reflects profits (revenues less the variable cost of production and inventory) adjusted by the marginal cost of production, namely  $k_2$ . In a similar vein, the bounds on the yield variable also have an economic pitch. That is, equations (20) and (21) essentially suggest that variable costs of production and inventory are discounted by the profit rates. Though subtle, this idea is seen in a simpler context in the next theorem. Note too that for Theorem 1, the demand process provides an effective upper bound for production. Also, the raw materials inventory level directly affects the production decision through the term  $\partial V(\cdot)/\partial I(t)$ . Anecdotally, when the relative price of raw materials rises sharply, the firm may exercise its shutdown option and sell the input. The term  $\partial V(\cdot)/\partial I(t)$  also has a “shadow cost” implication. In the current context, it can be interpreted as the raw material carrying cost rate barring the opportunity cost of capital which is implicitly integrated into the analysis through the shortfall term,  $\psi$ . However, this interpretation is limited to the finite resource case. In particular, as we have demonstrated below, when the supply of the input material is effectively infinite (i.e.  $\xi = 1$ ) the only relevant costs are the processing cost which is absorbed into  $K(q(t))$ .

**Corollary 1:**

Assume that  $\xi = 1$ . In this case, the optimal production policy  $\{q^*(t), q^* \in [0, Q]\}$ ,  $t \in [0, \tau]$  is given by:

$$q^*(t) = \text{Min}\left[q_c(t), \frac{D(t)}{Y(t)}\right] \quad (22)$$

where

$$q_c(t) = \begin{cases} Q & \text{if } Y(t) \geq \bar{Y}(t) \\ \tilde{q}(t) & \text{if } Y^*(t) < Y(t) < \bar{Y}(t) \\ 0 & \text{if } Y(t) \leq Y^*(t) \end{cases}$$

with

$$\tilde{q}(t) = \frac{\pi Y(t) - k_1}{2k_2} \quad (23)$$

$$\bar{Y}(t) = \frac{2k_2 Q + k_1}{\pi(t)} \quad (24)$$

$$Y^*(t) = \frac{k_1 + 2(k_0 k_2)^{1/2}}{\pi(t)} \quad (25)$$

**Proof:** See Appendix A

The above Corollary applies to production opportunities wherein the supply of levels cannot be fully depleted during the project’s life due to typically limited production capacities. Specifically, the whole of the available raw material cannot be used up during the production horizon. We note the following *comment*: It would be desirable to smooth the production process over time, since changes are costly. Such changes to the model make it impossible to find a closed form solution and are not pursued further at the theoretical level. However, we will incorporate this feature in the numerical example in section IV. The net effect of smoothing is to make policy changes gradually, rather than the sudden switches implied by the theoretical results.

**Theorem 2:**

Let  $k_2 = 0$  and  $I(0) > \tau Q(1 - \xi)$ , with  $0 \leq \xi < 1$ . The optimal production policy  $\{q^*(t), q^* \in [0, Q]\}$ ,  $t \in [0, \tau]$  is given by:

$$q^*(t) = \text{Min}\left[q_c(t), \frac{D(t)}{Y(t)}\right] \quad (26)$$

where

$$q_c(t) = \begin{cases} Q & \text{if } Y(t) \geq Y^*(t) \\ 0 & \text{if } Y(t) < Y^*(t) \end{cases}$$

with the critical yield factor,

$$Y^* = \frac{\frac{\partial V(\cdot)}{\partial I(t)}(1 - \xi) + k_1}{\pi(t)} \quad (27)$$

**Proof:** See Appendix A

Equations (26) and (27) characterize a “bang-bang” production policy in that we produce at the maximum feasible rate of production,  $Q$  only if the yield factor exceeds the “profit adjusted” variable cost of raw material inventory and production. To offer additional insight, recall that low realizations of the yield factor reduce profits relative to higher yield realizations. In effect, only when yield adjusted revenues exceed the variable cost of inventory and production, as shown by the numerator of equation (27), it becomes profitable enough to produce at the maximum rate; otherwise, a no-production mode is optimal. In the current context, the level of the raw material inventory is considered finite and therefore, the variable inventory cost or more appropriately “shadow cost”,  $\partial V(\cdot)/\partial I(t)$  has a direct bearing on the optimal production policy. As  $I(t)$  defines the level of a renewable resource, its shadow price in this context is interpreted as the holding (carrying) cost rate of the raw materials’ inventory, excluding the opportunity cost of capital. The opportunity cost of capital, as remarked earlier, has been indirectly incorporated into the analysis when considering the return shortfall rate,  $\psi = \alpha_w - \alpha_D$ , or  $\tilde{\psi} = \tilde{\alpha}_w - \tilde{\alpha}_D$  depending on which process is used to represent the demand uncertainty.

**Corollary 2:**

Assume that  $\xi = 1$ . Then, the optimal production policy  $\{q^*(t), q^* \in [0, Q]\}$ ,  $t \in [0, \tau]$  is given by:

$$q^*(t) = \text{Min}\left[q_c(t), \frac{D(t)}{Y(t)}\right] \quad (28)$$

where

$$q_c(t) = \begin{cases} Q & \text{if } Y(t) \geq Y^*(t) \\ 0 & \text{if } Y(t) < Y^*(t) \end{cases}$$

with the critical yield factor,

$$Y^*(t) = \frac{k_1}{\pi(t)} \quad (29)$$

**Proof:** Follows the proof of Corollary 1 and Theorem 2.

**Theorem 3:**

Let the demand process be defined by either stochastic process (2a) or (2b). Also, let  $V^* = \text{Max}_q [V(D, Y, I, t : q)]$ ,  $q(t) \in (0, Q)$ .  $V^*$  is unique.

**Proof:** See Appendix B

**Theorem 4:**

Consider two yield processes  $\{Y_j(t), t \geq 0, j = L, U\}$  such that  $Y_U(t) \geq Y_L(t)$  for all  $t \in [0, \tau]$ . Then for any demand process and for all  $t \in [0, \tau]$ ,  $V(D, Y_U, I, t : q) \geq V(D, Y_L, I, t : q)$ .

**Proof:** See Appendix C

The above theorem has a unique interpretation. In the current context, it can be specialized to reflect an “idealized project” where  $Y(t) = 1.0$ , or  $Y(t) < 1.0$  as we have assumed throughout the paper. The difference in value between the idealized project,  $V(D, Y = 1.0, I, t : q)$  and the actual project as we have considered in this paper,  $V(D, Y < 1.0, I, t : q)$  represents the maximum dollar amount that the producer should be willing to pay in order to alleviate its quantity risk, as defined herein. This may be interpreted as the total cost, in present value terms, of a finished goods inventory option. The inventory option would have the objective of meeting as much of demand as possible in light of spot shortages due to yield uncertainty. Alternatively, it can also be interpreted as the net worth, in present value terms, of a sequence of forward contracts for the delivery of the spot shortages resulting from the yield variability problem, in satisfying as much of the demand as possible.

The approach taken in this paper may be viewed as complementary to that taken in Boone, Ganeshan, Guo and Ord (2000). Those authors develop guidelines to determine suitable run times in the presence of imperfect production processes. Thus, the Boone et al. results are applicable when set-up costs are considerable. In the present research, we also recognize that production processes are imperfect, but operate in continuous time, making the decision at each point in time whether to continue production. In the manufacturing context, when the decision is to cease production, the opportunity exists to “retool”. That is, the production facility may be repaired or restored to a better level of operation, and production restarted under the guidelines provided by our solutions.

#### ( IV ) Numerical Results

For our model, the most appropriate numerical techniques are finite difference methods and lattice techniques.<sup>14</sup> The former methodology obtains numerical solutions through approximating the partial differential equation, which in our paper is the Bellman valuation equation. The lattice techniques provide results by approximating the stochastic evolution of the underlying sources of uncertainty. Given that there are two stochastic sources of variability, we decided to adopt a multinomial lattice approach. To that end, note that the stochastic process used here for defining the demand variable is given by expression (2a) which implies that conditional distribution of demand is lognormal. The yield process implies a doubly truncated normal distribution for the yield factor. To employ the intended multinomial lattice approach appropriately, a simple log-transformation is needed and thereafter the multinomial lattice can be constructed easily. The relationship between these two processes is well established in Karlin and Taylor (1981). With two state variables, two types of multinomial lattices may be used. Here, we use the 4-jump model of Boyle, Evnine and Gibbs (1989). An alternative approach is the 5-jump lattice model of Kamrad and Ritchken (1991). In general, a 5-jump lattice provides more accurate results. However, for our purposes a 4-jump lattice is sufficiently accurate for the analysis considered. As stated earlier, a backward recursion is used to dynamically superimpose our production control problem on a 4-jumps lattice.<sup>15</sup> We control for the upper and lower bound values on the yield process by establishing appropriate barriers at zero and one.

Our example considers demand, inventory and yield processes as specified in section II. We assume that price is constant and that the input is non-renewable. Table (2) below depicts the case parameters and functional forms. We should like to make the following observation with regard to this example. It will be observed that some of the constraints differ slightly between the theoretical model and the numerical example. We believe the constraints in the numerical example to be more realistic, and thus we preferred to perform the numerical analysis within this framework. However, the theoretical results in such circumstances would not yield closed-form solutions, so that the insights gained from the theory would be obscured. First, since our model does not allow for a finished goods inventory, we impose a (quadratic) penalty cost to regulate overages or underages. This penalty could be viewed as the cost of lost sales, or as the cost of buying in inventory from elsewhere to meet the extra demand. Likewise, if we overproduce the excess must be salvaged. This penalty function is employed instead of using demand as an upper limit. Second, since changes in the production level are costly, we introduce a switching cost to smooth the production plan. The penalty function also helps to smooth out production operations.

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<sup>14</sup> Smith and McCardle (1998) integrate stochastic dynamic programming and option theory, while Tsitsiklis and Van Roy (1999) demonstrate the link between Markov decision processes and options.

<sup>15</sup> Due to space limitation, lattice details are omitted. However, they are available upon request from the first author.



**Table (2): Base Case Parameters and Function Coefficients**

Production Cost	$K(q)$	$k_0 = 100.00; k_1 = 15.00; k_2 = 5.00$
Price	$\pi$	$\pi = \$300.00$ per unit
Initial Inventory	$I$	$I_0 = 20$ units
Initial Demand	$D_0$	$D_0 = 10$ units
Initial Yield	$Y_0$	$Y_0 = 0.70$ per annum
Demand Volatility	$\sigma_D$	$\sigma_D = 0.30$ per annum
Yield Volatility	$\sigma_Y$	$\sigma_Y = 0.2$ per annum
Interest Rate	$r$	$r = 0.08$ per annum
Average Output Yield	$\mu$	$\mu = 0.10$ per annum
Adjustments to drift shortfall	$\psi$	$\psi = 0.03$ per annum
Production Capacity	$Q$	$Q = 5$ units per period
Renewable Resource constant	$\xi$	$\xi = 0$
Switching Cost Function:	$1(q_i - q_{i-1})^2$	
Penalty Cost Function:	$H = 5(D_i - Y_i q_i)^2$	
Shutdown Cost Function:	$P(I, t_i) = 0$ , for $i = 0, 1, 2, \dots, n$	
Salvage Cost Function:	$C(I, \tau_n) = 0.50 I_n Y_n \pi_n$ given the state at time $t_n$	

The impact of increasing in the average yield on project value is depicted by Figure-1, showing the expected steady growth. The effect of increasing production capacity on the value function is shown in Figure-2 which demonstrates precisely the anticipated results. Namely, that the capacity constraint is relaxed (non-binding) beyond a certain point.

**[Figure-1] and [Figure-2]**

Of particular interest is the case where increased volatility in the yield process produces a corresponding increase in the value function. As  $\sigma_Y$  increases, so will the upside potential while the downside risks are truncated through a “no-production” or shutdown option. Thus, on an average basis the project’s value improves as the yield volatility increases, as shown in Figure-3. However, when the volatility of the demand is increased, the project’s value diminishes, as seen in Figure-4. This too is logical in light of the situation at hand. As  $\sigma_D$  increases, it becomes harder to satisfy demand due to limited production capacity and the non-existence of finished goods inventory. Beyond this capacity, any discrepancy between demand realization and production levels are also penalized as constrained in Table (2).

**[Figure-3] and [Figure-4]**

In Figure-5, the effect of  $\psi$  on the value function is shown. As  $\psi$  increases, the implied average growth rate of the demand process drops. All things being equal, and in light of our capacity constraint, it becomes that much easier to meet demand. This reduction in the total operating costs is manifested by a corresponding increase in the project's value.

**[Figure-5] and [Figure-6] and [Figure-7]**

Finally, given our choice of parameters, we consider the project's sensitivity toward the initial yield setting. As expected, the project's value is an increasing non-linear function of the yield factor and in that sense these findings echo the results in Figure 1. This has been illustrated in Figure 6. Of interest, however, is the project's value if yield uncertainty had not been an operational problem. That is, the "idealized project". In particular, we make a reference to Theorem 4 and provide in Figure 7 a numerical illustration of the maximum dollar value that management should be willing to expend to curb its quantity risk. That is, the value of perfect yield conditional on a particular yield realization: be it in the form of investment in finished goods inventory or by purchasing a sequence of forward contracts with various maturities to cover spot shortages.

**V. Conclusions**

The contributions of this paper are as follows. First, it introduces a framework for analyzing production based projects characterized by *both* market (demand) and process (yield) uncertainty. The framework introduced can be easily modified to accommodate other sources of market and/or process uncertainty. Second, the paper further extends the current literature's findings to a broader class of production problems. In particular, we examine manufacturing or other non-extraction based production control problems where the inputs may reflect renewable, partially renewable or non-renewable resources. Third, the class of production outputs considered herein reflects non-traded assets. Specifically, through employing an ICAPM approach, we have made some generalizations regarding the applicability of this model beyond that of commodities for which futures and other financial contracts trade directly. In that capacity, this paper also provides for future research opportunities in the operations arena using a CCA approach. Fourth, the base model developed draws from generalized stochastic processes (Brownian motion) to depict the sources of uncertainty considered and to obtain closed form results. Finally, and in light of quite robust numerical results, the model presented in this paper is sufficiently adaptable to allow for the inclusion of other sources of market or process uncertainty and the stochastic processes characterizing such uncertainty.

## **Appendix A**

### **Proof of Theorem 1:**

We have from equation (11),

$$\frac{\partial V}{\partial D}D(r-\psi) + \frac{\partial V}{\partial Y}\mu + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial D^2}D^2\sigma_D^2 + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\sigma_Y^2 + q(\pi Y - \frac{\partial V}{\partial I}(1-\xi)) - K(q) = rV \quad (\text{A1})$$

From the above equation (A1) the necessary and sufficient conditions imply that for maximization purposes,

$$\pi Y(t) - k_1 - 2k_2q - \frac{\partial V(\cdot)}{\partial I(t)}(1-\xi) = 0 \quad (\text{A2})$$

Solving for  $q$  we obtain,

$$\tilde{q}(t) = \frac{\pi Y(t) - (k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1-\xi))}{2k_2} \quad (\text{A3})$$

At  $q(t) = Q$ , (A2) becomes,

$$2k_2Q = \pi Y(t) - k_1 - \frac{\partial V(\cdot)}{\partial I(t)}(1-\xi) \quad (\text{A4})$$

implying that the minimum yield level to induce production at capacity is,

$$\bar{Y}(t) = \frac{2k_2Q + (k_1 + \frac{\partial V(\cdot)}{\partial I}(1-\xi))}{\pi} \quad (\text{A5})$$

Solving (A1) with  $\tilde{q}(t)$  results in

$$Y^*(t) = \frac{(k_1 + \frac{\partial V(\cdot)}{\partial I}(1-\xi)) + 2(k_0k_2)^{1/2}}{\pi} \quad (\text{A6})$$

This completes the proof.

### **Proof of Corollary 1:**

The proof follows from the above in a straightforward manner. Specifically,

$$\int_0^\tau q(t)dt \leq \tau Q < I_0 = \infty \Rightarrow \frac{\partial V(\cdot)}{\partial I(t)} = 0 \quad (\text{A7})$$

Substituting equation (A7) into equations (17)-(19) obtains the desired results.

### **Proof of Theorem 2:**

From the Bellman equation (11), it follows that  $V(D, Y, I, t; q)$  is maximized if  $q(t)$  is either zero or at maximum  $Q$  since by assumption the production cost function,  $K(q(t))$  is linear. Specifically,  $q^*(t) = Q$  so long as:

$$\pi Y(t) \geq k_1 + \frac{\partial V(\cdot)}{\partial I(t)}(1-\xi) \quad (\text{A8})$$

which results in equation (25). However, if the marginal operating revenues are less than the corresponding operating costs then,  $q^*(t) = 0$ .

## **Appendix B**

### **Proof of Theorem 3:**

To prove  $v^*$  is unique, we will prove the concavity of the value function  $V(\cdot)$  in  $q$ . Differentiating Bellman equation (11) successively with respect to  $q(t)$  we obtain,

$$r \frac{\partial V(\cdot)}{\partial q} = \pi Y - \left( \frac{\partial K(q)}{\partial q} + \frac{\partial V(\cdot)}{\partial I}(1-\xi) \right) \quad (\text{B1})$$

$$r \frac{\partial^2 V(\cdot)}{\partial q^2} = - \frac{\partial^2 K(q)}{\partial q^2} \quad (\text{B2})$$

The second derivative is negative by definition of  $K(q(t))$ . Consider (B1) where in perfect competition,

$$\pi Y = \frac{\partial K(q)}{\partial q} + \frac{\partial V(\cdot)}{\partial I} (1 - \xi)$$

and in the case of monopoly,

$$\pi Y > \frac{\partial K(q)}{\partial q} + \frac{\partial V(\cdot)}{\partial I} (1 - \xi)$$

Therefore, the RHS of (B1) is at least zero. Therefore, it can have at most one real maximum, that is  $v^*$ .

## **Appendix C**

### **Proof of Theorem 4:**

From Theorem 1, the optimal production policy at time  $t$  is,

$$q^*(t) = \text{Min}\left[q_c(t), \frac{D(t)}{Y(t)}\right] \quad (\text{C1})$$

which can be adapted for each choice of  $Y_j(t)$ ,  $j = U, L$ . Since  $Y_U(t) \geq Y_L(t)$ , then the optimal production policies satisfy of three conditions:

1.  $\frac{D(t)}{Y_U(t)} < \frac{D(t)}{Y_L(t)} < q_c(t)$ ,
2.  $\frac{D(t)}{Y_U(t)} < q_c(t) < \frac{D(t)}{Y_L(t)}$ ,
3.  $q_c(t) < \frac{D(t)}{Y_U(t)} < \frac{D(t)}{Y_L(t)}$ .

For case:

1. Demand is fully satisfied for both processes,  $Y_j(t)$ ,  $j = U, L$ . Also revenues are the same. However, since  $Y_U(t) \geq Y_L(t)$ , less is produced under process  $L$  and thus the value is increased.
2. Process  $U$  meets all demand. Process  $L$  does not even though more are produced. This increases the production cost and therefore the profit levels are lowered. Thus, the value under process  $L$  is lower relative to process  $U$ . Hence, if  $Y_U(t) > Y_L(t)$  at any time  $t$ , value is increased.
3. We produce the same amount regardless of the process ( $L$  or  $U$ ). However, production costs will be lower with regard to process  $U$ , and therefore a higher net value is at hand.

It follows that if  $\{Y_U(t) \geq Y_L(t), \forall 0 \leq t \leq \tau\}$  the project value must be higher for process  $U$ . In particular, we know that  $\{Y(t) \leq 1.0, \forall 0 \leq t \leq \tau\}$  so that  $Y(t) = 1.0$  for all  $t$  represents an upper bound.

The difference in value between the actual project and the idealized project (i.e.  $Y(t) = 1.0$ ) represents the maximum dollar amount that the producer should be willing to pay in order to alleviate its yield uncertainty or quantity risk. This may be interpreted as the cost of a finished goods inventory option to meet as much of the demand as possible in light of spot shortage or net cost of a sequence of forward contracts for the delivery of the spot shortages, all due to the yield variability problem.

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