Smooth pasting as rate of return equalisation: A note

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Abstract

In this short paper we further elucidate the smooth pasting condition that is behind the optimal early exercise condition of options. It is almost trivial to show that smooth pasting implies rate of return equalisation between the option and the levered position that results from exercise. This yields new economic insights into the optimal early exercise condition that the option holder faces.

1 Introduction

There has been considerable interest in the smooth pasting (or high-contact) condition associated with options and real options decisions, because of the implications and optimality of early exercise. It is well known that smooth pasting is a first-order condition for optimum, as proposed by Samuelson (1965), proven by Merton (1973) and discussed by Dumas (1991) and several others. Brekke and Øksendal (1991) also show that the condition is sufficient under weak constraints. Nonetheless, smooth pasting remains somewhat mysterious to both economists and practitioners, and it is apparently not very useful for many people except economic theorists. The popular introduction book to real options by Dixit and Pindyck (1994) saves the discussion of smooth pasting for a quite technical appendix, and no simple rules of thumb seem to exist for practitioners.

Dixit et al. (1999) aims at filling some of the gap from theory to practice when pointing at an analogy between optimal exercise of investment options of the McDonald and Siegel (1986) type, and optimal exercise of market power in the standard textbook model of monopoly. It turns out that optimal investment can be characterized by an elasticity-based markup which is analogous to the markup price chosen by a profit-maximizing monopolist.

In this short note we provide another, more intuitive and natural, explanation of the phenomenon; that of rate of return equalization between the option and its levered payoff. This hopefully allows a larger audience to appreciate and potentially implement smooth pasting techniques in a wider variety of situations. We also relate this result to the elasticity-based rules introduced by Dixit et al. (1999) and Sødal (1998).

2 Rates of return

A geometric Brownian diffusion can be written in the Risk Neutral Q or Real World P, in which case it will have drift $r-\delta$ or $\mu-\delta$ respectively (r, δ, μ, σ represent the continuous risk free, dividend, project return and volatility rates respectively)

$$\frac{dS}{S} = (r-\delta)\,dt + \sigma dW^Q \tag{1a}$$

$$\frac{dS}{S} = (\mu-\delta)\,dt + \sigma dW^P \tag{1b}$$

Local changes dC in the call price C (puts P can be analyzed as well) are given by the Ito expansion, furthermore no arbitrage requires that the Risk Neutral expectation $E^Q[dC]$ of these changes must be the risk free rate (alternatively the hedged position must yield the risk free rate of return)

$$dC = \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2 + \frac{\partial C}{\partial t}dt \tag{2}$$

and

$$E^Q[dC] = \frac{\partial C}{\partial S}S(r-\delta)\,dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}dt + \frac{\partial C}{\partial t}dt = rCdt \tag{3}$$

However, Real World returns depend on the risk premium $\mu-r$ through the expectation operator $E^P[dC]$, an expression which can be simplified through the use of the previous Risk Neutral, no arbitrage condition on $E^Q[dC]$

$$\begin{aligned} E^P[dC] &= \frac{\partial C}{\partial S}S(\mu-\delta)\,dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}dt + \frac{\partial C}{\partial t}dt \\ &= \frac{\partial C}{\partial S}S(\mu-r)\,dt + rCdt \end{aligned} \tag{4}$$

Thus, the well known local expected rate of return of the call option is given by (see Merton (1973))

$$r_C = \frac{1}{dt}\frac{E^P[dC]}{C} = r + \epsilon_C(\mu-r) \geqslant \mu \tag{5}$$

where the elasticity $\epsilon_C = \frac{\partial C}{\partial S}\frac{S}{C}$ has the interpretation of the relative beta of the option to the underlying.

3 Smooth pasting

This rate of return can be investigated at the point of optimal early option exercise $\overline{S}$ ($\overline{S} > X$ the exercise price). The two conditions necessary for this are value matching (payoff compensates for termination of option value) and smooth pasting (equality of slope between option value and payoff functions)

$$C\left(\overline{S}\right) = \overline{S} - X \tag{6a}$$
$$\left.\frac{\partial C}{\partial S}\right|_{S=\overline{S}} = 1 \tag{6b}$$

Thus at the critical exercise boundary $S = \overline{S}$ the call rate of return r_C is

$$r_C\left(\overline{S}\right) = r + \frac{\overline{S}}{C\left(\overline{S}\right)}(\mu - r) = \frac{\mu\overline{S} - rX}{\overline{S} - X} = r_{PO} \tag{7}$$

which is also the % rate of return on the levered payoff $\overline{S} - X$ (as a fraction of the payoff value PO itself) r_{PO}.

At early exercise, not only do the value of the option and payoff functions have the same value but smooth pasting and tangency implies that the expected rates of return on both positions are the exactly the same.

Risk neutral returns on both sides are not useful for determining early exercise since they are always equal (to r). Furthermore, any subjective estimate of the future risk premia, $\widehat{\mu}\ (> r)$, cancels out in the return equations, i.e. overestimating or underestimating $\widehat{\mu}$ has the same effect on subjective $\widehat{r}_C$ and $\widehat{r}_{PO}$ at $\overline{S}$. This means that every investor will exercise early when their estimated rates of return are the same, no matter what their individual belief $\widehat{\mu}$ about μ.

$$\widehat{r}_C - r = \left.\frac{\partial C}{\partial S}\right|_{S=\overline{S}} \frac{\overline{S}}{C\left(\overline{S}\right)}(\widehat{\mu} - r) = \frac{\overline{S}}{\overline{S} - X}(\widehat{\mu} - r) = \widehat{r}_{PO} - r \tag{8}$$

The analysis works equally well for puts P (evaluated at a lower threshold $\underline{S}$) whose expected rate of return can be negative:

$$r_P\left(\underline{S}\right) = r - \frac{\underline{S}}{P\left(\underline{S}\right)}(\mu - r) = \frac{rX - \mu\underline{S}}{X - \underline{S}} \gtrless 0 \tag{9}$$

4 Relationship to other approaches

The result that the rate of return on the option equates the return on the net payoff is closely related to other findings on smooth pasting. Dixit et al.

(1999) argues that the optimal exercise of a call option consists of maximizing the expected net present value

$$\begin{aligned} C &= \max D(S)(\overline{S} - X) && (10) \\ D(S) &= E^P\left[e^{-\rho T}\right] = E^Q\left[e^{-rT}\right] && (11) \end{aligned}$$

where T is the first-hitting time from the current value of the project, S_0, up to the value $\overline{S}$ at which the option is exercised. $E^P\left[e^{-\rho T}\right]$ is the expected discount factor[1].

That is, the objective is to maximize the expected, discounted value of the net pay–off $S - X$. For perpetual options, the discount rate, ρ, could either be adjusted for risk or the expectation for the discount factor be taken with respect to a risk-adjusted distribution $E^Q\left[e^{-rT}\right]$, all depending on what approach, (1a) or (1b), that is applied, setting

$$\rho = r + \epsilon_C\,(\mu - r)\,. \tag{12}$$

If the option is a perpetual option[2] (or can be regarded as such because S_0 is close to the optimal exercise point, i.e. the time partial is not important), the discount factor can be written equivalently *without reference to time* as a function $D(S)$, which will be decreasing in S as more discounting occurs by moving S away from S_0. Maximizing $C = D(S)(S - X)$ with respect to S, optimal exercise is easily found to be given by

$$\frac{\overline{S}}{\overline{S} - X} = \epsilon_D \tag{13}$$

where $\epsilon_D = -(\partial D/\partial S)/(D/S)$ is the magnitude of the elasticity of the discount factor with respect to S, evaluated at the optimal exercise point, $\overline{S}$. The expected value of the project relative to the net payoff equals $C/(S - X) = D(S)$, so the relationship to the elasticity, ϵ_D (> 1), is imminent: the elasticity measures the relative change in the relative net payoff following from a marginal change in S. This is a measure of returns, but

[1] Shackleton and Wojakowski (2002) show that perpetual calls (and puts) have constant rates of return, and that they are possibly the only option value for which discounted expectations can be taken in either the Risk Neutral Q, or Real World P.

[2] When the option is perpetual, the call option elasticity $\epsilon_C\,(\infty)$ does not depend on time

$$\epsilon_C\,(\infty) = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1$$

returns are measured per unit of S instead of per unit of time[3].

At the time of exercise, the elasticity coincides with the elasticity of the option value, $(\partial C/\partial S)/(C/S)$, where $C(S)$ is the option value. This holds as

$$\epsilon_C = \left(\frac{\partial C}{\partial S}\right)_{S=\overline{S}} \frac{\overline{S}}{\overline{C}} = 1 \cdot \frac{\overline{S}}{\overline{S}-X} = \epsilon_D \tag{14}$$

(The second equality follows from value matching (6a) and smooth pasting (6b), and the last equality from (13).) Thus the elasticity of the option, ϵ_C, measures the return on the option just like ϵ_D measures the return on the project relative to the net payoff. In the notation of previous sections, (5) implies

$$\epsilon_C = \frac{r_C - r}{\mu - r}(>1) \tag{15}$$

but note that the equivalence of return rates that has been pointed out, is not restricted to the geometric Brownian case, which is the focus of interest both in this paper and in practice.

Where Dixit et al. (1999) represents an approach to optimal exercise of options that does not hinge on smooth pasting, Sødal (1998) also uses the discount factor methodology, but for *deriving* value matching (6a) and smooth pasting (6b) by direct optimization. Both references point out the equivalence of the elasticities, but not the useful interpretation as measures of returns, which has motivated the writing of this paper.

We conclude this section by a short arbitrage argument that provides further intuition on smooth pasting. When exercising a call option, the true cost is really not just the strike price, X, but the sum of the strike price and the value of the foregone option. Assuming the option is optimally exercised, the option value, C, is a function of the project value, S. Starting from a slightly lower, fixed initial point, S_0, the true (net) expected present value of optimal exercise, including all opportunity costs, is therefore

$$D(S)(S - X - C(S)) \tag{16}$$

Maximizing this with respect to S by setting the derivative equal to zero, and noting that $D(\overline{S}) = 1$, gives

$$\left(\frac{\partial D}{\partial S}\right)_{S=\overline{S}} (\overline{S} - X - \overline{C}) = \left(\frac{\partial C}{\partial S}\right)_{S=\overline{S}} - 1 \tag{17}$$

[3]This can be illustrated with the deterministic case, for which $\rho = r$ and $D = e^{-rT}$. In that case T is a function of S, so the elasticity can be written as $\epsilon_D = \frac{(rdT)S}{dS}$. The numerator on the right-hand side is the interest payment on S over the short time period dT. This is divided by dS to obtain the return per unit of S.

The left–hand side of (17) equals zero when there is no value from arbitrage (value has already been maximised). This yields the value matching condition (6a). The smooth pasting condition (6b) follows by setting the right-hand side also equal to zero.

All this works equally well for put options.

5 Role of δ for calls (r for puts)

The dividend yield δ is important here since if it goes to zero, American prices tend toward zero dividend Black Scholes prices and early exercise smooth pasting and rate of return equalisation are all ruled out. That is to say that without an opportunity cost of waiting, early exercise never occurs. Thus it is important to understand the role of the dividend yield in determining the rate of return of the option, since without it rate of return equalisation is impossible.

First we look at the rates of return (sum of cashflow yield and capital gains) on the option, its replicating portfolio and finally its payoff at exercise to see if the magnitude of a dividend yield affects rate of equalisation. Secondly, we use an analytical approximation for American options to further demonstrate the effect that an increase in dividend yield has.

When hedging an option position, the current amounts of stock and borrowing required in the replication portfolio are often labelled Δ, κ (both positive, dynamic functions of S, T) so that the (perfect) local hedge requires[4]

$$C = \Delta S - \kappa X \tag{18}$$

The hedge parameters are given by partial derivatives with respect to S and X respectively

$$\Delta = \frac{\partial C}{\partial S} \qquad \kappa = -\frac{\partial C}{\partial X} > 0 \tag{19}$$

Note that the option has no interim cashflows (only payoffs) but the hedging position generates continuous dividends and interest expense on the hedge amounts $\delta \Delta S - r\kappa X$. Table 1 shows the expected capital gain and cashflow components to each of the option, hedge and payoff as a function of the stock itself (top row). Although the dividend yield changes the balance of capital gain to cashflow within the returns, it does not change the total

[4] When the call option is a linear combination in S, X of homogenous functions of S/X

$$C = \Delta\left(S/X\right)S - \kappa\left(S/X\right)X$$

smooth pasting also implies $\frac{\partial C}{\partial X}\Big|_{S=\overline{S}} = -1$.

Changes in dt :	**Exp.capital gain**	**Cashflow**	**Total return**
Stock S	$dS = (\mu - \delta)\, Sdt$	$\delta S dt$	$\mu S dt$
Option C	dC	0	dC
Hedge $\Delta S - \kappa X$	$(\mu - \delta)\, \Delta dS$	$(\delta \Delta S - r\kappa X)\, dt$	$(\mu \Delta S - r\kappa X)\, dt$
Payoff $S - X$	$(\mu - \delta)\, dS$	$(\delta S - rX)\, dt$	$(\mu S - rX)\, dt$

Table 1: Gains and flows to options holding, replication and payoffs

returns. Thus the influence of early exercise that dividends exert must be through other means.

As $\Delta, \kappa \rightarrow 1$ (at early exercise) the values and returns of the hedge portfolio and the payoff converge irrespective of δ. Thus the presence of δ in these return equations does not affect the returns, only the balance between gain and flow. Therefore as $\delta \rightarrow 0$ early exercise and smooth pasting become impossible because the deltas and elasticities are prevented from equalising, not because of a cashflow argument itself.

6 Analytical approximation

The effective role of the dividend yield on elasticities and deltas and therefore early exercise can be illustrated through the analytic approximation of Barone–Adesi Whaley (1987) and MacMillan (1986) (hereafter BAWM). By decomposing the *premium* of an American option over its corresponding Black Scholes (1973) value into multiplicative functions ($j(T)$ of time T and $k\left(S, 1 - e^{-rT}\right)$ of stock price S and $1 - e^{-rT}$) an approximate form is obtained

$$\begin{aligned} C \;&\approx\; Se^{-\delta T} N\left(d_1(S)\right) - Xe^{-rT} N\left(d_2(S)\right) \\ &\quad + \left(1 - e^{-\delta T} N\left(d_1\left(\overline{S}(T)\right)\right)\right) \frac{\overline{S}(T)}{\epsilon_C(T)} \left(\frac{S}{\overline{S}(T)}\right)^{\epsilon_C(T)} \end{aligned} \tag{20}$$

where the special, maturity dependent, elasticity parameter $\epsilon_C(T) > \epsilon_C(\infty)$ is defined by

$$\epsilon_C(T) = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\left(1 - e^{-rT}\right)\sigma^2}} \quad > \epsilon_C(\infty) \tag{21}$$

(note that $\epsilon_C(T)$ is greater than the infinite horizon value $\epsilon_C(\infty)$ and that it increases as δ increases). The maturity dependent critical stock price $\overline{S}(T)$ solves a value matching condition, which in turn implies an modified form of the critical threshold in the perpetual case above

$$\frac{\overline{S}(T)}{X}=\frac{1-e^{-rT}N\left(d_2\left(\overline{S}(T)\right)\right)}{1-e^{-\delta T}N\left(d_1\left(\overline{S}(T)\right)\right)}\frac{\epsilon_C(T)}{\epsilon_C(T)-1} \tag{22}$$

note that $\overline{S}(T)$ decreases as δ increases (some care must be taken since the approximation means that BAWM's $\overline{S}(T)$ can exceed the Merton $\overline{S}(\infty)$!). That is to say that the higher the dividend yield the earlier exercise will occur and the less waiting will occur. The American option price itself is sensitive to δ (it decreases as δ increases toward immediate payoff $\max(S-X,0)$) but that opportunity cost of waiting is at least partially mitigated by the early exercise feature and the lowering of $\overline{S}(T)$ with δ.

Under this approximation, the delta (and elasticity) of the American option is given by the sum of the elasticity of the Black Scholes and the elasticity of the premium, (even though this is only an approximation for C, it smooth pastes to the payoff at $\overline{S}(T)$)

$$\begin{aligned}\frac{\partial C}{\partial S} &\approx e^{-\delta T}N(d_1(S))+\left(1-e^{-\delta T}N\left(d_1\left(\overline{S}(T)\right)\right)\right)\left(\frac{S}{\overline{S}(T)}\right)^{\epsilon_C(T)-1}\\ \left.\frac{\partial C}{\partial S}\right|_{S=\overline{S}(T)} &\approx 1\end{aligned}$$

Now as δ increases, the Black Scholes component of the delta (and elasticity) decreases (both because of the exponent $e^{-\delta T}$ and the negative effect of δ on d_1). However the early exercise premium's elasticity *increases* both because the exponent $e^{-\delta T}$ and d_1 term decrease and because the critical threshold $\overline{S}(T)$ decreases with increasing δ (even though $\epsilon_C(T)$ increases), rendering the "discount factor" dependent on S greater. This is why the delta can reach unity. The effect of increased dividends makes the European option element delta smaller but the early exercise premium element rises with δ.

In fact for all $S>X$ there is a critical δ^* that would trigger early exercise (another implicit solution of Eq 22 for δ given $\overline{S}(T)=S$ instead of solving for $\overline{S}(T)$ given δ).

In summary it is the early exercise premium that mitigates opportunity cost of waiting and the same is true for the rates of return. It is the increasing rate of return on the early exercise premium that forces rate of return equalisation.

7 Implications and intuition

The results above have economic implications and intuition, particular for real option situations where it may be difficult to evaluate the option value function explicitly. Optimal early exercise of a real option is driven by two conditions, i) no loss (or gain) of value on exercise, ii) a rate of return equalization. Only if both conditions will be met will early exercise be optimal.

This gives a second, equivalent, but more intuitive decision condition to managers other than smooth pasting (which may be difficult to evaluate for some pricing problems). Do managers think that the time has come when the *rate of return* on the project (call) option has *fallen* to the same level as its underlying (levered) project? If so (and if the values of option and project are close also), they should exercise because this is equivalent to smooth pasting (and value matching).

Were the value function equal to the payoff function without rate of return equalization (smooth pasting) it would indicate that the option was not being managed optimally. Only if exercise is optimal will both conditions be achieved simultaneously.

For market participants who track prices of American or company investment options, two simultaneous features should persuade them to exercise, i) a call option being close to its payoff and ii) changes in the call option tracking changes in its payoff 1:1 (the slope coefficient before exercise will be greater than one). That is to say value matching and smooth pasting but now we understand that the call option % changes tracking the levered underlying changes 1:1 implies rate of return equalization. Thus empirically, even if the value function and its derivative are (theoretically) unknown, empirical rates of return can be used to determine the proximity of early exercise.

Finally, since the continuous case is not achievable (in a world with end of day discretization effects), our analysis contributes to the practise of early exercise by allowing the rate of return differential between option and payoff to be analyzed. Theoretically in a world where intraday exercise is possible, this should be zero but with end of day effects, exact equality will not be achieved.

8 References

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