Valuing the Surrender Options Embedded in a Portfolio of Italian Life Guaranteed Participating Policies: a Least Squares Monte Carlo Approach*

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Abstract

We price the surrender option embedded in two common types of guaranteed participating Italian life contracts and we adopt the Least Squares Monte Carlo approach following Longstaff and Schwartz (2001) giving a comparative analysis with the results obtained through a Recursive Tree Binomial approach according to Bacinello (2003). We present an application to a major Italian life policies' portfolio at two different market valuation dates. We use a Black&Scholes-CIR++ economy to simulate the reference fund; we estimate the fair value of portfolio's liabilities according to De Felice and Moriconi (2001), (2002) and Pacati (2000) extending the framework to price the embedded surrender options.

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Keywords: Surrender Option; Longstaff-Schwartz Least Squares Monte Carlo Approach; Black&Scholes-CIR++ Economy.

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1 Introduction

The most common types of life policies issued by Italian companies present two intimately linked faces: one actuarial and the other financial. From an actuarial point of view, these products provide a financial service to individuals that wish to insure themselves against financial losses which could be the consequence of death, sickness or disability. At the same time these products often include interest rate guarantees, bonus distribution schemes and surrender options that represent liabilities to the insurer. In the past, for example in the 1970's and 1980's when long term interest rates were high, some of these options have been viewed by insurers as far out of the money and were ignored in setting up reserves, but the value of these guarantees rose as long as term interest rates began to fall in the 1990's. If the rates provided under the guarantee are more beneficial to the policyholder than the prevailing rates in the market, the insurer has to make up the difference.

The problem of accurately identifying, separating and estimating all the components characterizing the guarantees and the participation mechanism has attracted an increasing interest both of researchers and practitioners from a risk management and option pricing point of view. In their seminal contributions, Brennan and Schwartz (1976),(1979b) and Boyle and Schwartz (1977) have employed the techniques of contingent claims analysis to provide a valuation framework in order to estimate the fair value of a guaranteed equity-linked contract.

According to the recent literature (Jensen, Jørgensen and Grosen (2001), Grosen and Jørgersen (2000) and Bacinello (2003), a life policy contract can be viewed as a participating American contract that can be splitted into a participating European contract and a surrender option. In the participating European contract the benefit is annually adjusted according to the performance of a reference fund, a bonus option, and a minimum return is guaranteed to the policyholder, minimum guarantee option; the literature is rich and we recall Norberg (1999) (2001), Bacinello (2001a), De Felice and Moriconi (2001), (2002), Pacati (2000), Consiglio, Cocco and Zenios (2001a) and (2001b).

The surrender option is defined as an American-style put option that enables the policyholder to give up the contract receiving the surrender value. Commonly surrenders can be modelled by actuarial methods using experience-based elimination tables. The rationality of exercise as for an American put option in the financial markets is assumed in the literature and we recall Albizzati and Geman (1994), Bacinello (2003), Jensen, Jørgensen and Grosen (2001) and Grosen and Jørgersen (2000). The behavior of the policyholder intuitively can be affected by other motivations where redemptions appears to be essentially driven by the evolution of personal consumption plans and the contract can be given up also if is not rationale from a strong financial point of view. In many practical situations the American options embedded in financial contracts turn out to be not rationally exercised as outlined by Schwartz and Torous (1989) referring to mortgage-backed securities and Brennan and Schwartz (1977) and Anathanarayanan and Schwartz (1980) referring to Canadian savings bonds. The surrender option may have significant value if it is not adequately penalized and is rationally exercised as we will give evidence in this paper. We are dealing with long term American put options which are intrinsically sensitive to the interest rate level and the asset allocation decisions achieved by the insurance company's management.

In addition, traditional Italian policies enable the policyholder to give up the contract either receiving the surrender value, a cash payment, or converting the surrender value into a guaranteed annuity, payable through the remaining lifetime and calculated at a guaranteed rate, which can be greater than market interest rate as outlined recently by Boyle and Hardy (2003) and Ballotta and Haberman (2002). Another factor added to the cost of these guarantees, according to Ballotta and Haberman (2003) and Lin and Tan (2003), is the following: the mortality assumption implicit in the guarantee did not take into account the improvement in mortality which took place in the last years.

In this paper our main purpose is to price the surrender option embedded in the Italian life guaranteed participating policies by Least Squares Monte Carlo approach proposed by Longstaff and Schwartz (2001) giving a comparative analysis with the results obtained by a Recursive Tree Binomial approach according to Bacinello (2003) without considering the actuarial uncertainty. Lattice or finite difference methods are naturally suited to coping with early exercise features, but there are limits in the number of stochastic factors they can deal with. These limits are due to the increase in the size of grid or the lattice which is used to discretize the space. On the contrary, one of the major strengths of Monte Carlo simulation is just the ability to price high-dimensional derivatives considering many additional random variables.

Our approach is to jointly take into account the term structure of interest rates and the stock index market making use of a Black&Scholes-CIR++ economy to simulate the reference fund, composed by equities and bonds. We present an application to a relevant portion of RAS SpA life policies' portfolio at the two different valuation dates, 31 December 2002 and 30 June 2003, characterized by significant different market conditions in terms of interest rates level and at-the-money cap implied volatilities. The policies analyzed are characterized by different premium payment styles (single and constant periodical) and are endowments including both a bonus option and a minimum guarantee option. We derive the fair value of portfolio's liabilities according to De Felice and Moriconi (2001), (2002) and Pacati (2000). We extend the Least Squares Monte Carlo approach considering the actuarial uncertainty according to Bacinello (2003) in order to price also the embedded surrender options. We analyze how the fair value of liabilities and the embedded options are affected by financial features as different composition of reference fund and different market interest rates conditions and actuarial features as bonus premia and surrender penalties. The results are purely indicative and the comments do not represent the views and/or opinion of RAS management.

Section 2 discusses the surrender option and the related literature. The Least Squares Monte Carlo approach proposed by Longstaff and Schwartz (2001) to price an American-style option is discussed also and a comparative analysis with the results obtained by a Recursive Tree Binomial approach according to Bacinello (2003) is presented. Section 3 describes the approach followed in the simulation of the reference fund and in the estimation of the fair value of liabilities. An extension of Least Squares Monte Carlo approach to derive the American contracts and to price the surrender option according to Bacinello (2003) is discussed. Then we proceed to analyze the numerical results. Finally, Section 4 presents conclusions.

2 A Least Squares Monte Carlo Approach to Price the Surrender Option

2.1 Surrender Option

Our purpose is to value the surrender option embedded in the endowment life Italian policies. The surrender option is an American-style put option that enables the policyholder to give up the contract and receive the surrender value. We implement a method that uses Monte Carlo simulation, adapting it, so that it can work also with products that present American-exercise features. In particular, we follow the Least Squares Monte Carlo approach presented by Longstaff and Schwartz (2001).

We make a comparative analysis, where only financial risks are treated, between the Least Squares Monte Carlo approach and the Binomial Tree approach adopted by Grosen and Jørgensen (2000) and Bacinello (2003). The effect of mortality is not considered and the riskless rate of interest is assumed to be constant.

We briefly summarize the problem analyzed: at time zero (the beginning of year one), the policyholder pays a single premium to the insurance company and thus acquires a contract of nominal value C_0. The policy matures after T years, when the insurance company makes a single payment to the policyholder. However, the contract can also be terminated depending on the policyholder's discretion before time T. The insurance company invests the trusted funds in an asset portfolio, that replicates a stock index, whose market value $A(t)$ is assumed to evolve according to a geometric Brownian motion,

$$dA(t) = \mu A(t)dt + \sigma A(t)dZ(t), \quad A(0) = A_0, \tag{1}$$

where μ, σ and $A(0)$ are constants and $Z(\cdot)$ is a standard Brownian motion with respect to the real-world measure. Under the risk neutral probability measure Q the evolution is given by

$$dA(t) = rA(t)dt + \sigma A(t)dZ^Q(t), \quad A(0) = A_0, \tag{2}$$

where $Z^Q(\cdot)$ is a standard Brownian motion under Q and r is the instantaneous spot rate. The rate credited to the policyholder once a year from time $t-1$ to time t, $t \in \{1, \ldots, T\}$, is denoted $r_C(t)$ and is guaranteed never to fall below s_{min}, the contractually specified guaranteed annual interest rate:

$$r_C(t) = \max\left(\frac{\beta I(t) - i_{tec}}{1 + i_{tec}}, s_{min}\right), \quad s_{min} = \frac{i_{min} - i_{tec}}{1 + i_{tec}}, \tag{3}$$

this is due to the policy holder at regular time dates defined by the contract (for example on annual or monthly base). We define i_{tec} as the technical interest rate that is used for reducing the rate of return given to the policyholder, s_{min} is the minimum rate guaranteed every time the return of reference fund is calculated and $\beta \in (0, 1]$ is the participation coefficient of the policy holder to the return of reference fund. Generally it assumes values from 80% to 95% and the difference 1-β is retained by the insurance company and provides an incentive to the insurance company on the asset allocation decisions achieved. The annual rate of return of the reference fund at time t, $I(t)$, is defined as:

$$I(t) = \frac{A(t)}{A(t-1)} - 1, \tag{4}$$

The nominal value C_0 grows according to the following mechanism:

$$C(t) = (1 + r_C(t)) \cdot C(t-1), \quad t \in \{1, 2, \ldots, T\}, \quad C(0) = C_0. \tag{5}$$

According to Grosen and Jørgensen (2000) and Bacinello (2003) we define two contract types: the European contract and the American contract. The former is simply the contract that pays $C(T)$ at the maturity date T, whereas the latter can be exercised depending on the policyholder's discretion at any time t in the set $\{1, 2, \ldots, T\}$. If the policyholder decides to exercise at time t, he receives $C(t)$. The surrender option value is given by the difference between the American contract value and the European contract value.

In order to price the American contract, Grosen and Jørgensen (2000) and Bacinello (2003) implement a binomial tree model á la Cox, Ross and Rubinstein (1979) while Jensen, Jørgensen and Grosen (2001) develop and implement a finite difference algorithm. Grosen and Jørgensen (2000) and Jensen, Jørgensen and Grosen (2001) use a different type of revaluation mechanism with respect to equation (3):

$$r_C(t) = \max\left\{\beta\left(\frac{B(t-1)}{C(t-1)} - \gamma\right), r_G\right\}, \tag{6}$$

where γ is the target buffer ratio, r_G is the contractually specified guaranteed annual interest rate similar to s_{min} and $B(t) = A(t) - C(t)$. C_0 grows according to the equation (5) as for the Italian mechanism, $C_0 = 100$ and $B_0 = 0$. Because of the dependence of the contract values on both $A(\cdot)$ and $C(\cdot)$, the size of the trees which keep track of these variables grows exponentially with T and Grosen and Jørgensen (2000) are forced to implement a recursive scheme using an annual step. This allows for $T+1$ final values of $A(t)$, 2^T different paths, and similarly (up to) 2^T different terminal values of $C(t)$. According to Grosen and Jørgensen (2000) the results cannot be so accurate.

Bacinello (2003) analyzes a life insurance product introduced in Italy at the end of 1970's and takes into account the presence of the surrender option employing a recursive binomial formula for describing the stochastic evolution of $A(t)$. Each policy year (the period that goes from a payment to the policy holder to the succeeding one) is divided into N subperiods of equal length. Let $\Delta = 1/N$, fix a volatility parameter $\sigma > \sqrt{\Delta} ln\,(1+r)$, set $u = exp\left(\sigma\sqrt{\Delta}\right)$ and $d = 1/u$. Then $A(t)$ can be observed at the discrete times $\delta = t + h\Delta, t = 0, 1, ...T; h = 0, 1, ..., N-1$ and $A(\delta+\Delta)$ can take only two possible values: $uA(\delta)$ ("up" value) and $dA(\delta)$ ("down" value). Under the risk-neutral measure, the probability of the event $A(\delta+\Delta) = uA(\delta)$ is given by

$$q = \frac{(1+r)^\Delta - d}{u-d} \tag{7}$$

while

$$1 - q = \frac{u - (1+r)^\Delta}{u-d} \tag{8}$$

represents the risk-neutral probability of the event $A(\delta+\Delta) = dA(\delta)$. The above assumptions imply that $I(t)$, $i = 1, 2, ..., T$ takes one of the following $N+1$ possible values:

$$I_j(t) = u^{N-j}d^j - 1, \quad j = 0, 1, ..., N \tag{9}$$

with risk neutral probability

$$Q_j = \binom{N}{j} q^{N-j}(1-q)^j, \quad j = 0, 1, ..., N. \tag{10}$$

At the same time the annual interest rate given to the policyholder can take $n+1$ possible values given by

$$r_{C_j}(t) = \frac{\beta I_j(t) - i_{tec}}{1 + i_{tec}} \quad j = 0, 1, ..., n \tag{11}$$

with probability Q_j and s_{min} with probability $1 - \sum_{j=0}^{n} Q_j$. We define n as the number of times that the values assumed by (11) are greater than s_{min}.

The European contract value is the expected value at time t of the terminal value and is defined as $E^Q(e^{-rT}C(T))$. The time in which the policy holder can exercise the option to surrender is $t = 1, 2, ..., T-1$ and the American contract value is computed by means of a backward recursive procedure operating from time $T-1$ to time 1. We observe that in each node at time T the value $F(T)$ of the whole contract coincides with $C(T)$, whereas at time $t < T$, in the backward procedure, in each node we compare the continuation value, that is the value deriving from staying in the contract, with the intrinsic value, that derives from immediately exercising the contract. The continuation value in each node is

$$F(t) = \max(C(t), E_t^Q[e^{-r}F(t+1)]). \tag{12}$$

2.2 Least Squares Monte Carlo Approach

We now briefly describe the method suggested in the paper by Longstaff and Schwartz (2001) in order to price American options by Monte Carlo simulation (LSM: Least Squares Monte Carlo approach). The mechanism underlying an option with American exercise features is the following: at any exercise time, the holder of an American option compares the payoff from immediate exercise, which we refer to as intrinsic value, with the expected payoff from continuation, and exercises if the immediate payoff is higher. In other words, at each simulated time instant, the value of the contract is the maximum between the intrinsic value and the continuation value. Thus, the optimal exercise strategy is determined by the discounted conditional expectation under the risk-neutral probability measure of the future cash flows, assuming an optimal exercise policy is adopted in the future. For example, in the case of an American put option written on a single non-dividend paying asset, the value (cashflow) $V(t, A(t))$ of the option at time t, conditional on the current asset price $A(t)$, considered step by step, is recursively given by

$$V(t, A(t)) = \max \left\{ I(t, A(t)), E_t^Q \left[e^{-r} V(t+1, A(t+1)) | A(t) \right] \right\}, \tag{13}$$

where $I(t, A(t))$ is the intrinsic value. The difficulty in using Monte Carlo derives from the fact that we should know the conditional expected value of the future option value, but this depends on the next exercise decisions.

The approach developed by Longstaff and Schwartz (2001) is that this conditional expectation can be estimated from the cross-sectional information in the simulation by using least squares, that is by regressing the discounted realized payoffs from continuation on functions of the values of the state variables (the current underlying asset price in this example). For example, the use of a quadratic polynomial would give

$$E_t^Q \left[e^{-r} V(t+1, A(t+1)) | A(t) \right] \approx a_1 + a_2 A(t) + a_3 A(t)^2. \tag{14}$$

While the generation of sampled paths goes forward in time, the regressions go backwards, starting from time T. At this time, the exercise strategy is trivial: the option is exercised if and only if it is in-the-money. If the strike is X, the cashflows for each path j are $\max\{X - A_j(T), 0\}$, provided that the option has not been exercised yet. Going backwards in time to time step $T-1$, if on a path the option is in-the-money, one may consider exercising it. The continuation value is approximated regressing the discounted cashflows only on the paths where the option is in-the-money. The important point is that the early exercise decision is based on the regressed polynomial, in which the coefficients are common on each path, and not on the knowledge of the future price along the same path.

2.3 Numerical Results

In the section devoted to numerical results, we analyze contracts for which $C_0 = 100$ and $T = 4$ years reporting the values of the American contract, European contract and surrender option. In Tables 1-4 we present the results we obtain through LSM, simulating

$400,000$ paths, and the Binomial Tree approach with a step $\Delta = 1/50$ according to Bacinello (2003) for different values of interest rate r, volatility σ, participation coefficient β and minimum guarantee s_{min}. The values of the American contract have been produced using different combinations of A, C and r_C as state variables and using different order for the polynomial. When we use the two state variables A and C together and the third-order polynomial, we involve cross-products such as AC, AC^2, A^2C and we regress according to

$$E_t^Q\left[e^{-r}F(t+1)\right] \approx a_1 + a_2A(t) + a_3A(t)^2 + a_4A(t)^3 + a_5C(t) + a_6C(t)^2 + \\ + a_7C(t)^3 + a_8A(t)C(t) + a_9A(t)^2C(t) + a_{10}A(t)C(t)^2. \qquad (15)$$

The reason for using more than one state variable is that, as pointed out by Longstaff and Schwartz (2001), if the regression involves all paths, more than two or three times as many basis functions may be needed to obtain the same level of accuracy as obtained by the estimator based on in-the-money paths. This is our case, since the intrinsic value is not, for example, the standard payoff of a put, but is given by the surrender value, so we cannot limit the number of values used in the regression to those where the option is in-the-money, but we have to consider all of them.

We assume σ=15%, r=5%, s_{min}=0% and β=45% and we vary the values of σ from 5% to 40% keeping constant r, s_{min} and β, and so for r from 0% to 10%, s_{min} from 0% to 4.5% and β from 40% to 100%. We base our regression using the state variables A, C and r_C and using a third order polynomial.

In general, our results do not produce a significant error, since the difference between the values of American and European contract obtained with the two approaches is in mean less than 2%; in addition, the differences are both positive and negative and irrelevant in most of the cases. We report the standard error deviation (s.e.) of our results. These results suggest that the LSM algorithm is able to approximate closely the Binomial Tree values. However the Binomial Tree values depend on the step adopted, $\Delta = 1/50$, and a higher number of steps in each year requires a large amount of CPU time; that is why we have to fix a low value for T and a high value for Δ.

We now look at the values in the Tables 1-4. As in Grosen and Jørgensen (2000) a higher participation coefficient, β, reduces the value of surrender option and the value of

American contract equals the value of European contract. A lower participation coefficient makes more incentive for the policyholder to give up the contract. This depends significantly on the values assumed by the minimum guarantee s_{min} that is not reduced by β. As expected, all the results reported in Table 2 are very sensitive with respect to the market interest rate r. The value of surrender option is increasing when r is increasing, revealing that the early exercise is never optimal when the market interest rate r is low and near to the minimum guarantee s_{min}. From Table 3 we observe that the minimum guarantee s_{min} has a significant influence in determining the value of the American contract and an increasing minimum guarantee s_{min} reduces the difference between the American and the European contract value. The volatility parameter σ affects significantly the results according to Table 4. A decreasing volatility produces an increase in the value of the surrender option making more incentive for the policyholder to give up the contract receiving the surrender value. As expected these results depend on the value of minimum guarantee s_{min}=0% and level of the market interest rate r=5%.

3 An Application to an Italian Life Policies' Portfolio

In this section we explain the assumptions and the method we follow to simulate the reference fund, to derive the fair value of the insurance life policies in terms of a European contract and to extend the European contract to an American contract pricing the surrender option. Finally we proceed to analyze the numerical results obtained at two different valuation dates.

3.1 Assets: Reference Portfolio

Our approach is to jointly model the term structure of interest rates and the stock index with stochastic processes governed by stochastic differential equations. Given the continuous-time perfect-market assumptions we assume that the reference fund evolves according to the following equation:

$$L(t) = \alpha A(t) + (1-\alpha)G(t), \quad 0 \leq \alpha \leq 1, \tag{16}$$

where $A(t)$ is a stock index and $G(t)$ is a bond index. De Felice and Moriconi (2001),

(2002) and Pacati (2000) model $G(t)$ as the cumulated results of a buy-and-sell strategy, with a fixed trading horizon δ, of stochastic zero coupon bonds with a fixed duration $D \geq \delta$ derived according to Cox, Ingersoll and Ross (CIR) model (1985).

We model interest rate uncertainty with the CIR++ model according to Brigo and Mercurio (2001), an extension of CIR model, yielding a short rate model that allows us (i) to obtain an exact fit of any observed term structure; (ii) to derive analytical formulae for bond prices, bond option prices, swaptions and caps prices; (iii) to guarantee positive rates without worsening the volatility calibration; (iv) to have the distribution of the instantaneous spot rate with fatter tails than in Gaussian case. The instantaneous interest rate (spot rate) $r(t)$ is defined as

$$r(t) = y(t) + \varphi(t). \tag{17}$$

where y is a process that evolves according to the CIR model

$$dy(t) = \kappa(\vartheta - y(t))dt + \eta\sqrt{y(t)}dZ^y(t), \quad y(0) = y_0, \tag{18}$$

where κ is the mean reversion coefficient, ϑ is the long term rate, η is the volatility parameter and y_0 is the initial spot rate. $\varphi(t)$ is a deterministic function, depending on the set Θ of parameters κ, ϑ, η, y_0 and integrable on closed intervals. The function φ can be chosen so as to fit exactly the initial term structure of interest rates: the model fits the currently observed term structure of discount factors if and only if

$$\exp\left[-\int_t^T \varphi(s)ds\right] = \frac{P^M(0,T)}{\Pi^y(0,T)}\frac{\Pi^y(0,t)}{P^M(0,t)}, \tag{19}$$

where $P^M(0,t)$ denotes the discount factor observed in the market for the maturity t and $\Pi^y(0,t)$ is the CIR closed-form formula, depending on the set Θ of parameters, for the price at time 0 of a zero-coupon bond maturing at t and with unit face value.

For the stock index $A(t)$, we use a Black&Sholes model having the following stochastic equation under the risk neutral probability measure:

$$dA(t) = r(t)A(t)dt + \sigma A(t)dZ^Q(t), \tag{20}$$

where σ is the volatility parameter. The model incorporates the correlation between the interest rate and the stock index. This is accomplished by explicitly introducing the correlation between the diffusion processes:

$$Corr(dZ^y(t), dZ^Q(t)) = \rho^{yQ}dt. \tag{21}$$

By the Markov property, the price of a traded security is a function of the state variables $r(t)$ and $A(t)$ at the time t:

$$V(t) = V(y(t), A(t), t). \tag{22}$$

The price $V(t)$ of security is determined by the risk adjusted probability measure where the drift of $r(t)$ and $A(t)$ are replaced by:

$$\kappa(\vartheta - y(t)) + \pi y(t) \tag{23}$$

$$r(t)A(t) \tag{24}$$

respectively and the aggregate parameters for the pricing are $\hat{\vartheta} = \vartheta\kappa$ and $\hat{\kappa} = \kappa - \pi$ and π is a constant and is associated with the market price of interest rate risk endogenously specified as the function:

$$q(y(t), t) = \frac{\pi\sqrt{y(t)}}{\eta}. \tag{25}$$

The model is computationally tractable in the sense that it can be implemented using Monte Carlo simulation: we generate a discrete sample path for $r(t)$ and $A(t)$. We calculate the price of zero coupon bonds with duration D along the sample path of $r(t)$ applying the short-term roll-over strategy fixing trading horizon δ and we derive the market value and the path of the reference fund $L(t)$. As we can observe from Figures 1 and 2, when D is short, the simulation of $G(t)$ generates paths with a high dispersion on the long period, whereas when D is long, the paths are characterized by a high local dispersion, but a reduced dispersion on the long period. We will see in the next section how interest rates and volatility structure, duration, weight and the volatility of equity affect the fair value of insurance life policies and its components. We will focus on the interest minimum guaranteed option and the surrender option.

3.2 Liabilities: European Contract

We consider two types of participating endowment policies, those paid by a single premium and those paid by constant annual premiums, part of a significant portion of a major Italian life portfolio. The premiums earned are invested into a reference fund. The benefit is annually adjusted according to the performance of the fund and a minimum return is guaranteed to the policyholder.

We define x as the age of the policyholder at the inception of the contract and n as the time length of the contract. We define a as the number of years between the inception date of the contract and our valuation date. We assume that a is integer and therefore the policy starts exactly a years before the valuation date, which we denote by t. $m = n - a$ is the time to maturity. We suppose that all the contractually relevant future events (premium payments, death and life at maturity) take place at the integer payment dates $a+1, \ldots, a+m$. In particular, benefit payments occur at the end of the year of death, if the policyholder dies within the remaining m years, or at the end of the m-th year, if he is alive after m years. Premium payments occur at the beginning of the year. Finally, we do not consider possible future transformations of the policies, such as reduction and guaranteed annuity conversion options.

Let $C(a)$ be the sum insured at the inception date $t - a$. We define $C(a+k)$ and $L(t+k)$ respectively as the benefit eventually paid at time $a+k$ and the market value of the reference fund, composed by a stock index and a bond index as specified in the previous section, at time $t+k$, $k = 1, \ldots, m$. As before i_{tec} is the technical rate, i_{min} is the minimum guaranteed and $\beta \in (0, 1]$ is the participation coefficient. We introduce i_{tr}, a minimum rate retained by the company, that reduces the rate credited to the policyholder. The annual rate of return of the reference fund at time $t+k$, $I(t+k)$, is defined as:

$$I(t+k) = \frac{L(t+k)}{L(t+k-1)} - 1, \tag{26}$$

and the readjustment measure is

$$r_C(t+k) = \max\left(\frac{\min(\beta I(t+k), I(t+k)-i_{tr}) - i_{tec}}{1+i_{tec}}, s_{min}\right) \tag{27}$$

that is different respect to (3) for the factor $\min(\beta I(t+k), I(t+k)-i_{tr})$. For $C(a+k)$, in the case of single premiums, we can write the following equation

$$C(a+k) = C(a+k-1)(1+r_C(t+k)) = C(a)\Phi(t,k), \tag{28}$$

$$\Phi(t,k) = \prod_{h=1}^{k}(1+r_C(t+h)), \tag{29}$$

(full readjustment rule), whereas in the case of constant annual premiums, we have, according to Pacati (2000),

$$\begin{aligned} C(a+k) &= C(a+k-1)(1+r_C(t+k)) - \frac{m-k}{n}C(0)r_C(t+k) \\ &= C(a)\Phi(t,k) - \frac{1}{n}C(0)\Psi(t,m,k), \end{aligned} \tag{30}$$

$$\Psi(t,m,k) = \sum_{h=0}^{k-1}(m-k+h)r_C(t+k-h) \cdot \prod_{l=k-h+1}^{k}(1+r_C(t+l)). \tag{31}$$

We value the European contract on first order bases using conservative probabilities excluding surrenders and considering net premiums. We denote by P the net constant premium paid by the policyholder in case of annual premiums; we suppose the valuation takes place soon after the premium payment. We define ${}_kp_{x+a}$ as the probability that the policyholder, with age $x+a$, is alive at time $a+k$, and ${}_{k-1|1}q_{x+a}$ as the probability that the policyholder, alive at time $a+k-1$, dies between $a+k-1$ and $a+k$. The policies paid by constant annual premiums that we consider are also characterized by the presence of a terminal bonus. We define b_D as the bonus paid in case of death and b_L as the bonus paid in case of life at maturity; both expressed as percentages of the benefit. The quantities $C(a+k)b_D$ and $C(a+m)b_L$ are also readjusted according to equation (30), but are multiplied by corresponding modifying survival probabilities with the persistency frequencies in the contract. We will denote the modified survival and death probabilities by ${}_kp^*_{x+a}$ and ${}_{k-1|1}q^*_{x+a}$ respectively. The net constant premium P is given by the sum of two components: $P = P_{tb} + P_m$, where P_{tb} is the component relative to the terminal bonus and P_m is the residual component of the premium.

According to the assumptions defined by De Felice and Moriconi (2001), (2002), the European contract value is the difference between the fair market value of the future benefits payable by the insurance company and the fair market value of the eventual future premiums payable by the policyholder, both multiplied for the corresponding survival probabilities:

$$V^E(t) = {}_mp_{x+a}C(a)V(t, \Phi(t, m)) + \sum_{k=1}^{m} {}_{k-1|1}q_{x+a}C(a)V(t, \Phi(t, k)) \tag{32}$$

for single premiums and

$$\begin{aligned} V^E(t) &= ({}_mp_{x+a} + {}_mp^*_{x+a}b_L)(C(a)V(t, \Phi(t, m)) - \frac{C(0)}{n}V(t, \Psi(t, m, m))) \\ &+ \sum_{k=1}^{m}({}_{k-1|1}q_{x+a} + {}_{k-1|1}q^*_{x+a}b_D)(C(a)V(t, \Phi(t, k)) - \frac{C(0)}{n}V(t, \Psi(t, m, k))) \\ &- \sum_{k=1}^{m-1}({}_kp_{x+a}P_m + {}_kp^*_{x+a}P_{tb})v(t, t + k) \end{aligned} \tag{33}$$

for constant annual premiums. We define $V(t, \Phi(t, k))$ and $V(t, \Psi(t, m, k))$ as the valuation factors

$$\begin{aligned} V(t, \Phi(t, k)) &= E^Q_t\left[\Phi(t, k)e^{-\int_t^{t+k} r(u)du}\right] \\ V(t, \Psi(t, m, k)) &= E^Q_t\left[\Psi(t, m, k)e^{-\int_t^{t+k} r(u)du}\right] \end{aligned} \tag{34}$$

derived through Monte Carlo approach after simulating the stock index A and the spot rate r that affect the reference fund L and so r_C as explained before. E^Q_t is the conditional expectation under the risk neutral measure Q. We denote $v(t, t+k)$ as the discount factor between times t and $t + k$

$$v(t, t + k) = e^{-\int_t^{t+k} r(u)du}. \tag{35}$$

We can apply the survival probabilities after calculating the fair financial values, since we suppose that actuarial and financial uncertainties are independent.

The fair value of the European contract can be viewed (see De Felice and Moriconi (2001), (2002) and Pacati (2000)) as a "put decomposition" and a "call decomposition" from a financial point of view. According to the "put decomposition", we observe in fact that the expression for $C(a+k)$, eq(28) and eq(30), can be decomposed into a base component and a put component. If we assume that it is always $\min(\beta I(t+k), I(t+k) - i_{tr}) \geq i_{min}$, we obtain the base component:

$$\mathcal{B}(a+k) = C(a) \prod_{h=1}^{k} \left(1 + \frac{\min(\beta I(t+h), I(t+h) - i_{tr}) - i_{tec}}{1 + i_{tec}}\right) \tag{36}$$

in the case of a single premium payment, and

$$\begin{aligned} \mathcal{B}(a+k) \quad &= \quad C(a) \prod_{h=1}^{k} \left(1 + \frac{\min(\beta I(t+h), I(t+h) - i_{tr}) - i_{tec}}{1 + i_{tec}}\right) \\ &- \quad \frac{C(0)}{n} \sum_{h=0}^{k-1} \left[(m-k+h) \frac{\min(\beta I(t+k-h), I(t+k-h) - i_{tr}) - i_{tec}}{1 + i_{tec}} \cdot \right. \\ &\cdot \quad \left. \prod_{l=k-l+1}^{k} \left(1 + \frac{\min(\beta I(t+l), I(t+l) - i_{tr}) - i_{tec}}{1 + i_{tec}}\right)\right] \end{aligned} \tag{37}$$

for constant annual premiums. Observing that $\mathcal{B}(a+k) \leq C(a+k)$, we can define the put component as the difference

$$\mathcal{P}(a+k) = C(a+k) - \mathcal{B}(a+k). \tag{38}$$

Equation (38) is the payoff of an European put option of annual *cliquet* type, that guarantees a consolidation of the results obtained year by year, with annual strike rate s_{min}, and where the underlying is the minimum between the return of the reference fund, multiplied by the participation coefficient, and the minimum retained by the company.

According to the "call decomposition", $C(a+k)$ can be viewed as the sum of a guaranteed component and a call component. Supposing that the sum initially insured grows at the minimum guaranteed level, we obtain the guaranteed component:

$$\mathcal{G}(a+k) = C(a)(1+s_{min})^k \tag{39}$$

in the case of a single premium payment, and

$$\mathcal{G}(a+k) = C(a)(1+s_{min})^k - \frac{C(0)}{n} s_{min} \sum_{l=0}^{k-1} l(1+s_{min})^l \tag{40}$$

for constant annual premiums. The difference

$$\mathcal{C}(a+k) = C(a+k) - \mathcal{G}(a+k) \tag{41}$$

is the call component: it is the payoff of an European call option of annual *cliquet* type and it represents the value of the extra-return of the reference fund with respect to the minimum guaranteed s_{min}.

The two decompositions also hold for the quantities $C(a+k)b_D$ and $C(a+m)b_L$.

3.3 Liabilities: American Contract and an Extension of Least Squares Monte Carlo Approach

We now describe how we apply LSM approach to our model. In considering actuarial uncertainty, we follow the approach adopted by Bacinello (2003). From a LSM point of view, the approach is the following (we recall that n denotes the term of the policy): at step $n-1$ (if the insured is alive) and along path j, the continuation value is given by

$$W_j(n-1) = v_j(t+n-1, t+n)C_j(n) \tag{42}$$

for single premium and

$$W_j(n-1) = v_j(t+n-1, t+n)C_j(n)(1+b_L) - P \tag{43}$$

for constant periodical premiums, since the benefit $C_j(n)$ is due with certainty at time n. P is the premium due at time $n-1$ and $v_j(t+n-1, t+n)$ is the discount factor between $n-1$ and n. The intrinsic value $R_j(n-1)$ (also referred to as surrender value in this context) is the benefit due in $n-1$ discounted with an annually compounded discount rate i_{sur} (eventually equal to zero) that can be considered as a surrender penalty applied to the policyholder benefit:

$$R_j(n-1) = \frac{C_j(n-1)}{(1+i_{sur})} \tag{44}$$

for a single premium and

$$R_j(n-1) = \left(C_j(n-1) - \frac{C(0)}{n}\right)\frac{1}{(1+i_{sur})} \tag{45}$$

for constant annual premiums. The value of the contract for path j is therefore the maximum between the continuation value and the surrender value:

$$F_j(n-1) = \max\{W_j(n-1), R_j(n-1)\}. \tag{46}$$

Assume now to be at time $a+k < n-1$ and along the path j: to continue means to immediately pay the premium P for constant annual premiums or nothing for single premium and to receive, at time $a+k+1$, the benefit $C_j(a+k+1)$, if the insured dies within one year, or to be entitled of a contract whose total random value (including the option of surrendering it in the future), equals $F_j(a+k+1)$, if the insured is alive. We suppose that the benefit received in $a+k+1$, in case of death between $a+k$ and $a+k+1$, is $C_j(a+k)$: we thus have a quantity known in $a+k$ and we can avoid taking the conditional discounted expectation of $C_j(a+k+1)$. The continuation value at time $a+k$ is then given by the following difference:

$$\begin{aligned} W_j(a+k) &= \Big\{{}_{1|1}q_{x+a+k} v_j(t+k, t+k+1) C_j(a+k) + (1 - {}_{1|1}q_{x+a+k}) \cdot \\ &\quad \cdot\ E^Q_{t+k}\left[e^{-\int_{t+k}^{t+k+1} r(u)du} F(a+k+1)\right]\Big\} \end{aligned} \tag{47}$$

for a single premium and

$$\begin{aligned} W_j(a+k) &= \Big\{{}_{1|1}q_{x+a+k} v_j(t+k, t+k+1) C_j(a+k)(1+b_D) + (1 - {}_{1|1}q_{x+a+k}) \cdot \\ &\quad \cdot\ E^Q_{t+k}\left[e^{-\int_{t+k}^{t+k+1} r(u)du} F(a+k+1)\right] - P\Big\}. \end{aligned} \tag{48}$$

for constant annual premiums. We use the regression to estimate the value

$$E^Q_{t+k}\left[e^{-\int_{t+k}^{t+k+1} r(u)du}F(a+k+1)\right]. \tag{49}$$

For the surrender value, we have

$$R_j(a+k) = C_j(a+k)(1+i_{sur})^{(k-m)} \tag{50}$$

in the case of a single premium, and

$$R_j(a+k) = \left(C_j(a+k) - C(0)\frac{m-k}{n}\right)(1+i_{sur})^{(k-m)} \tag{51}$$

in the case of constant annual premiums. At each step and for each path we compare the intrinsic value with the continuation value and take the optimal decision. For those paths where the optimal decision is to surrender, we memorize the time and the corresponding cashflow. When we arrive at time 1 (or the time the insured can start to surrender from), we have a vector with the exercise times for each path and a vector with the cashflows (benefits) corresponding to that time. At this point, taking into account that the policyholder can survive until the exercise time or die before, for each path we multiply the benefits due until exercise time by the corresponding survival or death probabilities, sum them all and finally calculate the average on the number of paths.

3.4 Application and Numerical Results

In Tables 5-18, we present the results we obtain analyzing 944 policies paid by single premium and 1,000 policies paid by constant annual premiums. The values are expressed in euro. We make the calculations on every policy without aggregating and we group the results by age layers. The two types of policies we consider are characterized by: i_{tec}=3%, i_{min}=4%, β=80%. We suppose that the policyholder can surrender after the first year from the inception of the contract and we assume that i_{tr}=1%.

We calibrate the CIR++ model on a cross-section of euro swap interest rates and implied at-the-money Euro cap volatilities for different maturities estimating the parameters under the risk neutral probability measure with Minimum Lest Squares approach. We consider two valuation dates: 31st December 2002 and 30th June 2003 and the estimated parameters are respectively $\hat{\kappa} = 0.2823$, $\hat{\vartheta} = 0.0437$, $\eta = 0.0833$, $y_0 = 0.0056$ for the first valuation date and $\hat{\kappa} = 0.3326$, $\hat{\vartheta} = 0.04742$, $\eta = 0.0981$, $y_0 = 0.0036$ for the second valuation date. According to the Figures 3 and 4, the market conditions are quite different: we observe a significant decrease of interest rates level in particular on the short maturities. The structure of at-the-money Euro cap volatilities shows a decrease shape at 31st December 2002 but the structure becomes humped and decreasing from the three year maturity on at 30th June 2003. We detect a significant increase of the implied volatilities from the one to five year maturities at 30th June 2003.

We simulate 10,000 paths for the reference fund, discretizing the equations for the spot rate and the stock index under the neutral probability measure according to the stochastic Euler scheme with a monthly step. We assume ρ^{yQ}=-0.1 and δ=3 months and analyze two different asset compositions obtained through different values of α, σ and D. We define the first asset composition as "conservative" with α=10%, σ=15%, D=5 and the second as "aggressive" with α=30%, σ=30%, D=10 years. For single premiums, in Tables 5-8 we suppose that the surrender penalty, expressed by the annually compound discount rate i_{sur}, is equal to zero. For constant annual premiums, in Tables 9-12 we set $b_L = 15\%$, $b_D = 10\%$ and $i_{sur} = 0\%$. Then we proceed to make different hypothesis for b_D, b_L and i_{sur} to make a sensitivity analysis to measure how actuarial features of the contracts affect the fair value of liabilities and the value of the surrender option for the constant annual premium policies: in Table 13-16 we set $i_{sur} = 0$ and $b_L = b_D = 0$; in Table 17 we set $b_V = 15\%$, $b_M = 10\%$ and $i_{sur} = 5\%$ if $m - k < 5$ and $i_{sur} = 4.5\%$ if $m - k \geq 5$ in Table 18 we set $b_L = 0\%$, $b_D = 0\%$ and $i_{sur} = 5\%$ if $m - k < 5$ and $i_{sur} = 4.5\%$ if $m - k \geq 5$. We choose to regress, for the valuation of the continuation value at time $t + k$, on three state variables: the value of the reference fund $L(t + k)$, the value of the benefit $C(a + k)$ and the revaluation mechanism $r_C(t + k)$, using a third order polynomial.

Looking at the values of the European contract, we observe that it increases from 31st December 2002 to 30th June 2003 (Tables 5-18) due to the decreasing of interest rates and the increasing at-the-money cap implied volatilities (Figures 3 and 4) keeping the asset composition fixed. The interest rate structure is very close to the minimum guarantee level: the values of put options increase too, meaning that the weight of the put component becomes greater. We observe that the European contract increases too as α, σ and D increase, so passing from the "conservative" to the "aggressive" asset composition. Put option values too, become more valuable with increasing uncertainty, due to increasing volatility of the underlying portfolio.

The value of the surrender option is relevant and decreases with a more "aggressive" composition of the reference fund (Tables 5-8) for single premium policies. This is pointed out also in our results (Table 4) and in Grosen and Jørgensen (2000) and is due to the fact that a more aggressive policy determines an advantage for the policyholder only, and so his incentive to prematurely exercise may be partly or fully reduced. Moreover this effect is reinforced when the minimum guarantee component is relevant. This pattern is significantly reinforced too by a decrease of interest rate structure and the increasing at-the-money cap implied volatilities as we observe from Figure 3 and 4.

The value of surrender option of constant annual premiums doesn't change in a significant way respect to the composition of reference fund and the interest rates market conditions and slightly increases with a more "aggressive" composition of the reference fund (Table 9-12). The results seems to be contradictive respect to the results obtained on single premium policies but these can be explained by the characteristics of the life contracts. The future premiums to be paid, according to eq(48), are constant and could play a significant role in leading the policyholder's decision smoothing the value of the surrender option. At the same time the bonus components embedded in the constant annual premium policies explain our previous results. When we set $b_L = b_D = 0$ (Table 13-16) we obtain lower values for European contracts and put options than in the presence of positive terminal bonus, and higher values for the surrender option deriving the same pattern of single premium's results in terms of reference fund's composition. This means that the terminal bonus is a strong incentive for the policyholder not to surrender and gives evidence how much important are the actuarial features embedded in the life contract and how much relevant is the value if surrender option if it is not adequately penalized. We observe, then, that the presence of the terminal bonus reduces significantly the effect of composition of the reference fund. Therefore the value of the surrender option significantly decreases also when we increase the surrender penalty from $i_{sur} = 0$ to $i_{sur} = 5\%$ if $m - k < 5$ and $i_{sur} = 4.5\%$ if $m - k \geq 5$ (Table 17-18): this is consistent with the fact that a positive value for i_{sur} determines a decrease of the surrender value.

4 Conclusions

In this paper we have presented a pricing application analyzing, in a contingent-claims framework, the two most common types of life policies sold in Italy. These policies, characterized by different premium payment styles (single and constant annual), are endowments including different types of options as the surrender option. We have proposed to price the surrender option by Least-Squares Monte Carlo (LSM) approach according to Longstaff and Schwartz (2001) giving a comparative analysis with the results obtained by a Recursive Tree Binomial approach. Lattice or finite difference methods are naturally suited to coping with early exercise features, but there are limits in the number of stochastic factors they can deal with. On the contrary, one of the major strengths of Monte Carlo simulation is just the ability to price high-dimensional derivatives considering many additional random variables. Our results are relevant: the differences between the Binomial Tree and LSM approach showed to be not significant.

We have then proceeded to present the application to a significant portion of a major Italian life policies' portfolio. We have adopted a Black&Scholes-CIR++ economy to simulate the reference fund and we have estimated the fair value of portfolio's liabilities, viewed as American contracts, pricing the single components. We have made our valuation at two different dates characterized by different interest rates and at-the-money cap implied volatilities.

The surrender option may assume significant value if it is not adequately penalized and is rationally exercised as we give evidence in this paper. We made evidence that we are dealing with long term American put options which are intrinsically sensitive to the interest rate level and the asset allocation decisions achieved by the insurance company's management. At the same time the characteristics of the contract from an actuarial point of view as surrender penalties and bonus components affect significantly the fair value of the surrender option and have to be adequately valued during the profit testing of the contract before issue.

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Table 1: σ=15%, r=5%, i_{min}=3%, i_{tec}=3%

	Binomial Tree			Simulation				
β	American	European	Surrender	American	s.e.	European	s.e.	Surrender
0.40	97.041	88.679	8.362	97.066	0.005	88.744	0.010	8.322
0.45	97.430	90.110	7.320	97.455	0.006	90.172	0.012	7.283
0.50	97.819	91.559	6.260	97.849	0.007	91.638	0.013	6.211
0.55	98.209	93.025	5.184	98.248	0.008	93.137	0.015	5.110
0.60	98.598	94.509	4.089	98.650	0.009	94.667	0.017	3.982
0.65	98.987	96.010	2.977	99.054	0.010	96.226	0.019	2.828
0.70	99.401	97.623	1.774	99.460	0.011	97.811	0.021	1.650
0.75	99.797	99.192	0.604	99.869	0.011	99.422	0.023	0.447
0.80	100.816	100.816	0.000	101.059	0.025	101.057	0.025	0.001
0.85	102.484	102.484	0.000	102.718	0.027	102.717	0.027	0.000
0.90	104.172	104.172	0.000	104.401	0.029	104.401	0.029	0.000
0.95	105.880	105.880	0.000	106.109	0.031	106.109	0.031	0.000
1.00	107.610	107.610	0.000	107.840	0.033	107.840	0.033	0.000

Table 2: σ=15%, β=45%, i_{min}=3%, i_{tec}=3%

	Binomial Tree			Simulation				
r	American	European	Surrender	American	s.e.	European	s.e.	Surrender
0.00	106.135	106.135	0.000	106.085	0.011	106.085	0.011	0.000
0.01	102.632	102.632	0.000	102.587	0.011	102.587	0.011	0.000
0.02	99.821	99.285	0.536	99.820	0.006	99.254	0.011	0.566
0.03	99.007	96.088	2.920	99.012	0.006	96.079	0.011	2.934
0.04	98.210	93.032	5.179	98.224	0.006	93.053	0.011	5.170
0.05	97.430	90.110	7.320	97.455	0.006	90.172	0.012	7.283
0.06	96.666	87.316	9.350	96.704	0.007	87.427	0.012	9.278
0.07	95.917	84.643	11.275	95.973	0.007	84.812	0.012	11.161
0.08	95.184	82.084	13.100	95.261	0.007	82.321	0.012	12.939
0.09	94.465	79.633	14.833	94.567	0.007	79.948	0.012	14.618
0.10	93.761	77.283	16.477	93.891	0.007	77.687	0.012	16.204

Table 3: σ=15%, β=45%, r=5%, i_{tec}=0%

	Binomial Tree			**Simulation**				
i_{min}	American	European	Surrender	American	s.e.	European	s.e.	Surrender
0.000	98.940	95.826	3.114	98.997	0.008	96.019	0.015	2.979
0.005	99.157	96.669	2.488	99.194	0.008	96.782	0.015	2.411
0.010	99.373	97.517	1.856	99.404	0.007	97.604	0.015	1.799
0.015	99.590	98.371	1.219	99.627	0.007	98.485	0.014	1.142
0.020	99.813	99.252	0.560	99.864	0.007	99.425	0.014	0.439
0.025	100.332	100.332	0.000	100.427	0.013	100.427	0.013	0.000
0.030	101.420	101.420	0.000	101.489	0.013	101.489	0.013	0.000
0.035	102.517	102.517	0.000	102.613	0.013	102.613	0.013	0.000
0.040	103.629	103.629	0.000	103.797	0.012	103.797	0.012	0.000
0.045	104.955	104.955	0.000	105.043	0.012	105.043	0.012	0.000

Table 4: β=45%, r=5%, i_{min}=3%, i_{tec}=3%

	Binomial Tree			**Simulation**				
σ	American	European	Surrender	American	s.e.	European	s.e.	Surrender
0.050	95.691	83.847	11.844	95.720	0.002	83.940	0.003	11.780
0.100	96.552	86.907	9.646	96.582	0.004	86.994	0.007	9.587
0.150	97.430	90.110	7.320	97.455	0.006	90.172	0.012	7.283
0.200	98.288	93.325	4.963	98.329	0.009	93.441	0.017	4.889
0.250	99.151	96.645	2.505	99.203	0.011	96.794	0.022	2.410
0.300	100.153	100.153	0.000	100.225	0.028	100.227	0.029	-0.003
0.350	103.714	103.714	0.000	103.741	0.035	103.739	0.035	0.001
0.400	107.335	107.335	0.000	107.329	0.043	107.328	0.043	0.001

Table 5: Single Premium 31 Dec. 2002, $\alpha = 10\%$**,** $\sigma = 15\%$**,** $D = 5$

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
15-20	4.11	45,207	3,538	48,241	3,034
21-25	5.02	190,660	24,079	228,973	38,313
26-30	5.55	653,932	59,767	726,902	72,970
31-35	5.38	990,696	80,971	1,078,529	87,832
36-40	5.25	772,306	72,413	853,635	81,328
41-45	4.46	1,031,450	81,029	1,110,294	78,843
46-50	3.66	1,321,146	86,669	1,389,242	68,095
51-55	3.37	416,766	26,073	436,038	19,273
56-60	3.34	521,434	29,846	541,140	19,705
61-65	2.75	105,277	4,926	107,376	2,099

Table 6: Single Premium 30 Jun. 2003, $\alpha = 10\%$**,** $\sigma = 15\%$**,** $D = 5$

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
15-20	4.11	46,128	4,447	48,528	2,400
21-25	5.02	196,735	30,019	230,331	33,596
26-30	5.55	669,199	74,843	731,214	62,015
31-35	5.38	1,011,675	101,619	1,084,927	73,251
36-40	5.25	791,267	91,051	858,699	67,431
41-45	4.46	1,052,570	101,781	1,116,880	64,310
46-50	3.66	1,343,317	108,529	1,397,483	54,166
51-55	3.37	423,438	32,634	438,625	15,187
56-60	3.34	529,017	37,307	544,350	15,333
61-65	2.75	106,486	6,104	108,013	1,527

Table 7: Single Premium 31 Dec. 2002, $\alpha = 30\%$**,** $\sigma = 30\%$**,** $D = 10$

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
15-20	4.11	48,940	8,097	49,366	426
21-25	5.02	223,164	62,771	234,257	11,092
26-30	5.55	725,880	146,485	743,651	17,771
31-35	5.38	1,084,042	193,961	1,103,127	19,085
36-40	5.25	856,452	174,242	873,506	17,055
41-45	4.46	1,121,078	189,997	1,135,860	14,782
46-50	3.66	1,410,859	196,538	1,421,050	10,191
51-55	3.37	443,320	58,641	446,014	2,694
56-60	3.34	551,061	66,174	553,432	2,371
61-65	2.75	109,750	10,414	109,823	73

Table 8: Single Premium 30 Jun. 2003, $\alpha = 30\%$**,** $\sigma = 30\%$**,** $D = 10$

Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
4.11	49,738	8,765	49,895	157
5.02	228,963	67,643	236,804	7,841
5.55	739,774	158,332	751,419	11,646
5.38	1,102,809	209,981	1,114,116	11,306
5.25	873,465	188,560	883,085	9,620
4.46	1,139,653	205,766	1,147,187	7,534
3.66	1,429,770	212,903	1,434,462	4,692
3.37	448,978	63,550	450,143	1,165
3.34	557,344	71,767	558,291	947
2.75	110,678	11,315	110,697	19

Table 9: Constant Annual Premiums 31 Dec. 2002, $\alpha = 10\%$**,** $\sigma = 15\%$**,** $D = 5$

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	408,019	41,375	413,116	5,097
21-25	5.95	1,757,021	203,888	1,784,167	27,146
26-30	6.00	2,295,533	261,410	2,330,870	35,337
31-35	5.77	2,717,336	295,450	2,756,106	38,770
36-40	5.28	2,665,631	273,319	2,701,130	35,498
41-45	4.33	2,329,687	199,112	2,354,263	24,576
46-50	3.49	980,596	68,296	988,235	7,639
51-55	3.81	342,582	26,584	345,820	3,238
56-60	2.75	107,609	5,959	108,235	626

Table 10: Constant Annual Premiums 30 Jun. 2003, $\alpha = 10\%$**,** $\sigma = 15\%$**,** $D = 5$

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	420,539	51,734	424,253	3,713
21-25	5.95	1,822,322	256,488	1,841,953	19,632
26-30	6.00	2,379,400	328,946	2,404,995	25,594
31-35	5.77	2,810,750	371,266	2,838,882	28,132
36-40	5.28	2,751,900	343,953	2,778,030	26,130
41-45	4.33	2,389,131	250,063	2,408,542	19,411
46-50	3.49	1,000,161	85,602	1,006,341	6,180
51-55	3.81	350,325	33,344	352,906	2,581
56-60	2.75	109,241	7,451	109,768	527

Table 11: Constant Annual Premiums 31 Dec. 2002, $\alpha = 30\%$, $\sigma = 30\%$, $D = 10$

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	457,141	100,382	462,601	5,460
21-25	5.95	2,002,762	500,353	2,032,247	29,485
26-30	6.00	2,610,091	641,170	2,648,452	38,361
31-35	5.77	3,070,136	720,998	3,112,075	41,939
36-40	5.28	2,984,700	659,451	3,023,091	38,392
41-45	4.33	2,546,291	462,520	2,572,631	26,340
46-50	3.49	1,051,772	155,197	1,059,904	8,132
51-55	3.81	370,798	60,990	374,246	3,447
56-60	2.75	113,484	13,182	114,145	661

Table 12: Constant Annual Premiums 30 Jun. 2003, $\alpha = 30\%$, $\sigma = 30\%$, $D = 10$

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	468,416	109,022	472,474	4,058
21-25	5.95	2,063,766	545,277	2,085,517	21,751
26-30	6.00	2,688,770	698,928	2,717,114	28,345
31-35	5.77	3,157,051	785,389	3,188,100	31,049
36-40	5.28	3,064,342	718,876	3,093,073	28,731
41-45	4.33	2,598,799	503,524	2,619,792	20,992
46-50	3.49	1,068,638	168,533	1,075,260	6,622
51-55	3.81	377,539	66,296	380,307	2,768
56-60	2.75	114,847	14,254	115,406	558

Table 13: Constant Annual Premiums 31 Dec. 2002, $\alpha = 10\%$, $\sigma = 15\%$, $D = 5$, no bonus

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	348,299	36,149	378,235	29,935
21-25	5.95	1,492,869	178,370	1,651,362	158,493
26-30	6.00	1,951,897	228,798	2,153,346	201,450
31-35	5.77	2,315,740	258,504	2,537,979	222,239
36-40	5.28	2,276,537	239,211	2,474,648	198,111
41-45	4.33	2,000,549	173,830	2,125,849	125,299
46-50	3.49	846,133	59,578	885,385	39,252
51-55	3.81	295,198	23,239	312,535	17,337
56-60	2.75	93,319	5,204	96,717	3,398

Table 14: Constant Annual Premiums 30 Jun. 2003, $\alpha = 10\%$, $\sigma = 15\%$, $D = 5$, no bonus

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	358,667	45,403	380,621	21,954
21-25	5.95	1,546,566	225,387	1,661,560	114,994
26-30	6.00	2,020,922	289,217	2,166,831	145,909
31-35	5.77	2,392,790	326,530	2,553,960	161,170
36-40	5.28	2,347,905	302,506	2,490,128	142,223
41-45	4.33	2,050,155	220,173	2,138,463	88,308
46-50	3.49	862,650	75,201	890,581	27,931
51-55	3.81	301,729	29,349	314,386	12,657
56-60	2.75	94,710	6,558	97,278	2,568

Table 15: Constant Annual Premiums 31 Dec. 2002, $\alpha = 30\%$, $\sigma = 30\%$, $D = 10$, no bonus

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	390,877	88,356	393,522	2,645
21-25	5.95	1,706,932	440,632	1,723,195	16,263
26-30	6.00	2,226,186	564,967	2,247,137	20,950
31-35	5.77	2,623,146	635,634	2,645,697	22,550
36-40	5.28	2,554,594	581,461	2,575,668	21,074
41-45	4.33	2,188,060	407,856	2,199,978	11,917
46-50	3.49	907,647	136,688	911,705	4,058
51-55	3.81	319,618	53,816	321,841	2,223
56-60	2.75	98,380	11,615	98,893	513

Table 16: Constant Annual Premiums 30 Jun. 2003, $\alpha = 30\%$, $\sigma = 30\%$, $D = 10$, no bonus

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	400,977	95,436	401,680	702
21-25	5.95	1,760,118	477,526	1,763,790	3,671
26-30	6.00	2,294,501	612,153	2,299,373	4,872
31-35	5.77	2,699,310	688,084	2,704,817	5,507
36-40	5.28	2,624,897	630,044	2,630,157	5,260
41-45	4.33	2,236,559	440,869	2,239,016	2,456
46-50	3.49	923,529	147,573	924,472	943
51-55	3.81	325,934	58,082	326,611	676
56-60	2.75	99,710	12,511	99,858	148

Table 17: Constant Annual Premiums 31 Dec. 2002, $\alpha = 10\%$**,** $\sigma = 15\%$**,** $D = 5$**,** $i_{sur} = 4.5\% - 5\%$

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	408,302	41,418	413,338	5,036
21-25	5.95	1,758,959	204,178	1,786,038	27,079
26-30	6.00	2,297,992	261,805	2,333,234	35,242
31-35	5.77	2,719,801	295,669	2,758,415	38,614
36-40	5.28	2,667,873	273,611	2,703,308	35,435
41-45	4.33	2,330,647	199,268	2,355,222	24,576
46-50	3.49	980,844	68,308	988,485	7,641
51-55	3.81	342,706	26,611	345,944	3,238
56-60	2.75	107,617	5,968	108,243	626

Table 18: Constant Annual Premiums 31 Dec. 2002, $\alpha = 10\%$**,** $\sigma = 15\%$**,** $D = 5$**,** $i_{sur} = 4.5\% - 5\%$**, no bonus**

Age	Time to Term (Average)	European Contract	Minimum Guarantee	American Contract	Surrender Option
16-20	5.39	348,126	36,323	349,595	1,470
21-25	5.95	1,492,237	179,109	1,500,451	8,215
26-30	6.00	1,951,109	229,778	1,961,749	10,640
31-35	5.77	2,314,510	259,616	2,326,049	11,540
36-40	5.28	2,275,587	240,243	2,285,857	10,270
41-45	4.33	1,999,281	174,955	2,005,031	5,751
46-50	3.49	845,761	59,946	847,383	1,623
51-55	3.81	295,056	23,388	295,782	726
56-60	2.75	93,282	5,240	93,394	111

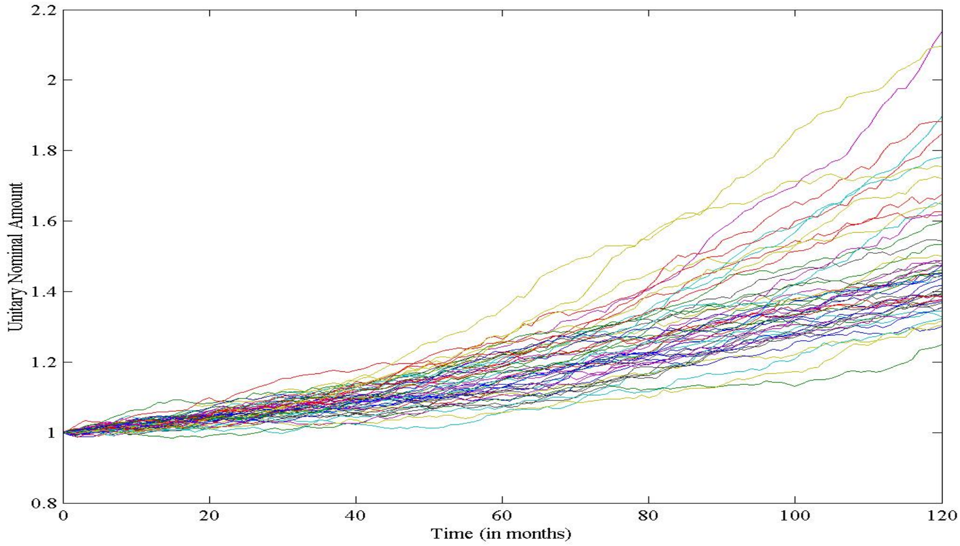

Figure 1: Evolution of the Reference Fund at 30/06/03, $\alpha = 10\%$, $\sigma = 15\%$, $D = .5$

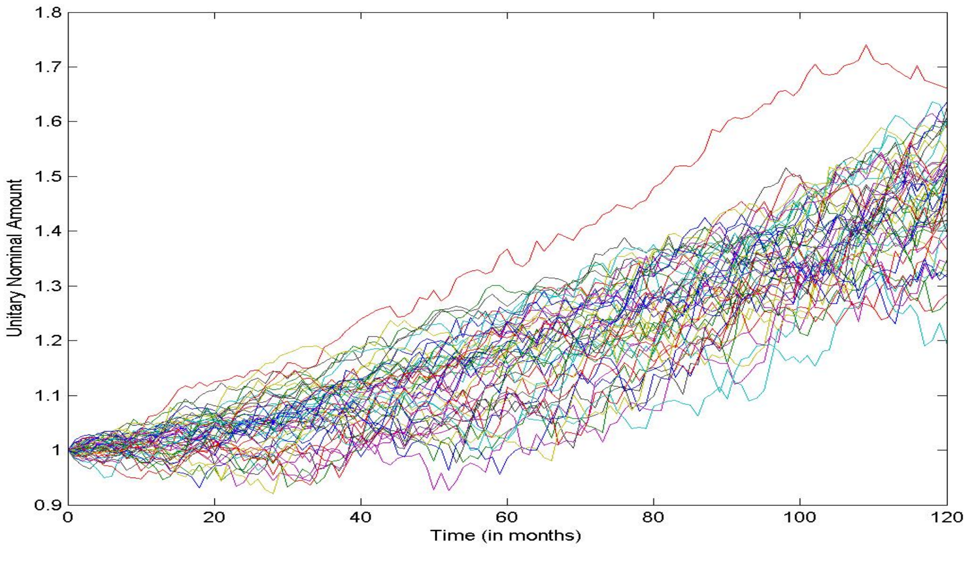

Figure 2: Evolution of the Reference Fund at 30/06/03, $\alpha = 10\%$, $\sigma = 15\%$, $D = 10$

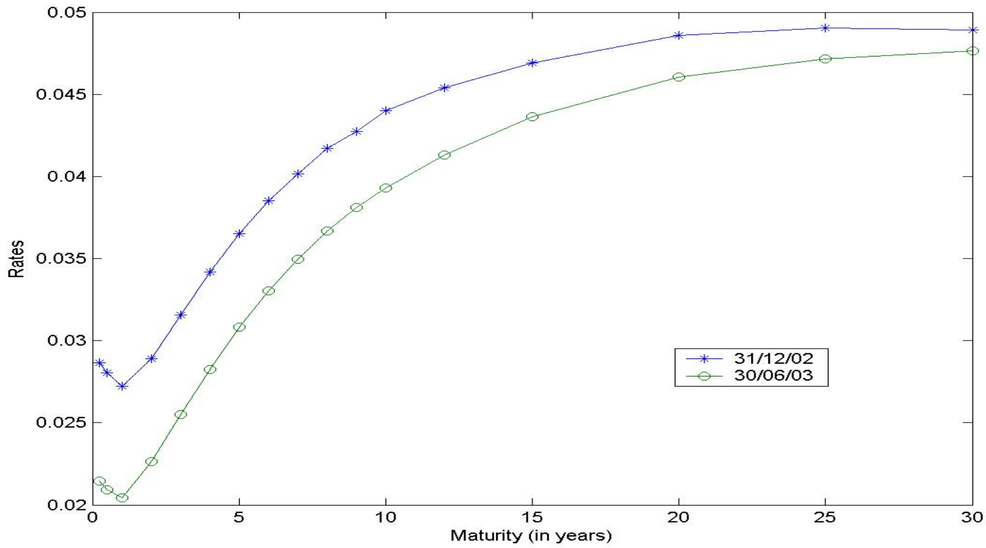

Figure 3: Term Structure of Euro Swap Interest Rates at 31/12/02 and 30/06/03

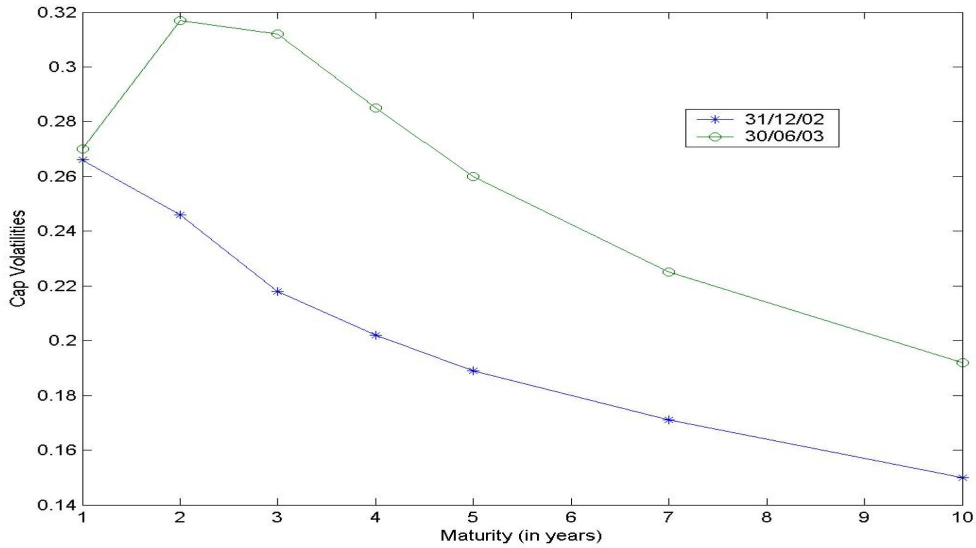

Figure 4: At-the-money Euro Cap Volatility Curve at 31/12/02 and 30/06/03