Real Options, Capital Structure, and Taxes

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Abstract

This paper presents a valuation approach for real options when the capital structure of the underlying project/firm is levered, assuming that the goal is the maximization of total firm/project value (i.e., under a first-best investment policy). We analyze also the effect of different financing schemes on the value of the real option and on the exercise policy. The main finding of this work is that a higher leverage reduce the time-value of the option to delay investment and increases the probability of exercising the options.

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1 Introduction

In this paper we present a valuation approach for real options when the financial structure of the real assets underlying the options is levered.

The effect of financing and capital structure decisions on the value of real options and on dynamic capital budgeting decisions is typically overlooked in the real options literature. Only few contributions and applications deal with the interaction between investment and financing decisions. An early contribution was given by Trigeorgis [17], who analyzed equityholders' option to default on debt payments, noting potential interactions with operating flexibility, but with no reference to tax benefits from debt financing. Another important contribution is Mauer and Triantis [9], who presented a real options model of a flexible production plant with a capital structure changing over time as a consequence of an optimal dynamic financing policy. Changes in capital structure entail recapitalization (repurchase and/or

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issuance) costs, so that an optimal capital structure is found and also the tax benefits of debt financing are included in their analysis. An important finding of Mauer and Triantis' research was that operating and financial flexibility are partial substitute and that operating flexibility have a positive effect on the cost of capital, since it reduces the probability to default on corporate debt, thus allowing to sustain a more levered capital structure. On the other hand, they do not find any influence of debt financing on the investment policy. A third contribution was given by Mauer and Ott [8] (but see also Childs, Mauer and Ott [3] for a discrete-time version of the model), who study the effect of agency costs of debt on the optimal investment policy according to the real options approach. Their work, following other existing models (e.g. Mello and Parsons [11]) studies the effect of agency problems of debt on the optimal investment policy for the firm's growth options (underinvestment or overinvestment) and provides a measures of the agency cost of debt.

All the above contributions lack a proper risk-neutral valuation approach that incorporates the effect of corporate and personal taxation on asset returns. It is well know (see Ross [13]) that an equivalent (equilibrium) martingale measure can be found assuming a convex tax schedule and that such equilibrium probability is different from the usual martingale measure embedded in financial asset prices, since the latter does not consider the tax shield. Hence, a valuation approach for real options which incorporates the effect of both risk and tax benefits is a necessary premise to any real options model dealing with financial flexibility.

With this aim, Sick [15] provides a risk-neutral valuation approach assuming personal and corporate taxes in a Miller equilibrium economy (see Miller [12]) under the simplifying assumption of a linear tax schedule. Hence, drawing on Sick's model, we present a valuation principle for real options when the decision to invest in a real asset entails also a decision on how to finance it. Our main result is a valuation formula for real options under a first-best (i.e., maximizing the total value of the firm/project) investment policy. Moreover, we study the effect of debt financing both on the value of the real options and on the investment policy. The main finding of this analysis is that a higher leverage for an infra-marginal firm/project (i.e., a firm/project which has an positive tax shield from leverage) increases the value of the option to delay investment and increases the probability of investing, thus reducing the time-value of the option to defer investment. Lastly, we extend the valuation formula above also to cases where more interacting options are available. This case is particularly interesting because it permits to address also the case where managerial flexibility (regarding investing decision) interacts with financial flexibility (regarding capital structure decision). Although the results are obtained for the prototypical case of an option to delay investment on a given real asset, the main results of our analysis can be easily implemented for any (simplex or complex) real options. Moreover, when possible the analysis is carried on by providing analytical solutions (or approximate analytical solutions) for option values and the probability of investing. When more complex situations are introduce, we resort to numerical approximations, using a log-transformed binomial lattice scheme as introduced in Appendix A.

The paper is organized as follows. In Section 2, we introduce the Miller economy with corporate and personal taxes, and the risk-neutral valuation approach for levered real assets, as of Sick [15]. In Section 3, we present a valuation formula for real options assuming that the goal of the firm is to keep the proportion of debt on real assets constant. Moreover, we discuss the effect of leverage both on the value of the option to invest and on the investment policy (here proxied by the probability of investing). Since this scheme presents some drawbacks in some extreme situations, in Section 4 we introduce a risk-neutral valuation formula for real options based on an APV approach. The main findings of the previous analysis are confirmed also under an APV approach. Lastly, in Section 5, we see how the approach introduced for simple real options can be extended to evaluate many interacting options.

2 Project valuation and capital structure in a Miller economy

The setting is an economy with financial markets with both personal and corporate taxes. Firms issue only bonds and stocks. Following Miller [12], we denote τ_c the marginal tax rate for a company; τ_{pb} the personal marginal tax rate for income from bonds, and τ_{pe} the personal marginal tax rate for income from share, with no distinction between capital gains or dividends. We assume that the financial market is in equilibrium (general tax equilibrium) and so there is no overall gain from debt in this economy although cross-sectionally corporate tax rates, τ_c, can be different. In particular, there can be supra- and infra-marginal firms, i.e., firms that have a gain and, respectively, a loss from leverage. So, denoted with τ_m the (marginal) tax rate for a marginal firm (i.e., a firm with no gain from leverage), so that

$$1 - \tau_m = \frac{1 - \tau_{pb}}{1 - \tau_{pe}},$$

an infra- (respectively, supra-) marginal company has a tax rate $\tau_c > \tau_m$ (respectively, $\tau_c < \tau_m$).

We will assume that all the above defined tax operators are linear (i.e., income and losses are taxed, for the same agent/firm, at the same rate). A linear tax code implies a symmetric tax system with full loss offset provisions. At a corporate level, this is the code used by Mauer and Ott [8]. On the contrary, Mauer and Triantis [9] assume an asymmetric tax system in which there are no loss offset provisions but discuss the effect of allowing for full loss offset provisions. The symmetry can be justified since it is a better approximation of the current tax systems in place in most countries. At a personal level, linearity is also a convenient assumption. In addition, τ_{pe} is used for the tax on equity income, without making a distinction between capital gains and dividends, when in real life tax systems both are treated differently. A possible solution could be to assume that individuals can avoid taxes on dividends by borrowing to create interest offsets or through tax-exempt investment vehicles.

As shown in Ross [13], under the assumption of linear[1] tax schedule there exists an equilibrium martingale measure that can be used to price cash flows at the market level taking into account personal taxation; i.e., *after corporate taxes and before personal taxes*, as if agents were risk-neutral:

$$p_t = \hat{\mathbb{E}}\left[e^{-r(T-t)}C_T|\mathcal{F}_t\right] \tag{1}$$

where:

$\hat{\mathbb{E}}$ is the expectation operator under the martingale measure. $\mathcal{F}_t$ is the information available at time t. Sometimes we will write also $\hat{\mathbb{E}}[\cdot|\mathcal{F}_t] = \hat{\mathbb{E}}_t[\cdot]$;

C_T is the after-corporate taxes and pre-personal taxes free cash flow from the project at time T. I.e., given the pre- corporate tax cash flow, X_T, $C_T = X_T(1-\tau_c)$;

r is the risk-free rate suited to discount cash flows from T to t;

p_t is the time t price of the cash flow from the project.

If we assume that the riskless rate is non-stochastic, then

$$\hat{\mathbb{E}}\left[e^{-r(T-t)}C_T|\mathcal{F}_0\right] = e^{-r(T-t)}\hat{\mathbb{E}}\left[C_T|\mathcal{F}_0\right]$$

and $\hat{\mathbb{E}}\left[C_T|\mathcal{F}_0\right] = \mathrm{CE}_t\left[C_T\right]$ is defined also as the certainty-equivalent operator at time t.

Given the same assumptions as introduced in Sick [15], in particular the above mentioned linearity of the tax operators, at least at the personal level, the following result holds:

Theorem 1 (Sick [15]). *The certainty-equivalent operators for pretax flows to debtholders, after-corporate tax flows to shareholders and after all taxes (both corporate and personal)* are the same*.*

[1]Actually, a sufficient condition for a tax-adjusted martingale measure to exist is that the tax operator is convex. See Ross [13, Proposition 3, Corollary 1].

This permits us to specialize equation (1): a cash flow to equity, C_T^e, has value $e^{-r_z(T-t)}\hat{\mathbb{E}}_t[C_T^e]$, where r_z is the certainty-equivalent rate of return on (risky) stocks;[2] a cash flow to bondholders, C_T^b, has value $e^{-r_f(T-t)}\hat{\mathbb{E}}_t[C_T^b]$, where r_f is the certainty-equivalent rate of return on bonds.[3] According to Sick [15, Proposition 6] under the same assumptions, CAPM/APT applies to the bond market with the same market price of risk as in the equity market, but with a different riskless rate. I.e., in general $r_z \neq r_f$ and is $r_z < r_f$. Only if $\tau_{pb} = \tau_{pe}$ we have $r_z = r_f$ (or, $\tau_m = 0$).

In Sections 3 and 4 we will present the valuation criterion for real options under the hypothesis of debt financing. For definiteness, we illustrate the case of an option to defer investment in a project financed both with debt (bond) and equity (stocks) with a prespecified capital structure. Throughout the paper, we assume that *bond and stocks are issued at the date the option to invest is exercised and the project is implemented*. The option is an American-like contingent claim on the gross value of the project; i.e., the underlying asset is the present value of the cash flows from operation starting at the implementation date. We will proceed from simpler to more complex situations. First we discuss, in Section 3, a model of real option valuation in an NPV/WACC environment with a constant debt proportion of company's total value, stressing that there could be a potentially unrealistic representation of corporate financing. Next, in Section 4, we provide a different financing scheme based this time not on the proportion but rather on the level of debt and hence a different valuation formula for an option to invest in a project in an APV environment.

In what remains of the paper we assume that the hypotheses for continuous-time valuation hold.

3 Real options valuation with a constant debt proportion

In the current and subsequent section, we will consider a prototypical problem of investment under uncertainty, in order to illustrate the basics of real options valuation when debt financing is introduced.

Let there be given a project,[4] with marginal (corporate) tax rate τ_c. For simplicity, the project is infinite lived, with value V and costs I to implement

[2] r_z can be thought of as the zero-beta rate of return or the intercept of the security market line for stocks in a CAPM framework.

[3] I.e., the risk free rate in a CAPM setting.

[4] Note that, for the time being, and in the sake of simplicity, our setting is different from the case of an ongoing company with its own capital structure and growth options. For the latter, different authors (see, among others, Mauer and Ott [8]) remark that this type of companies may not want to have much debt because that would prevent them from exercising the options. We will see that the converse may be true: a leverage company my find it easier to exercise its options.

it; the capital expenditure is assumed to be constant over time. We have the opportunity to delay the investment until date T. At the current date, $t < T$, we determine the debt proportion $L = B/V$, $0 \leq L < 1$, where B is the market value of debt. This means that, although at the valuation date we have an idea of the debt proportion to finance the project, since $B = LV$, the exact amount of debt raised will not be known until the exercise date, because V and the exercise date are stochastic. Hence, the funds to finance the capital expenditure will be raised (by issuing bonds and stocks in the desired proportions) only if investment takes place at the date the option is exercised. This assumption is realistic, since there would be no reason to raise capital before investment, so incurring in a (useless) opportunity cost of capital. The optimal exercise policy depends on V, the underlying asset, and consequently the date we will issue securities is a stopping time with respect to the information about the project value. We assume that, after the investment will be made, the debt proportion is kept constant. Myers has pointed out that if the firm maintains a constant debt ratio then debt is indeed riskless, since the firm must maintain the constant debt ratio by repurchasing debt when the value of the firm falls, in order to keep the debt ratio constant. Thus, there is no opportunity for the value of the firm to fall below the value of the debt.

Given a marginal investment project, we assume that its after corporate taxes (instantaneous) free cash flow, $\{C_t\}$, follows a geometric Brownian motion (under the actual/empirical probability measure)

$$dC_t = \alpha C_t dt + \sigma C_t dZ_t \tag{2}$$

where α is the expected growth rate. To compute the value of the project under the (tax-adjusted) martingale measure, or equivalently, using a (tax-adjusted) certainty-equivalent approach, we need to properly adjust the actual growth rate (see for instance, Constantinides [4]). To this aim, following Sick's [15] notation, let ρ_0 be the instantaneous expected rate of return for an all-equity-financed cash flow, C_T. Hence, the time t market value of this flow is $e^{-\rho_0(T-t)}\mathbb{E}_t[C_T]$ or, under the tax-adjusted and risk-neutral probability measure, $e^{-r_z(T-t)}\hat{\mathbb{E}}_t[C_T]$. According to Sick [15, Proposition 5], the tax- and risk-adjusted cost of capital (WACC) for a (after corporate taxes) free cash flow, C_T, is $\rho^* = \rho_0 - \tau^* r_f L$, so that the market value of this flow is

$$e^{-\rho^*(T-t)}\mathbb{E}_t[C_T].$$

Let define the weighted average cost of capital under the martingale measure (or alternatively, the certainty equivalent WACC) as

$$\rho = r_z - \tau^* r_f L = (1-L)r_z + L(1-\tau_c)r_f \tag{3}$$

where $\tau^* = \tau_c - \tau_m$ is the net tax shield per unit of interest. We can state the following proposition, a straightforward consequence of linearity of personal taxation and of the other Sick's [15] assumptions.

Proposition 2. *The risk-neutral growth rate of $\{C_t\}$, i.e., the drift under the (general tax equilibrium) martingale measure, is independent of capital structure. That is, the risk-neutral drift, $\hat{\alpha}$, is the empirical drift, α, less a risk premium, Φ, independent of capital structure:*[5]

$$\hat{\alpha} = \alpha - \Phi,$$

where $\Phi = \rho_0 - r_z = \rho^* - \rho$.

Proof. If the project is all equity financed, then $\rho_0 = r_z + \Phi$ and hence we can determine the risk premium, Φ. If the project is partially financed with debt, then $\rho^* = \rho + \Phi$. Hence

$$\Phi = (\rho_0 - \tau^* r_f L) - (r_z - \tau^* r_f L)$$

and this concludes. □

The result in Proposition 2 is expected since free cash flow only bear operational risk. Therefore, the appropriate growth rate remains unaltered with changes in financial risk due to changes in capital structure.

As a consequence, the dynamic for the cash flow from the project is, under the martingale measure,[6]

$$dC_t = \hat{\alpha} C_t dt + \sigma C_t dZ_t.$$

Since the project is infinite lived, its value is

$$\begin{aligned} V_t = V(C_t) &= \int_t^{\infty} \hat{\mathbb{E}}_t[C_s] e^{-\rho(s-t)} ds = \int_t^{\infty} C_t e^{\hat{\alpha}(s-t)} e^{-\rho(s-t)} ds \\ &= \frac{C_t}{\rho - \hat{\alpha}} = \frac{C_t}{\rho^* - \alpha} \end{aligned} \tag{4}$$

where we assumed $\rho > \hat{\alpha}$ (i.e., $\rho^* > \alpha$) for convergence.[7] As a consequence, the stochastic process for V under the tax-adjusted and risk-neutral probability measure is

$$\frac{dV_t}{V_t} = \hat{\alpha} dt + \sigma dZ_t.$$

Under the hypotheses of CAPM, the equilibrium relation on V is

$$\mathbb{E}\left[dV_t\right] = \rho^* V dt \tag{5}$$

[5] Shortly, we will specify Φ by introducing CAPM. For the time being, Φ is defined with no reference to any equilibrium model.

[6] With an abuse of notation, we will still denote by dZ_t the increment of the standard Brownian motion under the martingale measure.

[7] Following McDonald and Siegel [10], the difference $\delta = \rho^* - \alpha = \rho - \hat{\alpha} > 0$ is a rate of return shortfall with respect to the equilibrium rate of return on a liquid financial asset with the same systematic risk.

where ρ^* can now be specified as

$$\rho^* = \rho + \lambda\left[(1-L)\beta_E + L(1-\tau_c)\beta_B\right] = \rho + \lambda\beta_V \tag{6}$$

with λ is the market price of risk for equity cash flows, β_E is "beta" for equity cash flows, β_B is "beta" for bond cash flows, and ρ is the tax-adjusted discount rate under the martingale measure. In Equation (6), β_V is the right "beta" for a free cash flow for the project, with a debt proportion L.

Let Π denote the payoff at the exercise date, $\Pi(t, C_t) = \max\{V(C_t) - I, 0\}$, and let F denote the value of the investment project including the time-option to postpone the investment decision. Before going to the main result of this section, we remark the fact that issuance of debt is contingent on the decision to invest. Hence, the financing decision is influenced by the investment decision, in the sense that it happens when (and if) the investment is implemented. On the other hand, the investment decision is influenced by the financing decision, since the former is made if the expected free cash flow from the project can remunerate the cost of capital.

Proposition 3. *The value of the option to invest in a project is the NPV*[8] *at the optimal investment date, expected under the martingale probability measure, discounted at ρ:*

$$F(t, C_t) = \max_{s \in \mathcal{T}[t,T]} \left\{ e^{-\rho(s-t)} \hat{\mathbb{E}}\left[\Pi(s, C_s)\right] \right\} \tag{7}$$

where $\mathcal{T}[t,T]$ is the set of stopping times with respect to $\{\mathcal{F}_t\}$ and ρ is the weighted average cost of capital under the martingale measure, as in Equation (3)*.*

Proof. The option value, F, depends on V, the value of the project with debt proportion L. Hence, also the expected increment of F follows the equilibrium relation under the tax-adjusted and risk-neutral martingale measure:

$$\hat{\mathbb{E}}[dF] = \rho F dt. \tag{8}$$

On the other hand, by Itô's Lemma

$$\hat{\mathbb{E}}[dF] = \hat{\alpha} V F_V dt + \frac{1}{2}\sigma^2 V^2 F_{VV} dt + F_t dt \tag{9}$$

Equating the right-hand-sides of (8) and (9) we have

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + \hat{\alpha} V F_V + F_t - \rho F = 0. \tag{10}$$

Applying the usual boundary conditions, we obtain Equation (7). □

[8] Note that here we use the standard Free Cash-Flow discounted at WACC approach. Later on, in Section 4 we will move to an Adjusted Present Value Approach.

It is important to remark that the certainty equivalent cost of capital, ρ, is used to discount the expectation of the payoff even if the option to delay investment (i.e., an unexercised option) sustains an all-equity financial structure. In fact ρ depends solely on the capital structure after investment.

The above solution, together with equation (3), suggests that the option to invest in a marginal project (i.e., a project with corporate tax rate $\tau_c = \tau_m$ and no tax shield) is evaluated according to Black, Scholes, and Merton's approach, but using r_z instead of r_f. Note that in our setting, since a project cannot be all-debt financed, r_f is never used but when $\tau_{pb} = \tau_{pe}$ (which implies $\tau_m = 0$).

If $\tau_c \neq \tau_m$, we have to discuss separately the case of an infra- ($\tau_c > \tau_m$) from a supra-marginal ($\tau_c < \tau_m$) project.

Starting from Proposition 3, which states the valuation principle for real options assuming a levered capital structure, we wish to analyze the effect of debt and tax shield both on the value of the option to defer and on the investment policy. To pursue this aim, we provide an approximate solution for problem (7) applying the analytic approximation proposed by Barone-Adesi and Whaley [1]. According to this approach, the value of the option to invest in (7) can be approximated using the following expression

$$\widetilde{F}\left(t, C_t\right) = \begin{cases} f(t, C_t) + \kappa \left(\frac{C_t}{\rho - \hat{\alpha}}\right)^{\gamma} & \text{if } C_t < C_t^* \\ \frac{C_t}{\rho - \hat{\alpha}} - I & \text{otherwise} \end{cases} \tag{11}$$

where $f(t, C_t)$ is the value of the same option but with given exercise date T (i.e., a "European" claim):

$$f(t, C_t) = e^{-(\rho - \hat{\alpha})(T-t)} \mathcal{N}(d_1) \frac{C_t}{\rho - \hat{\alpha}} - e^{\rho(T-t)} \mathcal{N}(d_2) I$$

with

$$d_1 = \frac{\log \frac{C_t}{I(\rho - \hat{\alpha})} + \left(\hat{\alpha} + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

and $\mathcal{N}(\cdot)$ denoting the cumulative Normal distribution;

$$\gamma = \frac{1}{2} - \frac{\hat{\alpha}}{\sigma^2} + \sqrt{\left(\frac{\hat{\alpha}}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{\rho}{\sigma^2 h(T-t)}} \tag{12}$$

with $h(t) = 1 - e^{\rho t}$,

$$\kappa = \left(\frac{1}{\rho - \hat{\alpha}} - e^{(\rho - \hat{\alpha})(T-t)} \mathcal{N}(d_1(C_t^*))\right) (C_t^*)^{1-\gamma} \frac{(\rho - \hat{\alpha})^{\gamma}}{\gamma}$$

and C_t^* is a root of equation (in a neighborhood of $I(\rho - \hat{\alpha})$)

$$f(t, C_t^*) + \left(\frac{1}{\rho - \hat{\alpha}} - e^{-(\rho - \hat{\alpha})(T-t)} \mathcal{N}(d_1(C_t^*)\right) \frac{C_t^*}{\gamma} = \frac{C_t^*}{\rho - \hat{\alpha}} - I.$$

Note that, in equation (11), both γ and κ depend on t (and hence, also C_t^* depends on t). This means that all the above computations must be done at any point in time to define a time-dependent investment policy. Equation (11) can be use to discuss the influence of debt proportion, L, on the value of the investment opportunity, F.

It is interesting to analyze also the effect of a larger debt proportion on the investment policy by considering the probability of investing (assuming that currently the opportunity is still available) within the time horizon T.[9] This is equivalent to saying that, assuming that at t the option to defer has not been exercised yet, the stochastic process $\{C_s \mid s > t\}$ touches (from below) the investment threshold $\{C_s^* \mid t < s \leq T\}$ computed using the analytical approximation introduced above. According to Harrison [7, pp. 11–14] this probability is[10]

$$\widetilde{P}(C_t) = \mathcal{N}\left(p_1(C_t, C_t^*)\right) + \mathcal{N}\left(p_2(C_t, C_t^*)\right) \left(\frac{C_t^*}{C_t}\right)^{2\alpha/\sigma^2 - 1} \tag{13}$$

where

$$\begin{aligned} p_1(C_t, C_t^*) &= \frac{\log \frac{C_t}{C_t^*} + \left(\alpha - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \\ p_2(C_t, C_t^*) &= p_1(C_t, C_t^*) - \left(\frac{2\alpha}{\sigma^2} - 1\right)\sigma\sqrt{T-t} \end{aligned}$$

Since expressions (11) and (13) are not amenable for an analytic treatment, we will analyze the effect of debt both on the value of the option to defer and on the exercise policy by discussing a numerical example.

Before presenting the numerical examples, we observe that equations (11) and (13) simplifies when assuming that the horizon for the option to delay investment is infinite ($T \to \infty$). Indeed, letting $T \to \infty$, expression (11) becomes[11]

$$F^\infty(C_t) = \begin{cases} \kappa \left(\frac{C_t}{\rho - \hat{\alpha}}\right)^\gamma & C_t < C^\infty \\ \left(\frac{C_t}{\rho - \hat{\alpha}}\right) - I & C_t \geq C^\infty \end{cases} \tag{14}$$

where

$$C^\infty = \frac{\gamma}{\gamma - 1} I(\rho - \hat{\alpha}), \qquad \kappa = \frac{(\gamma - 1)^{\gamma - 1}}{\gamma^\gamma I^{\gamma - 1}},$$

[9] Note that the probability we are interested in is under the actual measure, and not under the martingale measure.

[10] Actually, the probability $\widetilde{P}$ is defined as the probability of a first passage time for C_t through C_t^*, with initial condition $C_t \leq C_t^*$. Nevertheless, C_t^* is the investment threshold. Since C_t^* is decreasing over time, $\widetilde{P}$ in equation (13) provides a lower bound of the probability of investing.

[11] Note that in this case the solution is exact and not approximated: $F^\infty = F$.

γ as in equation (12) with $h = 1$, and $C^{\infty} > C_t^*$ for all t. The above is consistent with the usual valuation model for a perpetual option to defer investment (see Dixit and Pindyck [5, Ch. 5]).

Also expression (13) considerably simplifies when $T \to \infty$. By straightforward algebra, the (actual) probability of investing becomes[12]

$$P^{\infty}(C_t) = \begin{cases} 1 & \text{if } \alpha - \sigma^2/2 \geq 0 \\ \left(\frac{C^{\infty}}{C_t}\right)^{\frac{2\alpha}{\sigma^2}-1} & \text{if } \alpha - \sigma^2/2 < 0 \end{cases} \tag{15}$$

where $C_t < C^{\infty}$.

The infinite horizon case permits us to present some general results. If the project is infra-marginal, the higher the debt proportion, L, the lower (than r_z) is the risk-neutral WACC, ρ. To check the effect of leverage on the the investment threshold, C^{∞}, let's note that γ is increasing with respect to ρ and $\gamma/(\gamma-1)$ is decreasing with respect to γ. Hence, C^{∞} is decreasing in L. Since the probability of investing (in the only interesting case, $2\alpha/\sigma^2 < 1$) is decreasing with respect to C^{∞}, then we can state a negative effect of L on C^{∞} and a positive effect on the probability of investing. The opposite is true for a supra-marginal project, since in that case ρ is increasing in L. We have so proved the following proposition.

Proposition 4. *When $T \to \infty$, for an infra-marginal project, the investment threshold, C^{∞} is decreasing and the probability of investing, P^{∞}, is increasing with respect to L. The opposite is true for a supra-marginal project.*

Unfortunately, even in the infinite horizon case, it is not easy to state by comparative statics if F is increasing or decreasing in L, and hence we will resort to a numerical example to assess the influence of L on the value of the option and to see if Proposition 4 is confirmed when $T < \infty$.

[Figure 1 about here]

[Figure 2 about here]

The base case parameters are[13] $C_t = 2$, $\hat{\alpha} = 0.04$, $\Phi = 0.1$, $\sigma = 0.15$, $r_f = 0.05$, $r_z = 0.07$, $I = 100$, $\tau^* = 0.15$ for an infra-marginal project and $\tau^* = -0.15$ for a supra-marginal project. By running a sensitivity of $\widetilde{F}$ and $\widetilde{P}$ (applying (11) and (13)) on the above parameters we observe that, if $\tau^* = 0$, $\widetilde{F}$ and $\widetilde{P}$ are not affected by L. Moreover, we have the following facts:

[12] Since the investment threshold, C^{∞}, is independent of t, the valuation formula for probability is exact: $P^{\infty} = P$.

[13] The behavior of $\widetilde{F}$ and $\widetilde{P}$ presented in Figures 1 and 2 are observed for a wide set of parameters.

1. for an infra-marginal project, at any value of the cash flow rate, C_t, the value of the option to delay is an increasing function of L. This is because the higher L, the lower ρ, and hence the higher the fundamental value of the project $V_t = C_t/(\rho - \hat{\alpha})$. The opposite is true for a supra-marginal project. See Figure 1;

2. for an infra-marginal project, at any value of σ, the value of the real option is increasing with respect to L for exactly the same reason as above. The opposite holds for a supra-marginal project. See Figure 1;

3. for an infra-marginal project, at any C_t, the probability of investing within T is an increasing function of L. The opposite is true for a supra-marginal project. See Figure 2;

4. for an infra-marginal project, at any σ, the probability of investing before T is increasing with respect to L and, at any L, decreasing w.r.t. σ. For a supra-marginal project, things are more involved since, for very low values of L, $\widetilde{P}$ is decreasing w.r.t. σ, and for a high L, $\widetilde{P}$ is increasing w.r.t. σ. The same fact can be noted when $\widetilde{P}$ is plotted against σ at different values for L: when L is high, the probability of investing can be increasing w.r.t. volatility. In any case, $\widetilde{P}$ is a decreasing function of L. See Figure 2.

The effect that, for an infra-marginal project the probability of investing is an increasing function the proportion of debt can be explained as a consequence of limited liability of equity financing: the higher L, the larger part of operational risk is born by debtholders and so the investment is implemented less prudentially (i.e., at lower NPV). In a sense, the effect of debt financing is to mitigate irreversibility of investment under uncertainty from shareholders standpoints.

The above results can be explained in terms of agency theory. Shareholders making self-interested investment decisions, as opposed to shareholders aiming to maximizing the total firm value, tend to underinvest by delaying exercise of growth options (and hence, by reducing the probability of investing). This has been explained by Mauer and Ott [8] and Childs, Mauer and Ott [3] with the motivation that, if the project is all equity financed, while levered equityholders bear the full cost of the investment, they share benefits (and especially, a reduction of probability of default) also with bondholders. Childs, Mauer and Ott prove that this agency issue between bondholders and equityholders turn into a positive agency cost (i.e., lower value of the firm) and higher cost of capital (because of higher cost for bonds). Moreover, Childs, Mauer and Ott shows that partially financing the firm growth options with debt could incentive management (acting on behalf of shareholders) to adopt an investment policy which maximizes total firm value (first-best) instead of shareholders value (second-best). In other words, the agency cost of underinvestment is reduced when investments is finance with debt.

Our results are in line with Child, Mauer and Ott's findings. Moreover, we remark that the prototypical example of an option to delay investment entails a first-best investment policy, since the object in problem (7) is to maximize the (total) net present value of the project $(V_t - I)$.

The result that the probability of investing is a decreasing function of asset volatility[14] is in line with Sarkar [14] and Cappuccio and Moretto [2]. We just want to stress that debt financing mitigates the effect of uncertainty, since for any σ the probability of investing is an increasing function of L.

Since the prototypical case of a new firm/project discussed in this section was aimed only to simplify the arguments, it is straightforward to extend Proposition 3 to the case of a marginal project held by an ongoing firm with a debt proportion L. The project is marginal in the sense that it does not change the capital structure, so that to finance the project with value V and cost I, new debt $D = LV$ is issued.

At the end of this section, we want to stress the implications of the valuation model presented above:

1. starting from the date of implementation of the project, the level of debt is changed over time because the debt proportion is kept constant and the value of the project changes randomly. This assumption is very restrictive for many real-life projects, and so in the next section we present a valuation approach that overcomes this limitation;

2. the amount of debt issued at the date of implementation is a proportion of the value of the project, V, and not of the capital expenditure, I. This can be seen as a long-term representation of the (desired) capital structure for the project/firm. Nevertheless, this approach is not fully satisfactory because, when V is much higher than I and L is relatively high, this implies that the capital expenditure, I, could be completely financed by debt, and this is not the case with most real-life projects;

3. a related issue is on the *compatibility between the flexibility of the investment decision and the rigidity of the financing decision.* This issue becomes more relevant when we discuss the case of financing a new venture, provided that a large part of its value is given by growth options. *Should it be financed proportionally to the value of its growth options or to the value of the capital expenditure needed to exercise them?* In the first case, debt and equity financing would be provided *before* the options are exercised, and proportionally to the value of the options. If (part of) those options are left unexercised, there would be

[14] Letting $\delta = \rho - \hat{\alpha} > 0$, Sarkar [14] Moretto and Cappuccio [2] shows that for some parameter values when $\delta < \rho$ the probability of investing can be increasing with respect to volatility.

a large opportunity cost of capital and no return. This suggests that a new venture should be financed with debt and equity as a proportion of the capital expenditure at the date the growth options are exercised; i.e., debt instruments to finance real options should be designed in order to be (at least) as flexible as the investment decisions they are aimed to finance.

For the above reasons, in the next section we present a valuation approach for real options based on Adjusted Present Value, instead of the approach based on NPV of free cash flows and WACC introduced above.

4 Real options valuation with a constant level of debt

In order to overcome the limitations of the option valuation approach based on free cash flows and WACC, we introduce also a valuation approach for real options based on Adjusted Present Value (APV), under more general (i.e., less rigid) assumptions on the capital structure.

As a prototypical problem, we consider the opportunity to delay investment in an infinite-lived project until date T. At the date the project is implemented, the capital expenditure, I, is partially financed with debt, $D < I$. Although the level of debt is prespecified, bonds are issued only if (and when) the project is implemented.

Given a project with free cash flow rate $\{C_t\}$ following the dynamics in Equation (2), the APV of the project at date t, with debt D is (Sick [15, Eq. (15)] in continuous-time)

$$W_t = W(C_t) = \frac{C_t}{r_z - \hat{\alpha}} + \frac{D\tau^* r_f}{r_z} \tag{16}$$

where, $\tau^* = \tau_c - \tau_m$, r_z is the discount rate for equity flows and r_f is the discount rate for debt flows under the martingale measure. The above model incorporates default on debt (i.e., the project value can fall below the value of debt) and so corporate debt is risky. Hence r_f is also the certainty equivalent of R, the risky rate of return on debt.[15] We assume $r_z > \hat{\alpha}$ for convergence. The APV approach in equation (16) allows us to overcome the before mentioned drawbacks due to its company value decomposition. Hence, the first addend on the right-hand-side is the present value of free cash flows as if the project were all equity financed avoiding any reference to capital structure; the second addend is the tax shield which depends on the level of debt, D, and not on the proportion, L. As noted in Sick [15], the debt tax shield is discounted at the rate of return for equity flows because it

[15] In details, $r_f = \hat{\mathbb{E}}[R]$, where R is defined so as to ensure that the cash flow to equityholders is nonnegative.

accrues to equityholders. Since the tax shield does not depend on the cash flow, and assuming D constant[16]

$$\frac{dW_t}{W_t} = \hat{\alpha}dt + \sigma dZ_t :$$

the dynamics of W is the same as the dynamic of V, but for a different current value.

The value of the option to invest, under a first-best investment policy (i.e., a policy aiming to maximize the total project/firm value) in this project depends on the evolution of W.

Proposition 5. *The value of the option to invest in a project is the NPV at the optimal investment date, expected under the martingale probability measure, discounted at* r_z*:*

$$F(t, C_t) = \max_{s \in \mathcal{T}[t,T]} \left\{ e^{-r_z(s-t)} \hat{\mathbb{E}} \left[\Pi(s, C_s) \right] \right\} \tag{17}$$

where $\mathcal{T}[t, T]$ *is the set of stopping times with respect to* $\{\mathcal{F}_t\}$*.*

Proof. Under the martingale measure, $\hat{\mathbb{E}}[dF] = r_z F dt$. On the other hand, applying Itô's Lemma,

$$\hat{\mathbb{E}}[dF] = F_C \hat{\alpha} C dt + F_t dt + \frac{1}{2}\sigma^2 F_{CC} C^2 dt.$$

Comparing the two equations, the usual valuation p.d.e. is obtained

$$\frac{1}{2}\sigma^2 F_{CC} C^2 + \hat{\alpha} F_C C + F_t - r_z F = 0.$$

From this equation, with the usual boundary condition, the result in (17) follows. □

Also in this case, to analyze the effect of debt on real option valuation and on the investment policy we take advantage of Barone-Adesi and Whaley [1] analytical approximation. Hence, the approximate value of the option to delay investment is

$$\widetilde{F}(t, C_t) = \begin{cases} f'(t, C_t) + \varphi \left(\frac{C_t}{r_z - \hat{\alpha}} + \frac{D\tau^* r_f}{r_z} \right)^{\eta} & \text{if } C_t < \bar{C}_t^* \\ \frac{C_t}{r_z - \hat{\alpha}} + \frac{D\tau^* r_f}{r_z} - I & \text{if } C_t \geq \bar{C}_t^* \end{cases} \tag{18}$$

where, in this case

$$f'(t, C_t) = e^{-(r_z - \hat{\alpha})(T-t)} \mathcal{N}(d_1') \frac{C_t}{r_z - \hat{\alpha}} - e^{r_z(T-t)} \mathcal{N}(d_2') \left(I - \frac{D\tau^* r_f}{r_z} \right)$$

[16] Below, in Section 5, we will model also the case of a debt level changing over time as an effect of financial flexibility.

with

$$d_1' = \frac{\log \frac{C_t/(r_z-\hat{\alpha})}{\left(I-\frac{D\tau^* r_f}{r_z}\right)} + \left(\hat{\alpha}+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2' = d_1' - \sigma\sqrt{T-t}$$

$$\varphi = \left(\frac{1}{r_z-\hat{\alpha}} - e^{(r_z-\hat{\alpha})(T-t)}\mathcal{N}(d_1'(\bar{C}_t^*))\right)\left(\frac{\bar{C}_t^*}{r_z-\hat{\alpha}} + \frac{D\tau^* r_f}{r_z}\right)^{1-\eta}\frac{(r_z-\hat{\alpha})}{\eta}$$

and $\bar{C}_t^*$ is a root of equation

$$\frac{\bar{C}_t^*}{r_z-\hat{\alpha}} + \frac{D\tau^* r_f}{r_z} - I = f'(t,\bar{C}_t^*) + \left(\frac{1}{r_z-\hat{\alpha}} - e^{-(r_z-\hat{\alpha})(T-t)}\mathcal{N}(d_1'(\bar{C}_t^*)\right)\left(\frac{\bar{C}_t^*}{r_z-\hat{\alpha}} + \frac{D\tau^* r_f}{r_z}\right)\frac{r_z-\hat{\alpha}}{\eta}.$$

and

$$\eta = \frac{1}{2} - \frac{\hat{\alpha}}{\sigma^2} + \sqrt{\left(\frac{\hat{\alpha}}{\sigma^2}-\frac{1}{2}\right)^2 + 2\frac{r_z}{\sigma^2 h'(T-t)}} > 1,$$

$h'(t) = 1-e^{-r_z t}$. We can compute also the probability of investing (assuming that currently the opportunity is still available) within the time horizon T, $\widetilde{P}(C_t)$ from equation (13) with $\bar{C}^*$ in place of C^*.

The infinite horizon case ($T \to \infty$) considerably simplifies the above equations. Following the usual argument, we have

$$F^\infty(C_t) = \begin{cases} \varphi\left(\frac{C_t}{r_z-\hat{\alpha}} + \frac{D\tau^* r_f}{r_z}\right)^\eta & C_t < \bar{C}^\infty \\ \frac{C_t}{r_z-\hat{\alpha}} + \frac{D\tau^* r_f}{r_z} - I & C_t \geq \bar{C}^\infty \end{cases}$$

where

$$\bar{C}^\infty = \left(\frac{\eta}{\eta-1}I - \frac{D\tau^* r_f}{r_z}\right)(r_z-\hat{\alpha}) \qquad \varphi = \frac{(\eta-1)^{\eta-1}}{\eta^\eta I^{\eta-1}}.$$

In the same way, also the probability of investing can be computed using equation (15) with $\bar{C}^\infty$ in place of C^∞.

The above equations, together with the fact that, in the infinite horizon case both η and φ are independent of D ($h' = 1$), imply the following proposition.

Proposition 6. *When $T \to \infty$, for an infra-marginal project*

- *the value of the option to invest, F^∞ is increasing w.r.t. the level of debt D;*
- *the investment threshold, $\bar{C}^\infty$ is decreasing, with respect to D;*

- *the probability of investment is an increasing function of D.*

The opposite is true for a supra-marginal project.

As in Section (3), we will analyze the effect of debt both on the value of the option to defer and on the exercise policy by discussing a numerical example.

[Figure 3 about here]

[Figure 4 about here]

The base case parameters are $C_t = 2$, $\hat{\alpha} = 0.04$, $\Phi = 0.1$, $\sigma = 0.15$, $r_f = 0.05$, $r_z = 0.07$, $I = 100$, $\tau^* = 0.15$ for an infra-marginal project and $\tau^* = -0.15$ for a supra-marginal project. By running a sensitivity of $\widetilde{F}$ and $\widetilde{P}$(applying (18) and (13)) on the above parameters we observe that, if $\tau^* = 0$, $\widetilde{F}$ and $\widetilde{P}$ are not affected by L.

Moreover, numerical results confirm what we observed under the assumption that the underlying project sustains a constant debt proportion.

5 Valuation of compound options and change in capital structure

In this section we generalize the result of Section 4, and in particular Proposition 5, to take into account the opportunity to expand the initial project after the first investment is made. We assume that expansion is financed also with debt, so that the debt level is changed because of expansion. As in the previous section, both the project underlying the first option and the project underlying the second options are infinite lived.

Although we will present the case of one option to invest followed by one option to expand, given the recursive nature of our argument, what we present can be extended to the case of several compound options just at the cost of more cumbersome notation. The relevant feature of the model we present below is that level of debt changed only as a consequence of the exercise of an option.[17]

The case we will present clearly deals with either operational or strategic flexibility of an investment project. Nevertheless, it is easy to see that this case entails also financial flexibility, namely the possibility to change the level of debt over time.

Let there be given two options to invest into two different (but not necessarily perfectly correlated) real assets. The first project has a free cash

[17] In this case, we deal with an option to expand, but the argument we can be easily extended to the case of an option to reduce (or abandon) with partial (or total) repayment of debt.

flow process $\{C_t^1\}$, and the second project provides a free cash flow $\{C_t^2\}$. We assume that the two stochastic processes follow the dynamic (under the actual/empirical probability measure)

$$dC^i = \hat{\alpha}_i C^i dt + \sigma_i C^i dZ^i, \qquad i = 1, 2$$

with $\mathbb{E}\left[dZ^1 dZ^2\right] = \theta dt$. We assume that (for some reason) *the investment in the first project is a necessary condition to invest in the second project.* Hence, the second option can be seen as a growth (or expansion) option with respect to the first project. Let D_1 be the level of debt after the first investment and D_2 the level of debt after the second investment. D_1 and D_2 are prespecified at the date of valuation, although, as in previous section, debt capital will be raised (or repaid, it depends on the sign of $D_2 - D_1$) only if the project and its expansion are implemented.

Following the APV approach, the value of the first real asset is

$$W_t^1 = W^1\left(C_t^1\right) = \frac{C_t^1}{r_z - \hat{\alpha}} + \frac{D_1 \tau^* r_f}{r_z} \tag{19}$$

and the incremental value of the second project is

$$W_t^2 = W^2\left(C_t^2\right) = \frac{C_t^2}{r_z - \hat{\alpha}} + \frac{(D_2 - D_1)\tau^* r_f}{r_z}. \tag{20}$$

The intrinsic value of the growth option is

$$\Pi_2(C_t^2) = \max\left\{W^2\left(C_t^2\right) - I_2, 0\right\}.$$

From Proposition 5, the value of the option to expand at t, assuming it has not been exercised yet, is

$$F_2(t, C_t^2) = \max_{s \in \mathcal{T}[t,T]} \left\{e^{-r_z(s-t)} \hat{\mathbb{E}}\left[\Pi_2(s, C_s^2)\right]\right\}. \tag{21}$$

The intrinsic value of the option to invest is

$$\Pi_1(C_t^1, C_t^2) = \max\left\{W^1\left(C_t^1\right) - I_1 + F_2(C_t^2), 0\right\}$$

and the value of this option is

$$F_1(t, C_t^1, C_t^2) = \max_{s \in \mathcal{T}[t,T]} \left\{e^{-r_z(s-t)} \hat{\mathbb{E}}\left[\Pi_1(s, C_s^1, C_s^2)\right]\right\}. \tag{22}$$

The above simple scheme can be used also to allow for financial flexibility within a given project, i.e., the possibility to change the level of debt (and hence the capital structure). This can be easily seen by putting $C_0^2 = 0$ in equation (20). In that case, the intrinsic value of the second option is given only by the change in the tax shield due to a change in the level of debt (from D_1 to D_2) and I_2 can be interpreted as a cost of changing the capital structure (i.e., the cost of repurchasing bonds or the cost of issuing new bonds). Also in this case, an extension to a sequence of compound options allowing for financial flexibility is straightforward.

Since neither problem (21) nor problem (22) can be solved analytically, we employ a numerical methodology. Hence, we will work on a base example, providing a sensitivity of the value of the whole project and of the investment policy on the level of debt. The base case parameters are in Table 1. The numerical solution is obtained by employing a log-transformed binomial lattice approach. An outline of the numerical approach is given in Section A.

Numerical results confirm all the properties we have already observe in section 4; i.e., the higher D_i the higher the value of the whole projects (and of the embedded options): see Figure 5 (above). Moreover, we analyze also how the level of debt affects the investment policy of the option to delay investment, whose value is influenced also by the value of the (subsequent) expansion option. To this aim, we analyze the investment threshold as follows. Given F_1, the (optimal) value (function) of the option to delay, as given in equation (22), the continuation region for the option to delay is

$$\mathcal{C} = \left\{(t, C_t^1, C_t^2) \in \mathbb{R}^3_{++} \mid F_1(t, C_t^1, C_t^2) > W^1\left(C_t^1\right) - I_1 + F_2(t, C_t^2)\right\}$$

and the stopping region $\mathcal{S} = \mathbb{R}^3_{++} \backslash \mathcal{C}$. Hence, there is an unknown investment threshold, $\Omega = \partial\mathcal{C} \cap \mathbb{R}^3_{++}$, given by the frontier of $\mathcal{C}$. Ω is a surface in $\mathbb{R}^3$. For simplicity, we will consider only the investment threshold at $t = 0$, $\Omega_0 = \Omega \cup \{t = 0\}$.[18] We will assume also that the debt proportion of the capital expenditure, D_i/I_i, $i = 1, 2$, is the same for both the option to invest and the option to expand. The results of this analysis are displayed in Figure 5 (below), where we provide three thresholds Ω_0, at different value of D_i. From Figure 5, it is clear that the continuation region $\mathcal{C}$ (i.e., the region below Ω_0) shrinks for larger levels of debt. Hence, the intuition that a higher level of debt increases the probability of investing is true also in the case of compound options on two underlyings.

[18]The investment threshold at any $0 \leq t \leq T$ can be obtained using the same methodology by changing the time to maturity of both options to $T_i - t$. The curves representing Ω_t are obtained by searching an approximate 0-level sections of $F_1(t, C_t^1, C_t^2) - W^1(C_t^1) - I_1 + F_2(t, C_t^1, C_t^2)$. In fact, the search for exact 0-level curve is very difficult because the exact 0-level curve degenerates in a region in $C^1_{\min}, C^1_{\max}] \times [C^2_{\min}, C^2_{\max}]$. In details, the graph show an ε-level section for $\varepsilon = 0.1$.

Table 1: Growth option: base case parameters

r_z	c.e. return for stocks	0.07	
r_f	c.e. return for bonds	0.05	
τ^*	net tax shield	0.15	
$\hat{\alpha}_1$	drift of C_t^1	0.05	
Φ_1	risk-premium for C_t^1	0.05	
$\hat{\alpha}_2$	drift of C_t^2	0.16	
Φ_2	risk-premium for C_t^2	0.1	
σ_1	volatility of C_t^1	0.1	
σ_2	volatility of C_t^2	0.2	
θ	correlation coefficient†	0	
T_1	expiry of first growth option	3	(years)
T_2	expiry of second growth option	5	(years)
I_1	capital expenditure for first option	150	($)
I_2	capital expenditure for second option	400	($)

† Note that the whole project value and the optimal investment policy are independent of θ. Hence, in the current case for convenience we assume $\theta = 0$.

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A Numerical methods

We summarize here the main features of the improved log-transformed binomial lattice approach, suited to price options with payoffs depending on a multidimensional log-Normal diffusion, as proposed in Gamba and Trigeorgis [6]. We specialize it to the two-dimensional setting of our valuation problem in Section 5. The log-transformed method maintains the stability feature of the one-dimensional approach (proposed by Trigeorgis [16]) while extending the lattice approach to a multidimensional setting. It can be proved that, according to the log-transformed approach, good approximations can be obtained also with few time steps. This is a very important feature for our purposes, since in our problem the results presented are computationally intensive.

Given the dynamics of free cash flows (under the martingale measure)

$$\begin{aligned} dC^1(t) &= \hat{\alpha}_1 C^1(t)dt + \sigma_1 C^1(t)dZ^1(t) \\ dC^2(t) &= \hat{\alpha}_2 C^2(t)dt + \sigma_2 C^2(t)dZ^2(t) \end{aligned} \tag{23}$$

with $\mathbb{E}\left[dZ^1 dZ^2\right] = \theta dt$, we take $X^i = \log C^i$, so that

$$\begin{aligned} dX^1(t) &= a_1 dt + \sigma_1 dZ^1(t), \\ dX^2(t) &= a_2 dt + \sigma_2 dZ^2(t), \end{aligned} \tag{24}$$

where $a_1 = \hat{\alpha}_1 - \sigma_1^2/2$ and $a_2 = \hat{\alpha}_2 - \sigma_2^2/2$. Let $a^\top = (a_1, a_2)$,

$$\Sigma = \begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}, \quad b = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \Omega = b\Sigma b^\top = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\theta \\ \sigma_1\sigma_2\theta & \sigma_2^2 \end{pmatrix}.$$

Define

$$\varrho_{1,2} = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 \mp \sqrt{\sigma_1^4 + 2(1-2\theta^2)\sigma_1^2\sigma_2^2 + \sigma_2^4}\right),$$

the diagonal matrix $\Lambda = (\varrho_i)$, and matrix

$$\Xi = \begin{pmatrix} \left(\frac{\varrho_1}{\sigma_1\sigma_2} - \frac{\sigma_2}{\sigma_1}\right)/(\theta c_1) & \left(\frac{\varrho_2}{\sigma_1\sigma_2} - \frac{\sigma_2}{\sigma_1}\right)/(\theta c_2) \\ 1/c_1 & 1/c_2 \end{pmatrix}$$

where

$$c_i = \sqrt{1 + \left(\frac{\varrho_i - \sigma_2^2}{\theta\sigma_1\sigma_2}\right)^2}.$$

Ξ is a matrix which provides a change of coordinates for the space $\left(X^1, X^2\right)$ so that the dynamics in (24) are uncorrelated. Hence, denoting $x^\top = (x^1, x^2)$ the vector of variables transformed through Ξ, the diffusion process of x is

$$\begin{aligned} dx^1 &= A_1 dt + B_{11} dZ_1 + B_{12} dZ_2 \\ dx^2 &= A_2 dt + B_{21} dZ_1 + B_{22} dZ_2 \end{aligned}$$

where $B = (B_{ij}) = \Xi^\top b$ and $A = \Xi^\top a$. The covariance matrix of dx is $dx dx^\top = \Lambda dt$; that is, dx^1 and dx^2 are uncorrelated.

We approximate (dx^1, dx^2) with a discrete process: given the time interval $[0, T]$, where $T = \max_i \{T_i\}$ from Table 1, and n, we consider subintervals of width $\Delta t = T/n$. The discrete process is $(\widetilde{x}^1, \widetilde{x}^2)$ with dynamics

$$\begin{aligned} \widetilde{x}^1(t) &= \widetilde{x}^1(t - \Delta t) + \ell_1 U_1(t) \\ \widetilde{x}^2(t) &= \widetilde{x}^2(t - \Delta t) + \ell_2 U_2(t) \end{aligned} \tag{25}$$

$t = 1, \ldots, n$ where (U_1, U_2) is a bi-variate i.i.d. binomial random variable:

$$(U_1, U_2) = \begin{cases} (1,1) & \text{with probability } p_1 \\ (1,-1) & \text{w.p. } p_2 \\ (-1,1) & \text{w.p. } p_3 \\ (-1,-1) & \text{w.p. } p_4 \end{cases}$$

and $\sum_{i=1}^4 p_i = 1$. We assign parameters

$$k_i = A_i \Delta t, \quad \ell_i = \sqrt{\varrho_i \Delta t + k_i^2}, \quad K_i = k_i/\ell_i$$

$i = 1, 2$ and probabilities

$$p(s) = \frac{1}{4} \left(1 + \Gamma_{12}(s) K_1 K_2 + \Gamma_1(s) K_1 + \Gamma_2(s) K_2\right) \quad s = 1, 2, 3, 4, \tag{26}$$

where

$$\Gamma_i(s) = \begin{cases} 1 & \text{if variable } i \text{ jumps up} \\ -1 & \text{if variable } i \text{ jumps down} \end{cases}$$

for $i = 1, 2$, and $\Gamma_{12}(s) = \Gamma_1(s)\Gamma_2(s)$.

We want to evaluate an option whose payoff, Π, is a non-linear function of $(C^1(t), C^2(t))$. According to the change of variable $X^i = \log C^i$, the payoff becomes

$$\Pi(C^1(0)e^{X^1(t)}, C^2(0)e^{X^2(t)}).$$

We make the payoff dependent on $x^\top = (x^1, x^2)$ by changing the payoff function as follows:

$$\Pi'\left(x^1(t), x^2(t)\right) = \Pi\left(C^1(0)e^{(\Xi x(t))_1}, C^2(0)e^{(\Xi x(t))_2}\right),$$

where $(\Xi x(t))_i$ is the i-th component of vector $\Xi x(t)$. The risk-neutral price of Π', denoted F', is equal to the risk-neutral price of Π, denoted F according to equation (17) (we refer to Gamba and Trigeorgis [6] for further details):

$$\begin{aligned} F'\left(x^1(t), x^2(t)\right) &= e^{r_z(T-t)} \mathbb{E}'\left[\Pi'\left(x^1(T), x^2(T)\right)\right] \\ &= e^{r_z(T-t)} \mathbb{E}\left[\Pi(C^1(T), C^2(T))\right] = F(C^1(t), C^2(t)) \end{aligned}$$

where $\mathbb{E}'[\cdot]$ denotes the risk neutral expectation with respect the martingale probability of process $\{(x^1, x^2)\}$, and $\mathbb{E}[\cdot]$ is the expectation w.r.t. the martingale probability of process $\{(C^1, C^2)\}$.

In order to compute the value of the growth option when a closed-form formula is not available, we exploit the above illustrated extended log-transformed binomial lattice approximation of the diffusion in (23). Hence, by approximating (x^1, x^2) with $(\widetilde{x}^1, \widetilde{x}^2)$, as of equation (25), the value of the second growth option is obtained by backward induction recursively applying equation

$$F_2'\left(\widetilde{x}^1(t), \widetilde{x}^2(t)\right) = \\ = \max\left\{\Pi_2\left(\widetilde{x}^1(t), \widetilde{x}^2(t)\right), e^{-r_z \Delta t}\widetilde{\mathbb{E}}_t'\left[F_2'\left(\widetilde{x}^1(t+\Delta t), \widetilde{x}^2(t+\Delta t)\right)\right]\right\}$$

where

$$\Pi_2\left(\widetilde{x}^1(t), \widetilde{x}^2(t)\right) = W\left(C^2(0)e^{(\Xi x(t))_2}\right) - I_2$$

from equation (19) and $\widetilde{\mathbb{E}}_t'[\cdot]$ denotes conditional expectation, at t, according to the discrete probability in (26). The same can be done to compute the value of the first growth option:

$$F_1'\left(\widetilde{x}^1(t), \widetilde{x}^2(t)\right) = \\ = \max\left\{\Pi_1\left(\widetilde{x}^1(t), \widetilde{x}^2(t)\right), e^{-r_z \Delta t}\widetilde{\mathbb{E}}_t'\left[F_1'\left(\widetilde{x}^1(t+\Delta t), \widetilde{x}^2(t+\Delta t)\right)\right]\right\}$$

where

$$\Pi_1\left(\widetilde{x}^1(t), \widetilde{x}^2(t)\right) = W\left(C^1(0)e^{(\Xi x(t))_1}\right) - I_1 + F_2'\left(\widetilde{x}^1(t), \widetilde{x}^2(t), t\right).$$

as in equation (20).

The extension of this numerical methodology to a multidimensional problem (i.e., with more than two underlying assets) and more compounded growth options is straightforward.

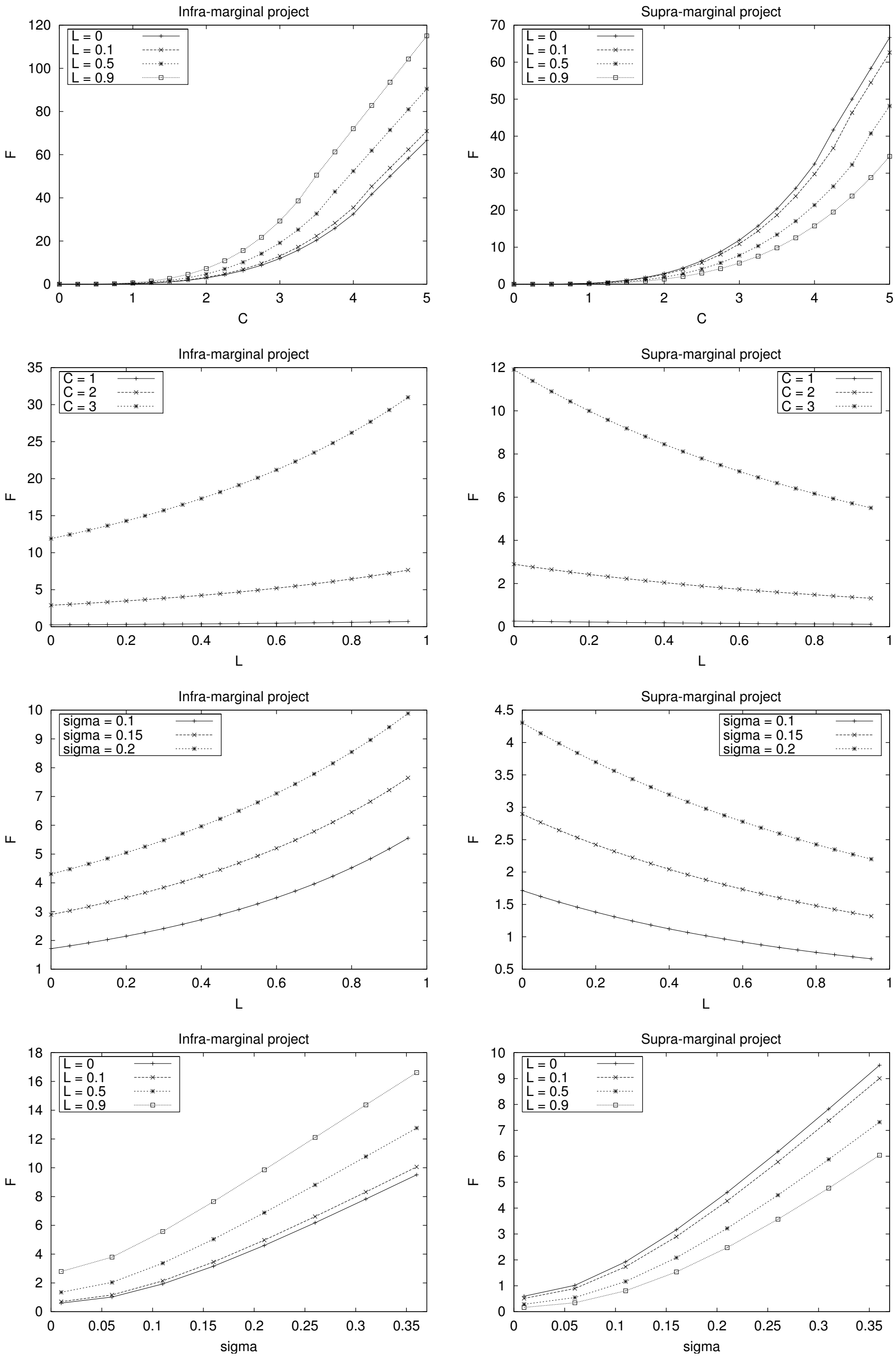

Figure 1: **Constant debt proportion**. Value of the option to invest, $\widetilde{F}$, for different debt proportions, L, different free cash flow rates, C, and for different volatilities, σ for an infra- ($\tau^* = 0.15$) and a supra-marginal ($\tau^* = -0.15$) project (other parameters: $\hat{\alpha} = 0.04$, $\sigma = 0.15$, $r_f = 0.05$, $r_z = 0.07$, $I = 100$, $T = 5$).

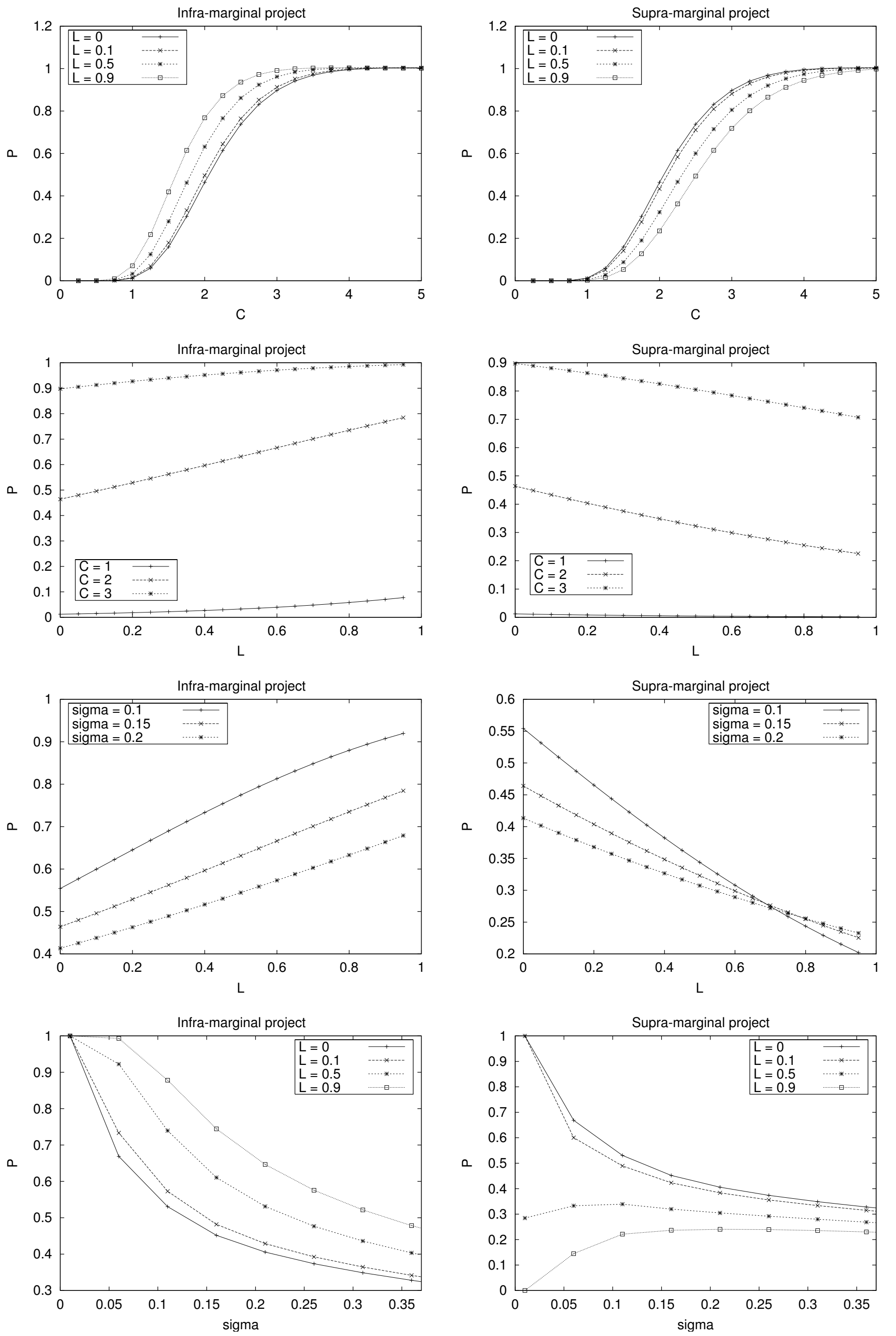

Figure 2: **Constant debt proportion**. Probability of investing, $\widetilde{P}$, for different debt proportions, L, for different free cash flow rates, C, and for different volatilities, σ, for an infra- ($\tau^* = 0.15$) and a supra-marginal ($\tau^* = -0.15$) project (other parameters: $\hat{\alpha} = 0.04$, $\sigma = 0.15$, $r_f = 0.05$, $r_z = 0.07$, $I = 100$, $T = 5$).

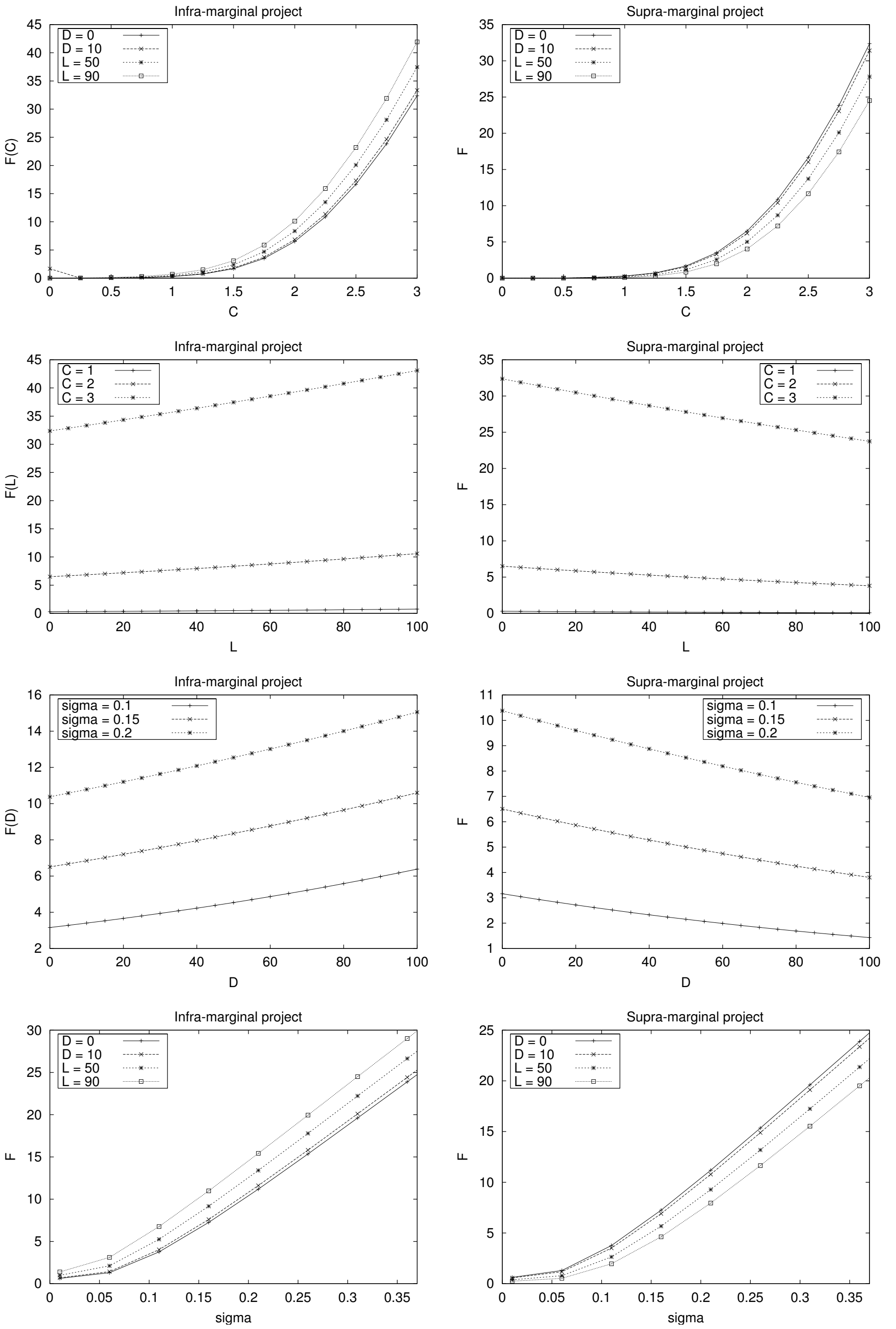

Figure 3: **Constant level of debt**. Value of the option to invest, $\widetilde{F}$, for different debt levels, D, different free cash flow rates, C, and for different volatilities, σ for an infra- ($\tau^* = 0.15$) and a supra-marginal ($\tau^* = -0.15$) project (other parameters: $\hat{\alpha} = 0.04$, $\sigma = 0.15$, $r_f = 0.05$, $r_z = 0.07$, $I = 100$, $T = 5$).

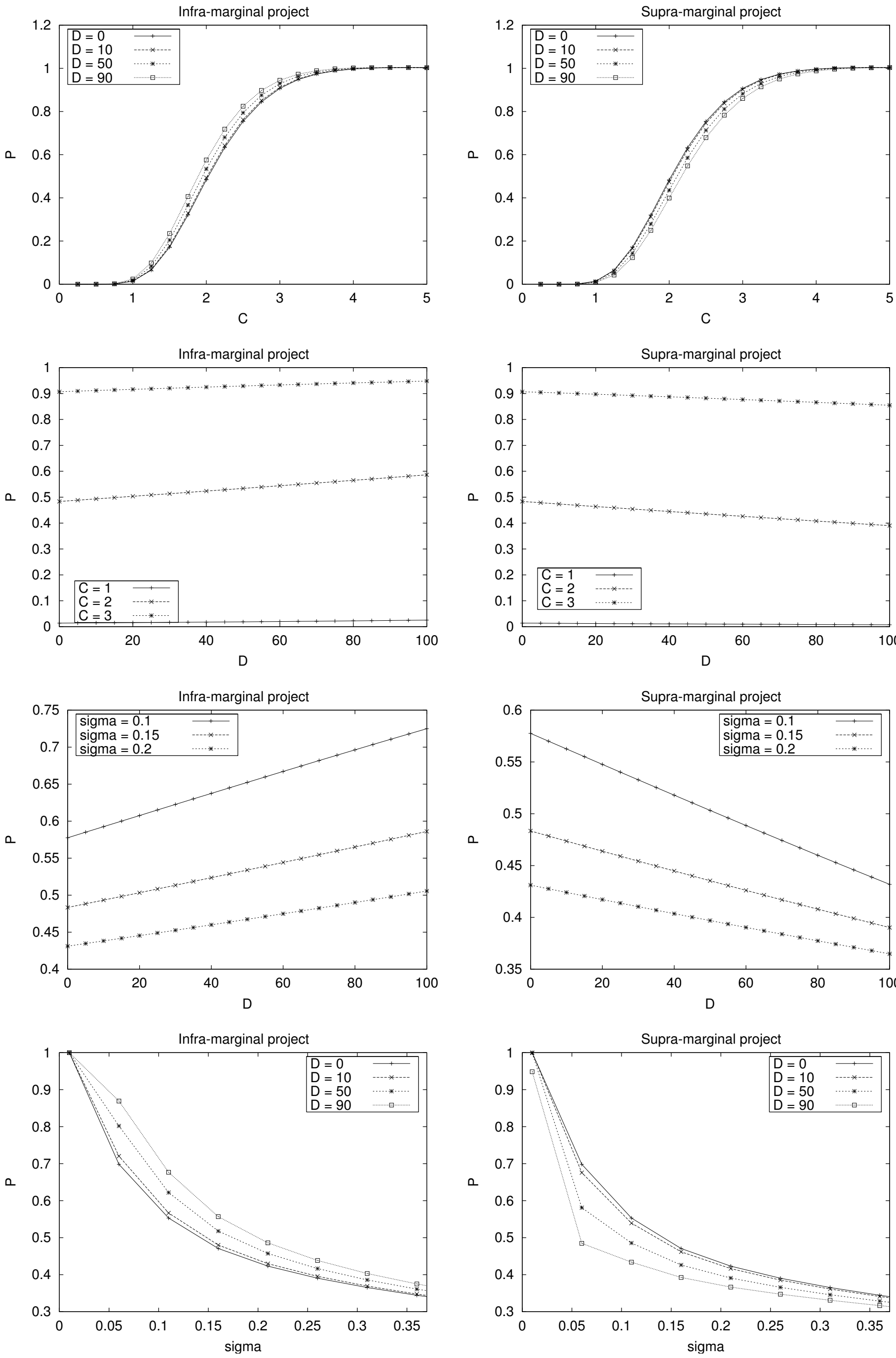

Figure 4: **Constant level of debt**. Probability of investing, $\widetilde{P}$, for different debt levels, D, for different free cash flow rates, C, and for different volatilities, σ, for an infra- ($\tau^* = 0.15$) and a supra-marginal ($\tau^* = -0.15$) project (other parameters: $\hat{\alpha} = 0.04$, $\sigma = 0.15$, $r_f = 0.05$, $r_z = 0.07$, $I = 100$, $T = 5$).

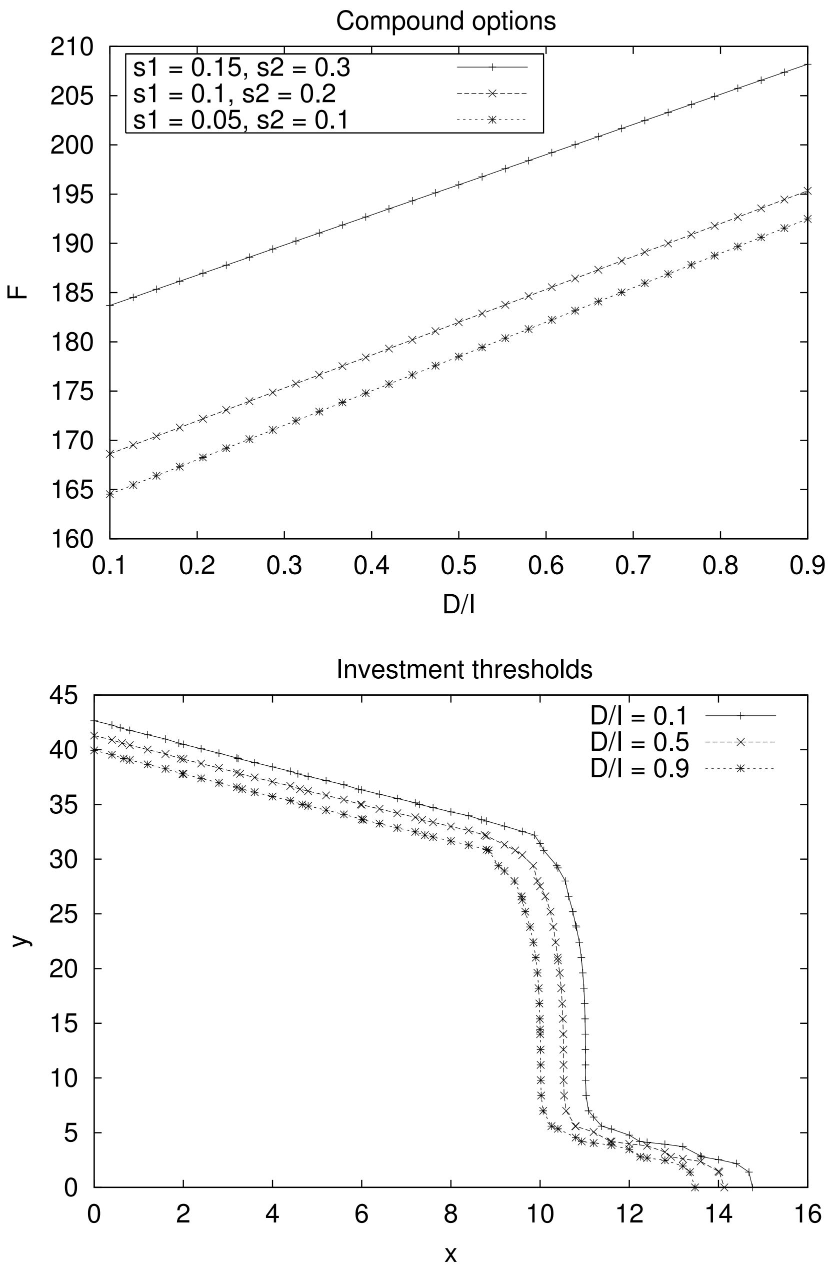

Figure 5: Compound options for an infra-marginal project (Case parameters are given in Table 1.): (above) Sensitivity of the value of the option to delay investment, F_1, on debt level D_i for different volatilities of the underlying real assets. (below) Sensitivity of investment threshold at time $t = 0$, Ω_0, on debt levels D_i for option to delay.