Real Options Analysis and the Assumptions of the NPV Rule[1]

Tom Arnold
Assistant Professor of Finance
E.J. Ourso College of Business Administration
2163 CEBA
Louisiana State University
Baton Rouge, LA 70803
O: 225-578-6369
F: 225-578-6366
tmarnol@lsu.edu

Richard L. Shockley, Jr.
Assistant Professor of Finance
Ford Motor Company Teaching Fellow
Kelley School of Business
Indiana University
1309 E. 10th Street, BU370
Bloomington, IN 47405-1701
O: 812-855-3407
F: 812-855-5875
rishockl@indiana.edu

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Introduction

DCF analysis, the NPV rule, and the maximiziation of shareholder value tenet are considered by most practitioners to be axioms of finance, when in fact they are actually *results* of theoretical economic constructions which rely on several critical assumptions. The objective of this paper is to reveal these assumptions and to show that the valuation technique known as *option pricing* is built on the exact same economic foundation. The import of this is straightforward: if a manager is willing to make the assumptions necessary to apply the NPV rule to a potential investment, then the manager has already made all of the assumptions necessary to also price *options* on that project. In other words, *real options analysis* is perfectly valid in any situations where DCF/NPV is applied *without further assumptions.*

A common objection to real option analysis is that option pricing models require certain assumptions that are not met in real asset markets. For example, one often hears the ritual protest that options on real assets cannot be priced because the real asset is not traded, and hence cannot be held in an arbitrage portfolio. We'll show that this objection is completely unfounded in any situation where DCF and the NPV rule can be applied: as long as you are willing to make the assumptions necessary for application of the NPV rule to an illiquid asset, you have made assumptions that are sufficiently strong for application of option pricing to value that asset – even though the asset is not traded. To put it another way, if you reject real options analysis in a corporate setting due to the illiquidity of the project, you must also reject DCF and the NPV rule as well.

The point of valuation in corporate finance is to ascertain what value a new project would have *if it were currently available in the financial markets*. All that the financial markets care about are the timing and risk of the cash flows from the investment, and if the corporate manager can 'purchase' those cash flows in the real asset market (by investing in a new project) more cheaply than investors could purchase those cash flows in the financial asset market, then the new project makes existing shareholders wealthier.

We'll demonstrate that the common foundation of both the DCF/NPV model and the option pricing model is *valuation by arbitrage*. Both models use the prices of existing assets, which are determined in equilibrium, to value new assets. In either approach, the objective is to find an existing asset or portfolio that exactly mimics the 'new' item; since the new item can be replicated by the existing assets, the new item must have the same price as the reference asset or portfolio. This makes the common assumption of the two models transparent: both DCF/NPV and option pricing models (regardless of their application) rely critically on the assumption that any new proposals are really "old wine in new bottles" – their cash flows can be recreated in the financial markets . This is the assumption of *complete markets*.[2]

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[2]The DCF rule requires market completeness because it uses equilibrium rates of return on *existing* assets to calculate the values of *new* assets. If markets are not complete, then the introduction of the new asset could change the equilibrium rates of return on the existing assets – which destroys the validity of the procedure. This points out a further assumption of the DCF rule: the new asset cannot change aggregate consumption in a material way (else the same effect would occur). If these assumptions are not met, then

But if DCF/NPV and option pricing are fundamentally the same, why do we teach two different valuation methods? Pedagogical simplicity.

The advantage of the DCF approach is that it is easy to teach, particularly if the teacher is willing to take shortcuts that simplify the presentation (at the expense of understanding the real economic explanation). The downside of the DCF approach is that it is extremely difficult to apply in an important set of situations. These are the situations of flexibility, and most managers know that their investments have flexibility features that are valuable but are not being captured in their DCF/NPV analysis. Option pricing techniques work in this broader set of situations; the problem is simply that option pricing methodology is slightly more difficult to teach and understand.

In the following sections, we'll set up a simple economy where there's no storage of assets allowed. We will use this economy to demonstrate the 'textbook' development of the NPV rule and, along the way, expose an extreme shortcut in the standard presentation that not only circumvents a large portion of the economic intuition behind the approach but also renders the pricing of derivatives impossible. We will then remove the shortcut and show how option pricing works in the same economy.

The Green Acres Economy

Let's consider the simplest possible model of risk. The community of Hooterville is a hamlet of N individuals who are identical in preferences, tastes, endowments and beliefs about the future. The residents of Hooterville are farmers, and because of the geography of the local area, the only crop is corn. Furthermore, the people of Hooterville are completely isolated from the outside world. Hence, the only consumptive commodity is corn; the residents can consume only what they produce, no more and no less.

To keep things simple, suppose that the Hooterville economy lasts only one year, and at the end of the year the individuals share equally in the (random) output of the year's crop. In other words, all N individuals have identical contingent claims on the crop at the end of the year: 200 bushels of corn if there is a banner crop ($y_{banner} = 200$) but only 80 bushels if the crop is poor ($y_{poor} = 80$). Moreover, all of the individuals agree that 'banner' and 'poor' harvests are equally likely: $\pi_{banner} = \pi_{poor} = .5$.

Each individual is endowed with 100 bushels of 'present' corn ($y_0 = 100$). Hooterville is a pure-exchange economy, so it is impossible for the residents to change their endowments by planting or storing; however, they can change their consumption patterns by trading. Trading takes place at Sam Drucker's general store. Drucker's market allows the residents to alter their consumption across time: those wishing to consume more than

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maximization of firm value may not be the desired goal of the shareholder: situations would exist when shareholders would like for the firm to adopt negative NPV projects that create positive changes to the rates of return on everything else in their portfolios. See Baron (1979) for an excellent review of what is known as the 'unanimity' literature.

100 bushels of corn today may 'borrow' against their future corn endowment, while those with 'extra' corn today may 'lend'.

We reiterate that Hooterville is a pure-exchange economy – there is no investment and *storage is not allowed* (that is, the corn is perishable). So the total current crop (100 bushels of corn times *N* individuals = *100N* bushels) must be consumed today. Similarly, the entire future crop, whatever it may be, must be consumed at the end of the period. The expected crop is .5x200 bushels + .5x80 bushels = 140 bushels per person, times *N* individuals = *140N* bushels. Individuals may trade current consumption for claims on future consumption so that some individuals may consume more than their endowments today while others consume less, but across the entire economy the consumption today must be *100N* bushels, and the expected consumption at the end of the year must be *140N* bushels (total consumption will be *200N* bushels of corn if the 'banner' crop appears and *80N* bushels if the 'poor' crop appears, regardless of the trades made).

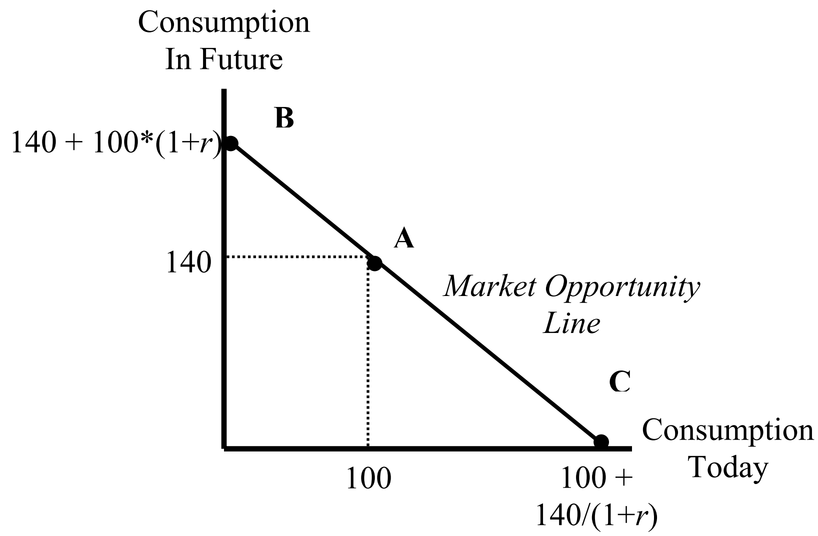
The Textbook Presentation of the Financial Market

Virtually every Corporate Finance textbook begins with a chapter (usually labeled 'advanced', and sometimes skipped by instructors), which demonstrates that individuals can use financial markets (like Drucker's store) to adjust their patterns of consumption over time. The ultimate point of this is to show that financial markets can provide a 'benchmark' for investment decisions, and this serves as the introduction to discounted cash flow and the NPV rule.

For the sake of pedagogy, a huge shortcut is taken: the intertemporal model with risk (the future crop is uncertain) is simplified into a world of *certainty* (the future crop is known). This simplified sort of presentation works well to provide a motivation for the NPV rule, but we'll demonstrate that the insight about NPV gained through the simplification comes at a cost: any insight about what it means for projects to have different risks is completely lost. This in turn makes option pricing impossible to teach in the same way.

In Hooterville, the shortcut changes the problem from one where the future harvest could be banner or poor to one where the 'expected' crop is used as a substitute. In other words, any distinction between the welfare of the population in the 'banner crop' state and in the 'poor crop' state is completely lost. This turns out to be extremely important, because *risk* is really a function of how payoffs differ across different states.[3] The intertemporal consumption opportunities available at Drucker's market for each individual are presented in the typical Finance text graphically in the following picture.

[3] For example, a 100-bushel loan to a resident of Hooterville differs in risk from a 20-bushel loan, because the small loan could always be repaid from future crop whereas the large loan defaults in the poor-crop state. The standard textbook treatment provides no distinction between the two because the future is collapsed to one state (a crop of 140); thus, both loans are incorrectly treated as having the same risk.



In the above graph, the point A is the initial endowment of Alf Monroe, a representative resident of Hooterville who (in partnership with his sister Ralph) provides carpentry services as a hobby. The horizontal axis presents Alf's current consumption of corn, and the vertical axis measures his future consumption of corn. The line that connects the two axes is the 'market opportunity line' which represents all of the possible baskets of current and future consumption that Alf could achieve by trading in the market at Sam Drucker's; the slope of the market opportunity line represents, in equilibrium, how much current corn the residents of Hooterville are willing to give up in order to get one additional unit of future corn. Mathematically,

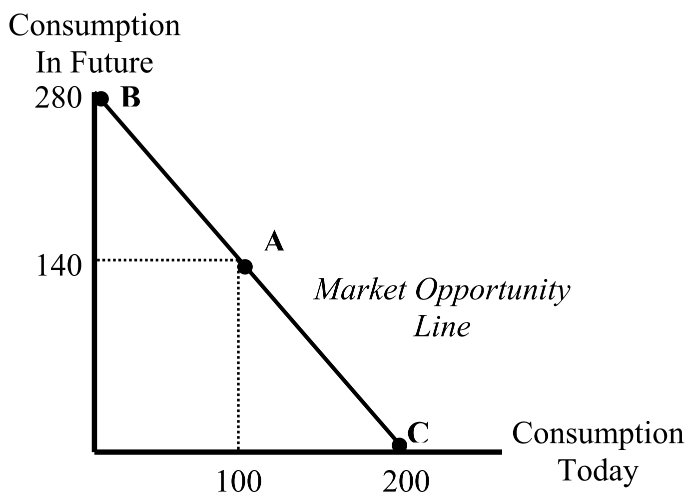
$$-1\cdot\left(slope\ of\ \ market\ opportunity\ line\right)=\left(1+rate\ of\ \ return\right)$$

Point B on the chart represents the maximum amount of corn Alf could consume in the future by lending away all 100 bushels of his current endowment of corn at the market rate of interest (140 of future endowment plus $100(1+r)$ of return on the loan of the current endowment). Point C represents the maximum amount of corn Alf could consume immediately by borrowing against his expected future endowment (100 bushels of current endowment plus $140/(1+r)$ of borrowing against the future endowment).

We stress the point that even though this is a pure-exchange economy, there is a market rate of interest. An interest rate is simply the price of consumption in a future period relative to the price of consumption in an earlier period. Here, the market rate of interest is the equilibrium marginal rate of substitution between consumption of corn today and in the future. So even though the residents of Hooterville can't save or invest, they can use Drucker's market to observe the equilibrium interest rate and to trade claims on future consumption for claims on current consumption.

In addition to trading, the residents of Hooterville can use the equilibrium outcome of Drucker's market to attach values to new deals that might be offered. In other words, the residents of Hooterville can use the NPV rule.

To see this, suppose that the equilibrium market rate of interest on corn is 40% (we'll demonstrate how this is derived in the next section). Using the calculations mentioned earlier, the chart below shows Alf's market opportunities available from trade at Sam Drucker's.



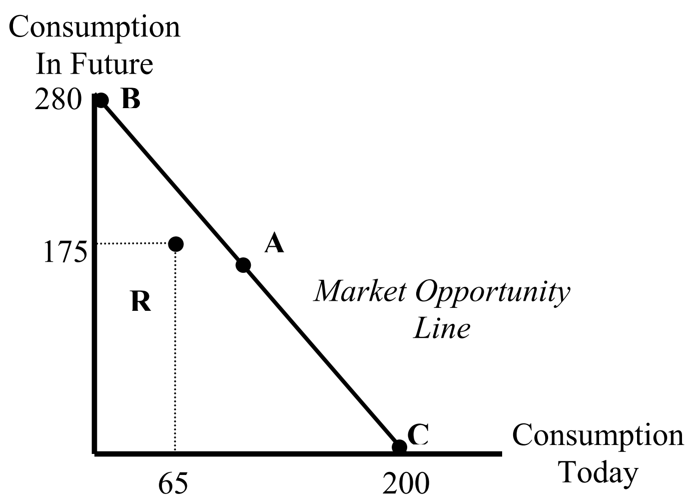
Again, Point A represents Alf's current (before-trading) consumption bundle: 100 bushels of corn today and 140 bushels of corn in the future. Point B represents Alf's opportunity for maximum consumption in the future. If Alf somehow wanted to consume zero corn today and a maximal amount in the future, he could lend his entire 100 bushel current endowment at a 40% rate and get back 140 bushels of corn in the future. So the most Alf could consume in the future is 140 bushels of future endowment plus 140 bushels return on his current-endowment loan equals 280 bushels of corn. Point B represents this consumption bundle for Alf: zero consumption today, 280 bushels in the future. Simliarly, since the most Alf can repay in the future on any loan is 140, the most Alf can borrow for enhanced current consumption is $140/(1+r) = 140/(1.40) = 100$ bushels of corn. So Alf's maximum current consumption of corn would be 100 bushels of current endowment plus 100 bushels of borrowing against future endowment equals 200 bushels; point C represents a consumption bundle for Alf of 200 bushels of corn today and zero in the future.

Armed with this information, Alf can examine any other opportunities that might arise. For example, suppose that one afternoon, Alf's sister Ralph offers to sell to Alf exactly one-fourth of her future corn endowment in exchange for 35 bushels of Alf's current endowment.

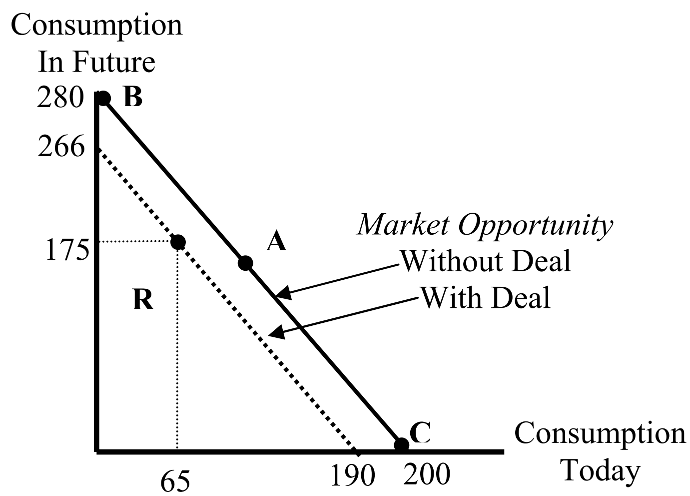
Alf examines this opportunity in the following way. The expected payoff on the deal, one-fourth of Ralph's future endowment of corn, is $\frac{1}{4}(.5\times 200)+\frac{1}{4}(.5\times 80)=35$. In the textbook treatment, this is considered risk-free (even though it is not) and Alf is satisfied thinking of it as an exchange of 35 bushels of corn today for 35 in the future.

	Today	Future
Alf's Original Endowment	100	140
Accept Ralph's Deal	-35	35
Alf's New Wealth	65	175

If Alf gives up 35 bushels of current corn in exchange for 35 bushels of 'expected' corn in the future, his consumption pattern is given by point R in the graph below: 65 bushels of corn today, and 175 bushels of 'expected' corn in the future.



Alf can determine the value of this deal in several different ways, all of which lead to the same result. First, Alf could re-calculate his market opportunity line under the assumption that he accepts Ralph's deal. Using the same calculations as before, one can easily generate the next picture which compares Alf's market opportunities *with* the deal versus those *without*.



In terms of maximum possible current consumption, Ralph's deal makes Alf strictly worse off by 10 bushels of corn. In terms of maximum possible future consumption, Ralph's deal makes Alf strictly worse off by 14 bushels of corn. It should not be surprising that 14/1.40 = 10.

A second way for Alf to analyze Ralph's deal would be to ask the question slightly differently. Suppose that Alf were to go to Drucker's market and find a way to 'replicate' Ralph's deal by trading. Ralph's deal promises 35 in the future and costs 35 today. If Alf can buy 35 bushels of future corn at Drucker's, how much more or less than Ralph's price would it cost?

At Drucker's, Alf can 'purchase' 35 bushels of expected future corn by lending 35/1.4 = 25 bushels of current corn. So, the same deal in the financial markets costs Alf 10 bushels of current corn *less* than the price Ralph is asking.

A third way that Alf could analyze Ralph's deal is similar to the second. Suppose that Alf were to accept Ralphs deal (by giving Ralph 35 bushels today and taking the promise of 25 bushels in the future), and then immediately go to Drucker's and borrow against the future corn promised by Ralph. The 35 future bushels of corn promised by Ralph would support a 25 bushel loan today at the market rate of 40%. How would this change Alf's wealth today?

	Today	Future
Alf's Original Endowment	100	140
Accept Ralph's Deal	-35	35
Borrow 25 Bushels @ 40% Interest	25	-35
Alf's New Wealth	90	140

Whichever way you look at it, Ralph's deal makes Alf 10 bushels of current corn worse off than he would be if he avoided it. –10 turns out to be the NPV of Ralphs' proposal (from Alf's perspective), because it cost 10 bushels more than a similar deal in the financial markets that provides the same return. Alf could have calculated the NPV directly: the present value of the promised inflows from Ralph's deal is 35/1.40 = 25, and the cost of entering Ralphs' deal is 35, so the NPV is 25 – 35 = -10.

What this demonstrates is that the NPV calculation is actually an arbitrage valuation. The present value of the inflows, 35/(1.40) = 25, is the *financial market price* of the inflows promised in the deal, while the present value of the outflows (35) is the cost of entering the deal *outside the financial market* (i.e. Ralph's asking price). The NPV, 25 – 35 = -10, represents how much wealthier or poorer the investor (Alf) becomes *relative to buying the same deal in the financial markets*. To repeat, the NPV is the difference between the cost of a newly proposed deal and the cost of an identical deal that can be replicated in the financial market.

The mechanics of NPV were developed to price fixed income investments, and there are two assumptions and one restriction implicit in the construction of the NPV rule. The *restriction* is that the payoff on the new security is proportional to aggregate consumption (or to the market portfolio) – we will explain this more clearly below. The critical *assumptions* are 1) that any 'new' cash flow stream can be exactly replicated by some combination of securities that already exist in the financial markets - in other words, the financial markets are *complete* – and 2) that the financial markets allow no arbitrage opportunities. When we value corporate investments using the NPV rule, we are implicitly making the market completeness assumption as well as the proportionality assumption.[4]

The situations where DCF is known to fail – situations of flexibility – are exactly the places where the *proportionality* restriction does not hold. We will show in the next section that when the cash flows from a project are not proportional to the payoffs on the market portfolio, then the NPV rule is impossible to implement and we need to turn to option pricing techniques (i.e. real options analysis). Option pricing, whether in financial or real asset markets, requires only the assumption of market completeness. Option pricing techniques can value all assets when markets are complete; DCF can value assets only when markets are complete and the proportionality assumption holds. We can't get around the completeness assumption, unless we are willing to do valuations by equilibrium analysis.

The More Complete Presentation of the Financial Market

What we'll show now is that option pricing can only be motivated from the more precise (but more complex) multiple-state model of the financial market, where risk is explicitly recognized. The textbook presentation above lumps all future state-contingent flows into one 'expected' consumption. If we want to examine a new proposal whose payoff is not proportional to aggregate consumption across all states, we have to keep track of the individual future states. Here we'll set up an equilibrium and derive the prices of current and future consumption across all future states.

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[4] If a risk-free asset were to exist, the restriction would be that the payoff on the new security must be *linear* in the market portfolio.

Let the utility of consumption function for every individual be

$$U = \ln\left(c_0 \cdot c_{banner}^{.5} \cdot c_{poor}^{.5}\right) \ .$$

Where c_0, c_{banner} and c_{poor} represent consumption in the present, banner and poor states (respectively).

We'll solve for those prices by nominating 'present' corn as the numeraire. Hence, each unit of 'present corn' has a price of 1(ϕ_0 = 1) , and we want to solve for the price of corn in the 'banner' state (ϕ_{banner})and 'poor' state (ϕ_{poor}) denominated in units of present corn. In the appendix, we derive the resulting pure-exchange equilibrium and show that ϕ_{banner} = .25 and ϕ_{poor} = .625. What this means is that every 1 extra bushel of corn consumption *in the banner crop state* costs .25 bushels of present corn consumption, while every 1 extra bushel of corn *in the poor crop state* costs .625 bushels of present corn. These are the prices of *state-contingent claims*.[5]

From this, we can ascertain several things. First, consider Eb Dawson – a simple but honest handyman. Eb wants to know his total wealth – that is, the total value of his current endowment plus the *current* value of his *future* endowment. How should Eb calculate the current value of his future endowment? Well, he can simply go to the financial market at Drucker's and see how much others would pay today for his future corn in each crop state. Since the price of corn in the banner state is .25, Eb could sell his banner-state corn for 200*.25 = 50 bushels of corn today; similarly, Eb could sell his poor-state corn for 80*.625 = 50 bushels of corn today. So the current (or present) value of Eb's future risky corn endowment is 50 + 50 = 100 current bushels of corn.

	Today	Future	
		'Banner' State	'Poor' State
Eb's Original Endowment	100	200	80
Sell Future 'Banner Crop' Endowment	50	-200	
Sell Future 'Poor Crop' Endowment	50		-80
Eb's Current Wealth	200	0	0

"Gollygee", Eb declares to his scarecrow friend Stuffy, "My current wealth is 200 – I've got 100 bushels of corn in the crib, and my future corn is worth 100 on the market".

This leads to a second point. Since Eb can find the current value of his future endowment of corn, then there must be an equilibrium discount rate for an investment in corn whose risk is exactly the same as the risk of Eb's endowment. Since the market value of Eb's

[5] The prices of consumption are different in the two states, even though the likelihood of the two states are identical, because aggregate consumption is different in the banner and poor crop states. Prices are determined in equilibrium by marginal utility, and if agents are risk-averse then marginal utility is high when consumption is low. Hence, the forward price of consumption in the low-endowment state (the poor harvest) is substantially higher than the forward price of consumption in the high-endowment state (the banner harvest). In lay terms, you're willing to pay more for a meal when you are hungry than when you are full.

future expected endowment is 100, and Eb's expected future endowment is .5x200 + .5x80 = 140 bushels of corn, the equilibrium rate of interest (or 'pure' rate of interest) is the r_{corn} that solves

$$PV\left(Eb's\ Future\ Endowment\right) = 100 = \frac{.5\times200+.5\times80}{\left(1+r_{corn}\right)} = \frac{140}{\left(1+r_{corn}\right)} \qquad so\ r_{corn} = 40\%$$

In other words, the risky discount rate associated with the risk of the corn crop in this economy (which is the market rate of interest) is 40%.

Eb can use this rate of return to calculate the NPV of new opportunities. Suppose Arnold Ziffle offers to pay Eb 105 bushels of corn today in exchange for Eb's entire allocation of corn, whatever it may be, in the future. From Eb's perspective, the future cash flows are negative and the current cash flow is positive, and Eb can calculate his NPV from entering the deal as follows:

$$NPV\left(Arnold\ Ziffle's\ Deal\right) = \frac{.5\left(-200\right)+.5\left(-80\right)}{1.40} + 105 = -100 + 105 = 5$$

To show that this is actually an arbitrage valuation, note that Eb could enter this deal with Arnold and give up his entire future consumption, use the proceeds from the deal to repurchase his future risky endowment in the financial market, and come out wealthier:

	Today	Future	
		'Banner' State	'Poor' State
Eb's Original Endowment	100	200	80
Sell Future Endowment to Arnold Ziffle	105	-200	-80
Buy 200 bu 'Banner' state corn @ .25/bu	-50	200	
Buy 80 bu 'Poor' state corn @ .626/bu	-50		80
Eb's New Wealth	105	200	80

From Eb's perspective, the arbitrage value of Arnold's proposal is 5. Note that this is exactly the NPV calculated above. The NPV is simply the arbitrage profit that can be earned by "buying low and selling high" between the financial market (Drucker's) and the private market (Arnold Ziffle's pigsty).

So now suppose an enterprising individual in Hooterville proposes a very unusual deal. Mr. Haney, an accomplished salesman, wants to sell <u>all but 50 bushels</u> of his future corn endowment, no matter what happens. In other words, he wants to assure himself 50 bushels for consumption in either future crop-state, but he's willing to turn over everything else (150 bushels in the banner crop state, or 30 bushels in the poor crop state) to for the right price today: 60 bushels of corn. He approaches Oliver Wendell Douglass, the smartest man in Hooterville: "Mr. Douglass, this is your lucky day".

Mr. Douglass was prepared to evaluate the deal. While attending a prestigious eastern Law school, Oliver took time to audit an introductory Finance course. He recalled from his Mealey and Bryers text that he should discount any risky future corn flow using a required rate of return that is commensurate with corn risk. He explained his reasoning to his wife in the following way: "Lisa, since the discount rate implied by the current valuation of everyone's future risky corn crop is 40%, then I must use 40% as the discount rate when purchasing risky claims on future corn." Mr. Douglas thus valued the Haney deal as follows:

$$NPV\left(Mr.\,Haney's\;Deal\right)\overset{?}{=}\frac{.5(150)+.5(30)}{1.40}-60\overset{?}{=}64.29-60\overset{?}{=}4.29$$

"We're rich, Lisa. With this deal, we can turn that scoundrel Haney into a money pump". So Mr. Douglas entered the deal and paid Mr. Haney 60 bushels of corn. Mr. Haney then went straight to Drucker's market and bought back the future corn allotment he had just sold to Mr. Douglas. And here's what happened:

	Today	Future	
		'Banner' State	'Poor' State
Mr. Haney's Original Endowment	100	200	80
Sell all but 50 of future endowment to Douglas	60	-150	-30
Buy 150 bushels 'Banner' state corn @ .25/bu	-37.5	150	
Buy 30 bushels 'Poor' state corn @ .626/bu	-18.75		30
Mr. Haney's New Wealth	103.75	200	80

Mr. Haney's wealth today increased by 3.75 while his future wealth remained the same. In other words, Mr. Haney became richer by exactly 3.75. But this is a zero-sum deal – whatever Mr. Haney makes, Mr. Douglas loses. So contrary to what Mr. Douglas believes, Mr. Haney has actually gotten the better end of this deal. The *true* value of the deal to Mr. Douglas is –3.75, and not +4.29. Where did Mr. Douglas make an error?

What Mr. Douglas didn't realize is that the deal he bought from Mr. Haney does not have the same risk as the overall corn market. The fact that Mr. Haney made money at Mr. Douglas' expense tells us that Mr. Douglas *overvalued* the deal – in other words, the discount rate that Mr. Douglas used was too low. How can this be true? What should have been the right discount rate?

To answer these questions, we need to start with an easier one: in what situations would it be proper for Mr. Douglas to use 40% as the proper discount rate on risky corn deals?

First, we've already established that 40% is the correct discount rate to use on the corn crop as a whole (i.e. on aggregate consumption). If Mr. Douglas prices his original future endowment using a 40% discount rate, there are no arbitrage profits available:

$$PV\left(Endowment\right)=\frac{.5(200)+.5(80)}{1.40}=100$$

$$Arbitrage\,Value(Endowment) = 200 \cdot (.25) + 80 \cdot (.625) = 100$$

Suppose Mr. Douglas were to sell one-half of his future endowment (100 bushels of corn in the banner crop state, and 40 in the poor crop state). Does a 40% discount rate work? The valuation of this risky corn deal using DCF is

$$PV(Mr.\,Douglas'\,Deal) = \frac{.5(100) + .5(40)}{1.40} = 50$$

and the value of an equivalent deal in at Drucker's market is

$$Arbitrage\,Value(Mr.\,Douglas'\,Deal) = 100 \cdot (.25) + 40 \cdot (.625) = 50.$$

So 40% is the correct discount rate for this particular deal.

Now suppose the Bradley sisters (Billy Jo, Bobby Jo and Betty Jo) want to pool their future endowments and sell them to Uncle Joe (a total of 600 banner-state bushels and 240 poor-state bushels). Would the 40% discount rate give an arbitrage-proof price? The DCF valuation would be

$$PV(Bradley\,Sisters'\,Deal) = \frac{.5(3 \times 200) + .5(3 \times 80)}{1.40} = 300\,.$$

One could re-create the same future corn flows at Drucker's market at a cost of

$$Arbitrage\,Value(Bradley\,Sisters'\,Deal) = 3 \times 200 \cdot (.25) + 3 \times 80 \cdot (.625) = 300\quad.$$

They are the same; hence the DCF at 40% again gives the right answer.

What is the point here? The point is this: the 40% discount rate being applied to the original endowments is the correct discount rate for risky deals for future corn **if and only if the promised future flows in deals for future corn are proportional to the original endowments.** In other words, you can use 40% as the right discount rate on a risky deal only when the corn flows on the deal are a constant multiple of future consumption (or the 'market portfolio'). This is what we mean by proportionality: the NPV rule is applicable only when the state-dependent cash flows from an investment are strictly proportional to aggregate consumption. The following table makes this clear.

	Deal Payoff		Proportion of Aggregate Consumption	
	'Banner' State	'Poor' State	'Banner' State	'Poor' State
Aggregate Consumption	*200N*	*80N*		
Eb's Endowment	200	80	1/N	1/N
Bradley Sisters' Deal	*600*	*240*	*3/N*	*3/N*
Mr. Douglas' Deal	*100*	*40*	*.5/N*	*.5/N*

Arnold Ziffle's Deal	*-200*	*-80*	*-1/N*	*-1/N*
Mr. Haney's Deal	*150*	*30*	*.75/N*	*.375/N*

No matter what happens, the Bradley Sisters' Deal pays off *3/N* times aggregate consumption. Similarly, the Mr. Douglas' Deal pays off *.5/N* times aggregate consumption, and Arnold Ziffle's Deal pays off *–1/N* times aggregate consumption. This is what we mean by proportionality: 40% is the correct discount rate to use on a deal *as long as the deal's payoff is a nonzero constant times aggregate consumption, no matter what state occurs*. Mr. Haney's deal is *not* proportional to aggregate consumption across all states: it pays off *.75/N* times aggregate consumption in the 'Banner' state but only *.375/N* times aggregate consumption in the 'Poor' state. When project payoffs are not proportional to aggregate consumption, the 'market rate' is not the correct rate of return for discounting.[6]

To give the simplest illustration of a deal with future flows that are not proportional to consumption, consider Hank Kimball, the county agent. Mr. Kimball is quite conservative, and he is looking to buy a deal that will deliver him 15 bushels of corn in the future no matter what the crop turns out to be. This future corn flows on this deal are obviously risk-free, but there is no risk-free asset available in Hooterville, so you might be tempted to use the corn interest rate of 40% in your valuation. If you were to apply the 40% discount rate to 'price' this deal, you would be willing to sell the deal to Mr. Kimball for

$$PV\left(Mr.\ Kimball's\ Deal\right) \stackrel{?}{=} \frac{.5\times 15 + .5\times 15}{\left(1.40\right)} \stackrel{?}{=} 10.71\,.$$

But the same deal could be sold in the financial market for

$$Arbitrage\ Value\left(Mr.\ Kimball's\ Deal\right) = 15\cdot\left(.25\right) + 15\cdot\left(.625\right) = 13.125\,.$$

So if you priced this deal using the 40% discount rate and sold it to Mr. Kimble for 10.71, he could immediately take the claims on 15 bushels of corn in each future state and sell them at Drucker's market for 13.125, giving himself an arbitrage profit of 13.125-10.71 = 2.415.

[6] The point here is that the discount rate on a reference portfolio taken from the financial market is the appropriate discount rate only for deals with payoffs that are strictly proportional to those of the reference portfolio. In our world, there is only one risky asset – corn – so the only available reference portfolio is simply aggregate consumption of corn (so it is by default our 'market portfolio'); the risk-adjusted discount rate for corn is only appropriate for deals with payoffs that are strictly proportional to aggregate corn consumption. If we had assumed the existence of a risk-free asset, then there would be an infinite number of reference portfolios that would combine the risk-free asset with the market portfolio (aggregate corn consumption). This would be a CAPM equilibrium. In a CAPM world, we can price any deals with payoffs that are *linear* in the market portfolio by creating a reference portfolio that combines the risk-free rate with the market portfolio. Still, this discount rate would be appropriate only for deals with payoffs that are strictly proportional to the reference portfolio.

	Today	Future	
		'Banner' State	'Poor' State
Mr. Kimball's Original Endowment	100	200	80
Buy the Deal for Fixed Corn Flow @ 10.71	-10.71	15	15
Sell 15 bushels 'Banner' state corn @ .25/bu	3.75	-15	
Sell 15 bushels 'Poor' state corn @ .625/bu	9.375		-15
Mr. Kimball's New Wealth	102.415	200	80

The arbitrage opportunity arises because we are applying the wrong discount rate for the deal. In this case, the corn flows on the deal are ***not*** proportional to the economy's overall consumption, and hence the corn discount rate is not the right rate.

But what is the correct discount rate? There's no way to know, without doing the arbitrage valuation first. There is no risk-free asset in this economy (recall that there is no storage), so Mr. Kimball's desired claim is actually a derivative – it is a futures contract for 15 bushels of corn. This is the general problem with valuation of derivatives and assets with derivative-like payoffs: NPV would work, if you knew the correct discount rate, but there is no way to know the right discount rate without knowing the value of the derivative in the first place. To value derivatives, we have to rely on arbitrage arguments – we create a portfolio of existing claims from the financial markets that exactly replicate the payoff on the derivative; the value of that package of claims is the value of the derivative. If that sounds to you like the way we described NPV above, then you are paying attention. DCF and option pricing methodology are built on the same foundation – arbitrage – and hence rely on the same underlying assumption – complete markets.

The financial market price of the deal Mr. Kimball desires is 13.125. This is the arbitrage-free price of the forward (or 'risk-free') claim. Hence, we can derive the correct discount rate to use on deals identical to Mr. Kimball's:

$$PV\left(Mr.\ Kimball's\ Deal\right) = \frac{.5(15)+.5(15)}{\left(1+r_F\right)} = 13.125 \quad so \quad r_F = 14.3\%$$

When is 14.3% the correct discount rate to use on a corn deal? Whenever the promised corn payments are *invariant* to the state of the crop. Hence, 14.3% is the risk-free rate of return in this economy – even though there is no risk-free asset available for purchase directly.

So this takes us back to Mr. Haney's deal. Remember, Mr. Haney offered Mr. Douglas 150 'banner-state' bushels of corn and 30 'poor-state' bushels of corn. Mr. Haney's deal is a derivative; it is actually a European call option on his entire future endowment with strike price equal to 50. The payoff on this deal is not invariant to the future state, so the risk-free rate is not the appropriate discount rate. Moreover, these payments are not proportional to aggregate consumption, so 40% is not the correct discount rate.
So what is the correct discount rate to use to value Mr. Haney's deal? The answer might bother you: **there is no way to know the right discount rate without first knowing the**

arbitrage-free value of the deal. However, we do have a way to value the deal – by looking at the price of existing claims in the financial market. In order to price derivatives this way, we must assume that the market is *complete*. That is, Mr. Haney's deal, or any other deal that can be proposed, can be recreated by trading at Drucker's market.

Mr. Haney's deal can be replicated in the financial markets by buying 150 bushels of 'Banner' state corn @ .25/bushel, and 30 bushels of 'Poor' state corn at .625/bushel. So the present value of Mr. Haney's deal (the price which gives it a zero NPV) is

$$Arbitrage\,Value\left(Mr.\,Haney's\ Deal\right)=150\cdot\left(.25\right)+30\cdot\left(.625\right)=37.5+18.75=56.25\,.$$

If the deal can be purchased for less than this, the purchaser has found a positive NPV opportunity. Mr. Douglas, on the other hand, paid 60 for the deal – a negative NPV investment!

Now that we know the value of Mr. Haney's deal, we can calculate the discount rate that *would have* given us the correct answer:

$$PV\left(Mr.\,Haney's\ Deal\right)=\frac{.5\left(150\right)+.5\left(30\right)}{1+r_{Mr.Haney}}=56.25\quad so\ r_{Mr.Haney}=60\%$$

The right discount rate to use on Mr. Haney's deal is 60% - much higher than the pure rate of interest on corn. This correct discount rate show's Mr. Douglas' true NPV from entering Mr. Haney's Deal:

$$NPV\left(Mr.\,Haney's\ Deal\ @\ 60\right)=\frac{.5\left(150\right)+.5\left(30\right)}{1.60}-60=56.25-60=-3.75$$

which shows that the value destruction caused to Mr. Douglas by entering Mr. Haney's deal is exactly the same as Mr. Haney's arbitrage profits from selling (shorting) the deal.

Of course, there is no way that we could have ascertained the 60% discount rate on Mr. Haney's deal without finding the value of the deal in the first place. This is the difficulty with options: NPV will price them if you know the right discount rate, but there is no way to know the right discount rate a priori.

You might be wondering why the discount rate on Mr. Haney's deal is so high. This is a standard result in Finance: Mr. Haney's deal is a call option on his future corn endowment (with a strike price of 50), and call options are always riskier than the underlying asset on which they are written. This is often explained to students as a leverage effect, because the hedge portfolio that mimics the option is a levered position in the underlying asset. But the story is much deeper.

To see why derivatives have different risk from their underlying assets in general, it is necessary to look at two very special derivatives. The "Banner Crop Special" pays off 1 bushel of corn in the banner crop state and nothing in the poor crop state, while the "Poor Crop Special" pays off 1 bushel of corn in the poor crop state and nothing in the banner crop state.[7]

$$Arbitrage\ Value\left(Banner\ Crop\ Special\right) = .25\left(1\right) + .625\left(0\right) = .25$$

$$PV\left(Banner\ Crop\ Special\right) = \frac{.5\left(1\right) + .5\left(0\right)}{1 + r_{Banner}} = .25 \quad so\ r_{Banner} = 100\%$$

$$Arbitrage\ Value\left(Poor\ Crop\ Special\right) = .25\left(0\right) + .625\left(1\right) = .625$$

$$PV\left(Poor\ Crop\ Special\right) = \frac{.5\left(0\right) + .5\left(1\right)}{1 + r_{Poor}} = .625 \quad so\ r_{Poor} = -20\%$$

The participants at Drucker's market impose a very high discount rate on the Banner Crop Special, and a negative discount rate on the Poor Crop Special. Why? One of the important results from microeconomics is that marginal utility of consumption is highest when total consumption is lowest. In other words, people value an additional unit of consumption in the low-endowment state (the poor crop state) much more highly than they do in the high-endowment state (the banner crop state), so when markets open at Druckers, the price of the Poor Crop Special comes out higher than the price of the Banner Crop Special *even though they have the exact same expected payoff* (.5 bushels of future corn).[8] In this economy, people value consumption in the poor crop state so much that the price of the Poor Crop Special is above its expected payoff (and hence is being discounted at a negative discount rate). The Poor Crop Special is actually an insurance contract that insures the buyer against the poor consumption state, and the fact that risk-averse people are willing to pay a price higher than expected value for an insurance contract is the economic reason that the insurance industry exists.

Note what else this means: each future state of nature has its own unique discount rate. When flows from a proposed deal are proportional to aggregate consumption in the economy, the aggregated discount rate of 40% is appropriate; however, when flows from a deal are more heavily weighted towards one of the states, then the market-wide rate of 40% is *not* appropriate.

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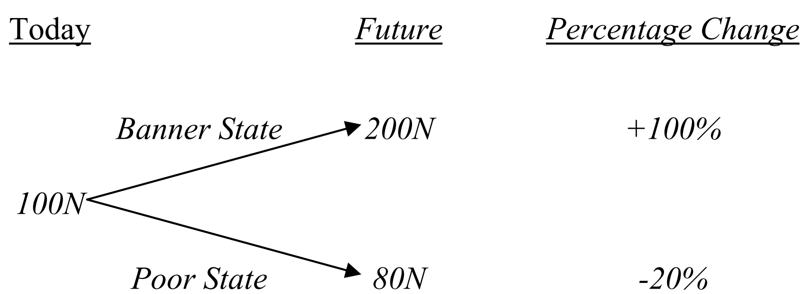
[7] The "Banner State Special" is a call on someone's future endowment at a strike price of 199, while the "Poor State Special" is a put on the same endowment with a strike price of 81.

[8] This is not a trick. As long as aggregate consumption is different across the two states, risk-averse individuals in the economy will wish to hedge against the low-consumption state and will value consumption in that state more highly.

A More Familiar Approach To Option Pricing

At this point, you may be questioning our interpretation of options in the simple economy and you might think that 'textbook' approaches to option valuation won't get the same answers. We'll show that this is not the case. The familiar binomial option pricing model of Cox, Ross and Rubenstein (1979), which gives the famous Black-Scholes (1973) model in the limit as time steps become small and the number of steps become large, gives the exact same results that we've just derived.

The Hooterville economy is easy to place in a binomial framework. Just let aggregate consumption be the nodes of the binomial tree:



The percentage change in consumption in the banner crop state (i.e. the size of the 'up' step in the binomial model) is (200*N*/100*N*) – 1 = +100%; similarly, the percentage change in consumption in the poor crop state (i.e. the size of the 'down' step in the binomial model) is (80*N*/100*N*) – 1 = -20%. Remembering that the risk-free payoff can be achieved by buying Mr. Kimball's deal above, and that the market price of this deal implied a risk-free rate of return of 14.3%, we can calculate the familiar 'risk-neutral probability' for this economy:

$$q = \frac{r-d}{u-d} = \frac{.143-(-.20)}{1.00-(-.20)} = .286$$

The risk-neutral probability q, along with the risk-free discount rate, prices all assets (including derivatives) in this economy:

$$Current\ Value = \frac{q\cdot(payoff\ in\ banner\ state)+(1-q)(payoff\ in\ poor\ state)}{1+r_F}$$

For example, Eb Dawson's Future Endowment:

$$Value(Eb's\ Future\ Endowment) = \frac{.286(200)+.714(80)}{1.143} = 100$$

which is the exact answer we got before. Or, Mr. Haney's Deal:

$$Value\left(Mr.\ Haney's\ Deal\right) = \frac{.286\left(150\right)+.714\left(30\right)}{1.143} = 56.25$$

again, precisely the same answer as before. The binomial option pricing model will price any asset in the economy properly.

Why Did The Binomial Model Work?

We reiterate that standard option pricing techniques work in our Hooterville economy *even though there is no storage*. Mr. Haney's deal, which is a call option on corn, can be priced by the binomial option pricing model **even though corn cannot be held in a hedge portfolio**. Why?

The answer is the *complete markets assumption*. A market is said to be *complete* when any future state that individuals care about can be hedged. A market will be complete when there are at least as many unique securities as states (as in Hooterville). These do not need to be primary securities. When there are less primary securities than states, the market can be completed by constructing options on the market portfolio.[9]

By construction, our Hooterville economy is complete – the market at Drucker's allows the residents to buy 'banner state corn' independently of 'poor state corn'. This in turn allows for valuation by arbitrage (which can be either DCF or option pricing). To see why, note that any proposed deal can be created in the financial market by creating a portfolio of Banner State Specials (with price $\phi_{Banner} = .25$) and Poor State Specials (with price $\phi_{Poor} = .625$): just buy 1 Banner State Special for every unit of payoff in the banner state, and 1 Poor State Special for every unit of payoff in the poor state. The value of this arbitrage portfolio will be

$$Value = \phi_{Banner} \cdot \left(Payoff\ in\ Banner\ State\right) + \phi_{Poor}\left(Payoff\ in\ Poor\ State\right)$$

For example, in Mr. Haney's Deal the payoff in the banner state is 150 and the payoff in the poor state is 30, so

$$Value\left(Mr.\ Haney's\ Deal\right) = .25\left(150\right)+.625\left(30\right) = 56.25$$

But note that we can re-arrange the binomial model valuation equation and get something very similar:

$$Value = \frac{q}{1+r_F}\left(Payoff\ in\ Banner\ State\right) + \frac{1-q}{1+r_F}\left(Payoff\ in\ Poor\ State\right).$$

[9] N.B.: Such options that are created to complete an incomplete market can only be priced by constructing the equilibrium – standard option pricing techniques will not apply.

Using Mr. Haney's deal again as the example:

$$Value\left(Mr.\ Haney's\ Deal\right)=\frac{.286}{1.143}\left(150\right)+\frac{.714}{1.143}\left(30\right)=56.25\ .$$

The similarity between the 'state claim' valuation of Mr. Haney's deal and the 'binomial option model' valuation of the same deal should strike you, because

$$\frac{q}{1+r_F}=\phi_{Banner} \quad and \quad \frac{1-q}{1+r_F}=\phi_{poor}$$

(check for yourself!). The famous 'risk-neutral probabilities' from option pricing (the *N(d)* terms in the Black-Scholes model) are actually prices for state securities times scaled by the risk-free return factor: $q=\left(1+r_F\right)\phi_{Banner}$, and $1-q=\left(1+r_F\right)\phi_{Poor}$

This is not a coincidence – it is one of the most important results in all of Finance. It was given to us by Harrison and Kreps (1979), and while the development of the result is very difficult to demonstrate, the actual statement of the result is easy and is twofold: 1) *if markets are complete and free of arbitrage opportunities*, then there exists a unique set of 'risk-neutral probabilities' that can be used, along with the risk-free rate of return, to price options (and all other assets in the economy) using textbook option pricing techniques; and 2) textbook option pricing techniques can be used to price options *only if markets are complete and free of arbitrage opportunities*.

The second part of the Harrison and Kreps result is the most important: we can only apply our standard option pricing technology (pricing by arbitrage) if we are willing to assume that markets are complete. When markets are complete, it is not necessary to form a hedge portfolio by holding the physical underlying asset (like corn) in a hedge portfolio – the hedge portfolio can be formed by holding a portfolio of state securities, which are themselves financial claims. Why must markets be complete for the Black-Scholes hedge to work? Because if markets are not complete, then introduction of the new derivative will change all of the equilibrium prices of the assets in the economy, so their prices *before introduction of the option* would tell us nothing about the option's value *after its introduction*.

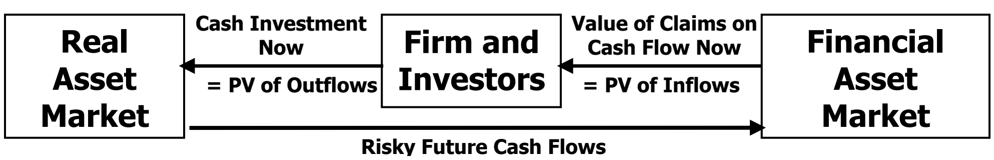
This turns out to be the reason that people insist on assuming that the underlying asset be tradeable in order to price options – because the tradeability assumption is actually equivalent to a market completeness assumption. If we assume that the underlying asset and a risk-free bond are continuously tradeable, then over any very short time interval a binomial model exists and the market is necessarily complete (two states, two assets). So as perverse as it sounds, many people reject the market completeness assumption – and then proceed to make it anyway through the tradeability assumption.

And that brings us full circle. DCF analysis of new propositions is only applicable when markets are complete, and when markets are complete we can use option pricing to value new propositions. And thus the point of this paper: if you are willing to make the assumptions necessary to perform a DCF analysis on an illiquid new investment, you have already made all the assumptions necessary for application of option pricing techniques to that investment.

Corporate Capital Budgeting Redux

Since we've come full circle, we might as well fit this entire discussion into the definition of corporate capital budgeting. The firm and its managers occupy a position between the real asset market (the market for new, specialized projects) and the financial asset market (the market for claims on real assets). Investors provide capital to firms because firms have special features that give them unique access to assets in the real asset market.

Valuation of a new project is accomplished by asking the following question: if we adopt the new project and immediately sell the financial claims on the cash flows to the financial markets, what would they be worth? The present value of the cash flows from a project is the value that the capital markets would pay for those cash flows immediately. The NPV is the difference between the price actually paid for the new real assets (the PV of the outflows) and the price that could be received in the financial market for the cash flows on the new assets (the PV of the inflows).



This illustrates the NPV rule: firm value grows when managers invest in positive NPV projects. It also illustrates the shareholder maximization rule: by investing in positive NPV projects, the existing shareholders capture the gain. Finally, it illustrates the procedure: we look to the financial markets to see how they currently value the risky future cash flows on a project.

The fact that managers don't actually sell cash flows from every new project to the financial markets is irrelevant. Using internally generated cash flow to fund a project is equivalent to selling those new cash flows to the existing equityholders, who capture the NPV regardless. The existing shareholders only care that the amount invested by the firm is less than the value of the risky cash flows *if they were sold to the financial markets*.

So the whole procedure of capital budgeting rests on figuring out what the risky cash flows of a new project are currently worth in the financial markets. The assumption of complete markets, which underlies both DCF/NPV and option pricing, is necessary to ensure that an exactly similar stream of cash flows is currently available in the capital markets (by some portfolio strategy). Once completeness is assumed, we can use the capital market's valuation methodologies to attach values to new, illiquid investments. That's the entire point of this article.

Summary

Corporate capital budgeting is the process of using *financial market prices* to determine the value of a new corporate investment opportunity. The NPV rule and the goal of maximizing shareholder value depend on the manager's ability to use financial market prices to value a new project; this in turn requires that the risky cash flows from the new project be replicable in some way in the financial markets. In other words, the financial markets must be *complete*.

When financial markets are complete and free of arbitrage opportunities, DCF and option pricing procedures are equally applicable, regardless of whether the new project is traded in a liquid market.

Appendix: Derivation of the Pure-Exchange Equilibrium

The consumer's optimization problem is

$$\max_{\{c_0, c_{banner}, c_{poor}\}} U\left(c_0, c_{banner}, c_{poor}\right)$$
$$s.t. \quad c_0 + \phi_{banner} c_{banner} + \phi_{poor} c_{poor} = y_0 + \phi_{banner} y_{banner} + \phi_{poor} y_{poor} \quad .$$

From the Lagrangian

$$\ell = U\left(c_0, c_{banner}, c_{poor}\right) + \lambda\left(c_0 + \phi_{banner} c_{banner} + \phi_{poor} c_{poor} = y_0 + \phi_{banner} y_{banner} + \phi_{poor} y_{poor}\right)$$

the first-order conditions are

$$\frac{\partial \ell}{\partial c_0} = \frac{\partial U}{\partial c_0} - \lambda = 0$$
$$\frac{\partial \ell}{\partial c_{banner}} = \frac{\partial U}{\partial c_{banner}} - \lambda \phi_{banner} = 0$$
$$\frac{\partial \ell}{\partial c_{poor}} = \frac{\partial U}{\partial c_{poor}} - \lambda \phi_{poor} = 0 \quad .$$

Along the isoutility curve,

$$\frac{\partial c_{banner}}{\partial c_0} = -\frac{1}{\phi_{banner}}$$

$$\frac{\partial c_{poor}}{\partial c_0} = -\frac{1}{\phi_{poor}} \quad .$$

In our problem,

$$\frac{\partial U}{\partial c_0} = \frac{1}{c_0 \cdot c_{banner}^{.5} \cdot c_{poor}^{.5}} \cdot c_{banner}^{.5} \cdot c_{poor}^{.5} = \frac{1}{c_0}$$

$$\frac{\partial U}{\partial c_{banner}} = \frac{1}{c_0 \cdot c_{banner}^{.5} \cdot c_{poor}^{.5}} \cdot c_0 \cdot c_{poor}^{.5} \cdot 5 c_{banner}^{-.5} = \frac{.5}{c_{banner}}$$

$$\frac{\partial U}{\partial c_{poor}} = \frac{1}{c_0 \cdot c_{banner}^{.5} \cdot c_{poor}^{.5}} \cdot c_0 \cdot c_{banner}^{.5} \cdot 5 c_{poor}^{-.5} = \frac{.5}{c_{poor}} \quad \text{, so}$$

$$-\frac{1}{\phi_{banner}} = \left.\frac{\partial c_{banner}}{\partial c_0}\right|_U = -\left.\frac{\partial U / \partial c_0}{\partial U / \partial c_{banner}}\right|_U = -\left.\frac{1 / c_0}{.5 / c_{banner}}\right|_U = -\left.\frac{c_{banner}}{.5 c_0}\right|_U$$

$$-\frac{1}{\phi_{poor}} = \left.\frac{\partial c_{poor}}{\partial c_0}\right|_U = -\left.\frac{\partial U \big/ \partial c_0}{\partial U \big/ \partial c_{poor}}\right|_U = -\left.\frac{1 \big/ c_0}{.5 \big/ c_{poor}}\right|_U = -\left.\frac{c_{poor}}{.5c_0}\right|_U \quad .$$

Since all of the individuals have identical beliefs, preferences, endowments and productive opportunities (none), the corn market must establish a set of prices such that each individual is satisfied to hold his or her original endowment. Evaluating the above at $c_0 = y_0$; $c_{banner} = y_{banner}$; $c_{poor} = y_{poor}$, we find the equilibrium prices

$$-\frac{1}{\phi_{banner}} = -\left.\frac{c_{banner}}{.5c_0}\right|_U = --\left.\frac{y_{banner}}{.5y_0}\right|_U = -\frac{200}{.5(100)} = -4 \quad so \quad \phi_{banner} = .25$$

$$-\frac{1}{\phi_{poor}} = -\left.\frac{c_{poor}}{.5c_0}\right|_U = -\left.\frac{y_{poor}}{.5y_0}\right|_U = -\frac{80}{.5(100)} = -1.6 \quad so \quad \phi_{poor} = .625 \quad .$$