A Real Options Model of Patent Litigation[*]

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Abstract

We build a compound real options model to study the dynamic strategic interactions between a patent-owning incumbent firm and a challenger firm in a patent dispute. New to the literature, we consider the challenger's option to exit the market during litigation because of the high litigation cost, and the incumbent's option to withdraw from an ongoing litigation or to force the challenger out of the market by a threat of litigation. Our model uncovers the two firms' similar willingness to pay for the litigation facilitates their settlement (in the form of royalty payment). We find novel results regarding how the product market characteristics and patent rules affect the likelihood, timing, and terms of settlement. The challenger's profit gain relative to the incumbent's loss of profits due to the alleged infringement ("gain-to-loss ratio") has to be high for settlements to be possible and settlement is less likely in more volatile product markets. In addition, the *English rule* (of loser pays) shifts the effective bargaining power from the incumbent to the challenger, compared to the *American rule* (of each party pays). Such a shift can cause an opposite effect of patent validity on settlement likelihood for countries using the English rule (positive effect) vs. the American rule (negative effect). Our model generates new testable implications for the litigation rate and the settlement rate in patent disputes from a finance perspective.

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1 Introduction

Intellectual property lawsuits are economically significant for the corporations involved. Empirical evidence suggests that patent litigation is typically complex and time-consuming, and it costs millions of dollars on average for public firms (Bessen and Meurer, 2005; Lanjouw and Schankerman, 1997; Bessen and Meurer, 2012). However, existing evidence is mostly based on filed lawsuits, which is only the tip of the iceberg of intellectual property disputes among corporations. In particular, the strategy of demanding licensing agreements with a threat of litigation is far more common than lawsuits (National Research Council, 2003), and private firms quite often get involved in these disputes too. Obviously, the likely resolution of potential patent disputes (e.g., litigation vs. licensing, and the royalty rate in a licensing agreement) can have a crucial impact on firms' ex-ante R&D activity and market entry decisions. Thus, it is surprising that little is known about the key determinant(s) of patent dispute outcomes, which we believe is a big missing piece in understanding corporate innovation in the literature.

In this paper, we aim to fill the gap using a theory framework, and provide answers to the following questions: (1) What are the key drivers of patent dispute outcomes? (2) How do the characteristics of the product market and the legal system affect firms' strategies in patent dispute and the resolutions? We focus on firms' tradeoff in paying for the ongoing litigation cost if a patent lawsuit were to start. The litigation cost includes the direct costs such as attorney or administrative fees, and the economic costs associated with the business disruption. To keep the issue at its core, we abstract from complications such as third-party litigation financing (Antill and Grenadier, 2023), information asymmetry (Bebchuk, 1984), or patent trolls (Cohen, Gurun, and Kominers, 2016).

To answer the research questions, we build a real options model of two value-maximizing firms with demand uncertainty. The *incumbent* ("I") extracts monopoly profits based on its patented technology until a *challenger* ("C") enters the market and starts to earn duopoly profits based on an alleged infringement, after which the incumbent's profits drop. If the incumbent exercises its call option to sue the challenger using its patent (*I-litigate*), then both firms incur ongoing costs during the litigation process. With a probability p that is common knowledge, the court rules in favor of the incumbent whose monopoly profits resume afterwards. Otherwise, the duopoly status quo remains. Because the continuous litigation cost can be high, it may be optimal for the firm(s) to terminate the litigation, such as the incumbent withdraws the lawsuit (*I-withdraw*), the challenger exits the market (*C-exit*), or the firms settle pre-trial (*ex-post settlement*). We model the ex-post settlement as a licensing agreement, in which the challenger pays a fixed royalty rate on its future cash flow to the incumbent and the duopoly sustains. Alternatively, firms can resolve their patent dispute by settling without a lawsuit (*ex-ante settlement*).

By backward induction, we find that the firms' relative willingness to pay for the litigation is a key determinant of patent dispute outcomes. If the incumbent can credibly commit to pay the litigation cost and remain in the litigation process much longer than the challenger, then the royalty rate it demands to settle the dispute can be too high for the challenger to accept. Likewise, if the challenger has a much stronger willingness to pay for the litigation, then the royalty rate that it is willing to pay can be too low for the incumbent to consider settling. In both cases, the possibility of settlement breaks down due to the strategic consideration from the financing asymmetry. With this novel determinant uncovered from the model, we obtain the main findings of our study as follows.

First of all, settlements only happen if the challenger's profit gain relative to the incumbent's loss of profit due to the challenger's market entry (in short, the "gain-to-loss ratio") is high enough. When the product market associated with the patented technology features a low gain-to-loss ratio, the incumbent litigates and firms do not settle. A low gain-to-loss ratio implies a significant reduction in the total market profits, and it happens, for example, if the challenger's products are close substitutes to the incumbent's. Because a low gain-to-loss ratio is tied to the challenger's restrictive capacity and willingness to pay for any litigation cost (and royalty fees), the incumbent optimally proceeds with the litigation to exhaust the challenger's resource, and enhance its possibility to restore its monopolistic market power. If, instead, the challenger's entry features a high gain-to-loss ratio, for example when its products are complements to the incumbent's, then settlement becomes likely and firms settle as their profits drop to a low level. A high ratio induces the incumbent to accept a low royalty rate in a settlement, as its best alternative of battling to the end and winning the lawsuit only brings moderate profit recovery. Meanwhile, a high ratio makes the challenger more willing to pay a high royalty rate in order to avoid the complete loss of profits from an adverse court order. Together, firms are more likely to settle with a higher gain-to-loss ratio. Moreover, we show that the incentive of saving the expensive litigation cost makes both firms willing to settle even if the total market profits shrink as a result of the challenger's entry.

Secondly, a similar willingness to pay for litigation between the two firms not only facilitates settlement, as we have argued, but also accelerates the settlement. When the incumbent has a higher willingness to pay for the litigation, compared to the challenger, it has an additional option to force the challenger out of the market at a low demand level via a threat of litigation. As such asymmetry of willingness to finance expands, the incumbent's forcing-out option becomes more valuable, making settling ex-ante less appealing. Moreover, a higher gain-to-loss ratio moves settlement from ex-post to ex-ante. For the incumbent, a higher ratio implies a lower loss from the challenger's entry, which reduces its incentive to start the costly process of litigation. For the challenger, a higher ratio means it gains more from the alleged infringement, thus it is better to resolve the patent dispute via an earlier settlement without an actual litigation lawsuit.

Thirdly, both the demand volatility and the incumbent's winning probability in the court ruling reduce settlement likelihood.[1] A higher demand volatility increases option value, and more so for the ex-post settlement option than the non-settlement options, making ex-post settlement more likely. However, this effect is dominated by the opposing force that a more volatile demand also prompts litigation and makes ex-ante settlement less likely. A higher demand volatility raises the incumbent's firm value during litigation but does not affect its ex-ante settlement payoff, making the ex-ante settlement option less attractive for the incumbent and the ex-ante settlement negotiation more difficult. Meanwhile, our model recognizes the compound options feature of the strategies: as the incumbent exercises the litigation option, it activates further options such as ex-post settlement. Thus, the option value of litigation is higher than without the sequential option consideration. This feature induces the incumbent to require a higher value in an ex-ante settlement to avoid litigation when its winning probability is high, leading to a less likely ex-ante settlement.

We extend the baseline model to further explore the impact of firms' relative willingness to finance litigation. In particular, we focus on the effect of the cost allocation rules in legal systems on the litigation outcomes, that is, the American rule (each party pays its own cost, as in the baseline) vs. the English rule (the loser pays). We argue that the English rule shifts the effective bargaining power from the incumbent to the challenger in the patent litigation setting, taking the American rule as the baseline. It is perhaps counter-intuitive that the financial disadvantage of the challenger drives the shift in the English rule, which is in favor of the challenger. As a result, the royalty rate in an ex-post settlement is lower under the English rule than under the American rule. We conjecture that, with everything else being equal, the incumbent's lower effective bargaining power in the English rule reduces its ex-ante innovation incentive, compared to an incumbent in the American rule. We also find that the incumbent's winning probability has opposite effects on the settlement likelihood in the two rules. Our model

[1] A higher "likelihood" refers to a wider range of parameters for which the firms choose a particular strategy.

generates empirical implications on litigation rate, settlement rate and settlement terms, and policy implications regarding patent protection and litigation process.

This paper is one of the first theory studies that examine strategic interactions of patent litigation from a finance perspective. We contribute to the law and economics literature by establishing the importance of product market characteristics, and re-examining the effect of the legal system. We believe that firms' relative willingness to finance litigation is a first-order factor in determining firms' litigation strategies, similar to the information asymmetry and the judgment amount that have been discussed in the literature (e.g. Bebchuk, 1984; Hughes and Snyder, 1995; Spier, 2007). This paper adds to the recent discussion of how financing considerations affect patent litigation (e.g. Aoki and Hu, 1999a; Cohen, Gurun, and Kominers, 2016; Choi and Spier, 2018). A number of papers have modeled both generic litigation and patent enforcement using litigation in real options models (e.g., Grundfest and Huang, 2005; Marco, 2005; Jeon, 2015). New to this literature, our model incorporates the possibility that the defendant may exit during litigation because of the high ongoing litigation cost. Our paper is also related to the finance literature that studies litigation risk in general (Hassan, Houston, and Karim, 2021, Guan et al., 2021, Liu, Si, and Miao, 2022), but differentiates from that literature due to the strategic nature of the game between the two product firms in our setting.

This paper proceeds as follows. In section 1, we provide an overview of the background and related literature on patent litigation. In section 2, we describe the model. In section 3, we present and discuss the model solutions. In Section 4, we present comparative statics of patent dispute outcomes. Section 5 discusses the model extensions and robustness. Section 6 concludes.

Background and the Related Literature Patent litigation is typically costly and time-consuming with great uncertainties. According to Bessen and Meurer (2012), patent litigation lawsuits in the 1980s and 1990s cost alleged public infringers about $28.7 million (1992 dollars) in the mean and $2.9 million in the median, and the figures are much higher for those reported in major media. From the beginning of 2000 to the end of 2020, over 80 thousand patent infringement lawsuits were filed in the U.S. District Courts, and the number of companies that involved reached 68 thousand and 105 thousand as plaintiffs and defendants respectively.[2] In the US, it takes three to five years from filing a case to judgment, based upon a jury trial. The empirical literature shows that the litigation cost in a patent infringement case can be too high for many firms to afford (Meurer, 1989; Elleman, 1996; Lanjouw and Schankerman,

[2] Source: Lex Machina. The majority of these companies are not non-practicing entities.

2001; Chien, 2008), and it is especially difficult for small and medium enterprises to benefit from patent litigation (Hu, Yoshioka-Kobayashi, and Watanabe, 2017). Moreover, on average, plaintiffs only have a 50% chance of winning a patent infringement case as the patent may be found to be invalid in court (Allison and Lemley, 1998; Moore, 2000).

Due to the aforementioned characteristics of patent litigation, the incumbent may prefer to settle or eventually abandon the lawsuit rather than fight a full legal battle. A challenger faces a heavy burden of litigation costs which can result in significant losses and even bankruptcy (Bessen and Meurer, 2012). Settlement is the most common way to resolve the patent dispute and avoid the high litigation costs (Burton, 1980; Kesan and Ball, 2006). The majority of patent lawsuits are terminated with settlement, and most settlement occurs soon after the suit is filed, or often before the pre-trial hearing is held (Cohen and Merrill, 2003; Cotropia, Kesan, and Schwartz, 2018). Besides the ex-post settlement, another kind of settlement is pointed out by Choi and Gerlach (2017): an ex-ante settlement that happens before a complaint is filed.

The economic analysis of litigation and settlement naturally stemmed from the law and economics literature (Landes, 1971; Gould, 1973; Shavell, 1982; P'ng, 1983; Bebchuk, 1984; Choi, 1998; Bessen and Meurer, 2006). The focus was on the legal aspects (such as the probability of conviction by trial and the trial versus settlement costs) or the information asymmetry between litigants. There is a recent surge of empirical studies in the economics and finance literature that examine patent litigation (e.g., Lerner, 2002; Claessens and Laeven, 2003; Lee, Oh, and Suh, 2021; Mezzanotti, 2021; Caskurlu, 2022; Suh, 2023; Acikalin et al., 2023; Emery and Woeppel, 2023; Giebel, 2023). They recognize the imperfectness of patent protection and the critical link between the patent litigation risks and firms' innovation incentives but are limited by the lack of theoretical guidance. Our work offers a theoretical perspective to understand the dynamic interactions between firms involved in patent disputes, which expand beyond patent litigation to include resolutions via licensing agreements without actual lawsuits (ex-ante settlement in our model). We focus on the intersection of the product market and the legal environment.

The theory models on patent litigation have mainly adopted the game theory approach. Besides some of the aforementioned early work in the law and economics literature, Meurer (1989), Crampes and Langinier (2002), and Choi and Gerlach (2015) use a non-cooperative game approach and study the sequential decisions related to a patent dispute. Aoki and Hu (1999a) use a cooperative approach to investigate litigation and settlement. We do not consider information friction as Bebchuk (1996), but we follow his work in recognizing that litigation costs are generally not incurred all at once but rather over time, which plays a crucial strategic role. Our model instead follows the real options approach for its advantage of analyzing the timing of decisions with uncertainty in an analytical framework, and we add the strategic considerations. The intersection of real options and game theory provides powerful new insights into the behavior of economic agents under uncertainty (Grenadier, 2000).

This paper is not the first to use a real options model to study litigation or patent value. However, previous work (Schwartz, 2004, Marco, 2005, Grundfest and Huang, 2005, Jeon, 2015) does not incorporate the compound real options feature of sequential decisions. That is, the fact that later strategies (such as ex-post settlement) are activated by the exercising of earlier options (such as starting a litigation). We contribute to this literature by incorporating the option value of later decisions into the payoffs of earlier decisions, which we believe leads to a more comprehensive analysis of the outcomes of patent disputes. In addition, our analysis hinges on the firms' relative willingness to finance litigation, as a result of firms' strategic considerations in the litigation process (in the spirit of Lambrecht, 2001), which has been absent in the current literature.

Last but not least, this paper contributes to the broader literature on the interaction of finance and innovation or intangible assets (Claessens and Laeven, 2003; Acemoglu et al., 2010; Choi and Spier, 2018; Li, Qiu, and Wang, 2019; Falato et al., 2022; Acikalin et al., 2023; Lin, 2023). Our model takes an initial step of studying what determines the patent dispute outcomes from a finance perspective, to help advance the understanding of corporate innovation from the overlooked aspect of post-innovation intellectual property protection.

2 The Model

As in Jeon (2015), the incumbent (denoted by "I") is a patent holder. By using its patented technology, the incumbent earns instantaneous monopoly profits of $\pi_1 x_t$, with the market demand x_t that follows a geometric Brownian motion:

$$dx_t = \mu x_t dt + \sigma x_t dW_t. \tag{1}$$

The growth rate of demand $\mu < r$ (the risk-free rate). σ represents the demand volatility, and W_t represents a standard Brownian motion. A challenger (denoted by "C") enters the market with products that are based on a technology similar to the patented technology and is thus suspected of infringing the incumbent's patent.[3] Upon the challenger's market entry, firms receive duopoly profit flows of $\pi_2^I x_t$ and $\pi_2^C x_t$, respectively. Denote the total duopoly profit as $\pi_2 x_t \equiv (\pi_2^I + \pi_2^C)x_t$, and the total market profit may increase or decrease ($\pi_2 \gtreqless \pi_1$) based on the alleged infringement. $\Delta\pi$ denotes the change in the profit parameter due to the challenger's entry, and

$$\Delta\pi^I \equiv \pi_2^I - \pi_1 < 0, \quad \Delta\pi^C \equiv \pi_2^C - 0 > 0, \quad \Delta\pi \equiv \pi_2 - \pi_1 = \Delta\pi^I + \Delta\pi^C \gtreqless 0. \tag{2}$$

Figure 1 depicts the timeline, starting from the challenger's market entry. Firms' strategies take the form of optimal timing decisions and are equivalent to threshold strategies under the standard assumptions. In what follows, we separate the model setup into before- and after- litigation.

[Insert Figure 1 here]

Before litigation Given the suspected infringement, the firms can take certain actions, which are equivalent to them exercising a call option. Such options include (1) *I-litigate*: the incumbent files an infringement lawsuit against the challenger at time $\inf\{t : x_t \geq x_l\}$ where x_l denotes the incumbent's litigation threshold. The game proceeds to the next stage, termed *during litigation*, if the litigation happens. Starting a litigation is assumed to be costless. (2) *ex-ante settlement*: the two firms can settle their patent dispute by signing a royalty agreement. Following Lukas, Reuer, and Welling (2012), we model settlement as follows. The incumbent proposes a running royalty rate, denoted as θ_a, which is the fraction of the challenger's future profit $\pi_2^C x_t$ payable to the incumbent. The challenger determines the ex-ante settlement threshold x_a at which it accepts the royalty agreement offer and the royalty payment starts. Both firms incur a one-time ex-ante settlement cost C_s^I and C_s^C, such as the cost to draft the settlement agreement. The settlement cost is minimal compared to the litigation cost. $\Delta\pi_a$ is used to denote the change in the profit parameters due to the ex-ante settlement, compared to the status quo of duopoly:

$$\Delta\pi_a^I \equiv \theta_a \pi_2^C > 0, \quad \Delta\pi_a^C \equiv -\theta_a \pi_2^C < 0. \tag{3}$$

The ex-ante settlement happens at time $\inf\{t : x_t \geq x_a\}$ if ever, and the firms' profit flows are $\Big((\pi_2^I + \Delta\pi_a^I)x_t, (\pi_2^C + \Delta\pi_a^C)x_t\Big)$ afterwards. (3) *forcing-out*: the incumbent makes a threat of litigation to force the challenger out of the market, without settling. This option is only relevant if the threat is credible and

[3] We are primarily interested in inadvertent infringement rather than piracy, which Bessen and Meurer (2013) shows is the dominant cases.

the market demand follows a certain pattern of evolution, as discussed in Section 3.2. The two firms' profit flows will be $(\pi_1 x_t, 0)$ afterward.

During litigation Once the incumbent starts the litigation, we assume both firms incur ongoing litigation costs, $c^I dt$ and $c^C dt$, such costs include attorneys' fees and the indirect costs of litigation.[4] We simplify the setting of legal procedure and assume the court has no intermediate decisions and only just rules the case at a random time τ, which follows an exponential distribution with a parameter λ. Before the court rules and before firms take any actions detailed below, firms keep operating in duopoly. The expected time to the court ruling is $E(\tau) = \frac{1}{\lambda}$, and the expected total litigation costs are denoted as $C_l^I \equiv \frac{c^I}{r+\lambda}$ and $C_l^C \equiv \frac{c^C}{r+\lambda}$, which are the present value of the full litigation costs supposing the case runs the course till judgment. We define the relative litigation cost as

$$\Lambda = \frac{C_l^C}{C_l^I}. \tag{4}$$

With probability p, which is common knowledge as in Lemley and Shapiro (2005), the court rules that the incumbent wins and the challenger has to leave the market.[5] As a result, the incumbent-monopoly is restored and the future profit flows are $(\pi_1 x_t, 0)$ for the two firms.[6] With probability $1 - p$, the court rules non-infringement and the firms keep sharing the duopoly profits $(\pi_2^I x_t, \pi_2^C x_t)$.[7] Therefore, the expected profits without any firm actions are $\Big((p\pi_1 + (1-p)\pi_2^I)x_t, (1-p)\pi_2^C x_t\Big)$, or equivalently, $\Big((\pi_2^I - p\Delta\pi^I)x_t, (\pi_2^C - p\Delta\pi^C)x_t\Big)$ for the two firms.

Taking any action during the litigation is equivalent to the firm(s) abandoning the litigation, and thus can be regarded as exercising a put option. The firms have three options to terminate the litigation: (1) *I-withdraw*: the incumbent withdraws from the litigation as the market demand drops, at time $\inf\{t$:

[4] According to Bessen and Meurer (2012), examples of indirect costs include the time managers and researchers spend on producing documents, testifying in depositions, discussing strategies with lawyers, and appearing in court, all of which also disrupt business. Litigation also strains the inter-party relationship and may jeopardize technology collaboration. In addition, firms in a weak financial position can experience soaring credit cost because of possible bankruptcy risk created by patent litigation. Alleged infringers face additional costs, such as the shutdown of production and sales due to preliminary injunctions during litigation. Customers may stop buying an alleged infringer's product. We assume the litigation cost does not impact the probability of winning or losing, which is consistent with the empirical evidence as in Friedman (1969). The ongoing litigation costs not only include attorney fees and administrative fees such as court costs and deposition fees, but also economic costs associated with reputational loss.

[5] p can be interpreted as the possibility of the alleged infringement being convicted, or the possibility of the incumbent's patent being invalidated in court because patent rights are probabilistic (Choi and Gerlach, 2015).

[6] After AIA in 2011, US Federal district courts are unlikely to grant permanent injunctions to non-practicing entities but are willing to do so where the claimant practices its invention and is a direct market competitor of the defendant and the patented technology is at the core of its business.

[7] In reality, the challenger can file a counterclaim against the plaintiff and challenge the validity of the incumbent's patent. We extract from all possible complications and assume the court rulings follow the simply binary form.

$x_t \le x_w\}$ where x_w denotes the incumbent's withdraw threshold. The firms keep sharing the market profits afterward; that is, their profits will be $(\pi_2^I x_t, \pi_2^C x_t)$. (2) *C-exit*: the challenger stops selling the products and exits the market, at its exit threshold x_e, which happens at time $\inf\{t : x_t \le x_e\}$.[8] The two firms' profits will be $(\pi_1 x_t, 0)$ afterward. (3) *ex-post settlement*: the firms settle by signing a royalty agreement (Lukas, Reuer, and Welling, 2012). The incumbent proposes a running royalty rate for an ex-post settlement, denoted as θ_p, which is the proportion of the challenger's future profits that will be transferred to the incumbent. The challenger determines the settlement threshold x_p. Define $\Delta\pi_p^i$ as the change in Firm i's profits due to the ex-post settlement, compared to the status quo of duopoly:

$$\Delta\pi_p^I \equiv \theta_p \pi_2^C, \quad \Delta\pi_p^C \equiv -\theta_p \pi_2^C. \tag{5}$$

The settlement happens at time $\inf\{t : x_t \le x_p\}$, and the two firms' profits are $\Big((\pi_2^I + \Delta\pi_p^I)x_t, (\pi_2^C + \Delta\pi_p^C)x_t\Big)$ afterward. To settle, both firms pay one-time ex-post settlement costs, which we assume are equal to the ex-ante settlement costs C_s^I and C_s^C. We further assume it is less costly to settle than paying the full litigation cost, that is, $C_s^i < C_l^i$ for $i \in \{I, C\}$.[9] Note that the cost saving $\Delta C^i \equiv C_l^i - C_s^i > 0$ gives both firms the incentive to settle.

3 Model Solution

We use backward induction for the analysis and start by examining the *likely outcome during litigation*, which is the outcome after litigation starts, provided that firms take actions before the court rules.

3.1 After the litigation starts ("During litigation")

To study firms' strategies after litigation starts, we first obtain the general forms of firm values during litigation (Proposition 1). Next, we examine the unilateral decisions of I-withdraw and C-exit (Corollaries 1 and 2) in the absence of ex-post settlement, and determine the *likely non-settlement outcome during litigation* (Lemma 2). After that, we figure out firms' ex-post settlement strategy (Corollary 3 and Theorem 1), and then investigate whether the likely outcome during litigation is ex-post settlement or non-settlement (Theorem 2). To ease propositions, we represent the deferred perpetual factor of the

[8] In reality, the challenger can file for bankruptcy during the patent litigation process and leave the market. Technically, incorporating both I-withdraw and C-exit as possible outcomes is necessary to examine the strategic interactions between the two firms.

[9] Ex-post settlement has the same cost as Ex-ante settlement is a simplification assumption. By distinguishing the two, we may get different royalty rates in the ex-ante settlement, but our main results on non-/settlement regions remain the same. See Appendix A.15 for the discussion.

demand stream that starts from the court ruling as δ, and the corresponding equivalent perpetual cash flow rate that starts from the current time as ω:

$$\delta \equiv \frac{1}{r-\mu} - \frac{1}{r+\lambda-\mu}, \quad \omega \equiv \delta(r-\mu) \in (0,1), \tag{6}$$

with $\frac{1}{r-\mu}$ being the perpetuity factor and $\frac{1}{r+\lambda-\mu}$ the annuity factor for the profit flow that stops at the court ruling. In other words, the present value of a perpetual stream $\{x_t\}_{t=\tau}^{\infty}$ that starts from the court ruling is δx_0, where x_0 is the current demand. A perpetual stream of $\{\omega x_t\}_{t=0}^{\infty}$ that starts instantly has the same present value as δx_0. Thus, $p\delta x_0$ can be used to value a perpetual stream of $\{x_t\}_{t=\tau}^{\infty}$ which starts from a court ruling conditional on the incumbent wins, and $\{p\omega x_t\}_{t=0}^{\infty}$ is the corresponding cash flow stream that has the same present value.

The general forms of during-litigation value functions At any time in the litigation process, the expected change in the firm value equals the required return on its assets. The incumbent's during-litigation firm value $V^I(x)$ satisfies the following ordinary differential equation (*ODE*), with $\mathbb{E}_t dV^I = \left(\mu x \frac{\partial V^I}{\partial x} + \frac{1}{2}\frac{\partial^2 V^I}{\partial x^2} x^2 \sigma^2\right) dt$:

$$\mathbb{E}_t dV^I + (\pi_2^I x - c^I)dt + \lambda\left(\frac{p\pi_1 x + (1-p)\pi_2^I x}{r-\mu} - V^I\right)dt = rV^I dt.$$

The first term on the left is the expected instantaneous change of the incumbent's firm value associated with the demand fluctuation. The next two terms represent the incumbent's net profit flow. The last term represents the expected change in the firm value due to the potential court ruling. The challenger's during-litigation value V^C satisfies a similar condition, with $\mathbb{E}_t dV^C = \left(\mu x \frac{\partial V^C}{\partial x} + \frac{1}{2}\frac{\partial^2 V^C}{\partial x^2} x^2 \sigma^2\right) dt$:

$$\mathbb{E}_t dV^C + (\pi_2^C x - c^C)dt + \lambda\left(\frac{(1-p)\pi_2^C}{r-\mu} x - V^C\right)dt = rV^C dt.$$

We can write the ODEs on both firms' values uniformly as

$$\mathbb{E}_t dV^i + (\pi_2^i x - c^i)dt + \lambda\left(\frac{\pi_2^i - p\Delta\pi^i}{r-\mu} x - V^i\right)dt = rV^i dt, \quad \text{for } i \in \{I, C\}. \tag{7}$$

One boundary condition for Equation (7) comes from the fact that taking any action is not optimal when the demand is high, which suggests $\lim_{x\to\infty} V^i(x) = \left(\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i\right)x - C_l^i$, for $i \in \{I, C\}$. Thus, we obtain firm values as shown in Proposition 1, with the proof in Appendix A.1.

Proposition 1. *The during-litigation firm values for the incumbent and the challenger can be written as*

$$V^i(x) = \Big(\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i\Big)x - C_l^i + B^i x^{\beta_\lambda}, \quad i \in \{I, C\}, \tag{8}$$

where $\beta_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0$, $C_l^i = \frac{c^i}{r+\lambda}$*, the constants* B^I *and* B^C *depend on the relevant strategies.*

The first two terms on the right of Equation (8) represent the firm value if both firms take no actions once litigation starts and wait for the court to rule. Specifically, the first term $\frac{\pi_2^i}{r-\mu}x$ represents the present value of future profits under the status quo of duopoly, and the second term $-p\delta\Delta\pi^i x$ is the expected value of the court ruling. Only when the court rules in favor of the incumbent, which happens with the probability p, the firm profits depart from the status quo and revert to the pre-entry level. The change in profit is $-\Delta\pi^i x_t$, with the deferred perpetual factor δ. The third term $-C_l^i$ subtracts the present value of litigation costs. Our analysis below focuses on the last term $B^i x^{\beta_\lambda}$, which represents the option value associated with firms' strategies during litigation.

3.1.1 The non-settlement during litigation

Burdened by the high litigation cost, the incumbent may optimally withdraw from the lawsuit, and the challenger may exit the market after the litigation starts. To compare firms' willingness to keep financing the patent litigation once it starts, we define the *gain-to-loss ratio* Φ as follows.

$$\text{The gain-to-loss ratio:} \quad \Phi \equiv \frac{\Delta\pi^C}{|\Delta\pi^I|} = \frac{\pi_2^C}{\pi_1 - \pi_2^I} > 0. \tag{9}$$

Φ represents the challenger's profit gain relative to the incumbent's loss of profit, as a result of the challenger's market entry.[10] We focus on the case where the total market profits drop in order to investigate whether and why settlement can occur in this non-trial case.[11] That is, $\Phi \leq 1$.

[10] It is also the ratio between the challenger's stake of winning the lawsuit and the incumbent's stake of winning the lawsuit, thus mirrors the *asymmetric stake* or *strategic stake* in the management literature (Somaya, 2003, Harhoff and Reitzig, 2004).

[11] $\Phi \leq 1$ is equivalent to $\Delta\pi \leq 0$: $\Phi = \frac{\Delta\pi^C}{-\Delta\pi^I} = \frac{\Delta\pi - \Delta\pi^I}{-\Delta\pi^I} = \frac{\Delta\pi}{-\Delta\pi^I} + 1$. If $\Phi > 1$, the total market expands as the challenger enters. Intuitively, it implies the two firms can optimally settle and transfer profits between themselves to sustain the higher total market profits, as opposed to the incumbent recovering its monopoly profits via a successful lawsuit. Thus, it is less interesting than the non-trivial case of $\Phi \leq 1$. Relatedly, our model analysis is not directly applicable to patent trolls whose business models rely on extracting settlement payment from the threat of patent lawsuits (Cohen, Gurun, and Kominers, 2016).

I-withdraw Once the incumbent withdraws from the litigation, the two firms remain to share the duopoly profits. Equation (10) shows the firms' payoffs at I-withdraw, and Corollary 1 shows the firm values with the I-withdraw option and the associated withdrawal threshold with its proof in Appendix A.2.

$$\hat{V}_w^i(x) = \frac{\pi_2^i x}{r-\mu}, \quad i \in \{I, C\}. \tag{10}$$

Corollary 1. ***(I-withdraw)*** *The firm values with the I-withdraw option, V_w^i for $i \in \{I, C\}$, follow Equation (8) in Proposition 1, with $B_w^i = [C_l^i + p\delta\Delta\pi^i x_w]x_w^{-\beta_\lambda}$ and the I-withdraw threshold $x_w = \frac{\beta_\lambda}{\beta_\lambda - 1}\frac{C_l^I}{p\delta|\Delta\pi^I|}$.*

The incumbent's option value of withdraw ($B_w^I x^{\beta_\lambda}$) is the product of its net present value of withdraw and a stochastic discount factor, that is, $\left(C_l^I + p\delta\Delta\pi^I x_w\right)(\frac{x}{x_w})^{\beta_\lambda}$. By withdrawing, the incumbent saves all the future litigation costs but gives up the potential reversion of its future profits to the monopoly level, so the net payoff is worth $C_l^I + p\delta\Delta\pi^I x_w$. The stochastic discount factor $(\frac{x}{x_w})^{\beta_\lambda}$ values the future payoffs that materialize at the time of I-withdraw, taking into account that the court ruling kills the I-withdraw option. Meanwhile, as a result of the I-withdraw, the challenger not only saves its future litigation cost, but also saves its expected loss in a court ruling. The challenger's value associated with the I-withdraw option is $B_w^C x^{\beta_\lambda} = \left(C_l^C + p\delta\Delta\pi^C x_w\right)(\frac{x}{x_w})^{\beta_\lambda}$. Corollary 1 also indicates that the incumbent waits until its cost of staying in the litigation exceeds the benefit of staying in the litigation by a sufficient amount before exercising its withdraw option, that is, $\frac{C_l^I}{p\delta|\Delta\pi^I|x_w} = \frac{\beta_\lambda - 1}{\beta_\lambda} > 1$ (with $\beta_\lambda < 0$). The incumbent's decision to wait beyond the break-even point is driven by the irreversibility of its withdraw option and the uncertainty regarding the future market demand.

C-exit As the challenger exits the market, it no longer makes any profit while the incumbent restores its monopoly. With the C-exit payoffs shown in Equation (11), we present Corollary 2 with the proof in Appendix A.2:

$$\hat{V}_e^i(x) = \frac{\pi_2^i - \Delta\pi^i}{r-\mu}x, \quad i \in \{I, C\}. \tag{11}$$

Corollary 2. ***(C-exit)*** *The firm values with the C-exit option, V_e^i for $i \in \{I, C\}$, follow Equation (8) in Proposition 1, with $B_e^i = [C_l^i + (p\delta - \frac{1}{r-\mu})\Delta\pi^i x_e]x_e^{-\beta_\lambda}$ and the C-exit threshold $x_e = \frac{\beta_\lambda}{\beta_\lambda - 1}\frac{C_l^C}{(\frac{1}{r-\mu} - p\delta)\Delta\pi^C}$.*

With the challenger's market exit, the litigation ceases, and the incumbent saves its future litigation cost and regains its monopoly profits immediately, instead of having to wait for an uncertain court ruling,

With patent trolls, the alleged infringement increases the total market profits from the patented technology significantly, which implies $\Phi \gg 1$.

resulting in the present value of $C_l^I + (\frac{1}{r-\mu} - p\delta) \times (-\Delta\pi^I)x_e$ at the time of C-exit. By exiting, the challenger saves the future litigation cost but loses all the the duopoly profits. Taken into account that if the court rules the infringement and forces the challenger out of the market, the challenger loses duopoly profits upon the ruling anyways, exercising the C-exit option gives a payoff of $C_l^C + (p\delta - \frac{1}{r-\mu})\Delta\pi^C x_e$ to the challenger. The option values of C-exit for the two firms ($B_e^I x^{\beta_\lambda}$, $B_e^C x^{\beta_\lambda}$) are the products of the aforementioned present value at the C-exit and the stochastic discount factor $(\frac{x}{x_e})^{\beta_\lambda}$. Similar to the incumbent's withdrawal decision, the challenger also waits until the cost of staying in the litigation exceeds the benefit of doing so for a sufficient amount before exercising the exit option.

I-withdraw vs. C-exit We compare the two firms' *reservation thresholds* (Lambrecht, 2001) during litigation to determine whether I-withdraw or C-exit is the likely non-settlement outcome.[12] A reservation threshold is the lowest demand level at which the firm is willing to stay in the litigation and continue paying the litigation cost. Specifically, the incumbent's reservation threshold, denoted as x_I, is the demand level at which its during-litigation firm value including the option to withdraw at x_w equals its firm value if the challenger exits the market earlier, in which case the incumbent would lose its withdrawal option but restore its monopoly. Likewise, the challenger's reservation threshold x_C is the demand level at which its during-litigation firm value including its exit option at x_e equals its firm value if the incumbent withdraws earlier, in which case it would lose its exit option but remain in a duopoly. We express the conditions on the reservation thresholds x_I and x_C as $V_w^I(x = x_I; x_w) = \hat{V}_e^I(x = x_I)$ and $V_e^C(x = x_C; x_e) = \hat{V}_w^C(x = x_C)$, respectively. With the proof in Appendix A.3:

Lemma 1. *The reservation threshold of the incumbent x_I satisfies $\frac{1}{1-\beta_\lambda}\left(\frac{x_I}{x_w}\right)^{\beta_\lambda} - \frac{\beta_\lambda}{\beta_\lambda - 1}\frac{(1-p\omega)}{p\omega}\frac{x_I}{x_w} = 1$. The reservation threshold of the challenger x_C satisfies $\frac{1}{1-\beta_\lambda}\left(\frac{x_C}{x_e}\right)^{\beta_\lambda} - \frac{\beta_\lambda}{\beta_\lambda - 1}\frac{p\omega}{(1-p\omega)}\frac{x_C}{x_e} = 1$. The likely non-settlement outcome during litigation is I-withdraw if $x_I > x_C$, and it is C-exit if $x_I < x_C$.*

The firm with a lower reservation threshold is willing to continue financing the litigation for longer. The firm with the higher reservation threshold thus stops paying the litigation cost at their optimal threshold (x_w for the incumbent or x_e for the challenger), and in turn, the remaining firm benefits from ceasing to pay litigation costs at the rival's optimal threshold too. Thus, the likely non-settlement outcome is the one associated with the higher reservation threshold, that is, I-withdraw if $x_I > x_C$ and

[12] Because both firms act strategically knowing the other firm also has the option to leave the litigation, a direct comparison of the two firms' leaving thresholds x_w vs. x_e from Corollaries 1 and 2 is not appropriate to determine whether s_{ns} = (I-withdraw) or s_{ns} = (C-exit).

C-exit if $x_C > x_I$. Lemma 2 details sufficient conditions for the likely non-settlement outcome during litigation s_{ns}, with its proof in Appendix A.3.

Lemma 2. *The likely non-settlement outcome during litigation s_{ns} = (I-withdraw) if $p\omega > 0.5$ and $\Phi > \frac{p\omega}{1-p\omega}\Lambda$, and s_{ns} = (C-exit) if $p\omega < 0.5$ and $\Phi < \frac{p\omega}{1-p\omega}\Lambda$. If $p\omega = 0.5$, then s_{ns} = (I-withdraw) if $\Phi > \Lambda$ and s_{ns} = (C-exit) if $\Phi < \Lambda$.*

Lemma 2 indicates that the likely non-settlement outcome during litigation is non-trivial: it is I-withdraw even if the incumbent gets a high expected cash flow from staying in the litigation (i.e., $p\omega > 0.5$), as long as the gain-to-loss ratio Φ sufficiently exceeds the litigation cost ratio Λ (with a multiplier of $\frac{p\omega}{1-p\omega} > 1$).[13] Likewise, C-exit is likely in non-settlement even if the incumbent gets a low expected cash flow by staying in the litigation (i.e., $p\omega < 0.5$), as long as the gain-to-loss ratio is sufficiently lower than the litigation cost ratio. Compared to the industry organization literature where the exit order in a duopoly depends solely on the relative benefit-versus-cost, this lemma demonstrates the additional consideration of court ruling in our setting, via both the lengthiness of litigation and the court ruling probability (or the term $p\omega$).

Summary of non-settlement Based on Corollaries 1–2 and Equations (10)–(11), we summarize the non-settlement firm values during litigation V^i_{ns}, the non-settlement payoffs $\hat{V}^i_{ns}$, the arbitrary constants in the option value of non-settlement B^i_{ns} and the non-settlement threshold x^i_{ns}, $i \in \{I, C\}$, as:

$$\left(V^i_{ns}(x)^i, \hat{V}^i_{ns}(x), B^i_{ns}, x_{ns}\right) = \begin{cases} \left(V^i_w(x), \hat{V}^i_w(x), B^i_w, x_w\right), & s_{ns} = \text{(I-withdraw)} \\ \left(V^i_e(x), \hat{V}^i_e(x), B^i_e, x_e\right), & s_{ns} = \text{(C-exit)} \end{cases} \quad i \in \{I, C\}. \tag{12}$$

3.1.2 The settlement during litigation ("Ex-post settlement")

For ex-post settlement, we assume the incumbent makes an offer of the licensing agreement with its proposal of the royalty rate once the litigation commences.[14] The challenger chooses when to accept the settlement offer, and ex-post settlement happens once the challenger accepts the offer, as long as

[13]The condition $\Phi > \frac{p\omega}{1-p\omega}\Lambda$ is equivalent to $\frac{\Phi}{\Lambda} = \frac{\Delta\pi^C}{C^C_l}\left(\frac{-\Delta\pi^I}{C^I_l}\right)^{-1} > \frac{p\omega}{1-p\omega}$, which is greater than 1 given $p\omega > 0.5$. In other words, the challenger's gain of entering the market relative to its litigation cost sufficiently exceeds the incumbent's loss caused by the opponent's entry relative to its litigation cost.

[14]The litigation outcome is the same regardless of when the incumbent makes the settlement offer, as long as it is before the market demand drops to the challenger's optimal acceptance threshold. Thus, it is optimal for the incumbent to offer a settlement immediately after it launches the litigation, if ever.

the court has not yet ruled. By settling, the firms save the litigation cost C_l^i and only pay the settlement cost $C_s^i(< C_l^i)$. Define the ratio of the challenger's cost saving, from settling rather than continuing with litigation, and the incumbent's cost saving as

$$\text{The relative-cost-saving:} \quad \Gamma \equiv \frac{\Delta C^C}{\Delta C^I} = \frac{C_l^C - C_s^C}{C_l^I - C_s^I}. \tag{13}$$

When the litigation cost is much higher than the settlement cost, or the settlement cost is proportional to the litigation cost, the relative-cost-saving is approximately the same as the relative litigation cost in Equation (4), i.e., $\Gamma \to \Lambda$. Ex-post settlement happens when both firms agree on the royalty rate θ_p and the settlement threshold x_p.[15] The ex-post settlement payoffs $\hat{V}_p^i(x;\theta_p)$ can be expressed as Equation (14), after paying the one-time settlement cost C_s^i. We present Corollary 3 with the proof in Appendix A.4.[16]

$$\hat{V}_p^i(x;\theta_p) = \frac{\pi_2^i + \Delta\pi_p^i}{r-\mu}x, \quad i \in \{I, C\}. \tag{14}$$

Corollary 3. ***(Ex-post settlement)*** *The firm values with the ex-post settlement option V_p^I and V_p^C follow Equation (8). Given a royalty rate θ_p, the arbitrary constants are $B_p^I = \left[\Delta C^I + \left(p\delta\Delta\pi^I + \frac{\theta_p \Delta\pi^C}{r-\mu}\right)x_p\right]x_p^{-\beta_\lambda}$, $B_p^C = \left[\Delta C^C + \left(p\delta - \frac{\theta_p}{r-\mu}\right)\Delta\pi^C x_p\right]x_p^{-\beta_\lambda}$ and the settlement threshold is $x_p = \frac{\beta_\lambda}{\beta_\lambda - 1}\frac{\Delta C^C}{(\frac{\theta_p}{r-\mu} - p\delta)\Delta\pi^C}$.*

The option value of settlement $B_p^i x^{\beta_\lambda}$ is intuitive. By settling, both firms save the future litigation cost but pay the settlement cost, with the net cost savings worth ΔC^i at the settlement. Meanwhile, by settling, the incumbent gives up the potential recovery of monopoly profits from a favorable court ruling but gains the assured future royalty payment, whilst the challenger avoids losing its duopoly profit caused by an adverse court decision and pays the assured royalty fee. Such cash flow considerations worth $\left(p\delta\Delta\pi^i + \frac{\Delta\pi_p^i}{r-\mu}\right)x_p$ for firm $i \in \{I, C\}$ at the settlement. $(\frac{x}{x_p})^{\beta_\lambda}$ is the stochastic discount factor for cash flow that happens at the settlement $\inf\{t : x_t \le x_p\}$. Corollary 3 also indicates that the challenger waits until its settlement benefit exceeds the cost for a sufficient amount before settling. Furthermore, a higher θ_p leads to a delay in settlement, which is evident from $\frac{\partial x_p}{\partial \theta_p} = -\frac{\beta_\lambda}{\beta_\lambda - 1}\frac{\Delta C^C(r-\mu)}{\Delta\pi^C}\frac{1}{(\theta_p - p\omega)^2} < 0$. There-

[15] Following Lukas, Reuer, and Welling (2012) to model settlement is simple but not without critique. Alternatively, the incumbent can make a take-it-or-leave-it offer on both the royalty rate and a demand threshold below which the settlement offer is off the table. A critique of it regards the credibility of the incumbent's commitment of not accepting a lower settlement threshold. Or, the two firms negotiate both the royalty rate and the settlement threshold via a Nash bargaining game. We argue that the bargaining power in such a game arises from how the firms split the market, and our model effectively endogenizes the bargaining power.

[16] Alternatively, the arbitrary constant in the firm value with the ex-post settlement option V_p^i for $i \in \{I, C\}$ can be written as $B_p^i = \left[\Delta C^i + \left(p\delta\Delta\pi^i + \frac{\Delta\pi_p^i}{r-\mu}\right)x_p\right]x_p^{-\beta_\lambda}$, where the settlement threshold $x_p = \frac{\beta_\lambda}{\beta_\lambda - 1}\Delta C^C(-p\delta\Delta\pi^C - \frac{\Delta\pi_p^C}{r-\mu})^{-1}$.

fore, when deciding the royalty rate to offer, the incumbent trades off a direct positive effect on royalty payment versus an indirect negative effect via settlement delay.

Maximizing the incumbent's firm value with the ex-post settlement option $V_p^I(x_p, \theta_p)$, taking into account the challenger's settlement strategy $x_p(\theta_p)$, we get the following result, with its proof in Appendix A.5.

Theorem 1. *The incumbent's optimal royalty rate in an ex-post settlement is* $\theta_p^* = p\omega\big(1 - g(\Gamma)\big) + \frac{p\omega}{\Phi} g(\Gamma)$*, and the corresponding settlement threshold is* $x_p^* = \frac{\beta_\lambda}{\beta_\lambda - 1} \cdot \frac{\Delta C^C}{p\delta g(\Gamma)(-\Delta\pi)}$*, where* $g(\Gamma) = (\frac{\beta_\lambda}{\beta_\lambda - 1} + \frac{1}{\Gamma})^{-1} > 0$.

Theorem 1 demonstrates the strategic interaction between the two firms in ex-post settlement. The royalty rate that maximizes the incumbent's option value of settlement θ_p^* takes the general form of a weighted average of two royalty rates, $p\omega$ and $\frac{p\omega}{\Phi}$, with the weights $(1 - g(\Gamma), g(\Gamma))$ as functions of the relative-cost-saving Γ. The former rate $p\omega \in (0, 1)$ is the one that makes the challenger indifferent between paying the royalty fees and continuing with the litigation versus settling ex-post, in the absence of any cost considerations. That is, $\frac{\pi_2^C(1-\theta)x}{r-\mu} = \frac{\pi_2^C x}{r-\mu} - p\delta\Delta\pi^C x$. The latter rate $\frac{p\omega}{\Phi} > 0$ is the one that makes the incumbent indifferent between getting the assured stream of royalty payments and continuing with the litigation, in the absence of any cost considerations. That is, $\frac{\pi_2^I x}{r-\mu} + \frac{\theta\Delta\pi^C x}{r-\mu} = \frac{\pi_2^I x}{r-\mu} - p\delta\Delta\pi^I x$. The weights $(1 - g(\Gamma), g(\Gamma))$ adjust the indifferent royalty rates with the cost-saving considerations from settlement. If the challenger does not save any litigation cost by settling while the incumbent does ($\Gamma \to 0$), then the challenger has little incentive to settle and is thus more empowered in the settlement and effectively has the full bargaining power in settlement. This leads to the royalty rate being the challenger's indifference rate. On the contrary, if the challenger saves more by settling than the incumbent, and the relative-cost-saving equals $1 - \beta_\lambda$, then the royalty rate is the incumbent's indifference rate. Corollary 4 lists the comparative statics of settlement timing and terms with its proof in Appendix A.6.

Corollary 4. $\frac{\partial\theta_p^*}{\partial p} > 0$, $\frac{\partial\theta_p^*}{\partial\sigma^2} > 0$, $\frac{\partial\theta_p^*}{\partial\Phi} < 0$, $\frac{\partial\theta_p^*}{\partial\Gamma} > 0$. $\frac{\partial x_p^*}{\partial p} < 0$, $\frac{\partial x_p^*}{\partial\sigma^2} < 0$, $\frac{\partial x_p^*}{\partial\Delta\pi} > 0$, $\frac{\partial x_p^*}{\partial\Delta C^i} > 0$.

3.1.3 Non-settlement vs. settlement during litigation

To determine the likely outcome during litigation, we compare the firm values with the settlement option versus with the non-settlement option. The firms only agree to settle if their values with the option of settling (discussed in Section 3.1.2) are at least as high as their values of not settling (discussed in Section 3.1.1). Specifically, the incumbent is only willing to offer a settlement if its value of settling is at least as high as its value with the non-settlement option, and the challenger is only willing to accept to settle under a similar condition. These conditions narrow down the range of royalty rates for which the likely outcome during litigation is ex-post settlement:

$$V_p^I(x_p, \theta_p) \geq V_{ns}^I(x_p) \quad \text{and} \quad V_p^C(x_p, \theta_p^C) \geq V_{ns}^C(x_p). \tag{15}$$

The incumbent's condition in Expression (15) implies both a lower bound θ_p^{Imin} and an upper bound θ_p^{Imax} on the royalty rate. A minimum rate is required to compensate the incumbent for not continuing with the litigation, and a maximum rate is required to prevent further delay of the settlement which makes settlement suboptimal for the incumbent. Between the two effects of θ_p on the incumbent's firm value, the positive impact through gaining a higher proportion of the challenger's profits dominates at low θ_p and the negative impact via settlement delay dominates at high θ_p. The challenger's condition in Expression (15) implies an upper bound θ_p^{Cmax}, above which the challenger rejects the settlement offer, regardless of the demand condition, because it does not retain enough profits to make settlement worthwhile. With the proof in Appendix A.7, we show:[17]

Theorem 2. *An ex-post settlement is the likely outcome during litigation, as opposed to non-settlement, if and only if* $\theta_p^* \in \left[\theta_p^{Imin}, \min\{\theta_p^{Cmax}, \theta_p^{Imax}\}\right]$, *with* θ_p^{Imin}, θ_p^{Imax} *and* θ_p^{Cmax} *specified in Equations* (A.6) *and* (A.7).

Summary of during-litigation values and strategies During litigation, the firm values V^i, their payoffs after exercising the relevant option $\hat{V}^i$, the arbitrary constants in option values B^i, and the lawsuit termination threshold X are as follows, with elements based on Corollary 3, Theorem 1, Equations (12) and (14):

$$\left(V^i(x), \hat{V}^i(x), B^i, X\right) = \begin{cases} \left(V_p^i(x), \hat{V}_p^i(x), B_p^i, x_p\right) & \text{if the Theorem 2 condition holds,} \\ \left(V_{ns}^i(x), \hat{V}_{ns}^i(x), B_{ns}^i, x_{ns}\right) & \text{otherwise.} \end{cases} \tag{16}$$

[17] Our numerical exercises suggest that $\theta_p^{Imax} \leq \theta_p^{Cmax}$ in C-exit, but $\theta_p^{Imax} > \theta_p^{Cmax}$ in I-withdraw. The challenger's high willingness to pay for the royalty fee in C-exit, reflected in $\theta_p^{Imax} \leq \theta_p^{Cmax}$, is probably driven by the motive to avoid its own exit. On the contrary, I-withdraw is a better outcome for the challenger than its own exit, thus its willingness to pay for the royalty fee to avoid I-withdraw is not as high, reflected in $\theta_p^{Cmax} < \theta_p^{Imax}$.

3.2 Before the litigation possibly starts ("Before litigation")

We view the firms' before-litigation problems as compound real options problems, given their sequential nature with respect to the during-litigation strategies. We first obtain the general forms of before-litigation firm values in Proposition 2, and then examine the incumbent's litigation decision alone in Corollary 5, taking into account of what happens if litigation starts. After that, we investigate firms' optimal ex-ante settlement strategy in Corollary 6, and decide whether firms settle ex-ante or enter a patent litigation against each other (Theorem 3). Regardless of ex-ante settlement or I-litigate, the before-litigation strategy resembles the exercise of a call option.

The general forms of before-litigation value functions After the alleged infringement and before firms take any action(s), the firm values V_0^i follow the ODEs:

$$\text{For } i \in \{I, C\}: \quad \mathbb{E}_t dV_0^i + \pi_2^i x dt = rV_0^i dt, \text{ where } \mathbb{E}_t dV_0^i = \left(\mu x \frac{\partial V_0^i}{\partial x} + \frac{1}{2}\frac{\partial^2 V_0^i}{\partial x^2} x^2 \sigma^2\right) dt \tag{17}$$

With the proof in Appendix A.8, we show the following general form of before-litigation firm values:

Proposition 2. *After the alleged infringement and before firms take any action(s), firms' value functions V_0^i, for $i \in \{I, C\}$, depend on the likely non-settlement outcome s_{ns} during litigation:*

$$V_0^i = \begin{cases} \frac{\pi_2^i x}{r-\mu} + A^i x^\alpha, & s_{ns} = (\textit{I-withdraw}) \\ \frac{\pi_2^i x}{r-\mu} + a^i x^\alpha + b^i x^\beta. & s_{ns} = (\textit{C-exit}) \end{cases} \tag{18}$$

where $\alpha, \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$ ($\alpha > 1$, $\beta < 0$). The constants A^i, a^i, and b^i depend on the relevant strategies.

The firm values V_0^I and V_0^C include the additional terms $b^I x^\beta$ and $b^C x^\beta$ if C-exit is the likely non-settlement outcome during litigation, compared to I-withdraw. These terms are driven by the incumbent's extra option to force the challenger out of the market (the incumbent's *forcing-out* strategy), which is relevant if (1) the market demand drops to the C-exit threshold x_e before bouncing back up to the litigation threshold x_l or the ex-ante settlement threshold x_a, and (2) C-exit is the likely non-settlement outcome during litigation. The incumbent can exercise the forcing-out option via a litigation threat at time $\inf\{t : x \leq x_e\}$ if the challenger refuses to leave the market then. Recognizing the possibility of a forcing-out strategy is economically important and technically necessary.[18] Forcing-out is effective in driving the challenger to exit because if C-exit is the likely outcome during litigation, then the challenger would optimally leave at x_e after litigation starts anyways. On the contrary, if ex-post settlement is the likely outcome during litigation, then the incumbent can credibly commit not to offer ex-post settlement. By exercising the forcing-out option at x_e, the incumbent immediately regains the monopoly profits without getting involved in costly litigation, instead of having to wait until x_l to start the litigation before the challenger exits at x_e or to wait and settle ex-ante at x_a.

3.2.1 Litigation by the incumbent ("I-litigate")

In the absence of ex-ante settlement consideration, firm payoffs after the incumbent exercises the litigation option is the during-litigation firm values V^i (see Equation (16)). With the proof in Appendix A.9, we show

Corollary 5. ***(I-litigate)*** *After the alleged infringement, the firm values with the incumbent's litigation option, V_l^i for $i \in \{I, C\}$, follow Equation (18) in Proposition 2, with $A_l^i = \left[-C_l^i - p\delta\Delta\pi^i x_l + B^i x_l^{\beta_\lambda} \right] x_l^{-\alpha}$, and a_l^i and b_l^i specified in Equation (A.11). The litigation threshold x_l satisfies $(\alpha-1)p\delta\Delta\pi^I x_l + (\beta_\lambda - \alpha)B^I x_l^{\beta_\lambda} + \alpha C_l^I + \frac{\partial B^I}{\partial x_l} x_l^{\beta_\lambda+1} = 0$ in I-withdraw with B^I defined in Equation (16), and it satisfies Equation (A.12) in C-exit.*

From Corollary 5, the incumbent's option value of litigation in I-withdraw $A_l^I x^\alpha$ can be expressed as the product of the net payoff of exercising the litigation option and the stochastic discount factor $(\frac{x}{x_l})^\alpha \in (0,1)$, with the market demand $x < x_l$ before exercising this call option. By exercising the litigation option, the incumbent expects to pay the litigation cost C_l^I, recovers its profit loss $-\Delta\pi^I x$ with the probability of p and discounted using δ which is worth $-p\delta\Delta\pi^I x_l$ at the I-litigate threshold and gains the opportunity to exercise the follow-on options during litigation which worths $B^I x_l^{\beta_\lambda}$ at the commencement of litigation. We can further verify that the incumbent waits until the benefit of litigation exceeds the cost for a sufficient amount before exercising the litigation option. The arbitrary constants and the action threshold for C-exit are more complicated for a straightforward interpretation due to the forcing-out option, so we leave the expressions in the appendix as Corollary 5 indicates.

[18] Technically, the forcing-out option leads to value matching conditions for both firms at x_e before litigation, see Equation (A.10) in the appendix.

3.2.2 Settling without litigation ("Ex-ante settlement")

In an ex-ante settlement, the incumbent proposes a royalty rate θ_a, and the challenger decides whether and when to accept the royalty agreement offer. The ex-ante settlement payoffs $\hat{V}_a^i(x;\theta_a)$, after the firms pay the one-time settlement cost C_s^i, can be expressed as Equation (19). We present Corollary 6 with the proof in Appendix A.10.

$$\hat{V}_a^i(x;\theta_a) = \frac{\pi_2^i + \Delta\pi_a^i}{r-\mu}x. \tag{19}$$

Corollary 6. ***(Ex-ante settlement)*** *The firm values with the ex-ante settlement option, V_a^i for $i \in \{I, C\}$, follow Equation (18) in Proposition 2, with $A_a^i = (\frac{\Delta\pi_a^i}{r-\mu}x_a - C_s^i)x_a^{-\alpha}$, $a_a^i = \frac{\frac{\Delta\pi_a^i}{r-\mu}x_a^{1-\beta} + \frac{\Delta\pi^i}{r-\mu}x_e^{1-\beta} - C_s^i x_a^{-\beta}}{x_a^{\alpha-\beta} - x_e^{\alpha-\beta}}$, $b_a^i = \frac{\frac{\Delta\pi_a^i}{r-\mu}x_a^{1-\alpha} + \frac{\Delta\pi^i}{r-\mu}x_e^{1-\alpha} - C_s^i x_a^{-\alpha}}{x_a^{\beta-\alpha} - x_e^{\beta-\alpha}}$, and the ex-ante settlement threshold $x_a = x_l$.*

In the absence of forcing-out, the incumbent's option value of the ex-ante settlement strategy ($A_a^I x^\alpha$) equals the payoff of exercising the option at the time of settlement, that is $\frac{\Delta\pi_a^I}{r-\mu}x_a - C_s^I$, multiples the stochastic discount factor $(\frac{x}{x_a})^\alpha$. $(\frac{x}{x_a})^\alpha \in (0,1)$ provided that $x < x_a$ before exercising the call option. Absent of the I-litigate option, the challenger's option value of the ex-ante settlement is negative ($A_a^C x^\alpha < 0$) because it has to pay the future royalty fees and the ex-ante settlement cost once settled ex-ante, but gets no positive cash flows in return. Meanwhile, because the challenger's value of settlement increases with the settlement threshold, the challenger waits for as long as possible before signing the licensing agreement, if ever. Similar reasoning applies to the situation in which C-exit is the likely non-settlement outcome during litigation where the forcing-out option becomes relevant, and the analytical solutions become more difficult to interpret in a straightforward way.

3.2.3 I-litigate vs. ex-ante settlement before litigation

The firms settle ex-ante, as opposed to getting involved in the litigation if (1) the challenger has a higher value by accepting the settlement offer, instead of rejecting to settle and proceeding to the litigation, and (2) the incumbent's firm value of ex-ante settlement is higher than its value with the litigation option. Together:

$$\hat{V}_a^C(\theta_a, x_a) - C_s^C \geq V_l^C(x_a) \Rightarrow \theta_a \leq \theta_a^{Cmax}, \quad \hat{V}_a^I(\theta_a, x_a) - C_s^I \geq V_l^I(x_a) \Rightarrow \theta_a \geq \theta_a^{Imin}. \tag{20}$$

Theorem 3 describes the conditions for the before-litigation strategies, with its proof in Appendix A.11.

Theorem 3. *After the alleged infringement, firms settle ex-ante if and only if $\theta_a^{Imin} \leq \theta_a^{Cmax}$ with θ_a^{Imin} and θ_a^{Cmax} specified in Equation (A.13), and the settlement royalty rate $\theta_a = \theta_a^{Cmax}$. Otherwise, the incumbent litigates at x_l.*

Beyond the challenger's highest acceptable royalty rate θ_a^{Cmax}, the challenger would rather get involved in the litigation and face uncertainties than settle ex-ante. The same consideration applies to the incumbent if the royalty rate is below its minimum required rate θ_a^{Imin}. Unlike in the ex-post settlement, the incumbent has no maximum required royalty rate in an ex-ante settlement. This is because the concern that the challenger delays settlement further by a higher royalty rate, which is valid for the ex-post settlement, is irrelevant here. Ex-ante settlement happens at the litigation threshold regardless. Therefore, the incumbent proposes the royalty rate θ_a^{Cmax} provided that ex-ante settlement is better than I-litigate for both firms.

Summary of before-litigation strategies and value functions Based on Corollaries 5 and 6, we represent the before-litigation firm values V_0^i, the payoff functions $\hat{V}_0^i$ at any demand $x < \min\{x_l, x_a\}$, the arbitrary constants in the before-litigation firm values, and the strategy threshold X_0, for $i \in \{I, C\}$, as

$$\left(V_0^i(x), \hat{V}_0^i(x), A^i, a^i, b^i, X_0\right) = \begin{cases} \left(V_a^i(x), \hat{V}_a^i(x), A_a^i, a_a^i, b_a^i, x_a\right) & \text{if } \theta_a^{Imin} \leq \theta_a^{Cmax} \text{ in Theorem 3,} \\ \left(V_l^i(x), \hat{V}_l^i(x), A_l^i, a_l^i, b_l^i, x_l\right) & \text{otherwise.} \end{cases} \tag{21}$$

We expect the before-litigation threshold to exceed the during-litigation threshold ($X_0 = \{x_a, x_l\} \geq X = \{x_{ns}, x_p\}$) for at least three reasons: (1) the before-litigation strategy resembles the exercising of a call option, and the firms take actions when demand rises to the threshold from below at $\inf\{t : x_t \geq X_0\}$, (2) the during-litigation strategy resembles the exercising of a put option, and the firms take actions when the demand drop from above at $\inf\{t : x_t \leq X\}$, and (3) exercising the litigation option activates the latter options. For example, as the market demand booms after the alleged infringement, the incumbent starts the litigation, and then as the demand declines, the two firms settle ex-post. This chain of events indicates $x_l > x_p$, as expected.

New complications arise if $X > X_0$. In particular, if ex-post settlement is the likely outcome during litigation, and $x_p \geq x_l$, then firms immediately settle upon the commencement of litigation (we call it the *immediate settlement*[19]). Recognizing the possible immediate settlement outcome, the firms

[19] Technically, we have two types of immediate settlement. In a *constrained immediate settlement*, the optimal ex-post settle-

re-optimize and settle ex-ante to avoid litigation in the first place. We leave the details of the immediate settlement solution and the discussion of its feasibility in the Appendices A.12 – A.14. With our assumption that there is no cost to start a litigation and settlement cost is the same for ex-ante and ex-post settlement, if settlement occurs immediately once litigation starts, then ex-ante settlement always occurs beforehand.[20]

4 Quantitative Analysis

Given any model parameter values associated with the product market ($\mu, \sigma, \pi_1, \pi_2^I, \pi_2^C$), the legal system ($p, \lambda, c_l^I, c_l^C, C_s^I, C_s^C$) and the risk free rate r, we solve the model with the detailed steps in Section 3. Most of the benchmark parameter values in Table 1 follow the existing literature such as Jeon (2015). The mean and the volatility of the demand shock are $\mu = 2\%$ and $\sigma = 30\%$, respectively. The risk-free rate is $r = 0.05$. The expected time to court ruling is two-and-a-half years ($\lambda = 0.4$), in line with the empirical evidence on patent litigation cases in the U.S. The probability that the incumbent wins the patent infringement lawsuit is $p = 0.5$.

[Insert Table 1 here.]

At the benchmark, we use the numerical method and find that the likely outcome is I-litigate followed by ex-post settlement, with the litigation threshold of $x_l = 1.72$ and the ex-post settlement threshold of $x_p = 0.76$. Figure 2 illustrates the sequential actions at the benchmark on a realized demand path. The red dot represents the time of litigation, which is the first time that the demand increases beyond x_l from below. The green dot represents the ex-post settlement time, which is the first time that the demand drops below x_p from above, with the assumption that the court has not yet ruled then.

[Insert Figure 2 here.]

Using the numerical method repeatedly by varying one model parameter value at a time, we map the likely outcome of patent disputes to the determinants, assuming the market demand reaches the relevant threshold at some point. The potential outcomes include *ex-ante settlement*, *litigation followed*

ment royalty rate exceeds the maximum that the challenger is willing to pay, that is, $\theta_p^* > \theta_p^{Cmax}$, in which case the royalty rate is θ_p^{Cmax}. In an *unconstrained immediate settlement*, $\theta_p^* < \theta_p^{Cmax}$, in which case the royalty rate is θ_p^*. The strategy of settling immediately when litigation occurs is rarely modeled in theoretical work but is uncovered by numerous empirical studies in patent litigation literature. According to Kessler (1996) the frequency of settlements decreases as the length of negotiations increases. In some bargaining models with incomplete information, most settlements occur at the beginning of litigation (Spier, 1992; Fanning, 2016; Vasserman and Yildiz, 2019).

[20] The ex-ante settlement royalty rate is the same as the royalty rate in immediate settlement, as shown in Appendix A.15.

by ex-post settlement, or *litigation followed by I-withdraw or C-exit*. The determinants uncovered by our model analysis include the gain-to-loss ration ($\Phi = \frac{\Delta\pi^C}{|\Delta\pi^I|}$), the relative-cost-saving ($\Gamma = \frac{\Delta C^C}{\Delta C^I}$), the possibility of an infringement ruling (p), the market demand volatility (σ), and the court ruling intensity (λ).

The gain-to-loss ratio Φ is a product market characteristic associated with the patented technology. Recall it measures the challenger's profit gain relative to the incumbent's profit loss due to the alleged infringement. A high ratio indicates a substantial profit gain of the challenger relative to the incumbent's profit loss. For example, the entrant uses patented technology to produce complements of the incumbent's products. In contrast, a low ratio indicates a substantial profit loss of the incumbent relative to the challenger's profit, which is typical for a Cournot or Bertran competition of homogeneous products from the new entrant. It corresponds to a significant decline in the industry's total profits upon the challenger's market entry. Meanwhile, we regard the relative-cost-saving Γ as the combination of product market and legal system characteristics. It captures the extent to which settling the dispute saves litigation costs that differ for the two firms. A high Γ indicates a substantial litigation cost saving via a settlement for the challenger, relative to the incumbent. For example, the non-infringement claim is very costly for the challenger to prove while the patent infringement claim is straightforward for the incumbent to construct. The opposite is true for a low Γ.

4.1 The effects of Φ and Γ on patent dispute outcomes

In Figure 3, we plot the outcome regions of a patent dispute with respect to the gain-to-loss ratio and the relative-cost-saving ($\Phi \times \Gamma$).[21] The outcomes are conditional on the occurrence of alleged infringement and the market demand reaching the relevant thresholds before the court rules. Among the three likely outcomes of patent disputes, the blue region represents ex-ante settlement, the green region represents patent litigation followed by ex-post settlement, and the white region represents patent litigation followed by I-withdraw or C-exit.

[Insert Figure 3 here.]

We highlight three findings illustrated by Figure 3, which remain consistent in many unreported analyses. First, settlements (either ex-ante or ex-post) are more likely as the $\Phi \times \Gamma$ region gets closer to the dashed line, which marks the equal willingness of the two firms to continue financing the litigation

[21] We vary Φ by changing the monopoly profit multiplier π_1, and vary Γ by changing the incumbent's litigation cost c_l^I.

once the lawsuit starts. As the firms' willingness to finance becomes asymmetric, that is, the region is further away from the dashed line, the outcome shifts from settlement to non-settlement. The shift is due to the strategic consideration of the two firms. Above the dashed line, the likely non-settlement outcome during litigation (see Section 3.1.1) is C-exit, and below the line, it is I-withdraw.[22] In the C-exit area, the challenger is in a weaker position to finance the litigation costs compared to the incumbent, which presses the challenger to accelerate ex-post settlement if litigation were to start. In response, the incumbent strategically refuses to offer a settlement, knowing that the challenger has to exit soon after litigation starts. Likewise, in the I-withdraw area, the incumbent is in a weaker position to finance the litigation costs, which incentivizes the challenger to refuse a settlement offer, knowing that the incumbent has to withdraw soon after the litigation starts. This finding contributes to the literature by formalizing the idea that the asymmetry in strategic stakes drives non-settlement (Somaya, 2003), specifically from a finance angle.

Second, Figure 3 demonstrates that settlements (either ex-ante or ex-post) only happen if the gain-to-loss ratio Φ is high enough. The intuition for a high gain-to-loss ratio being a necessary condition of settlement comes from both firms' finance-related considerations. A high Φ indicates a relatively low profit loss for the incumbent due to the challenger's market entry, which makes the incumbent more willing to accept a low royalty rate in the settlement, knowing its best alternative of battling to the end and winning the lawsuit only brings moderate profit recovery. For the challenger, a high Φ indicates a more significant profit gain from its market entry, making it more willing to pay a high royalty rate in settlement to avoid the scenario of losing the profits completely in non-settlement. As a result of both firms' considerations, a higher Φ makes it more likely that the challenger's maximum acceptable royalty rate in settlement exceeds the incumbent's minimum required royalty rate, which leads to the mutual agreement on settlement. In a way, this finding formalizes the idea in the literature that non-settlement can result from the "infringing" firm's inability to adequately compensate the patentee because monopoly prices cannot be sustained with two firms in the market (Lanjouw and Lerner, 1998), and we generalize it to include the patent owning firm's consideration explicitly. Relatedly, Figure 3 also demonstrates that with the high litigation cost considered, settlement is possible even when the total market profits shrink upon the challenger's market entry, that is, $\Phi < 1$.

Third, Figure 3 shows that ex-ante settlement occurs in a region with a higher gain-to-loss ratio

[22] Consistent with Lemma 2 and with the gain-to-loss ratio being close to the relative litigation cost ($\Gamma \to \Lambda$), the boundary draws from (Φ, Γ) = (low, low) to (high, high). $\Gamma \to \Lambda$ can be true if, for example, the litigation cost is much higher than the settlement cost for both firms or the settlement cost is proportional to the litigation cost for both firms.

Φ, compared to ex-post settlement. This result on settlement timing can be explained by both firms' tendency to settle earlier (that is, settle ex-ante instead of ex-post) as $\Phi \uparrow$. From a Φ that induces ex-post settlement, a higher Φ means the challenger gains even more from the alleged infringement, which gives it a stronger incentive to keep operating in the market and resolving the patent dispute via an earlier settlement, as opposed to getting involved in costly yet uncertain litigation. Meanwhile, with a higher Φ, the incumbent loses less (relatively) from the alleged infringement, thus it has less to recover from a successful but costly lawsuit, making it more willing to reach an earlier settlement and avoid paying any litigation cost. Besides the dependence of settlement timing on Φ, Figure 3 also shows that firms are more likely to settle ex-post in the C-exit area, compared to the I-withdraw area at a fixed Φ, say $\Phi = 0.6$. The explanation comes from the incumbent's incentive to delay settlement in the C-exit region. The incumbent not only is in a financially advantageous position once the litigation starts, but it also has the additional option to force the challenger to exit the market with the threat of litigation. As a result, the incumbent optimally waits to litigate and then settle ex-post or waits to exercise its forcing-out option if the demand keeps declining, as opposed to reaching an ex-ante licensing agreement with the challenger. This explanation also answers why firms settle ex-post while they have the option to settle ex-ante.

4.2 The Impact of p, σ, and λ on patent dispute outcomes

Figure 4 shows how the incumbent's winning probability p, the market demand volatility σ, and the expected time to court ruling $\frac{1}{\lambda}$ further shape the outcomes of patent disputes.[23] The top panel demonstrates a shrinkage of the settlement region as p increases from 0.3 to 0.7, indicating that settlement is more difficult to reach as the probability of court ruling in favor of the incumbent increases (a more incumbent-friendly court). This is because as $p \uparrow$, the challenger's during-litigation value drops while the incumbent's during-litigation value increases ($V^C \downarrow$ and $V^I \uparrow$, see Section 3.1), which shifts the relative willingness to finance from the challenger to the incumbent. Consistent with the implication of Lemma 2, the increase in p moves the equal-willingness-boundary downwards from Figure 4 (a).i to (a).ii, replacing some of the I-withdraw area by the C-exit area, which directly reduces the ex-post settlement likelihood.

[Insert Figure 4 here.]

[23] In contrast to Lemley (2001) and Lemley and Shapiro (2005), which suggest that examining patents more thoroughly or reducing patent litigation uncertainty ($p \downarrow$) may not be cost-effective from the perspective of designing the patent system.

The negative impact of p on the settlement that we find is in contrast with the existing literature (Lemley, 2001; Lemley and Shapiro, 2005; Jeon, 2015). In particular, Jeon (2015) suggests a higher probability of patent validity incentives for both firms to settle because the incumbent can use litigation as a more credible threat to induce the challenger to accept an ex-ante settlement. Two model features contribute to our contrary finding. Firstly, we recognize the compound option feature of the I-litigate decision: the exercise of I-litigate option activates new options of ex-post settlement or non-settlement options of I-withdraw and C-exit. As a result, ceteris paribus, the value with the option to litigate is higher in our model, which makes the incumbent require a higher ex-ante settlement value to avoid litigation, making ex-ante settlement less likely. Secondly, we consider the option of C-exit during litigation which drives the negative impact of p on the likelihood of settlement. Our model helps reveal that the negative impact is stronger than the positive effect of p on the likelihood of settlement via the I-withdraw option during litigation, which the existing literature has focused on.

The middle panel of Figure 4 shows a negative impact of demand volatility σ on settlement likelihood. This negative impact is the aggregate of a small positive effect on the ex-post settlement likelihood and a larger negative effect on the ex-ante settlement likelihood. The firm value of ex-post settlement is more sensitive to the demand volatility change than the value of non-settling during litigation, thus a higher volatility increases the ex-post settlement value more than the non-settlement value, making ex-post settlement more likely. A more volatile demand condition has a second effect of inducing litigation and making ex-ante settlement less likely. This is because, by increasing the during-litigation firm value for the incumbent but not affecting its ex-ante settlement payoff, a higher σ makes the ex-ante settlement option less attractive. As a result, a higher σ raises the incumbent's required minimum ex-ante royalty rate and decreases the challenger's acceptable maximum ex-ante royalty rate, making the ex-ante settlement more difficult to achieve. Indeed, empirical studies, such as Lowry and Shu (2002), find that higher litigation risks are associated with more volatile cash flows. The bottom panel of Figure 4 confirms the intuition that a shorter span of litigation (equivalent to a larger λ, Figure 4 (c).i to (c).ii) induces litigation in the C-exit region but induces settlement in the I-withdraw region.

5 Model Extension, Discussion, and Implications

5.1 The English Rule vs. American Rule

Following the existing literature (Bebchuk, 1996; Aoki and Hu, 1999b), we extend the model to study the implications of cost allocation rules in legal systems on firms' strategies related to patent disputes. Under the *English rule* (also called the "loser pays rule"), the litigant who loses in court has to pay both parties' legal costs. In contrast, the *American rule* states that each party is generally responsible for its own legal cost, as in our baseline model.

We argue that, compared to the American rule, the English rule weakens the incumbent's position in its strategic interaction with the challenger, and shifts the outcomes of patent dispute at least for two reasons. Firstly, the incumbent faces a much higher direct cost of losing the lawsuit under the English rule than under the American rule. This is driven by the asymmetry of financial resources between the incumbent and the challenger. The incumbent can use the duopoly profits to pay (at least partially) the challenger's litigation cost if it loses the lawsuit under the English rule, but the challenger is financially incapable of paying the incumbent's litigation cost if it loses.[24] Secondly, it is only under the English rule that the incumbent may find itself being forced to liquidate, and as a result, the challenger assumes the role of a new monopoly (and its profits become $\pi_1^C x_t$). This happens if the incumbent loses the lawsuit but is lack of sufficient funds to cover the challenger's litigation costs.[25] Relatedly, we separate the analysis into Case A: *the incumbent remains a going-concern* vs. Case B: *the incumbent may liquidate*. Both cases are relevant for the English rule, but only Case A is relevant for the American rule.

Define $\bar{x} = \frac{C_l^C(r-\mu)}{\pi_2^I}$ as the liquidation threshold for the incumbent under the English rule. It is the level of the market demand such that the incumbent's firm value, if it loses the lawsuit then, equals the challenger's expected litigation cost ($\frac{\pi_2^I \bar{x}}{r-\mu} = C_l^C$). Case A is equivalent to $x_d \geq \bar{x}$, where $d = \{w, e, p\}$ represents *I-withdraw*, *C-exit*, and *ex-post settlement*, whichever is the likely outcome during litigation. This is because the court rules either before firms take actions ($\tau \leq t_d$) or after ($\tau > t_d$). The condition $x_d \geq \bar{x}$ in the former scenario, provided that demand is high at litigation due to the call option nature of the I-litigate option, implies $x_\tau > x_d \geq \bar{x}$, which means the incumbent's value at the time of court ruling

[24] Our assumption of no other revenues for the challenger, besides the profits from suspected infringement, directly leads to such difference. However, what matters is the asymmetric impact of the English rule on the two firms.

[25] The zero-scrap-value assumption of the incumbent is reasonable in this scenario (i.e., the incumbent loses the lawsuit), because the incumbent's patent validity becomes questionable and it is unclear whether the incumbent can sell the patent in any market.

is sufficient to cover its liability of the challenger's litigation cost. The condition $x_d \geq \bar{x}$ in the latter scenario implies that the lawsuit ends before the court rule and the liquidation is irrelevant. Together, the condition $x_d \geq \bar{x}$ guarantees we are in Case A. The condition of Case B is $x_d < \bar{x}$.[26]

Case A - *The incumbent remains a going-concern* The analyses for this case, which apply to both the American rule and the English rule, resemble Section 3 of the baseline model, with the expected total litigation cost modified as follows. We replace the present value of the total litigation costs (C_l^I, C_l^C) in Section 3 by their general forms: $\bar{C}_l^I = C_l^I + \frac{\mathbb{1}_{\mathrm{E}} \cdot c_l^C (1-p)\lambda}{r+\lambda}$ and $\bar{C}_l^C = C_l^C - \frac{\mathbb{1}_{\mathrm{E}} \cdot c_l^C (1-p)\lambda}{r+\lambda}$, where $\mathbb{1}_{\mathrm{E}}$ is the indicator for the English rule. Relatedly, we define the generalized cost saving as $\Delta\bar{C}^i = \bar{C}_l^i - C_s^i$ and the generalized relative-cost-saving as $\bar{\Gamma} \equiv \frac{\Delta\bar{C}^C}{\Delta\bar{C}^I}$, which we can show is lower under the English rule compared to the American rule.[27] We prove the following result in Appendix A.16.1.

Theorem 4. *Ceteris paribus, the ex-post settlement royalty rate is lower in the English rule than in the American rule.*

Theorem 4 demonstrates the importance of financing considerations in understanding the impact of legal systems. Perhaps surprisingly, the challenger's weaker financing position, relative to the incumbent, gives it a competitive edge under the English rule. The consequence of losing the lawsuit is different for the two firms in the following sense: while the incumbent pays the challenger's litigation cost upon losing the lawsuit, the challenger does not have the funds to pay the incumbent's litigation cost. This asymmetry shifts the willingness to finance litigation towards the challenger. As a result, the English rule weakens the incumbent's effective bargaining power during litigation and lowers the royalty rate in ex-post settlement. The English rule also changes the likely non-settlement outcome during litigation from C-exit to I-withdraw for a range of parameter values.

Case B - *The incumbent may liquidate* Given the complexity of this case, we leave the model derivation details in the appendix and summarize the firm values in Appendices A.16.2 (during litigation) and A.16.3 (before litigation).

[26] The court can rule (1) before firms take actions whilst at a demand lower than the liquidation cutoff ($\tau < t_d$ and $x_\tau < \bar{x}$), or (2) before firms take actions whilst at a demand higher than the liquidation cutoff ($\tau < t_d$ and $x_\tau \geq \bar{x}$), or (3) after firms' actions ($\tau > t_d$). With $x_d < \bar{x}$, the incumbent liquidates in Scenario (1) if the court rules against the incumbent. Note the incumbent still remains a going-concern in the other two scenarios during litigation under the condition.

[27] It is obvious from $\Delta\bar{C}^C(\mathbb{1}_{\mathrm{E}} = 1) < \Delta\bar{C}^C(\mathbb{1}_{\mathrm{E}} = 0)$, $\Delta\bar{C}^I(\mathbb{1}_{\mathrm{E}} = 1) > \Delta\bar{C}^I(\mathbb{1}_{\mathrm{E}} = 0)$, which gives us $\bar{\Gamma}(\mathbb{1}_{\mathrm{E}} = 1) < \bar{\Gamma}(\mathbb{1}_{\mathrm{E}} = 0)$.

5.2 Outcome comparisons: the English rule vs. American rule

Figure 5 shows the patent dispute outcomes on a range of the gain-to-loss ratio and the relative-cost-saving ($\Phi \times \Gamma$) under the English system, with varying p, σ, or λ in each row of the figure. Comparing Plot i.(b) of Figure 5 for the English rule at the benchmark and Figure 3 for the American rule, we confirm that the English rule enlarges the I-withdraw area and reduces the C-exit area. This is consistent with the shift of the willingness to finance litigation from the incumbent to the challenger (discussed after Theorem 4).

[Insert Figure 5 here.]

Figure 5 also confirms the general patterns of settlement (or non-settlement) that we find in the baseline are preserved under the English rule. For example, the closeness to the dashed line (which marks the firms' equal willingness to keep financing the litigation) leads to settlement; Settlement is only likely with a high enough gain-to-loss ratio Φ; market demand volatility reduces settlement likelihood and the expected litigation span increases settlement likelihood. However, Figure 5 is not informative of the settlement timing, simply because the C-exit area decreases dramatically under the English rule. C-exit, being the likely nonsettlement outcome, triggers the incumbent to litigate, as opposed to settle ex-ante with the challenger.

In stark contrast to the negative impact of the incumbent's winning probability p on the settlement likelihood at the baseline, Plot (i).(a)–(c) of Figure 5 and numerous unreported ones show its positive effect on settlement likelihood under the English rule. The striking difference is driven by the weakened effective bargaining power of the incumbent under the English rule, reflected in a larger I-withdraw area compared to the American rule on the outcome figures. Under both legal systems, the challenger's incentive to refuse settlement increases with p whilst the incumbent's incentive to not offer settlement reduces with p. In the American rule, the second effect of a higher p on non-settlement dominates because of the incumbent's higher willingness to finance litigation (or the larger area of C-exit compared to I-withdraw in the figure), leading to an overall increase in the likelihood of non-settlement. In the English rule, the first effect of a higher p on non-settlement dominates because of the challenger's higher willingness to finance litigation (that is, a larger area of I-withdraw compared to C-exit in the figure), leading to a reduced likelihood of non-settlement, or equivalently, a higher settlement likelihood.

5.3 Testable Implications

Our model generates a few testable implications. (1) The litigation rate of patent lawsuits[28] is higher with more volatile demand or shorter expected litigation span, and if the allegedly infringing products are close substitutes to the incumbent's, compared to complementary products. It increases with patent protection under the American rule but decreases with patent protection under the English rule. (2) The settlement rate[29] decreases with demand volatility and the asymmetry in firms' willingness to finance litigation but increases with the expected span of litigation. It decreases with patent protection in the American rule but increases with patent protection in the English rule. (3) Conditional on ex-post settlement, the royalty rate is higher for more stringent patent approval or stronger patent protection, with more volatile demand, and with higher relative-cost-saving or lower gain-to-loss ratio between the two firms. The settlement is delayed by higher demand volatilities and stronger patent protection, but accelerated by each firm's saving of the litigation cost via settlement (Corollary 4).

5.4 Model Discussion

We made several simplifying assumptions to focus on the basic financing considerations. Allowing richer descriptions of financing options and more realistic consideration of patent litigation can be fruitful. However, we argue the main insights from the model are not driven by such assumptions, and relaxing them may only change the exact outcome of a patent dispute for a given set of parameter values. For example, there is no damage ruling (modeled in previous work such as Bebchuk, 1984 and Lanjouw and Lerner, 1998), or the incumbent does not get any compensation for the lost profits of a convicted infringement. This assumption is justifiable in practice. Compared to the litigation cost, the damage award is insignificant in most of the patent lawsuits that end with judgment. One may argue that there is a selection bias in the sense that the lawsuits with potentially high damage have likely been settled. We suggest adding a damage parameter in the model essentially shifts the willingness to finance from the challenger to the incumbent, but it makes the model much less tractable. Previous work such as Crampes and Langinier (2002) suggests that settlement can be optimal for both firms even if the penalty paid by the infringer when it loses is high. We also assume firms' cash flows are based solely on selling products associated with the patented technology, and firms do not use debt financing or external litigation financing. If, for example, the challenger has other sources of cash flow to finance its litigation, then

[28] It corresponds to the size of the white and green areas in outcome figures such as Figures 3 – 5.
[29] It corresponds to the size of the blue and green areas in outcome figures such as Figures 3 – 5.

as long as such cash flow is independent of the profits in the model and the decision on such cash flow is based on value-maximization, we expect the firms' strategies are not affected. Thus, the outcomes of patent litigation in the baseline model remain.

6 Concluding Remarks

In this paper, we develop a dynamic model to analyze the strategic interactions of two firms in a patent dispute and examine the likely outcome from a financial perspective. By using the real options approach on a sequential game with time-varying market demand, we find that by influencing the two firms' relative willingness to keep financing a lawsuit (if it starts), the *gain-to-loss ratio* and the *relative-cost-saving* significantly affect firms' settlement likelihood, settlement terms, and settlement timing. In the model extension, we find that the English rule of loser pays shifts the effective bargaining power from the patent-owning firm to the allegedly infringing entrant. Thus, the rule leads to a lower royalty rate in ex-post settlement compared to the American rule of each party paying. We also find the opposite effects of the patent-owning firm's winning probability in a court ruling on the settlement likelihood under the two rules. Our model generates testable implications on the litigation rate, the settlement rate, and the royalty rate with respect to the product market and the legal environment characteristics.

A few directions for related future research can be promising. One regards firms' market entry decisions and their innovation incentives. Another potential regards the possible extension to study industry equilibrium with multiple entrants, and the examination of the welfare implications of patent litigation.

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Figure 1: Timeline of events

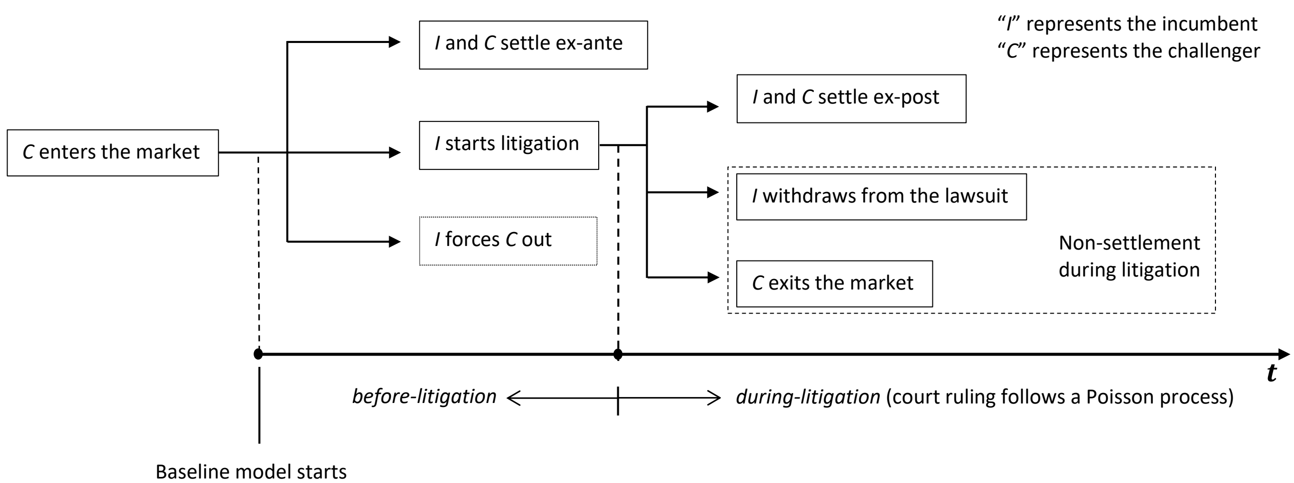


Figure 2: An example of a realized demand path and the strategies

This figure illustrates a scenario of the likely outcome being the incumbent litigates and the firms then settle ex-post. The parameter values are set at the benchmark in Table 1. The blue line is a realized demand path x_t that starts at $x_0 = 1$ and follows a stochastic process specified in Equation (1). The litigation threshold is $x_l = 1.76$ and the ex-post settlement threshold is $x_p = 0.72$. The red dot represents the time of litigation, which is when the demand reaches x_l from below for the first time. The green dot represents the ex-post settlement time, which is when the demand reaches x_p from above for the first time, assuming the court has not yet ruled then.

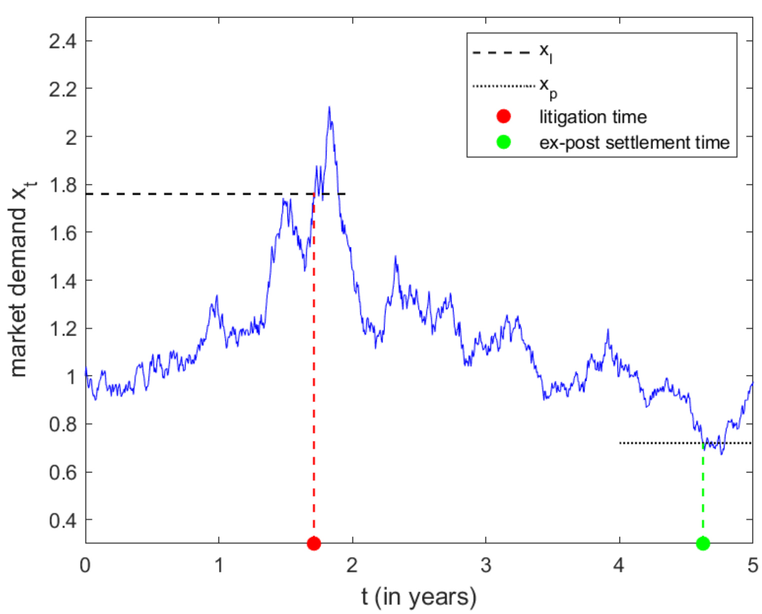


Figure 3: Likely outcomes in Baseline

The incumbent and the challenger's strategic interaction, illustrated in this graph, starts when the demand reaches the I-litigate threshold ($x \geq x_l$). In the white region, the incumbent litigates and the likely outcome during litigation is either C-exit as $x \leq x_e$ or I-withdraw as $x \leq x_w$. In the green region, the incumbent litigates and the likely during-litigation outcome is ex-post settlement as the demand drops $x \leq x_p$. In the blue region, the firms sign royalty agreement as $x \geq x_a = x_l$ without getting involved in litigation. The black dashed line marks the boundary of I-withdraw and C-exit as the likely non-settlement outcome if litigation happens. However, if $\inf\{t : x_t \leq x_e\} < \inf\{t : x_t \geq x_l\}$, then the incumbent exercises the forcing-out option in the C-exit area, but remain unchanged in the I-withdraw region until $x \geq x_l$ which makes this outcome graph relevant again. $\Phi = \frac{\Delta\pi^C}{|\Delta\pi^I|}$ and $\Gamma = \frac{\Delta C^C}{\Delta C^I}$. Benchmark parameter values are given in table 1.

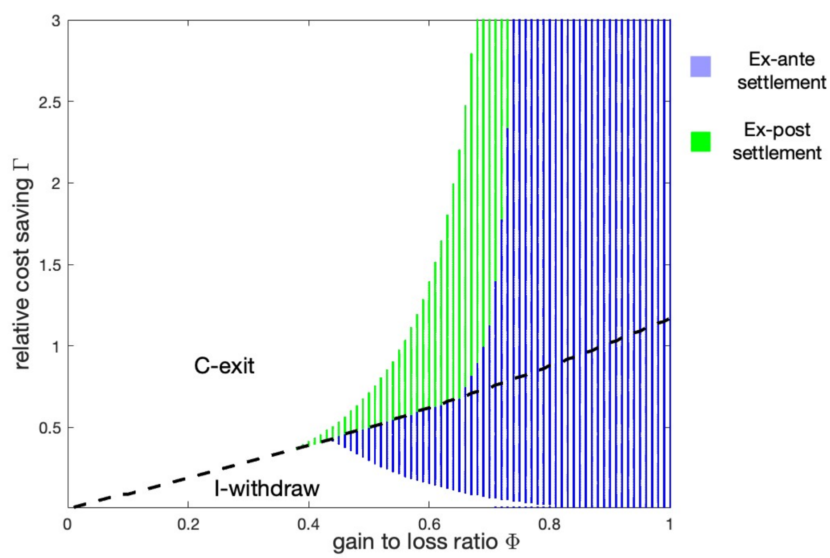
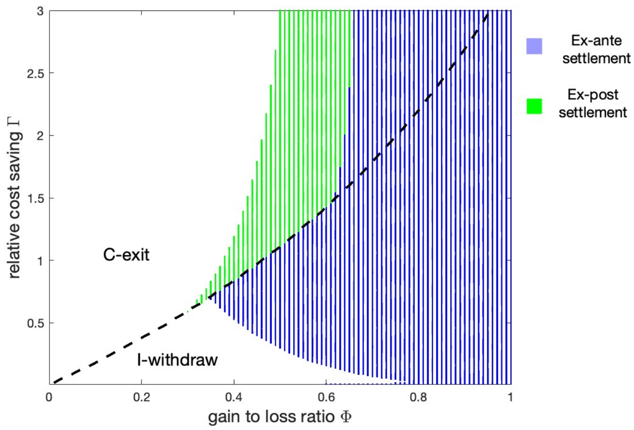
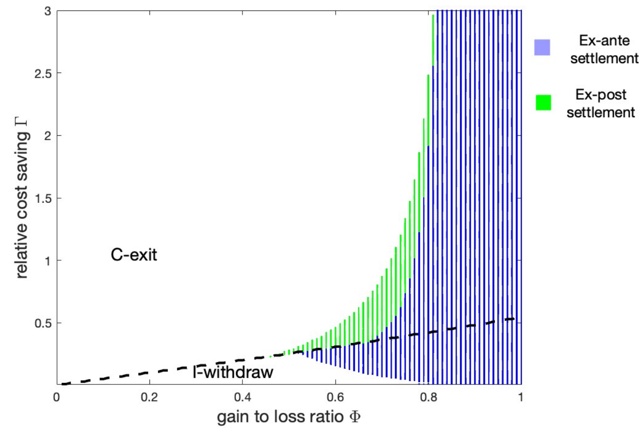


Figure 4: Likely outcomes in baseline with changes to p or σ

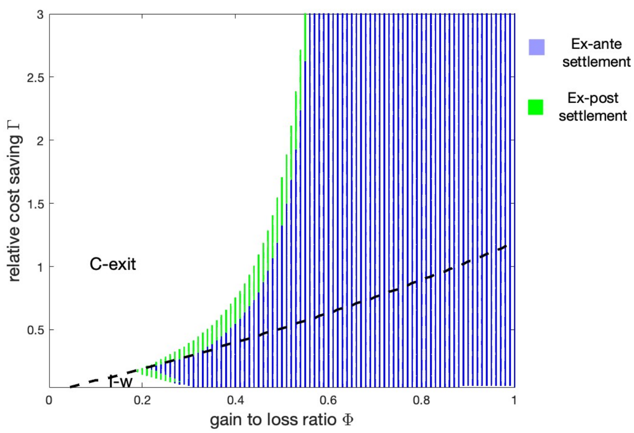
This graph illustrates the likely litigation outcomes at the benchmark parameter values as in Table 1, with changes in π_1 and c_l^I. The green area is the feasible region for ex-post settlement. In the blue area, firms settle ex-ante, that is, they sign royalty agreement without getting involved in litigation. The black dashed line marks the boundary of C-exit vs. I-withdraw as the likely non-settlement outcome during litigation.



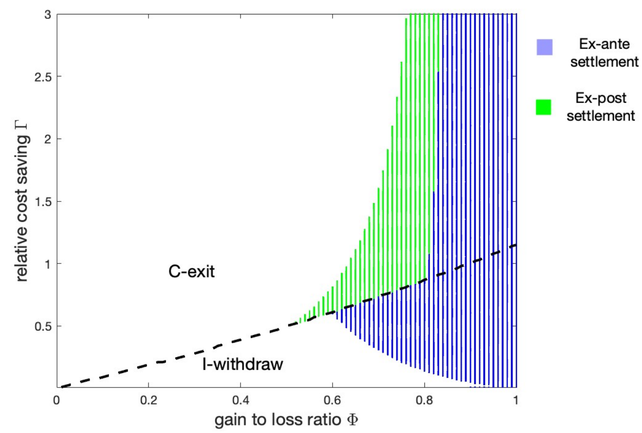
(a).i. low p ($p = 0.3$)



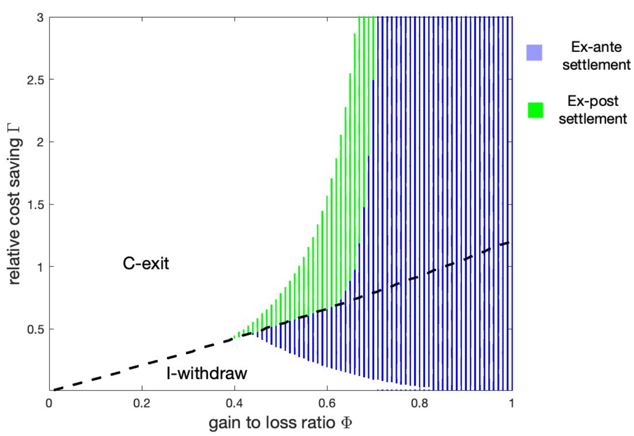
(a).ii. high p ($p = 0.7$)



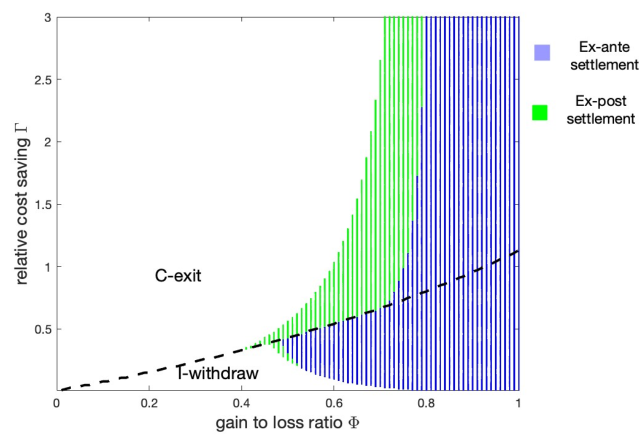
(b).i. low σ ($\sigma = 0.1$)



(b).ii. high σ ($\sigma = 0.5$)



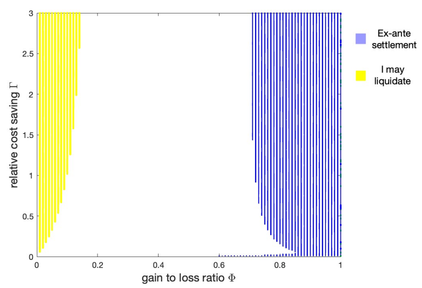
(c).i. low λ ($\lambda = \frac{1}{3.5}$)



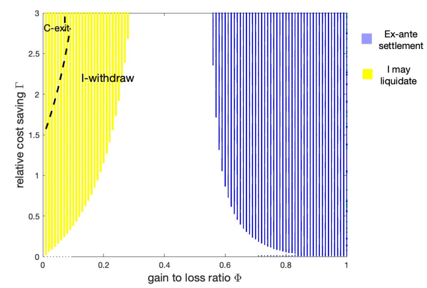
(c).ii. high λ ($\lambda = \frac{1}{1.5}$)

Figure 5: Likely outcomes under the English rule

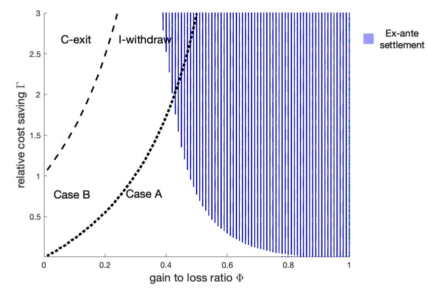
This graph illustrates the likely litigation outcomes under the English rule. The black dashed line marks the boundary of C-exit vs. I-withdraw as the likely non-settlement outcome during litigation. In the blue area, firms sign royalty agreements without getting involved in litigation. The yellow area indicates Case B (the incumbent may liquidate) while the non-yellow area indicates Case A (the incumbent remains a going-concern), with the exceptions of Plot i.(c) and Plot ii.(a), in which the dotted line separates Case B and Case A. Parameter values are set at the benchmark as shown in Table 1.



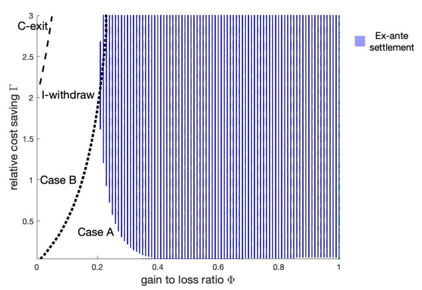
i.(a) low p ($p = 0.3$)



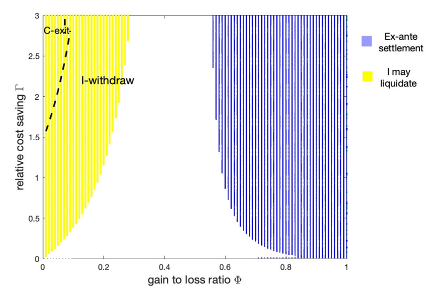
i.(b) baseline p ($p = 0.5$)



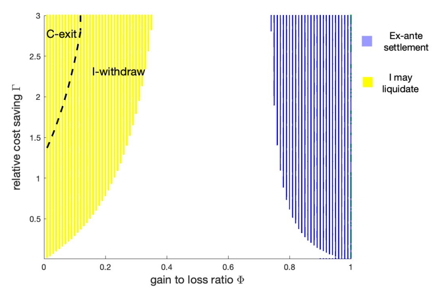
i.(c) high p ($p = 0.7$)



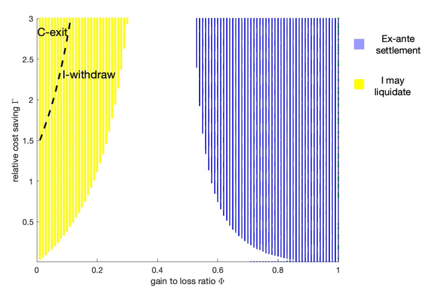
ii.(a) low σ ($\sigma = 0.1$)



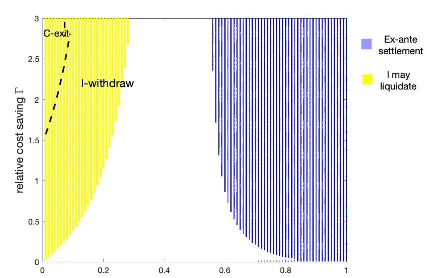
ii.(b) baseline σ ($\sigma = 0.3$)



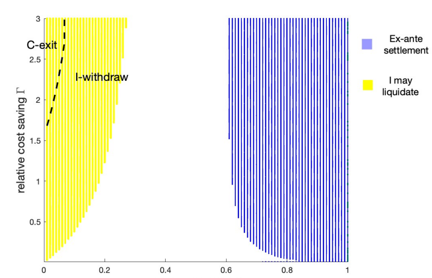
ii.(c) high σ ($\sigma = 0.5$)



iii.(a) low λ ($\lambda = \frac{1}{3}$)



iii.(b) baseline λ ($\lambda = \frac{1}{2.5}$)



iii.(c) high λ ($\lambda = \frac{1}{2}$)

Table 1: Benchmark parameter values in the baseline model

Parameters	Symbol and value
Basics	
Risk-free rate	$r = 0.05$
Arrival rate of court ruling	$\lambda = \frac{1}{2.5}$
Arrival rate of R&D success	$\epsilon = \frac{1}{5}$
Probability of patent validity	$p = 0.5$
Growth rate/volatility of the demand shock	$\mu = 0.02, \sigma = 0.3$
profit multipliers (profit = πx)	$\pi_1 = 1.2, \pi_2^I = 0.7, \pi_2^C = 0.3$
Flow litigation cost	$c_l^i = 1$
One-time settlement costs	$C_s^i = 0.5$
Ratios	
gain-to-loss ratio	$\Phi = \frac{\Delta\pi^C}{\lvert\Delta\pi^I\rvert} = \frac{\pi_2^C}{\pi_1 - \pi_2^I} = 0.6$
relative-cost-saving	$\Gamma = \frac{\Delta C^C}{\Delta C^I} = \frac{\frac{c_l^C}{r+\lambda} - C_s^C}{\frac{c_l^I}{r+\lambda} - C_s^I} = 1$
Other Greeks	$\delta = \frac{1}{r-\mu} - \frac{1}{r+\lambda-\mu} = 31.01$
	$\omega = \delta(r-\mu) = 0.93$
Strategy thresholds	$x_l/x_a = 1.76, \quad x_p = 0.72$
	$x_e = 0.31, \quad x_w = 0.21$
	$x_C = 0.17, \quad x_I = 0.12$

Table 2: Notations for model solutions

Notation	Interpretation (I - incumbent; C - challenger)
Thresholds	
x_l / x_a	I-litigate / ex-ante settlement threshold
x_e / x_w/x_p	C-exit / I-withdraw / ex-post settlement threshold
Royalty rates	
θ_a / θ_p	ex-ante / ex-post settlement royalty rate
Value functions	
V_0	before-litigation firm value that includes option values
V_l / V_a	with I-litigate / ex-ante settlement option
V	during-litigation firm value which includes option values
V_p / V_{ns}	with /without the ex-post settlement option
V_e / V_w	with C-exit / I-withdraw option
$\hat{V}$	payoff after option exercising

Appendices

Appendix A Proofs

A.1 Proof of Proposition 1

Proof. The general solution of the ODE Equation (7) is as follows:

$$V^i(x) = A^i x^{\alpha_\lambda} + B^i x^{\beta_\lambda} + V_p^i,$$

where

$$\beta_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0, \quad \alpha_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1$$

are the solutions to $\frac{1}{2}x(x-1)\sigma^2 + \mu x - (r+\lambda) = 0$. As mentioned in the main text, a boundary condition during litigation is

$$\lim_{x\to\infty} V^i(x) = \Big(\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i\Big)x - C_l^i.$$

Thus $A^i = 0$ and $V^i(x) = B^i x^{\beta_\lambda} + V_p^i$. Using a linear guess of $V_p^i(x)$ for Equation (7), one particular solution is $V_p^i = \Big(\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i\Big)x - C_l^i$, from which we obtain Equation (8) in the proposition. □

A.2 Proof of Corollaries 1 and 2

Proof. **I-withdraw** The incumbent maximizes its firm value by choosing x_w. Thus, we apply the value-matching conditions (VM) on both firms' value functions during litigation with respect to their payoff upon I-withdraw, as well as the smooth-pasting condition (SP) on the incumbent's value function at the threshold x_w. That is, for $i \in \{I, C\}$:

$$\text{VM:} \quad V_w^i(x_w) = \hat{V}_w^i(x_w) \quad \Rightarrow \Big(\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i\Big)x_w - C_l^i + B_w^i x_w^{\beta_\lambda} = \frac{\pi_2^i}{r-\mu}x_w, \quad i \in \{I, C\}.$$

$$\text{SP:} \quad \frac{\partial V_w^I(x)}{\partial x}\bigg|_{x_w} = \frac{\partial \hat{V}_w^I(x)}{\partial x}\bigg|_{x_w} \quad \Rightarrow \frac{\pi_2^I}{r-\mu} - p\delta\Delta\pi^I + \beta_\lambda B_w^I x_w^{\beta_\lambda - 1} = \frac{\pi_2^I}{r-\mu}.$$

From the above three equations, we can solve the three unknowns B_w^I, B_w^C and x_w, as expressed in Corollary 1.

C-exit Similarly, the challenger maximizes its firm value during litigation by choosing x_e. Thus, we apply the value-matching conditions on both firms' value functions with the exit option during litigation with respect to the C-exit payoff, and the smooth-pasting condition on the challenger's value function at x_e:

$$\text{VM:} \quad V_e^i(x_e) = \hat{V}_e^i(x_e) \quad \Rightarrow \Big(\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i\Big)x_e - C_l^i + B_e^i x_e^{\beta_\lambda} = \frac{\pi_2^i - \Delta\pi^i}{r-\mu}x_e, \quad i \in \{I, C\}.$$

$$\text{SP:} \quad \frac{\partial V_e^C(x)}{\partial x}\bigg|_{x_e} = \frac{\partial \hat{V}_e^C(x)}{\partial x}\bigg|_{x_e} \quad \Rightarrow \frac{\pi_2^C}{r-\mu} - p\delta\Delta\pi^C + \beta_\lambda B_e^C x_e^{\beta_\lambda - 1} = \frac{\pi_2^C}{r-\mu}.$$

From the above three equations, we can solve the three unknowns B_e^I, B_e^C and x_e, as expressed in Corollary 2. □

A.3 Proofs of Lemmas 1 and 2

Proof. We calculate the "reservation threshold" as in Lambrecht (2001) for each firm, i.e., the demand level for which a firm is "indifferent between leaving first at their optimal exit/withdrawal threshold and waiting until the rival leaves". For the incumbent, the condition of its reservation threshold x_I, with $\omega = p\delta(r-\mu) < 1$ defined, can be written and rearranged as

$$V_w^I(x = x_I; x_w) = \hat{V}_e^I(x = x_I) \quad \Rightarrow \quad \Big(\frac{\pi_2^I}{r-\mu} - p\delta\Delta\pi^I\Big)x_I - C_l^I + (C_l^I + p\delta\Delta\pi^I x_w)\Big(\frac{x_I}{x_w}\Big)^{\beta_\lambda} = \frac{\pi_1 x_I}{r-\mu}$$

$$\Rightarrow \quad \Delta\pi^I\big(\frac{1}{r-\mu} - p\delta)x_I + \frac{C_l^I}{1-\beta_\lambda}\Big(\frac{x_I}{x_w}\Big)^{\beta_\lambda} - C_l^I = 0$$

Divide by C_l^I, replace x_I by $x_I \cdot \dfrac{x_w}{x_w}$ $\quad \Rightarrow \quad z_1 = \dfrac{x_I}{x_w}$ satisfies the following condition, as shown in the lemma:

$$y_1(z_1) = 1 - \frac{1}{1-\beta_\lambda}z_1^{\beta_\lambda} + \frac{\beta_\lambda}{\beta_\lambda - 1}\frac{(1-p\omega)}{p\omega}z_1 = 0. \tag{A.1}$$

Note that $\beta_\lambda < 0$, $p\omega \in (0, p)$ and $\frac{(1-p\omega)}{p\omega} > 0$, thus the first-order derivative of $y_1(z_1)$ with respect to z_1 is negative, and the solution of $y_1(z_1) = 0$ is unique, which means there is one and only one solution of x_I. For the challenger, the condition of its reservation threshold x_C can be written and rearranged as

$$V_e^C(x = x_C; x_e) = \hat{V}_w^C(x = x_C) \quad \Rightarrow \quad \Big(\frac{\pi_2^C}{r-\mu} - p\delta\Delta\pi^C\Big)x_C - C_l^C + \Big(C_l^C + \Delta\pi^C x_e(p\delta - \frac{1}{r-\mu})\Big)\Big(\frac{x_C}{x_e}\Big)^{\beta_\lambda} = \frac{\pi_2 x_C}{r-\mu}$$

$$\Rightarrow \quad -\Delta\pi^C p\delta x_C + \frac{C_l^C}{1-\beta_\lambda}\Big(\frac{x_C}{x_e}\Big)^{\beta_\lambda} - C_l^C = 0$$

Divide by C_l^C, replace x_C by $x_C \cdot \dfrac{x_e}{x_e}$ $\quad \Rightarrow \quad z_2 = \dfrac{x_C}{x_e}$ is the unique solution of the following condition:

$$y_2(z_2) = 1 - \frac{1}{1-\beta_\lambda}z_2^{\beta_\lambda} + \frac{\beta_\lambda}{\beta_\lambda - 1}\frac{p\omega}{(1-p\omega)}z_2 = 0. \tag{A.2}$$

Equations (A.1) and (A.2) are equilvalent to the conditions in Lemma 1. They can be written more generally, that is, z_1 and z_2 are the solutions to $y(z; M) = 0$ where

$$y = 1 - \frac{1}{1-\beta_\lambda}z^{\beta_\lambda} + \frac{\beta_\lambda}{\beta_\lambda - 1}M \cdot z, \text{ with } M = \begin{cases} \dfrac{1-p\omega}{p\omega}, & \text{for } z_1 \\ \dfrac{p\omega}{1-p\omega}, & \text{for } z_2 \end{cases} \tag{A.3}$$

Because $\beta_\lambda < 0$, $\dfrac{\partial y}{\partial z} = \frac{\beta_\lambda}{\beta_\lambda - 1}z^{\beta_\lambda - 1} + \frac{\beta_\lambda}{\beta_\lambda - 1}M > 0$, $\lim_{z\to 0} y(z) = -\infty$, and $\lim_{z\to 1} y(z) > 0$, a unique solution of $z < 1$ is guaranteed. Thus $\frac{x_I}{x_w} < 1$ and $\frac{x_C}{x_e} < 1$, which implies $x_I < x_w$ and $x_C < x_e$. Apply the implicit function theorem on Equation (A.3), we get

$$\frac{\partial z}{\partial M} = -\frac{\dfrac{\partial y}{\partial M}}{\dfrac{\partial y}{\partial z}} = -\frac{z}{z^{\beta_\lambda - 1} + M} < 0.$$

Next, we compare z_1 and z_2 in order to compare x_C vs. x_I. There are three possibilities:

1. If $p\omega > 1/2$, then $M(z_1) < 1 < M(z_2)$. This means $\frac{x_I}{x_w} > \frac{x_C}{x_e} \Rightarrow \frac{x_I}{x_C} > \frac{x_w}{x_e} = \frac{1-p\omega}{p\omega}\frac{\Phi}{\Lambda}$. With an added condition of $\frac{1-p\omega}{p\omega}\frac{\Phi}{\Lambda} > 1$, we can have $x_I > x_C$, and the likely non-settlement outcome during litigation is I-withdraw.

2. If $p\omega < 1/2$, then $M(z_1) > 1 > M(z_2)$. This means $\frac{x_I}{x_w} < \frac{x_C}{x_e} \Rightarrow \frac{x_I}{x_C} < \frac{x_w}{x_e}$. With an added condition of $\frac{1-p\omega}{p\omega}\frac{\Phi}{\Lambda} < 1$, we can have $x_I < x_C$, and the likely non-settlement outcome during litigation is C-exit.
3. If $p\omega = 1/2$, then $M(z_1) = M(z_2) = 1$. This means $\frac{x_I}{x_w} = \frac{x_C}{x_e} \Rightarrow \frac{x_I}{x_C} = \frac{x_w}{x_e}$.
   (a) If $\frac{\Phi}{\Lambda} < 1$, then $x_I < x_C$, and the likely non-settlement outcome during litigation is C-exit.
   (b) If $\frac{\Phi}{\Lambda} > 1$, then $x_I > x_C$, and the likely non-settlement outcome during litigation is I-withdraw.
   (c) If $\frac{\Phi}{\Lambda} = 1$, then $x_I = x_C$.

□

A.4 Proof of Corollary 3

Proof. Given any proposed royalty rate θ_p, the challenger chooses the settlement threshold x_p to maximize its value with the settlement option. Thus, we use the value-matching conditions on both firms' values and the smooth-pasting condition on the challenger's firm value at x_p. That is, for $i \in \{I, C\}$:

$$\text{VM:} \quad V_p^i(x_p) = \hat{V}_p^i(x_p) - C_s^i \quad \Rightarrow \left(\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i\right)x_p - C_l^i + B_p^i x_p^{\beta_\lambda} = \frac{\pi_2^i + \Delta\pi_p^i}{r-\mu}x_p - C_s^C. \quad \text{(A.4)}$$

$$\text{SP:} \quad \left.\frac{\partial V_p^C(x)}{\partial x}\right|_{x_p} = \left.\frac{\partial(\hat{V}_p^C(x) - C_s^C)}{\partial x}\right|_{x_p} \quad \Rightarrow \frac{\pi_2^C}{r-\mu} - p\delta\Delta\pi^C + \beta_\lambda B_p^I x_p^{\beta_\lambda - 1} = \frac{\pi_2^C + \Delta\pi_p^C}{r-\mu}. \quad \text{(A.5)}$$

We can then solve x_p and the arbitrary constants B_p^i as shown in the corollary. □

A.5 Proof of Theorem 1

Proof. From Corollary 3, the first-order derivative $\frac{dx_p}{d\theta_p^*} = \frac{-x_p}{\theta_p^* - p\delta(r-\mu)}$. The first-order condition for the incumbent's optimal royalty rate is

$$\frac{dV_p^I}{d\theta_p^*} = \frac{dB_p^I}{d\theta_p^*} = \frac{\partial B_p^I}{\partial\theta_p^*} + \frac{\partial B_p^I}{\partial x_p}\frac{dx_p}{d\theta_p^*} = 0.$$

That is, $\beta_\lambda(\Delta C^I + \frac{\pi_2^C\theta_p}{r-\mu}x_p) - p\delta x_p(\Delta\pi^C + \Delta\pi^I(1-\beta_\lambda)) = 0$. For the simple case of $x_l > x_p$ (the litigation threshold is higher than the ex-post settlement threshold), the challenger optimally accepts the ex-post settlement offer at x_p, and the royalty rate choosen by the incumbent is the one that maximizes V_p^I. Thus,

$$\theta_p^* = \frac{p\delta(r-\mu)[\Delta C^I(\beta_\lambda - 1) + \Delta C^C(1 + \frac{\beta_\lambda - 1}{\Phi})]}{\Delta C^I(\beta_\lambda - 1) + \beta_\lambda\Delta C^C}.$$

Simplification leads to the expression in the theorem. We can plug the expression of $\theta_p = \theta_p^*$ in $x_p(\theta_p) = \frac{\beta_\lambda}{\beta_\lambda - 1}\frac{\Delta C^C}{(-p\delta + \frac{\theta_p}{r-\mu})\Delta\pi^C}$ (implied by Corollary 3), and get the expression of x_p^* in the theorem. □

A.6 Proof of Corollary 4

Proof. $\frac{\partial\beta_\lambda}{\partial\sigma^2} = \mu\sigma^{-4}\times\Big(1-(\frac{1}{2}-\frac{\mu}{\sigma^2}-\frac{r+\lambda}{\mu})\big((\frac{1}{2}-\frac{\mu}{\sigma^2})^2+\frac{2(r+\lambda)}{\sigma^2}\big)^{-1/2}\Big) > 0$. $\frac{\partial\beta_\lambda}{\partial\lambda} = -\frac{\big((\frac{1}{2}-\frac{\mu}{\sigma^2})^2+\frac{2(r+\lambda)}{\sigma^2}\big)^{-1/2}}{\sigma^2} < 0$. $\frac{\partial g(\Gamma)}{\partial\Gamma} = (\frac{\beta_\lambda}{\beta_\lambda}\Gamma+1)^{-2} > 0$, $\frac{\partial g(\Gamma)}{\partial\beta_\lambda} = (\beta_\lambda + \frac{\beta_\lambda - 1}{\Gamma})^{-2} > 0$. Using the expression of θ_p^* from Theorem 1 with $\Phi < 1$ and $g(\Gamma) > 0$:

$$\frac{\partial\theta_p^*}{\partial\Phi} = -\frac{p\omega g(\Gamma)}{\Phi^2} < 0, \quad \frac{\partial\theta_p^*}{\partial p} = \omega\Big(1 + g(\Gamma)(\frac{1}{\Phi} - 1)\Big) > 0,$$

$$\frac{\partial\theta_p^*}{\partial\Gamma} = \frac{\partial\theta_p^*}{\partial g(\Gamma)} \times \frac{\partial g(\Gamma)}{\partial\Gamma} = p\omega(\frac{1}{\Phi} - 1) \times (\frac{\beta_\lambda\Gamma}{\beta_\lambda - 1} + 1)^{-2} > 0,$$

$$\frac{\partial\theta_p^*}{\partial\sigma^2} = \frac{\partial\theta_p^*}{\partial g(\Gamma)} \times \frac{\partial g(\Gamma)}{\partial\beta_\lambda} \times \frac{\partial\beta_\lambda}{\partial\sigma^2} = p\omega(\frac{1}{\Phi} - 1)\frac{\partial g(\Gamma)}{\partial\beta_\lambda} \times \frac{\partial\beta_\lambda}{\partial\sigma^2} > 0.$$

We can write x_p^* in Theorem 1 as $x_p^* = \frac{\beta_\lambda}{\beta_\lambda-1}\cdot\frac{-\Delta C^C}{p\delta g(\Gamma)\Delta\pi} = \frac{\beta_\lambda}{\beta_\lambda-1}\cdot\frac{\Delta C^C}{-p\delta\Delta\pi}(\frac{\beta_\lambda}{\beta_\lambda-1}+\frac{1}{\Gamma}) = \frac{\beta_\lambda}{\beta_\lambda-1}\cdot\frac{1}{-p\delta\Delta\pi}\big(\frac{\beta_\lambda}{\beta_\lambda-1}\Delta C^C + \Delta C^I\big)$. Thus,

$$\frac{\partial x_p^*}{\partial\beta_\lambda} = -\frac{1}{p\delta|\Delta\pi|(\beta_\lambda - 1)^2}\big(\frac{2\beta_\lambda}{\beta_\lambda - 1}\Delta C^C + \Delta C^I\big) < 0 \Rightarrow \frac{\partial x_p^*}{\partial\sigma^2} = \frac{\partial x_p^*}{\partial\beta_\lambda} \times \frac{\partial\beta_\lambda}{\partial\sigma^2} < 0.$$

And $\frac{\partial x_p^*}{\partial\Delta\pi} > 0$, $\frac{\partial x_p^*}{\partial p} < 0$, $\frac{\partial x_p^*}{\partial\Delta C^i} > 0$. □

A.7 Proof of Theorem 2

Proof. From Equation (15), we use Corollary 3 for the expression of V_p^i, use Corollaries 1 and 2 and Expression (12) for the expressions of V_{ns}^i.

If s_{ns} =(I-withdraw):

$$\begin{aligned} V_p^I > V_w^I \quad &\Rightarrow \quad B_p^I > B_w^I \Rightarrow \big[\Delta C^I + \big(p\delta\Delta\pi^I + \frac{\theta_p\Delta\pi^C}{r-\mu}\big)x_p\big]x_p^{-\beta_\lambda} > \frac{C_l^I}{1-\beta_\lambda}x_w^{-\beta_\lambda} \\ &\Rightarrow \quad \frac{1-\beta_\lambda}{\Gamma} - \beta_\lambda\frac{\theta_p - p\omega/\Phi}{\theta_p - p\omega} > \big(\frac{\Delta C^C}{C_l^I}\big)^{\beta_\lambda - 1}\big(\frac{p\omega}{\Phi(\theta_p - p\omega)}\big)^{\beta_\lambda} \\ &\Rightarrow \quad \text{the implicit expression for } \theta_p^I \in [\theta_p^{Imin}, \theta_p^{Imax}] \end{aligned}$$

$$\begin{aligned} V_p^C > V_w^C \quad &\Rightarrow \quad B_p^C > B_w^C \Rightarrow \frac{\Delta C^C}{1-\beta_\lambda}x_p^{-\beta_\lambda} > (C_l^C + p\delta\Delta\pi^C x_w)x_w^{-\beta_\lambda} \\ &\Rightarrow \quad \frac{\Delta C^C}{(1-\beta_\lambda)C_l^C - \beta_\lambda\Phi C_l^I} > \Big(\frac{\Delta C^C p\omega}{\Phi(\theta - p\omega)}\Big)^{\beta_\lambda} \\ &\Rightarrow \quad \theta_p < \Big[\Big(\frac{(1-\beta_\lambda)\Lambda}{\Phi} - \beta_\lambda\Big)^{\frac{1}{\beta_\lambda}}\big(\frac{\Delta C^C}{C_l^I}\big)^{1-\frac{1}{\beta_\lambda}} + 1\Big]p\omega = \theta_p^{Cmax}. \end{aligned}$$

If s_{ns} =(C-exit):

$$\begin{aligned}
V_p^I > V_e^I \quad &\Rightarrow \quad B_p^I > B_e^I \Rightarrow \big[\Delta C^I + \big(p\delta\Delta\pi^I + \frac{\theta_p\Delta\pi^C}{r-\mu}\big)x_p\big]x_p^{-\beta_\lambda} > [C_l^I + (p\delta - \frac{1}{r-\mu})\Delta\pi^I x_e]x_e^{-\beta_\lambda} \\
&\Rightarrow \quad \frac{1-\beta_\lambda}{\Gamma} - \beta_\lambda\frac{\theta_p - p\omega/\Phi}{\theta_p - p\omega} > (\frac{\Delta C^C}{C_l^C})^{\beta_\lambda - 1}(\frac{1-p\omega}{\theta_p - p\omega})^{\beta_\lambda}(\frac{1-\beta_\lambda}{\Lambda} - \frac{\beta_\lambda}{\Phi}) \\
&\Rightarrow \quad \text{the implicit expression for } \theta_p^I \in [\theta_p^{Imin}, \theta_p^{Imax}]
\end{aligned}$$

$$\begin{aligned}
V_p^C > V_e^C \quad &\Rightarrow \quad B_p^C > B_e^C \Rightarrow \frac{\Delta C^C}{1-\beta_\lambda}x_p^{-\beta_\lambda} > \frac{C_l^C}{1-\beta_\lambda}x_e^{-\beta_\lambda} \Rightarrow \frac{x_e}{x_p} < \Big(\frac{C_l^C}{\Delta C^C}\Big)^{\frac{1}{\beta_\lambda}} \\
&\Rightarrow \quad \frac{\frac{C_l^C}{(\frac{1}{r-\mu} - p\delta)\Delta\pi^C}}{\frac{\Delta C^C}{-p\delta\Delta\pi^C - \frac{\Delta\pi_p^C}{r-\mu}}} < \Big(\frac{C_l^C}{\Delta C^C}\Big)^{\frac{1}{\beta_\lambda}} \Rightarrow \frac{C_l^C(-p\delta + \frac{\theta_p}{r-\mu})}{(\frac{1}{r-\mu} - p\delta)\Delta C^C} < \Big(\frac{C_l^C}{\Delta C^C}\Big)^{\frac{1}{\beta_\lambda}} \\
&\Rightarrow \quad \theta_p < \Big(\frac{C_l^C}{\Delta C^C}\Big)^{\frac{1}{\beta_\lambda} - 1}(1 - p\omega) + p\omega \equiv \theta_p^{Cmax}
\end{aligned}$$

We summarize $\theta_p^{Cmax}, \theta_p^{Imin}, \theta_p^{Imax}$ as follows:

$$\theta_p^{Cmax} = \begin{cases} \Big[\Big(\frac{(1-\beta_\lambda)\Lambda}{\Phi} - \beta_\lambda\Big)^{\frac{1}{\beta_\lambda}}\big(\frac{\Delta C^C}{C_l^I}\big)^{1-\frac{1}{\beta_\lambda}} + 1\Big]p\omega, & s_{ns} = \text{(I-withdraw)} \\ \Big(\frac{\Delta C^C}{C_l^C}\Big)^{1-\frac{1}{\beta_\lambda}}(1-p\omega) + p\omega, & s_{ns} = \text{(C-exit)}. \end{cases} \tag{A.6}$$

$\theta_p^I \in [\theta_p^{Imin}, \theta_p^{Imax}]$ satisfies

$$f(\theta_p) = A(\theta_p - p\omega)^{-\beta_\lambda} - \frac{1-\beta_\lambda}{\Gamma} + \beta_\lambda\frac{\theta_p - p\omega/\Phi}{\theta_p - p\omega} \leq 0, \tag{A.7}$$

where

$$A = \begin{cases} \big(\frac{\Delta C^C}{C_l^I}\big)^{\beta_\lambda - 1}\big(\frac{p\omega}{\Phi}\big)^{\beta_\lambda}, & s_{ns} = \text{(I-withdraw)} \\ \big(\frac{\Delta C^C}{C_l^C}\big)^{\beta_\lambda - 1}\big(\frac{1-\beta_\lambda}{\Lambda} - \frac{\beta_\lambda}{\Phi}\big)(1-p\omega)^{\beta_\lambda}, & s_{ns} = \text{(C-exit)}. \end{cases}$$

The first-order derivative can be represented as

$$f'(\theta_p) = \frac{\beta_\lambda}{\theta_p - p\omega}\Big(1 - A(\theta_p - p\omega)^{-\beta_\lambda} - \frac{\theta_p - p\omega/\Phi}{\theta_p - p\omega}\Big). \tag{A.8}$$

□

A.8 Proof of Proposition 2

Proof. The general form of the solutions of the ODEs in Equation (17) is as follows, with α and β specified in Proposition 2:

$$V_0^i(x) = \frac{\pi_2^i}{r-\mu}x + A_1^i x^\alpha + A_2^i x^\beta.$$

In I-withdraw, firms take no action when the market demand is low provided that any action is equivalent to exercising a call option, so we can impose the boundary conditions at zero demand:

$$\lim_{x\to 0} V_0^i(x) = 0 \Rightarrow A_2^i = 0 \tag{A.9}$$

In C-exit, if the market demand decreases to exit threshold x_e before the incumbent starts litigation, then the incumbent forces the challenger to exit at time $\inf\{t : x_t \le x_e\}$. Thus, we use the value-matching conditions at x_e as follows

$$V_0^I(x_e) = \frac{\pi_1 x_e}{r-\mu}, V_0^C(x_e) = 0$$

from which $A_2^i \neq 0$. To separate from I-withdraw, we use a^i and b^i to represent the arbitrary constants in C-exit.

□

A.9 Proof of Corollary 5

Proof. **I-withdraw** We apply (1) the value matching and smooth-pasting conditions on the incumbent's firm value at x_l, and (2) the challenger's value matching condition at x_l, where $V_l^i(x)$ is specified in Equation (18) of Proposition 2 and $V^i(x)$ is specified in Equation (8) of Proposition 1:

$$\text{VM:} \qquad V_l^i(x_l) = V^i(x_l) \qquad \Rightarrow \frac{\pi_2^i x_l}{r-\mu} + A_l^i x_l^\alpha = \left(\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i\right)x_l - C_l^i + B^i x_l^{\beta_\lambda}, \quad i \in \{I, C\}.$$

$$\text{SP:} \qquad \left.\frac{\partial V_l^I(x)}{\partial x}\right|_{x_l} = \left.\frac{\partial V^I(x)}{\partial x}\right|_{x_l} \qquad \Rightarrow \frac{\pi_2^I}{r-\mu} + \alpha A_l^I x_l^{\alpha-1} = \frac{\pi_2^I}{r-\mu} - p\delta\Delta\pi^I + \beta_\lambda B^I x_l^{\beta_\lambda - 1} + \frac{\partial B^I}{\partial x_l} x_l^{\beta_\lambda}.$$

We can then solve the arbitrary constants A_l^I, A_l^C and the litigation threshold x_l, as expressed in the corollary.

C-exit Besides the two VM and one SP conditions as follows

$$\text{VM:} \qquad V_l^i(x_l) = V^i(x_l) \qquad \Rightarrow \frac{\pi_2^i x_l}{r-\mu} + a_l^i x_l^\alpha + b_l^i x_l^\beta = \left(\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i\right)x_l - C_l^i + B^i x_l^{\beta_\lambda}, \quad i \in \{I, C\}$$

$$\text{SP:} \qquad \left.\frac{\partial V_l^I(x)}{\partial x}\right|_{x_l} = \left.\frac{\partial V^I(x)}{\partial x}\right|_{x_l} \qquad \Rightarrow \frac{\pi_2^I}{r-\mu} + \alpha a_l^I x_l^{\alpha-1} + \beta b_l^I x_l^{\beta-1} = \frac{\pi_2^I}{r-\mu} - p\delta\Delta\pi^I + \beta_\lambda B^I x_l^{\beta_\lambda - 1} + \frac{\partial B^I}{\partial x_l} x_l^{\beta_\lambda},$$

there is one additional value-matching condition for each firm, due to the incumbent's forcing-out option:

$$\left.\begin{aligned} V_l^I(x_e) = \tfrac{\pi_1 x_e}{r-\mu} &\Rightarrow \tfrac{\pi_2^I x_e}{r-\mu} + a_l^I x_e^\alpha + b_l^I x_e^\beta = \tfrac{\pi_1 x_e}{r-\mu} \\ V_l^C(x_e) = 0 &\Rightarrow \tfrac{\pi_2^C x_e}{r-\mu} + a_l^C x_e^\alpha + b_l^C x_e^\beta = 0 \end{aligned}\right\} a_l^i x_e^\alpha + b_l^i x_e^\beta = \frac{-\Delta\pi^i x_e}{r-\mu} \tag{A.10}$$

With the five equations, the arbitrary constants can be solved as:

$$a_l^i = \frac{-p\delta\Delta\pi^i x_l^{1-\beta} + \frac{\Delta\pi^i}{r-\mu} x_e^{1-\beta} + B^i x_l^{\beta_\lambda - \beta} - C_l^i x_l^{-\beta}}{x_l^{\alpha-\beta} - x_e^{\alpha-\beta}}, \quad b_l^i = \frac{-p\delta\Delta\pi^i x_l^{1-\alpha} + \frac{\Delta\pi^i}{r-\mu} x_e^{1-\alpha} + B^i x_l^{\beta_\lambda - \alpha} - C_l^i x_l^{-\alpha}}{x_l^{\beta-\alpha} - x_e^{\beta-\alpha}}, \tag{A.11}$$

and the litigation threshold x_l satisfies

$$\big[(\alpha-1)(\frac{x_e}{x_l})^{\beta}-(\beta-1)(\frac{x_e}{x_l})^{\alpha}\big]p\delta\Delta\pi^I x_l+\big[(\beta_\lambda-\alpha)(\frac{x_e}{x_l})^{\beta}-(\beta_\lambda-\beta)(\frac{x_e}{x_l})^{\alpha}\big]B^I x_l^{\beta_\lambda}$$
$$=(\alpha-\beta)\frac{\Delta\pi^I}{r-\mu}x_e-\Big(\alpha(\frac{x_e}{x_l})^{\beta}-\beta(\frac{x_e}{x_l})^{\alpha}\Big)C_l^I-\big[(\frac{x_e}{x_l})^{\beta}-(\frac{x_e}{x_l})^{\alpha}x_l^{\alpha-\beta}\big]\frac{\partial B^I}{\partial x_l}x_l^{\beta_\lambda+1}. \tag{A.12}$$

□

A.10 Proof of Corollary 6

Proof. **I-withdraw:** We first list the value-matching conditions for both firms at x_a:

$$\text{VM:}\quad V_a^i(x_a)=\hat{V}_a^i(x_a)-C_s^i\Rightarrow\frac{\pi_2^i x_a}{r-\mu}+A_a^i x_a^{\alpha}=\frac{\pi_2^i+\Delta\pi_a^i}{r-\mu}x_a-C_s^i,\quad i\in\{I,C\},$$

from which we get A_a^i as in the corollary. In particular, $A_a^C(x_a,\theta_a)=-(\frac{\theta_a\pi_2^C}{r-\mu}x_a+C_s^C)x_a^{-\alpha}<0$. We know $x<x_a$ before firms possibly settle ex-ante, and $\frac{\partial}{\partial x_a}A_a^C(x_a|\theta_a)=(\alpha-1)\frac{\theta_a\pi_2^C}{r-\mu}x_a^{-\alpha}+\alpha C_s^C x_a^{-\alpha-1}>0$ indicates that the challenger prefers x_a to be as high as possible. In other words, the challenger waits as long as possible before settling ex-ante. However, ex-ante settlement has to happen, if ever, before I-litigate to avoid the costly litigation. Thus $x_a=x_l$.

C-exit:

$$\begin{aligned}\text{VM at }x_a:\quad V_a^i(x_a)&=\hat{V}_a^i(x_a)-C_s^i\Rightarrow\frac{\pi_2^i x_a}{r-\mu}+a_a^i x_a^{\alpha}+b_a^i x_a^{\beta}=\frac{\pi_2^i+\Delta\pi_a^i}{r-\mu}x_a-C_s^i,\quad i\in\{I,C\},\\ \text{VM at }x_a:\quad V_a^I(x_e)&=\frac{\pi_1 x_e}{r-\mu}\text{ and }V_a^C(x_e)=0\Rightarrow a_a^i x_e^{\alpha}+b_a^i x_e^{\beta}=\frac{-\Delta\pi^i x_e}{r-\mu}.\end{aligned}$$

□

A.11 Proof of Theorem 3

Proof. For ex-ante settlement to be a better outcome than I-litigate for both firms, they must have higher firm values with the ex-ante settlement option than with the I-litigation option for any relevant demand level. Thus, if s_{ns} =(I-withdraw):

$$\begin{aligned}V_a^I\geq V_l^I\quad&\Rightarrow\quad\frac{\pi_2^I x}{r-\mu}+A_a^I x^{\alpha}\geq\frac{\pi_2^I x}{r-\mu}+A_l^I x^{\alpha}\Rightarrow A_a^I\geq A_l^I\Rightarrow(\frac{\Delta\pi_a^I}{r-\mu}x_a-C_s^I)x_a^{-\alpha}\geq A_l^I\\ &\Rightarrow\quad(\frac{\theta_a\pi_2^C}{r-\mu}x_a-C_s^I)x_a^{-\alpha}\geq A_l^I\Rightarrow\theta_a\geq\frac{(A_l^I x_a^{\alpha}+C_s^I)(r-\mu)}{\pi_2^C x_a}=\theta_a^{Imin}\end{aligned}$$

$$\begin{aligned}V_a^C\geq V_l^C\quad&\Rightarrow\quad\frac{\pi_2^C x}{r-\mu}+A_a^C x^{\alpha}\geq\frac{\pi_2^C x}{r-\mu}+A_l^C x^{\alpha}\Rightarrow A_a^C\geq A_l^C\Rightarrow(\frac{\Delta\pi_a^C}{r-\mu}x_a-C_s^C)x_a^{-\alpha}\geq A_l^C\\ &\Rightarrow\quad(\frac{-\theta_a\pi_2^C}{r-\mu}x_a-C_s^C)x_a^{-\alpha}\geq A_l^C\Rightarrow\theta_a\leq-\frac{(A_l^C x_a^{\alpha}+C_s^C)(r-\mu)}{\pi_2^C x_a}=\theta_a^{Cmax}\end{aligned}$$

If s_{ns} =(C-exit):

$$V_a^I \geq V_l^I \quad \Rightarrow \quad \frac{\pi_2^I x_a}{r-\mu} + a_a^I x_a^\alpha + b_a^I x_a^\beta \geq \frac{\pi_2^I x_a}{r-\mu} + a_l^I x_a^\alpha + b_l^I x_a^\beta \Rightarrow \frac{\theta_a \pi_2^C}{r-\mu} x_a - C_s^I \geq a_l^I x_a^\alpha + b_l^I x_a^\beta$$
$$\Rightarrow \quad \theta_a \geq (a_l^I x_a^\alpha + b_l^I x_a^\beta + C_s^I)\frac{r-\mu}{\pi_2^C x_a} = \theta_a^{Imin}$$

$$V_a^C \geq V_l^C \quad \Rightarrow \quad \frac{\pi_2^C x_a}{r-\mu} + a_a^C x_a^\alpha + b_a^C x_a^\beta \geq \frac{\pi_2^C x_a}{r-\mu} + a_l^C x_a^\alpha + b_l^C x_a^\beta \Rightarrow \frac{-\theta_a \pi_2^C}{r-\mu} x_a - C_s^C \geq a_l^C x_a^\alpha + b_l^C x_a^\beta$$
$$\Rightarrow \quad \theta_a \leq (a_l^C x_a^\alpha + b_l^C x_a^\beta + C_s^C)\frac{r-\mu}{-\pi_2^C x_a} = \theta_a^{Cmax}$$

Together, the feasible range of ex-ante settlement can be written as

$$\theta_a \in [\theta_a^{Imin}, \theta_a^{Cmax}] \begin{cases} \left[\frac{(A_l^I x_a^\alpha + C_s^I)(r-\mu)}{\pi_2^C x_a}, -\frac{(A_l^C x_a^\alpha + C_s^C)(r-\mu)}{\pi_2^C x_a}\right], & s_{ns} = \text{(I-withdraw)} \\ \left[\frac{r-\mu}{\pi_2^C x_a}(a_l^I x_a^\alpha + b_l^I x_a^\beta + C_s^I), -\frac{r-\mu}{\pi_2^C x_a}(a_l^C x_a^\alpha + b_l^C x_a^\beta + C_s^C)\right], & s_{ns} = \text{(C-exit).} \end{cases} \tag{A.13}$$

Regarding θ_a, we have $\frac{\partial A_a^I}{\partial \theta_a} > 0$ in I-withdraw, and $\frac{\partial V_a^I}{\partial \theta_a} = \frac{\partial (a_a^I x_a^\alpha + b_a^I x_a^\beta)}{\partial \theta_a} = \frac{\partial a_a^I}{\partial \theta_a} x^\alpha + \frac{\partial b_a^I}{\partial \theta_a} x^\beta$ in C-exit. Because $\frac{\partial x_a}{\partial \theta_a} = 0$ as proved in Appendix A.10, we can show that $\frac{\partial V_a^I}{\partial \theta_a}\Big|_{x=x_a} = \frac{\pi_2^C}{r-\mu} x_a > 0$. Hence, $\theta_a^* = \theta_a^{Cmax}$ regardless of the s_{ns}.

When I-withdraw, we have

$$\begin{aligned}
\theta_a &= -\frac{(A_l^C x_a^\alpha + C_s^C)(r-\mu)}{\pi_2^C x_a} = -\frac{(-p\delta\pi_2^C x_l + B_p^C x_l^{\beta_\lambda} - \Delta C^C)(r-\mu)}{\pi_2^C x_l} \\
&= -\frac{r-\mu}{\pi_2^C x_l}(-p\delta\pi_2^C x_l + (\Delta C^C + (p\delta - \frac{\theta_p^*}{r-\mu}\pi_2^C x_p)(\frac{x_l}{x_p})^{\beta_\lambda} - \Delta C^C) \\
&= (p\delta(r-\mu) - (p\delta(r-\mu) - \theta_p^*)(\frac{x_l}{x_p})^{\beta_\lambda - 1} + \frac{r-\mu}{\pi_2^C x_l}\Delta C^C(1 - (\frac{x_l}{x_p})^{\beta_\lambda}) \\
&= p\delta(r-\mu)(1 - (\frac{x_l}{x_p})^{\beta_\lambda - 1}) + \frac{r-\mu}{\pi_2^C x_l}\Delta C^C(1 - (\frac{x_l}{x_p})^{\beta_\lambda}) + \theta_p^*(\frac{x_l}{x_p})^{\beta_\lambda - 1}
\end{aligned}$$

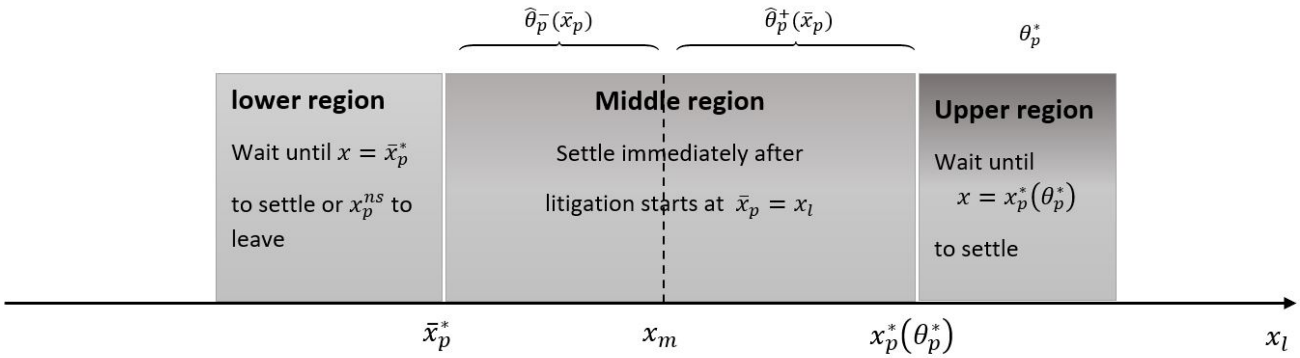
If $\theta_a < \theta_p$, then

$$\begin{aligned}
&p\delta(r-\mu)(1 - (\frac{x_l}{x_p})^{\beta_\lambda - 1}) + \frac{r-\mu}{\pi_2^C x_l}\Delta C^C(1 - (\frac{x_l}{x_p})^{\beta_\lambda}) + \theta_p^*(\frac{x_l}{x_p})^{\beta_\lambda - 1} < \theta_p^* \\
\Rightarrow &p\delta(r-\mu)(1 - (\frac{x_l}{x_p})^{\beta_\lambda - 1}) + \frac{r-\mu}{\pi_2^C x_l}\Delta C^C(1 - (\frac{x_l}{x_p})^{\beta_\lambda}) < \theta_p^*(1 - (\frac{x_l}{x_p})^{\beta_\lambda - 1}) \\
\Rightarrow &p\delta(r-\mu) - \theta_p^* + \frac{r-\mu}{\pi_2^C x_l}\Delta C^C \frac{1 - (\frac{x_l}{x_p})^{\beta_\lambda}}{1 - (\frac{x_l}{x_p})^{\beta_\lambda - 1}} < 0 \\
\Rightarrow &p\delta(r-\mu) - p\delta(r-\mu)\frac{\Delta C^I \pi_2^C(\beta_\lambda - 1) + \Delta C^C(\pi_2^C - (\beta_\lambda - 1)\Delta\pi^I}{\pi_2^C(\Delta C^I(\beta_\lambda - 1) + \beta_\lambda \Delta C^C)} + \frac{r-\mu}{\pi_2^C x_l}\Delta C^C \frac{1 - (\frac{x_l}{x_p})^{\beta_\lambda}}{1 - (\frac{x_l}{x_p})^{\beta_\lambda - 1}}
\end{aligned}$$

□

A.12 All scenarios of ex-post settlement

In the main text, we focus on the simple scenario in which the litigation threshold is higher than the ex-post settlement threshold ($x_l > x_p^*$, where x_p^* is expressed in Theorem 1). Thus, if ex-post settlement is the likely outcome during litigation, then it happens as the market demand x drops to or below x_p^* from a higher level. However, the litigation threshold can be low in principle, and such that $x_l < x_p^*$. We illustrate the full set of scenarios regarding the level of x_l that is relevant for the discussion of ex-post settlement, and we mark the corresponding strategies during litigation below.



Generally, the firm values with the ex-post settlement option can be expressed as follows:

$$V_p^i = \begin{cases} (\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i)x + B_p^i x^{\beta_\lambda} - C_l^i, & \text{if} \quad x \geq x_p^*(\theta_p^*) \\ \frac{\pi_2^i + \Delta\pi_p^i(\bar{\theta}_p)}{r-\mu} x - C_s^i, & \text{if} \quad x \in [\bar{x}_p^*, x_p^*(\theta_p^*)) \\ (\frac{\pi_2^i}{r-\mu} - p\delta\Delta\pi^i)x + \bar{A}_p^i x^{\alpha_\lambda} + \bar{B}_p^i x^{\beta_\lambda} - C_l^i. & \text{if} \quad x \in (x_p^{ns}, \bar{x}_p^*), \end{cases} \tag{A.14}$$

where α_λ and β_λ are defined in Proposition 1, B_p^I, B_p^C and $x_p^*(\theta_p^*)$ follow Corollary 3, and x_p^{ns} is a low threshold for the two firms to unilaterally stop the lawsuit.

The upper region ($x \geq x_p^*$) This is the simplest case and its solution is discussed in Section 3.1.2. Ex-post settlement in this region is termed as the "later ex-post settlement".

The middle region ($x \in [\bar{x}_p^*, x_p^*)$, immediate settlement) As $x_l < x_p^*(\theta_p^*)$, settlement happens immediately after the commence of litigation. We term the ex-post settlement in this region as the "immediate settlement". Use the bar-notation for immediate settlement. For example, $\bar{x}_p$ denotes the settlement threshold, and it equals the litigation threshold, i.e., $\bar{x}_p = x_l$. The determination of x_l is detailed in Section A.13. Denote the royalty rate in this region as $\bar{\theta}_p(\bar{x}_p)$. There are two sub-scenarios.

1. In the top part of the middle region, we obtain $\bar{\theta}_p(\bar{x}_p)$ by finding the royalty rate that maximizes the incumbent's firm value at $\bar{x}_p = x_l$. This is equivalent to inverting the expression of $x_p^*(\theta_p)$ in Corollary 3, and writing the royalty rate, denoted as $\bar{\theta}_p^+$, as a function of any given $\bar{x}_p$

$$\bar{\theta}_p^+(\bar{x}_p) = p\omega + \frac{\beta_\lambda \Delta C^C (r-\mu)}{(\beta_\lambda - 1)\pi_2^C \bar{x}_p}. \tag{A.15}$$

Because the incumbent's firm value from immediate settlement $\hat{V}_p^I(\bar{x}_p;\theta_p)$ increases linearly with the royalty rate, $\bar{\theta}_p^+(\bar{x}_p)$ is the maximum royalty rate for which x_l is in the challenger's immediate acceptance region.

2. In the bottom part of the middle region, the incumbent chooses the highest royalty rate that the challenger accepts, we use the smooth-pasting conditions for challenger at the threshold x_p^* or $\bar{x}_p^*$. Applying the smooth-pasting and value-matching conditions at x_p^{ns} and $\bar{x}_p^*$, we obtain the expression of royalty rate that it is optimal to settle immediately

$$\bar{\theta}_p^-(\bar{x}_p) = \frac{p\delta[\tau_a + \bar{k}(\bar{M} + \bar{N}) + \Phi(M + N) + \bar{k}\Phi n](r - \mu)}{\Phi(M + N + \tau_a)} \tag{A.16}$$

where $\bar{k} = \frac{x_p^{ns}}{\bar{x}_p}$, $\tau_a = \frac{n\tau_c + M\Gamma}{n + \bar{M}\tau_i} n$, $\Gamma = \frac{\Delta C^C}{\Delta C^I}$, $\tau_c = \frac{C_l^C}{\Delta C^I}$, $\tau_i = \frac{C_l^I}{\Delta C^I}$, $n = \alpha_\lambda - \beta_\lambda$, $N = \bar{k}^{\beta_\lambda} - \bar{k}^{\alpha_\lambda}$, $\bar{N} = \bar{k}^{-\beta_\lambda} - \bar{k}^{-\alpha_\lambda}$, $M = \beta\bar{k}^{\alpha_\lambda} - \alpha\bar{k}^{\beta_\lambda}$, and $\bar{M} = \beta\bar{k}^{-\alpha_\lambda} - \alpha\bar{k}^{-\beta_\lambda}$.

From a wide range of parameter values, it is always true that $\bar{\theta}_p^-(x) \geq \bar{\theta}_p^{Cmax}$, where $\bar{\theta}_p^{Cmax}$ follows Equation (A.19). Thus, for the incumbent to get the highest payoff in immediate settlement, the royalty rate is either (1) $\bar{\theta}_p = \bar{\theta}_p^{Cmax}$ if $\bar{\theta}_p^{Cmax} > \bar{\theta}_p^{Imin}$, or (2) $\bar{\theta}_p = \bar{\theta}_p^+(x)$, where $\bar{\theta}_p^+(x)$ is given by Equation (A.15) if $\bar{\theta}_p^+(x) \in [\bar{\theta}_p^{Imin}, \bar{\theta}_p^{Cmax}]$.

The lower region ($x \in (x_p^{ns}, \bar{x}_p^*)$) In this region, the incumbent chooses the optimal royalty rate and the challenger decides the settlement threshold $\bar{x}_p^*$. If both $V_p^I \geq V_{ns}^I$ and $V_p^C \geq V_{ns}^C$, then firms settle as market demand reaches $\bar{x}_p^*$ from below, and with royalty rate $\bar{\theta}_p^*$. Or, if such settlement is not feasible, one of the firms leave the lawsuit as a lower threshold x_p^{ns}. From VM and SP conditions at both $\bar{x}_p^*$ and x_p^{ns}, we get

$$\bar{A}_p^i = \frac{1}{(\bar{x}_p^*)^{\alpha_\lambda-\beta_\lambda} - (x_p^{ns})^{\alpha_\lambda-\beta_\lambda}}\Big[(\frac{\Delta\pi_p^i}{r-\mu} - C_s^i)(\bar{x}_p^*)^{-\beta_\lambda} + C_l^i((\bar{x}_p^*)^{-\beta_\lambda} - (x_p^{ns})^{-\beta_\lambda}) + p\delta\Delta\pi^i((\bar{x}_p^*)^{1-\beta_\lambda} - (x_p^{ns})^{1-\beta_\lambda}) + I_e\frac{\Delta\pi^i}{r-\mu}(x_p^e)^{1-\beta_\lambda}\Big],$$

$$\bar{B}_p^i = \frac{1}{(\bar{x}_p^*)^{\beta_\lambda-\alpha_\lambda} - (x_p^{ns})^{\beta_\lambda-\alpha_\lambda}}\Big[(\frac{\Delta\pi_p^i}{r-\mu} - C_s^i)(\bar{x}_p^*)^{-\alpha_\lambda} + C_l^i((\bar{x}_p^*)^{-\alpha_\lambda} - (x_p^{ns})^{-\alpha_\lambda}) + p\delta\Delta\pi^i((\bar{x}_p^*)^{1-\alpha_\lambda} - (x_p^{ns})^{1-\alpha_\lambda}) + I_e\frac{\Delta\pi^i}{r-\mu}(x_p^e)^{1-\alpha_\lambda}\Big],$$

$$\bar{x}_p^* = \begin{cases} \dfrac{\Delta C^I(\alpha_\lambda-\beta_\lambda) + C_l^I(\beta_\lambda k^{-\alpha_\lambda} - \alpha_\lambda k^{-\beta_\lambda})}{(-p\delta\Delta\pi^I - \frac{\theta_p\Delta\pi^C}{r-\mu})(\alpha_\lambda-\beta_\lambda) - p\delta\Delta\pi^I\left((\beta_\lambda-1)k^{1-\alpha_\lambda} - (\alpha_\lambda-1)k^{1-\beta_\lambda}\right)}, & s_{ns} = \text{(I-withdraw)} \\ \dfrac{\Delta C^C(\beta_\lambda-\alpha_\lambda) + C_l^C(\alpha_\lambda k^{-\beta_\lambda} - \beta_\lambda k^{-\alpha_\lambda})}{(p\delta - \frac{\theta_p}{r-\mu})\Delta\pi^C(\alpha_\lambda-\beta_\lambda) + (p\delta - \frac{1}{r-\mu})\Delta\pi^C\left((\beta_\lambda-1)k^{1-\alpha_\lambda} - (\alpha_\lambda-1)k^{1-\beta_\lambda}\right)}, & s_{ns} = \text{(C-exit)} \end{cases}.$$

where $k = \frac{x_p^{ns}}{\bar{x}_p^*} \in (0, 1)$ and I_e is an indicator function, $I_e = 1$ if $s_{ns} =$ (C-exit) and $I_e = 0$ if $s_{ns} =$ (I-withdraw).

A.13 Solution of the litigation threshold with immediate settlement

If the litigation is followed by an immediate settlement, the incumbent takes into account the outcome of immediate settlement when choosing x_l. Then, $\frac{\partial B^I}{\partial x_l} \neq 0$, and the following result gives us the corresponding x_l.

Corollary 7. *With immediate settlement during litigation, the litigation threshold* $x_l = [\frac{\Delta C^C}{(1-\beta_\lambda)B_w^C}]^{\frac{1}{\beta_\lambda}}$ *if* $s_{ns} =$ *(I-withdraw), and it satisfies* $p\delta\pi_2^C\left((\beta-1)(\frac{x_e}{x_l})^\alpha - (\alpha-1)(\frac{x_e}{x_l})^\beta\right) + \frac{\Delta\pi^I}{r-\mu}(\beta-\alpha)\frac{x_e}{x_l} + (\frac{\beta_\lambda}{\beta_\lambda-1}\Delta C^C - C_s^I)\left(\beta(\frac{x_e}{x_l})^\alpha - \alpha(\frac{x_e}{x_l})^\beta\right)x_l^{-1} = 0$ *if* $s_{ns} =$ *(C-exit).*

Proof. Denote firm values with the litigation option in the immediate settlement as $\bar{V}_l^i$. From Appendix A.12, the immediate settlement royalty rate $\bar{\theta}_p$ is either $\bar{\theta}_p^{Cmax}(x)$ or $\bar{\theta}_p^+$.

1. If $s_{ns} =$ (I-withdraw): VM conditions imply the following, where $\bar{V}_l^i$ is from Corollary (5), and $\hat{V}_p^i$ is from

Equation (14):

$$\text{VM at } x_l: \quad \bar{V}_l^i = \hat{V}_p^i(\bar{\theta}_p) - C_s^i \Rightarrow \frac{\pi_2^i}{r-\mu}x_l + \bar{A}_l^i x_l^\alpha = \frac{\pi_2^i + \Delta\pi_p^i(\bar{\theta}_p)}{r-\mu}x_l - C_s^i \Rightarrow \bar{A}_l^i = \Big(\frac{\Delta\pi_p^i(\bar{\theta}_p)}{r-\mu}x_l - C_s^i\Big)x_l^{-\alpha}.$$

(a) If $\bar{\theta}_p = \bar{\theta}_p^{Cmax}(x)$: for the optimality condition $\frac{\partial \bar{V}_l^I}{\partial x_l} = 0$ to satisfy, we obtain the implicit expression for $\bar{x}_p$, which is the same as x_l, as $\alpha(C_s^I - \Delta C^C) + p\delta\Delta\pi^C(1-\alpha)x_l + B_w^C x_l^{\beta_\lambda}(\alpha - \beta_\lambda) = 0$.

(b) If $\bar{\theta}_p = \bar{\theta}_p^+$, the first-order derivative gives $\frac{\partial \bar{V}_l^I}{\partial x_l} = \frac{\partial \bar{A}_l^I}{\partial x_l} = (1-\alpha)p\delta\pi_2^C x - \alpha[\frac{\beta_\lambda}{\beta_\lambda - 1}\Delta C^C - C_s^I] < 0$.

Therefore, the litigation threshold can be expressed as:

$$x_m = [\frac{\Delta C^C}{(1-\beta_\lambda)B_w^C}]^{\frac{1}{\beta_\lambda}}\bar{x}_p.$$

Our numerical exercises show $\bar{\theta}_p^+(x_m)$ gets the incumbent the highest firm value in immediate settlement. We further check the lower region numerically and find that the value of not settling is always higher than the value of settling. Therefore, the incumbent, knowing that immediate settlement is feasible when $x < x_p^*$, litigates at x_m and then settles immediately with the maximum immediate royalty rate $\bar{\theta}_p^+(x_m)$, where

$$\bar{\theta}_p^+(x_m) = \frac{\frac{\beta_\lambda \Delta C^C}{\beta_\lambda + 1}(r-\mu)}{-(\beta_\lambda - 1)^2(B_w^C)^{\beta_\lambda}\pi_2^C} + p\omega.$$

2. s_{ns} = (C-exit).

$$\text{VM at } x_l \qquad \frac{\pi_2^i}{r-\mu}x_l + a_l^i x_l^\alpha + b_l^i x_l^\beta = \frac{\pi_2^i + \Delta\pi_p^i(\bar{\theta}_p)}{r-\mu}x_l - C_s^i, \quad i \in \{I, C\}.$$

$$\text{VM at } x_e \qquad \frac{\Delta\pi^i}{r-\mu}x_e + a_l^i x_e^\alpha + b_l^i x_e^\beta = 0$$

$$\text{SP:} \qquad \frac{\pi_2^C}{r-\mu} + \alpha a_l^C x_e^{\alpha-1} + \beta b_l^C x_e^{\beta-1} = 0.$$

Using the optimality condition $\frac{\partial V_l^I}{\partial x_l} = 0$, we obtain the expression of $\bar{x}_p$:

(a) If $\bar{\theta}_p = \bar{\theta}_p^{Cmax}(x)$, $\bar{x}_p$ satisfies

$$(\alpha(\frac{x_e}{\bar{x}_p})^\beta - \beta(\frac{x_e}{\bar{x}_p})^\alpha - (\frac{x_e}{\bar{x}_p})^\beta + (\frac{x_e}{\bar{x}_p})^\alpha)(p\delta\pi_2^C - B_e^C\bar{x}_p^{\beta_\lambda - 1}) + (\beta(\frac{x_e}{\bar{x}_p})^\alpha - \alpha(\frac{x_e}{\bar{x}_p})^\beta)(C_s^I - \Delta C^C)\bar{x}_p^{-1} - (\beta - \alpha)(\frac{x_e}{\bar{x}_p})\frac{\Delta\pi^I}{r-\mu} = 0.$$

(b) If $\bar{\theta}_p = \bar{\theta}_p^+$, $\bar{x}_p$ satisfies

$$p\delta\Delta\pi^C((\beta_\lambda - 1)(\frac{x_e}{\bar{x}_p})^\alpha - (\alpha - 1)(\frac{x_e}{\bar{x}_p})^\beta) + \frac{\Delta\pi^I}{r-\mu}(\beta - \alpha)\frac{x_e}{\bar{x}_p}\bar{x}_p - (\frac{\beta_\lambda}{\beta_\lambda - 1}\Delta C^C - C_s^I)\big(\alpha(\frac{x_e}{\bar{x}_p})^\beta - \beta(\frac{x_e}{\bar{x}_p})^\alpha\big) = 0.$$

The numerical checks confirm it is optimal for the incumbent to settle with $\bar{\theta}_p^+(x_m)$ than $\bar{\theta}_p^{Cmax}$ in the immediate settlement region, and there is no feasible royalty rates in the lower region.

If $\bar{\theta}_p^+(\bar{x}_p) \in [\bar{\theta}_p^{Imin}(\bar{x}_p), \bar{\theta}_p^{Cmax}(\bar{x}_p)]$, firms settle immediately at $\bar{x}_p$ with royalty rate $\bar{\theta}_p^+(\bar{x}_p)$. Note if the ex-post settlement is feasible in both the middle region and the top region, the incumbent compares its firm values (V_l^I) for the two possibilities at $\bar{x}_p$ to decide whether to settle immediately or wait to settle at x_p^*. There can be only one type of settlement, and here is why. the first order condition leads to the x_l that satisfies

$(1-\alpha)p\delta\pi_2^C x_l - \alpha[\frac{\beta_\lambda}{\beta_\lambda - 1}\Delta C^C - C_s^I] = 0$. As long as the challenger's litigation cost C_l^C is not too low, $\frac{\partial A_l^I(\bar{\theta}_p, x_l)}{\partial x_l} < 0$. This suggests the incumbent's value with the option to litigate that is followed by immediate settlement decreases with the litigation/immediate settlement threshold, therefore, the incumbent chooses the lowest feasible x, denoted by x_m. Because the first derivative of royalty rate in unconstrained immediate settlement $\bar{\theta}_p(x_l)$ in Equation (A.15) with respect to x_l is negative, i.e., $\frac{\partial \theta_p(x_l)}{\partial x_l} = -\frac{\beta_\lambda(\Delta C^C)(r-\mu)}{(\beta_\lambda - 1)\pi_2^C} x_l^{-2} < 0$, choosing x_m allows the incumbent to settle immediately with the highest royalty rate. We find the feasible optimal royalty rate in immediate settlement by comparing the values of settling and the values of not settling for both firms. The lowest feasible x_m satisfies following condition

$$\bar{\theta}_p(x_m) = \bar{\theta}_p^{Cmax}(x_m) \tag{A.17}$$

where $\bar{\theta}_p^{Cmax}(x_m)$ follows the expression in Equation (A.19).

We figure out the feasibility of the (constrained) immediate settlement by checking whether

$$\bar{\theta}_p^{Cmax}(x_l) \geq \bar{\theta}_p^{Imin}. \tag{A.18}$$

Because the incumbent optimally chooses the lowest feasible threshold to maximise its value with the option to settle immediately, the immediate settlement threshold is the market demand x that makes the value of settling immediately equal to the value of not settling for the challenger, and at this threshold, the incumbent's value of settling immediately is at least as great as the value of not settling. Therefore, when x reaches x_m the challenger is indifferent between settling immediately or not settling immediately, but the incumbent is willing to settle immediately.

If the firms do not settle immediately, then the incumbent's option value during litigation $B^I x^{\beta_\lambda}$ does not vary with its choice of the litigation threshold x_l, i.e., $\frac{\partial B^I}{\partial x_l} = 0$. We obtain x_l by applying $\frac{\partial B^I}{\partial x_l} = 0$ in Corollary 5.

□

A.14 Feasibility of immediate settlement

Lemma 3. *Immediate settlement is feasible if $\bar{\theta}_p^{Imin}(\bar{x}_p) \leq \bar{\theta}_p^{Cmax}(\bar{x}_p)$, with $\bar{\theta}_p^{Imin}(\bar{x}_p)$ and $\bar{\theta}_p^{Cmax}(\bar{x}_p)$ specified in Equations (A.19) and (A.20). The royalty rate in the immediate settlement is specified in Equation (A.15) if $\bar{\theta}_p(\bar{x}_p) \in [\bar{\theta}_p^{Imin}(\bar{x}_p), \bar{\theta}_p^{Cmax}(\bar{x}_p)]$, and it is $\bar{\theta}_p^{Cmax}(\bar{x}_p)$ as in Equation (A.19) if $\bar{\theta}_p(\bar{x}_p) > \bar{\theta}_p^{Cmax}(\bar{x}_p)$.*

Proof. Immediate settlement is feasible if the settlement payoffs, i.e., $\hat{V}_p^I(\bar{x}_p;\theta_p)$ and $\hat{V}_p^C(\bar{x}_p;\theta_p)$ with θ_p substituted with $\bar{\theta}_p(\bar{x}_p)$ in Equation (A.15), are at least as high as the corresponding value with non-settlement, i.e., $V_{ns}(\bar{x}_p)$. As in the case of ex-post settlement, the feasibility constraint of the challenger in Condition (15) implies the royalty rate offered by the incumbent $\bar{\theta}_p$ is capped in immediate settlement

$$\bar{\theta}_p \leq \bar{\theta}_p^{Cmax}(\bar{x}_p) = \begin{cases} p\omega\left[1 - (\frac{\bar{x}_p}{x_w})^{\beta_\lambda - 1}\right] + \frac{r-\mu}{\pi_2^C \bar{x}_p}\left[\Delta C^C - C_l^C(\frac{\bar{x}_p}{x_w})^{\beta_\lambda}\right], & s_{ns} = \text{(I-withdraw)} \\ p\omega\left[1 - (\frac{\bar{x}_p}{x_e})^{\beta_\lambda - 1}\right] + \frac{r-\mu}{\pi_2^C \bar{x}_p}\left[\Delta C^C - C_l^C(\frac{\bar{x}_p}{x_e})^{\beta_\lambda}\right] + (\frac{\bar{x}_p}{x_e})^{\beta_\lambda - 1}, & s_{ns} = \text{(C-exit)} \end{cases} \tag{A.19}$$

$\bar{\theta}_p^{Cmax}(\bar{x}_p)$ is higher in C-exit than I-withdraw, evident from the extra third term in C-exit in Equation (A.19), implying that the challenger is willing to pay up to a higher maximum royalty if it were to exit earlier than the incumbent would withdraw in the absence of settlement. Thus, challenger's non-settlement value is higher in s_{ns} = (I-withdraw) than in s_{ns} = (C-exit). Unlike in the ex-post settlement, the feasibility constraint of the incumbent in Condition (15) leads only to a lower bound on the royalty rates in immediate settlement. The absence of an upper bound is due to the lack of the adverse effect from a higher royalty rate on the incumbent's firm value via delaying the settlement, given that the challenger accepts the settlement offer immediately. Without the adverse effect, the incumbent's settlement value monotonically increases with respect to the royalty rate and the value of settling lies below the non-settlement value only when the royalty rate is too low.

$$\bar{\theta}_p \geq \bar{\theta}_p^{Imin}(\bar{x}_p) = \begin{cases} \frac{p\omega}{\Phi}\left[1-(\frac{\bar{x}_p}{x_w})^{\beta_\lambda-1}\right]+\frac{r-\mu}{\pi_2^C\bar{x}_p}\left[C_l^I(\frac{\bar{x}_p}{x_w})^{\beta_\lambda}-\Delta C^I\right], & s_{ns}=\text{(I-withdraw)} \\ \frac{p\omega}{\Phi}\left[1-(\frac{\bar{x}_p}{x_e})^{\beta_\lambda-1}\right]+\frac{r-\mu}{\pi_2^C\bar{x}_p}\left[C_l^I(\frac{\bar{x}_p}{x_e})^{\beta_\lambda}-\Delta C^I\right]+\frac{1}{\Phi}(\frac{\bar{x}_p}{x_e})^{\beta_\lambda-1}. & s_{ns}=\text{(C-exit)} \end{cases} \tag{A.20}$$

If $\bar{\theta}_p^{Cmax}(\bar{x}_p) \geq \bar{\theta}_p^{Imin}(\bar{x}_p)$, then there exists some royalty rate that makes immediate settlement better than not to settle for both firms. There are two sub-cases under the feasibility: 1) in *unconstrained immediate settlement*: the royalty rate that maximizes the incumbent's value, i.e., $\bar{\theta}_p(\bar{x}_p)$ in Exp.(A.15), lies within the feasible range, so $\bar{\theta}_p(\bar{x}_p)$ is the royalty rate; 2) in *constrained immediate settlement*, the incumbent's optimal royalty rate $\bar{\theta}_p(\bar{x}_p)$ is too higher for the challenger to accept, but offering the maximum royalty rate that the challenger accepts is still better than non-settlement for the incumbent, thus it uses $\bar{\theta}_p^{Cmax}(\bar{x}_p)$ in its proposal. At the constrained immediate settlement, firm values for any demand x are $\hat{V}_p^I(x;\bar{\theta}_p^{Cmax}(x))$ and $\hat{V}_p^C(x;\bar{\theta}_p^{Cmax}(x))$. The challenger is indifferent between accepting settlement offer of $\bar{\theta}_p^{Cmax}(x)$ immediately at $\bar{x}_p$ and continuing with the litigation in which its value is $V_{ns}^C(\bar{x}_p)$. The incumbent's firm value with the immediate settlement option is greater than $V_{ns}^I(\bar{x}_p)$.

□

A.15 Proof of identical royalty rate in ex-ante and immediate settlement

Proof. The settlement thresholds, in both ex-ante settlement and the immediate ex-post settlement, are the same and equal x_l. Applying the value-matching conditions on the firm values before litigation and after settling at x_l

$$V_0^I(x_l) = \frac{\pi_2^I + \bar{\theta}_p\pi_2^C}{r-\mu}x_l - C_s^I = \frac{\pi_2^I + \theta_a\pi_2^C}{r-\mu}x_l - C_s^I, \tag{A.21}$$

$$V_0^C(x_l) = \frac{(1-\bar{\theta}_p)\pi_2^C}{r-\mu}x_l - C_s^C = \frac{(1-\theta_a)\pi_2^C}{r-\mu}x_l - C_s^C. \tag{A.22}$$

With the assumption that settlement cost is the same regardless of the settlement timing, we can get the royalty rate in ex-ante settlement and immediate settlement are the same, that is, $\bar{\theta}_p = \theta_a$. □

A.16 Analysis under the English rule

A.16.1 Proof of Theorem 4

Proof. From Theorem 1, the incumbent's optimal royalty rate in an ex-post settlement is $\theta_p^* = p\omega\big(1-g(\Gamma)\big) + \frac{p\omega}{\Phi}g(\Gamma)$, where $g(\Gamma) = (\frac{\beta_\lambda}{\beta_\lambda-1}+\frac{1}{\Gamma})^{-1} > 0$. Because $\bar{\Gamma}(\mathbb{1}_E=1) < \bar{\Gamma}(\mathbb{1}_E=0)$, we then have $g(\bar{\Gamma}(\mathbb{1}_E=1)) < g(\bar{\Gamma}(\mathbb{1}_E=0))$. Given $\Phi \leq 1$ in ex-post settlement, we get $\theta_p^*(\mathbb{1}_E=1) < \theta_p^*(\mathbb{1}_E=0)$. □

A.16.2 Analysis of during-litigation strategies under the English Rule

See the online appendix with the link here.

A.16.3 Analysis of before-litigation strategies under the English Rule

See the online appendix with the link here.