Competitive Real Option Risk

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Submitted to the ROC Bologna 2024, 4 March 2024 Revised 24 June 2024

JEL Classification: D81, G31, H25.

Key words: Real Option Risks; Duopoly; Analytical Partial Derivatives; Numerical Partial Derivatives; Hedging

Abstract

We evaluate the risk aspects of a simple portfolio of real options to invest for a duopoly. After summarizing the basic model, covering three sequences, two thresholds, and three strategic and rival options, we look at five risk elements: delta, vega, rho (the conventional option Greeks) along with epsilon (yield) and alpha (market share). The value functions of both the leader and follower is very sensitive to revenue (delta), interest rate (rho), yield (epsilon) and market share (alpha) variations, which we view in terms of sensitivities (to percentage changes), partial derivatives (analytical confirmed by numerical) and to a range of each of the input variables. Naturally, delta and rho hedging are plausible and appropriate risk avoidance actions. Maintaining final stage market share is particularly important for the follower.

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Highlights

Operating cost is proportional to market share while other authors focus on net revenue.

Leader is burdened by a **rival** option (negative value), reflecting the prospective loss of final market share.

There are analytical solutions for the five unknowns (two thresholds and three real option value coefficients), given five equations, two value matching and two smooth pasting equations for the basic investments in a non-pre-emptive duopoly with an exogeneous market.

Proofs are offered that the derived results solve the basic ODEs, VM and SP conditions. The delta and gamma derivatives are derived, and used for the proofs and for delta hedging to reduce risks.

A sensitivity analysis of the effect of a 1% change in each of the eight parameter values on the value functions for each of the three regimes (before investment, after only leader invests, after both have invested) indicates the importance and risk of each input.

Analytical **partial derivatives** are derived for each critical input. Ever finer numerical partial derivatives are shown to converge to the analytical partial derivatives, thereby confirming the analytical results.

The function representing the impact of a parameter change on the value function is semi-continuous for at least one of the players.

Vegas with sign-reversals vary with v and σ, so creating **vega neutrality** for the leader is problematic.

Delta hedging before considering transaction costs significantly reduces the variability of the values for both the leader and follower.

Acknowledgements: We thank in advance the discussant and participants in the Real Options Conference Bologna 2024 for helpful comments.

1 Introduction

What is the appropriate measure of risk for real options in a duopoly? We address this issue through studying (i) the sensitivities of changes in the value functions to 1% changes in the model input parameter values, (ii) through calculating the analytical partial derivatives for the thresholds, option coefficients and value functions (reconfirmed with numerical partial derivatives), and (iii) through calculating the changes in value functions across the regimes along a range of changes for each input parameter value[2]. Which is the most appropriate method for observing (and eventually managing) risk?

There is limited literature on most of these approaches. Both pre-emptive and non-pre-emptive duopoly real options usually require a numerical solution for the leader's threshold, and ignore risk exposure, partial derivatives and risk management. Few of the models allow for an operating cost. Few models offer the proofs that the differential equation is solved (or not), and that the value matching and smooth pasting conditions are satisfied. Few authors are concerned with market share derivatives, or with other risk assessments.

Fudenberg and Tirole (1985) created the foundations of real options in a competitive setting while developing a model of games of timing with a continuous time version of strategy equilibrium. Smets (1993) considered a strategic setting where firms can act under the fear of pre-emption, presented by Dixit and Pindyck (1994) Chapter 9.3. Joaquin and Butler (2000) considered the first mover advantage of lower operating costs. Smit and Trigeorgis (2001) modelled different investment strategies under quantity or reciprocating price competition.

Tsekrekos (2003) studied the sensitivity of the leader and follower value function to market share (with both temporary and pre-emptive permanent market share advantages for the leader), assumed to be constant after the follower enters. Paxson and Pinto (2003) modeled a leader with an initial market share advantage, which then evolves as new customers arrive (birth) and existing customers depart (death). Paxson and Pinto (2005) suggested a two-factor model with permanent quantity advantages accorded to the leader. Paxson and Melmane (2009) provided a two-factor model

[2] Not in this version.

where the leader starts with a larger but stochastic market share. Bobtcheff and Mariotti (2013) considered a pre-emptive game of two innovative competitors, whose existence may be revealed only by first-mover investment. Azevedo and Paxson (2014) reviewed the literature on developing such real option games. Huberts et al. (2019) showed that for a duopoly, entry may be deterred by competitive actions, possibly in a war of attrition or pre-emption, following interesting strategies. Adkins et al. (2022) provided quasi-analytical solutions for switching and divesting opportunities in a duopoly with mutually exclusive options, using the rival option concept.

We provide six innovations for basic once-off investment opportunities in a duopoly with variable operating costs: analytical solutions for all of the thresholds and option coefficients; analytical solutions for the partial derivatives for all of the inputs; confirming all of these solutions with numerical solutions; confirming that these solutions solve the conventional differential equations, and the value matching and smooth pasting conditions (except for the special case of crossing the follower's threshold); confirming that the sensitivities, partial derivatives and simulation of value functions across the basic revenue ranges are consistent; and finally showing how the delta partial derivative can be used for delta hedging to sharply reduce risk[3].

The rest of the paper is organized as follows. Section 2 derives the investment real options model for a duopoly with variable operating costs. Section 3 shows sensitivities of the value functions for each of the parameter inputs. Section 4 derives analytical results for each of the partial derivatives, and discusses some of the option coefficient characteristics. Section 5 shows delta hedging over one basic range. Section 6 summarizes and concludes and provides some suggestions for further research and applications.

2 Real Option Model for a Duopoly with Variable Operating Costs

We demonstrate the analytical procedure based on partial derivatives for determining the impact of input parameter variations on the value function of a leader and follower in a duopoly investment opportunity. We find that for one of the rivals in a duopoly model, the derivative of their.value function with respect to market share, volatility, interest rate and yield are semi-continuous

[3] Incomplete for this version.

functions with a jump, which can be both positive and negative, and varies according to the value of the state variable. The partial derivatives have similar characteristics, regarding thresholds and option coefficients, as opposed to the revenue (delta) derivative (where the thresholds and option coefficients remain the same).

In our basic model, a firm, with no current cash-flow, has a perpetual opportunity to invest in an operating asset that it intends to exercise and operate forever as soon as the asset's prevailing cash-inflow, denoted by v, is sufficiently high. The optimal policy is to retain the investment option for $0 < v < v_1$, where v_1 denotes the threshold cash-inflow, and to exercise the option for $v_1 \leq v < \infty$. While the cash-inflow remains within the inaction region, $v \in (0, v_1)$, the firm does nothing. Whenever v departs from the inaction region, where $v \notin (0, v_1)$, the firm makes the investment. We assume the state variable, the gross revenue, follows a geometric Brownian motion process, $\mathrm{d}v = \mu v \mathrm{d}t + \sigma v \mathrm{d}W$, where μ, σ denote the instantaneous drift and volatility, respectively, and $\mathrm{d}W$ an increment of the standardized Wiener process.

Then, we assume there is a simple duopoly where a first mover leader, and a follower share the final market. The leader's initial market share on entering the market is denoted by $m_L = 1$ from capturing the entire market. When the follower subsequently enters the market, its market share is denoted by $0 < m_{FL} < 1$ and simultaneously the leader's final market share reduces to $0 < m_{LF} < 1$ with $m_{LF} + m_{FL} = 1$ and $m_{FL} < m_{LF}$ due to the leader's first mover advantage. Then:

$$0 < m_{FL} < m_{LF} < m_L = 1, m_{LF} + m_{FL} = 1.$$

The nature of the duopoly game is that the leader always commits to a policy change ahead of the follower. By backwardation, we first examine the follower's value function. The value $G_F(v)$ of the follower's perpetual opportunity is:

$$G_F = \left\{ \begin{array}{l} g_{F1} = A_{F1} v^{\beta_1}, \quad v \in (0, v_{F1}), \\ g_{F2} = \dfrac{m_{FL}\, v}{\delta} - \dfrac{m_{FL}\, f}{r} - K, \quad v \notin (0, v_{F1}). \end{array} \right\} \text{F1,F2} \qquad (1)$$

where K denotes the investment cost, f the operating cash-outflow, and r the risk-free rate with the net adjusted return shortfall, or convenience yield δ=r-μ, with an unknown threshold, v_{F1}, an investment option coefficient, A_{F1}, and the option power parameter, β_1,. We assume that, where v=p*q, where q is a constant market volume quantity, p is stochastic, and f is equivalent to a fixed operating cost multiplied by the market share of the follower or leader. In (1), the term $A_{F1} v^{\beta_1}$ represents the real option value for the follower of eventually entering the market.

The value $G_L(v)$ of the leader's opportunity is:

$$G_L = \left\{ \begin{array}{l} g_{L1} = A_{L1} v^{\beta_1}, \quad v \in (0, v_{L1}), \\ g_{L2} = A_{L11} v^{\beta_1} + \dfrac{m_L\, v}{\delta} - \dfrac{m_L\, f}{r} - K, \quad v \in [v_{L1}, v_{F1}), \\ g_{L3} = \dfrac{m_{LF}\, v}{\delta} - \dfrac{m_{LF}\, f}{r} - K, \quad v \notin (0, v_{F1}). \end{array} \right\} \text{L1, L2, L3} \qquad (2)$$

In (2), the term $A_{L11} v^{\beta_1}$ represents the value for the leader of the **rival option** (negative value for the leader, when the follower enters the market). The coefficient A_{L11} is obtained from the value conserving condition $g_{L2}(v_{F1}) = g_{L3}(v_{F1})$. The solutions for the follower's entry threshold, v_{F1}, and coefficient, A_{F1}, the leader's entry threshold, v_{L1}, and coefficients, A_{L1}, A_{L11}, and β_1 are derived as follows.

From (1), the value-matching relationship and smooth-pasting condition[4] for the follower's value function are, respectively:

[4] The conventional approach to such an optimal stopping problem is that if v follows a geometric Brownian motion process, the solution G(v) must satisfy an ordinary differential equation,

$$\left.\begin{aligned} g_{F1}\left(v_{F1}\right)-g_{F2}\left(v_{F1}\right)&=A_{F1}v_{F1}^{\beta_1}-\frac{m_{FL}\,v_{F1}}{\delta}+\frac{m_{FL}\,f}{r}+K=0,\\ \frac{\partial\left(g_{F1}\left(v_{F1}\right)-g_{F2}\left(v_{F1}\right)\right)}{\partial v}&=\beta_1A_{F1}v_{F1}^{\beta_1-1}-\frac{m_{FL}}{\delta}=0.\end{aligned}\right\} \qquad \text{(3a, 3b)}$$

Solving for v_{F1} and A_{F1} yields:

$$\left.\begin{aligned} v_{F1}&=\frac{\left(m_{FL}\,f+rK\right)\beta_1\delta}{m_{FL}\left(\beta_1-1\right)r},\\ A_{F1}&=\frac{m_{FL}}{\beta_1\delta}\left[\frac{\left(m_{FL}\,f+rK\right)\beta_1\delta}{m_{FL}\left(\beta_1-1\right)r}\right]^{1-\beta_1}.\end{aligned}\right\} \qquad \text{(4a, 4b)}$$

From (2), the value-conserving condition for the leader when the follower exercises is:

$$g_{L2}\left(v_{F1}\right)-g_{L3}\left(v_{F1}\right)=A_{L11}v_{F1}^{\beta_1}+\frac{m_L\,v_{F1}}{\delta}-\frac{m_L\,f}{r}-\frac{m_{LF}\,v_{F1}}{\delta}+\frac{m_{LF}\,f}{r}=0. \qquad (5)$$

Solving for A_{L11} yields:

$$A_{L11}=-\left(m_L-m_{LF}\right)\left(\frac{v_{F1}}{\delta}-\frac{f}{r}\right)v_{F1}^{-\beta_1}. \qquad (6)$$

From (2), the value-matching relationship and smooth-pasting condition for the leader's value function are, respectively:

$$\left.\begin{aligned} g_{L1}\left(v_{L1}\right)-g_{L2}\left(v_{L1}\right)&=A_{L1}v_{L1}^{\beta_1}-A_{L11}v_{L1}^{\beta_1}-\frac{m_L\,v_{L1}}{\delta}+\frac{m_L\,f}{r}+K=0,\\ \frac{\partial\left(g_{L1}\left(v_{L1}\right)-g_{L2}\left(v_{L1}\right)\right)}{\partial v}&=\beta_1A_{L1}v_{L1}^{\beta_1-1}-\beta_1A_{L11}v_{L1}^{\beta_1-1}-\frac{m_L}{\delta}=0.\end{aligned}\right\} \qquad \text{(7a, 7b)}$$

Solving for v_{L1} and A_{L1} yields:

$$\left.\begin{aligned} v_{L1}&=\frac{\left(m_L\,f+rK\right)\beta_1\delta}{m_L\,r\left(\beta_1-1\right)r},\\ A_{L1}&=A_{L11}+\frac{m_L}{\beta_1\delta}\left[\frac{\left(m_L\,f+rK\right)\beta_1\delta}{m_L\,r\left(\beta_1-1\right)r}\right]^{1-\beta_1}.\end{aligned}\right\} \qquad \text{(8a, 8b)}$$

The power parameter β_1 is the positive root of the characteristic Q function:

$\frac{1}{2}\sigma^2v^2G''(v)+\mu vG'(v)-rG(v)=0$, pre-investment, along with the value matching and smooth pasting boundary (v*) conditions, 3a,b, 5, 7a,b, see Dixit and Pindyck (1994), page 141

$$Q(\beta_1) = \tfrac{1}{2}\sigma^2\beta_1(\beta_1 - 1) + \beta_1(r - \delta) - r = 0, \quad \beta_1 > 0. \tag{9}$$

In summary, all option coefficients and thresholds have an analytical solution, with the option coefficients simplified using the threshold expressions:

$$\left.\begin{aligned}
v_{F1} &= \frac{\beta_1\delta\left(m_{FL}\ f + rK\right)}{m_{FL}\left(\beta_1 - 1\right)r}, \\
A_{F1} &= \frac{m_{FL}}{\beta_1\delta}\left[v_{F1}\right]^{1-\beta_1}, \\
v_{L1} &= \frac{\beta_1\delta\left(m_L\ f + rK\right)}{m_L\left(\beta_1 - 1\right)r}, \\
A_{L1} &= A_{L11} + \frac{m_L}{\beta_1\delta}\left[v_{L1}\right]^{1-\beta_1}, \\
A_{L11} &= -\left(m_L - m_{LF}\right)\left(\frac{v_{F1}}{\delta} - \frac{f}{r}\right)v_{F1}^{-\beta_1} < 0, \\
\beta_1 &= \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) + \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.
\end{aligned}\right\} \tag{10}$$

Table 1A

Mathematica Thresholds & Option Coefficients for Duopoly Model

v_{L1}	v_{F1}	A_{L1}	A_{F1}	A_{L11}	β_1
12.95155	28.06170	1.93700	0.71636	-1.17607	1.71508

Table 1B ODE, VM & SP Conditions

	A	B	C	D
1	**CROR MODEL**		ODE	
2	INPUT			
3	v	5.00		
4	K	140.00		
5	σ	0.16		
6	r	0.05		
7	δ	0.03		
8	f	2.00		
9	mLF	0.60		
10	mFL	0.40		
11	OUTPUT			
12	F1(v)	11.3220	IF(B3<B14,B16*(B3^B19),B13)	1a
13	F2(v)	-89.3333	B10*(B3/B7-B8/B6)-B4	1b
14	vF1	28.0617	(B19*B7/(B19-1))*((B6*B4+B10*B8))/(B6*B10)	4a
15	vL1	12.9516	(B19*B7/(B19-1))*(B6*B4+B8)/B6	8a
16	AF1	0.7164	(B10/(B19*B7))*(B14^(1-B19))	4b
17	AL1	1.9370	B18+(1/(B19*B7))*(B15^(1-B19))	8b
18	AL11	-1.1761	(-B10*(B14/B7-B8/B6)*(B14^-B19))	6
19	β_1	1.7151	0.5-(B6-B7)/(B5^2)+SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2))	9
20	L(v)	30.6141	IF(B3<B15,B21,IF(AND(B3>B15,B3<B14),B22,B23))	2
21	L1(v)	30.6141	B17*(B3^B19)	2a
22	L2(v)	-31.9211	B18*(B3^B19)+(B3/B7-B8/B6)-B4	2b
23	L3(v)	76.0000	B9*((B3/B7)-B8/B6)	2c
24	Leader	Pre-Invest v=5		
25	ODE	0.0000	0.5*(B5^2)*(B3^2)*B27+(B6-B7)*B3*B26-B6*(B17*(B3^B19))	
26	F'(v)	10.5012	B19*B17*(B3^(B19-1))	
27	F''(v)	1.5018	B19*(B19-1)*B17*(B3^(B19-2))	
28	F(vL1)	156.6229	B17*(B15^B19)	VM1
29	V*-K	156.6229	B18*(B15^B19)+(B15/B7-B8/B6)-B4	VM1
30	SP1	0.0000	B19*B17*(B15^(B19-1))-(B19*B18*(B15^(B19-1))+1/B7)	SP1
31	Leader	Post-Invest L, Pre-Invest F v=14		
32	ODE	0.0000	0.5*(B5^2)*(B68^2)*B35+(B6-B7)*B68*B34-B6*B33+(B68-B8)	
33	F(v)	411.3222	B18*(B68^B19)+(B68/(B6-B7)-B8/B6)-B4	
34	F'(v)	36.6863	B19*B18*(B68^(B19-1))+1/(B6-B7)	
35	F''(v)	-0.6800	B19*(B19-1)*B18*(B68^(B19-2))	
36	V*	537.2340	B18*(B14^B19)+(B14/B7-B8/B6)	VM2
37	V**	537.2340	B9*(B14/B7-B8/B6)	VM2
38	SP2	-8.5566	B18*B19*(B14^(B19-1))+(1/B7)-B9*(1/B7)	SP2
39	Follower	Pre-Invest L&F v=5		
40	ODE	0.0000	0.5*(B5^2)*(B3^2)*B48+(B6-B7)*B3*B47-B6*B47	
41	FF(v)	11.3220	B16*(B3^B19)	
42	FF'(v)	3.8836	B19*B16*(B3^(B19-1))	
43	FF''(v)	0.5554	B19*(B19-1)*B16*(B3^(B19-2))	
44	Follower	Post-Invest L, Pre-Invest F v=14		
45	ODE	0.0000	0.5*(B5^2)*(B68^2)*B56+(B6-B7)*B68*B55-B6*B54	
46	F(v)	66.1966	B16*(B68^B19)	
47	FF'(v)	8.1095	B19*B16*(B68^(B19-1))	
48	FF''(v)	0.4142	B19*(B19-1)*B16*(B68^(B19-2))	
49	FF(vF)	218.1560	B16*(B14^B19)	VM3
50	V*-K	218.1560	B10*((B14/B7-B8/B6))-B4	VM3
51	SP3	0.0000	B19*B16*(B14^(B19-1))-B10/B7	SP3

Figure 1A

v	0	5	10	15	20	25	30	35	40
F(v)	0.00	11.32	37.17	74.51	122.04	178.94	244.00	310.67	377.33
L(v)	0.00	30.61	100.51	197.67	286.31	359.56	436.00	536.00	636.00

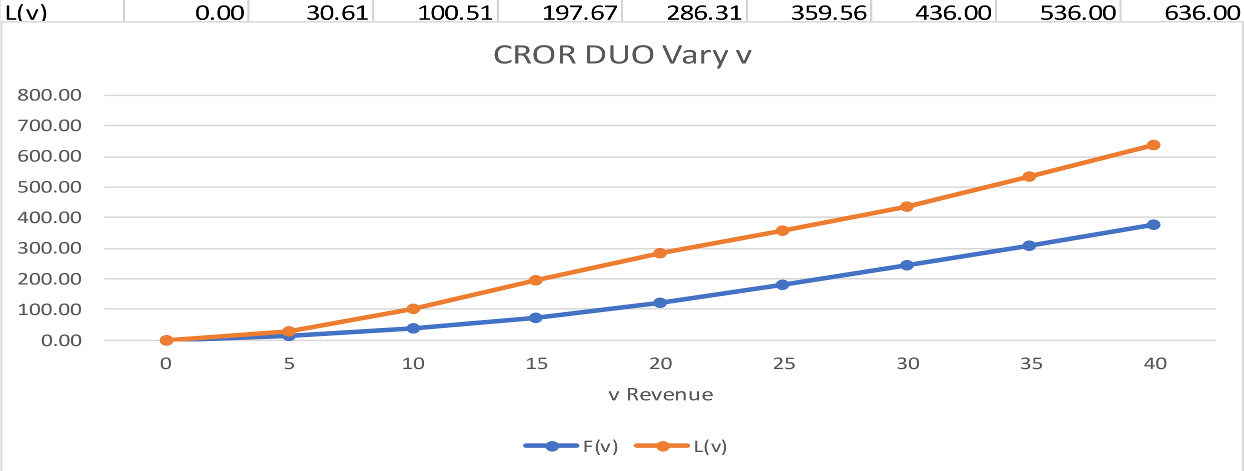
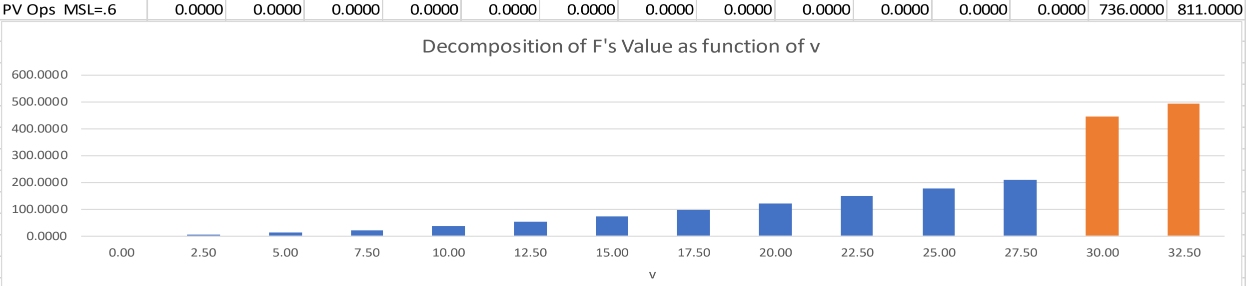
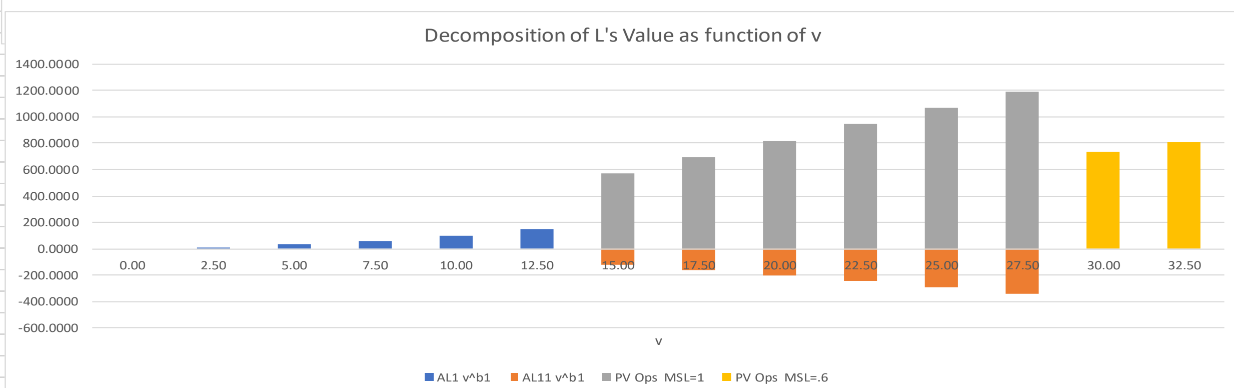


Figure 1A shows that the value functions are an almost linear function of increasing v, with a slightly greater slope for the leader, despite the leader market share falling to 60% after the follower invests.

Figure 1B

AF1 v^b1	0.0000	3.4485	11.3220	22.6952	37.1718	54.5033	74.5118	97.0610	122.0410	149.3608	178.9429	210.7204	0.0000	0.0000
PV Ops MSF=.4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	444.0000	494.0000
VF L	0.0000	9.3246	30.6141	61.3668	100.5110	147.3746	447.6708	535.6509	619.6401	699.7880	776.2218	849.0514	736.0000	811.0000
AL1 v^b1	0.0000	9.3246	30.6141	61.3668	100.5110	147.3746	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
AL11 v^b1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-122.3292	-159.3491	-200.3599	-245.2120	-293.7782	-345.9486	0.0000	0.0000
PV Ops MSL=1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	570.0000	695.0000	820.0000	945.0000	1070.000	1195.000	0.0000	0.0000
PV Ops MSL=.6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	736.0000	811.0000





A decomposition of the value functions in Figure 1B shows that the follower's value function (blue) option to invest steadily increases with v. The leader's value is split into the PV of operations after investing (grey) less the negative value of the rival option (orange); when the follower invests, the leader's value is entirely the PV of operations (60% of the market).

3 Sensitivities

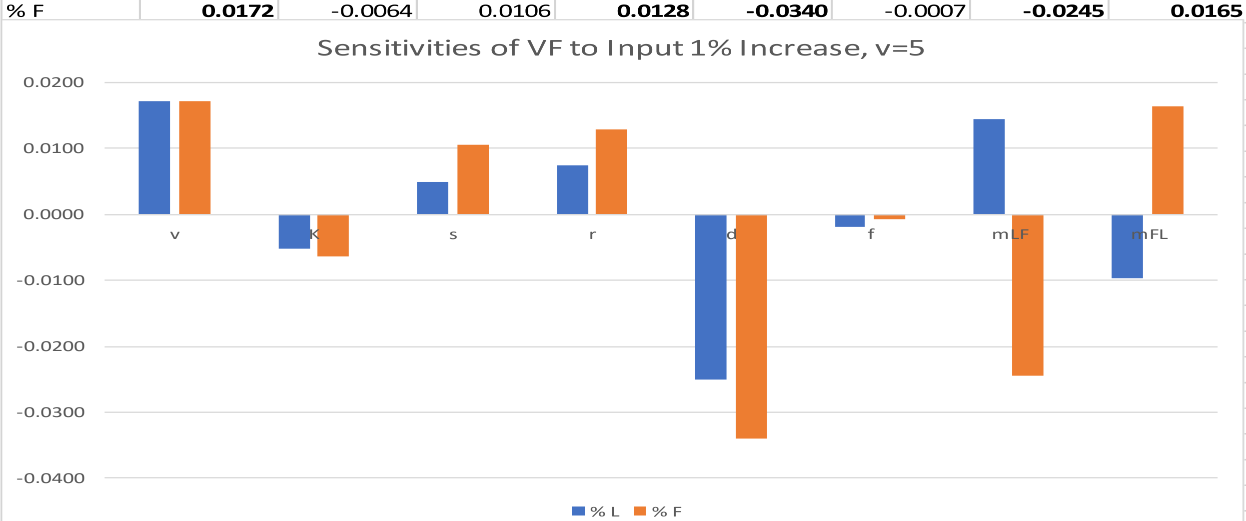
Figure 2 (v=5, regime L1 before either has invested) shows a quick and easy way to assess the sensitivity of the leader and follower value functions to changes in each of the eight parameter values separately. Significance (more than 1%) is indicated in bold. Thresholds are not very sensitive generally to changes in the parameter values (except for the follower's threshold to the leader's final market share).

Table 1C

Percentage Change in Thresholds, Option Coefficients & Value Functions

for a 1% Increase in the Parameter Value

	Numerical	K	σ	r	δ	f	mLF	mFL
	vF1	0.0090	0.0050	0.0047	0.0018	0.0010	**0.0137**	-0.0089
	vL1	0.0078	0.0050	0.0035	0.0018	0.0022	0.0000	0.0000
	AF1	-0.0064	**0.0206**	**0.0242**	**-0.0495**	-0.0007	**-0.0245**	**0.0165**
	AL1	-0.0053	**0.0149**	**0.0188**	**-0.0407**	-0.0019	**0.0145**	-0.0098
	AL11	0.0060	**0.0172**	**-0.0207**	**0.0443**	0.0011	**0.0239**	**-0.0161**
	β_1	0.0000	-0.0036	-0.0040	0.0059	0.0000	0.0000	0.0000
VF, v=5	**v**	**K**	**σ**	**r**	**δ**	**f**	**mLF**	**mFL**
% L	**0.0172**	-0.0053	0.0050	0.0075	**-0.0250**	-0.0019	**0.0145**	-0.0098
% F	**0.0172**	-0.0064	0.0106	**0.0128**	**-0.0340**	-0.0007	**-0.0245**	**0.0165**



All sensitivities are logical, with the value functions of both the leader and follower having the same sign, except for the increase in m_{LF}=1-m_{FL}, that is the leader's final market share. Naturally, each value function increases with an increase in v, and decreases with an increase in K. Consistent with expected sensitivity for a call option (investment opportunity), each value function increases with increases in volatility (before investing) and in the interest rate, but decreases with increases in yield, which is the most significant in percentage terms. Changes in the operating costs do not make much of a difference.

4 Partial Derivatives

Market Share Partials

In Appendix B, we present the partial derivative solutions pertaining with respect to the market share parameter, $m_{LF} = 1 - m_{FL}$. Because of independence, this confirms that $\partial\beta_1/\partial m_{LF} = 0$ and $\partial v_{L1}/\partial m_{LF} = 0$, and reveals that $\partial A_{L1}/\partial m_{LF} = \partial A_{L11}/\partial m_{LF}$ showing that the change in the leader's market share after the follower's entry impacts the leader's option values identically before and after the leader's exercise. It represents an anticipatory change that values the future consequences benefiting the leader when the follower exercises, which has an identical effect whether or not the leader has exercised.

From (1), the impact of the market share m_{LF} change on the follower's opportunity value is:

$$G_{F,m_{LF}}' = \left\{ \begin{array}{ll} \dfrac{\partial g_{F1}}{\partial m_{LF}} = \dfrac{\partial A_{F1}}{\partial m_{LF}} v^{\beta_1}, & v \in (0, v_{F1}), \\ \dfrac{\partial g_{F2}}{\partial m_{LF}} = -\dfrac{v}{\delta} + \dfrac{f}{r}, & v \notin (0, v_{F1}). \end{array} \right\} \quad (11a, 11b)$$

From (2), the impact of the market share m_{LF} change on the leader's opportunity value is:

$$G_{L,m_{LF}}' = \left\{ \begin{array}{ll} \dfrac{\partial g_{L1}}{\partial m_{LF}} = \dfrac{\partial A_{L1}}{\partial m_{LF}} v^{\beta_1}, & v \in (0, v_{L1}), \\ \dfrac{\partial g_{L2}}{\partial m_{LF}} = \dfrac{\partial A_{L11}}{\partial m_{LF}} v^{\beta_1}, & v \in [v_{L1}, v_{F1}), \\ \dfrac{\partial g_{L3}}{\partial m_{LF}} = \dfrac{v}{\delta} - \dfrac{f}{r}, & v \notin (0, v_{F1}). \end{array} \right\} \quad (12, 13, 14)$$

The derivation of the partial derivatives for each variable with respect to the leader's market share after the follower's entry, m_{LF}, follows the procedure described in Appendix A & C.

$$\frac{\partial v_{L1}}{\partial m_{LF}} = 0, \qquad (15)$$
$$\frac{\partial v_{F1}}{\partial m_{LF}} = \frac{v_{F1}^{1-\beta_1}}{(\beta_1 - 1)\beta_1 A_{F1}} \left(\frac{(\beta_1 - 1) v_{F1}}{\delta} - \frac{\beta_1 f}{r} \right),$$

$$\left.\begin{aligned} \frac{\partial A_{L1}}{\partial m_{LF}} &= \frac{-v_{F1}^{1-2\beta_1}(m_L - m_{LF})}{\delta A_{F1}} \left(\frac{v_{F1}}{\delta \beta_1} - \frac{f}{r(\beta_1 - 1)} \right) \\ &\quad + \frac{v_{F1}^{-\beta_1}(A_{F1} - A_{L11}) v_{F1}}{\delta A_{F1}} - \frac{v_{F1}^{-\beta_1} f \delta A_{F1}}{r \delta A_{F1}} + \frac{v_{F1}^{-\beta_1} f A_{L11} \beta_1}{r A_{F1}(\beta_1 - 1)}, \end{aligned}\right\} \qquad (16)$$

$$\frac{\partial A_{F1}}{\partial m_{LF}} = -v_{F1}^{-\beta_1} \left(\frac{v_{F1}}{\delta} - \frac{f}{r} \right) \qquad (17)$$

$$\left.\begin{aligned} \frac{\partial A_{L11}}{\partial m_{LF}} &= \frac{-v_{F1}^{1-2\beta_1}(m_L - m_{LF})}{\delta A_{F1}} \left(\frac{v_{F1}}{\delta \beta_1} - \frac{f}{r(\beta_1 - 1)} \right) \\ &\quad + \frac{v_{F1}^{-\beta_1}(A_{F1} - A_{L11}) v_{F1}}{\delta A_{F1}} - \frac{v_{F1}^{-\beta_1} f \delta A_{F1}}{r \delta A_{F1}} + \frac{v_{F1}^{-\beta_1} f A_{L11} \beta_1}{r A_{F1}(\beta_1 - 1)}, \end{aligned}\right\} \qquad (18)$$

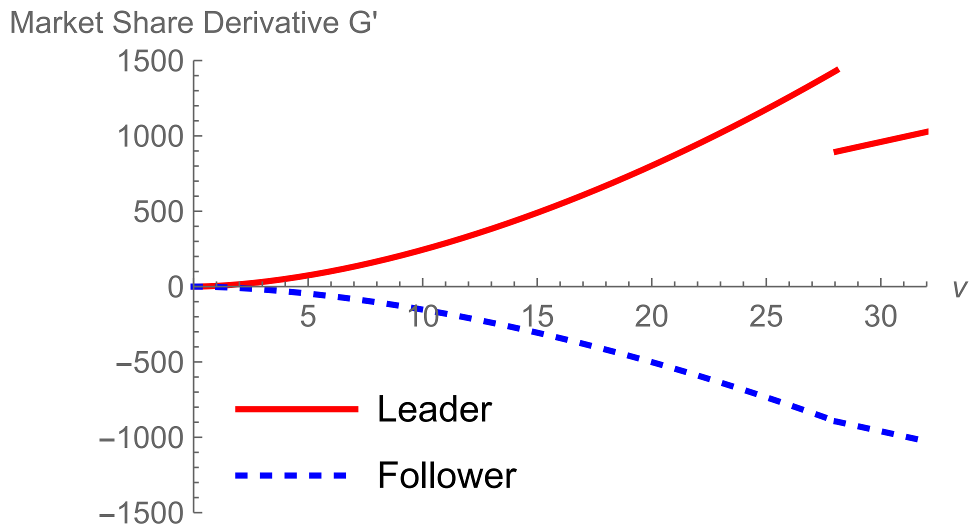
$$\frac{\partial \beta_1}{\partial m_{LF}} = 0. \qquad (19)$$

Table 2
Leader's Market Share Derivative Values

$\frac{\partial v_{L1}}{\partial m_{LF}}$	$\frac{\partial v_{F1}}{\partial m_{LF}}$	$\frac{\partial A_{L1}}{\partial m_{LF}}$	$\frac{\partial A_{F1}}{\partial m_{LF}}$	$\frac{\partial A_{L11}}{\partial m_{LF}}$	$\frac{\partial \beta_1}{\partial m_{LF}}$
0.0	62.95895	4.709147	-2.940183	4.709147	0.0

The plots of $G_{L,m_{LF}}{}'$ and $G_{F,m_{LF}}{}'$ are presented in Figure 2. Since a positive change in m_{LF} benefits the leader but produces a loss for the follower, the value function derivatives with respect to m_{LF} are positive and negative, respectively. The follower's piecewise function $G_{F,m_{LF}}{}'$ is continuous as the join occurs at the follower's exercise point, while for the leader $G_{L,m_{LF}}{}'$ is semi-continuous with a downward jump at $v = v_{F1}$, but elsewhere it is continuous.

Figure 2
Impact of Market Share Change on the Leader and Follower Value Functions



So, at all levels of v, with these parameter values, the leader will benefit from an increase of m_{LF}, except at the follower's threshold v_{F1}, when there is a sudden drop in the leader's value function, which, however, is still positive as a function of v thereafter.

Volatility Partials

The impact of volatility changes on the follower's opportunity value is found from:

$$G_F{}' = \begin{cases} \dfrac{\partial g_{F1}}{\partial \sigma} = \dfrac{\partial A_{F1}}{\partial \sigma} v^{\beta_1} + \dfrac{\partial \beta_1}{\partial \sigma} A_{F1} v^{\beta_1} \log(v), \quad v \in (0, v_{F1}), \\ \dfrac{\partial g_{F2}}{\partial \sigma} = 0, \quad v \notin (0, v_{F1}). \end{cases} \qquad \text{(20a, 20b)}$$

$$G_L' = \left\{ \begin{array}{l} \frac{\partial g_{L1}}{\partial \sigma} = \frac{\partial A_{L1}}{\partial \sigma} v^{\beta_1} + \frac{\partial \beta_1}{\partial \sigma} A_{L1} v^{\beta_1} \, log\left(v\right), \quad v \in \left(0, v_{L1}\right), \\ \frac{\partial g_{L2}}{\partial \sigma} = \frac{\partial A_{L11}}{\partial \sigma} v^{\beta_1} + \frac{\partial \beta_1}{\partial \sigma} A_{L11} v^{\beta_1} \, log\left(v\right), \quad v \in \left[v_{L1}, v_{F1}\right), \\ \frac{\partial g_{L3}}{\partial \sigma} = 0, \quad v \notin \left(0, v_{F1}\right). \end{array} \right\} \quad \text{(21a, 21b, 21c)}$$

The derivatives expressed in (20) and (21) are determined in Appendix C, and their solutions are presented below:

$$\frac{\partial v_{L1}}{\partial \sigma} = \frac{v_{L1} \sigma \beta_1}{\left(r + \frac{1}{2}\beta_1^2 \sigma^2\right)}, \tag{22}$$

$$\frac{\partial v_{F1}}{\partial \sigma} = \frac{v_{F1} \sigma \beta_1}{\left(r + \frac{1}{2}\beta_1^2 \sigma^2\right)}, \tag{23}$$

$$\begin{aligned} \frac{\partial A_{L1}}{\partial \sigma} = \frac{\left(m_L - m_{LF}\right) v_{F1} \beta_1 \sigma}{\left(r + \frac{1}{2}\beta_1^2 \sigma^2\right) \delta v_{F1}^{\beta_1}} + \frac{A_{L11} \sigma \beta_1^2 \left(1 + \left(\beta_1 - 1\right)\left(log\left[v_{L1}\right] - log\left[v_{F1}\right]\right)\right)}{r + \frac{1}{2}\beta_1^2 \sigma^2} \\ - \frac{A_{L1} \sigma \beta_1^2 \left(\beta_1 - 1\right) log\left[v_{L1}\right]}{r + \frac{1}{2}\beta_1^2 \sigma^2}, \end{aligned} \tag{24}$$

$$\frac{\partial A_{F1}}{\partial \sigma} = \frac{A_{F1} \beta_1^2 \left(\beta_1 - 1\right) \sigma \, log\left[v_{F1}\right]}{r + \frac{1}{2}\beta_1^2 \sigma^2}, \tag{25}$$

$$\frac{\partial A_{L11}}{\partial \sigma} = \frac{\left(m_L - m_{LF}\right) v_{F1} \beta_1 \sigma}{\left(r + \frac{1}{2}\beta_1^2 \sigma^2\right) \delta v_{F1}^{\beta_1}} + \frac{A_{L11} \sigma \beta_1^2 \left(1 - \left(\beta_1 - 1\right) log\left[v_{F1}\right]\right)}{r + \frac{1}{2}\beta_1^2 \sigma^2} \tag{26}$$

$$\frac{\partial \beta_1}{\partial \sigma} = -\frac{\sigma \beta_1^2 \left(\beta_1 - 1\right)}{\left(r + \frac{1}{2}\beta_1^2 \sigma^2\right)}. \tag{27}$$

The partial derivatives of the value functions with respect to volatility are:

$$\frac{\partial g_{F1}}{\partial \sigma} = \frac{A_{F1} v^{\beta_1} \sigma \, \beta_1^2 \left(\beta_1 - 1\right)\left(log\left[v_{F1}\right] - log\left[v\right]\right)}{r + \frac{1}{2}\beta_1^2 \sigma^2}, \quad v \in \left(0, v_{F1}\right), \tag{28}$$

and:

$$\left.\begin{aligned}
\frac{\partial g_{L1}}{\partial\sigma} &= -\frac{\left(m_L - m_{LF}\right) v_{F1}\beta_1\sigma}{\left(r+\frac{1}{2}\beta_1^2\sigma^2\right)(\delta) v_{F1}^{\beta_1}} v^{\beta_1} + \frac{A_{L11}\sigma\beta_1^2\left(\beta_1-1\right) log\left[v_{F1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2} v^{\beta_1} \\
&\quad + \frac{\left(A_{L1} - A_{L11}\right)\sigma\beta_1^2\left(\beta_1-1\right) log\left[v_{L1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2} v^{\beta_1} - \frac{A_{L11}\sigma\beta_1^2}{r+\frac{1}{2}\beta_1^2\sigma^2} v^{\beta_1} \\
&\quad - \frac{A_{L1}\sigma\beta_1^2\left(\beta_1-1\right)}{r+\frac{1}{2}\beta_1^2\sigma^2} v^{\beta_1} log\left[v\right], \quad v\in\left(0, v_{L1}\right), \\
\frac{\partial g_{L2}}{\partial\sigma} &= -\frac{\left(m_L - m_{LF}\right) v_{F1}\beta_1\sigma}{\left(r+\frac{1}{2}\beta_1^2\sigma^2\right)(\delta) v_{F1}^{\beta_1}} v^{\beta_1} + \frac{A_{L11}\sigma\beta_1^2\left(\beta_1-1\right) log\left[v_{F1}\right]}{r+\frac{1}{2}\beta_1^2\sigma^2} v^{\beta_1} \\
&\quad - \frac{A_{L11}\sigma\beta_1^2}{r+\frac{1}{2}\beta_1^2\sigma^2} v^{\beta_1} - \frac{A_{L11}\sigma\beta_1^2\left(\beta_1-1\right)}{r+\frac{1}{2}\beta_1^2\sigma^2} v^{\beta_1} log\left[v\right], \quad v\in\left[v_{L1}, v_{F1}\right).
\end{aligned}\right\} \quad (29)(30)$$

$G_F{}'$ is a continuous but not a smooth function for $v\in\mathrm{R}^+$. The value of the function $\partial g_{F1}/\partial\sigma$ is non-negative for $v\in\left(0, v_{F1}\right)$, equals zero at its two end-points, $v=0, v\geq v_{F1}$, and exhibits a point of maximum at $v = v_{F1}\mathrm{Exp}\left[-1/\beta_1\right]$.

Figure 3
Impact of Volatility Changes on the Leader and Follower Value Functions

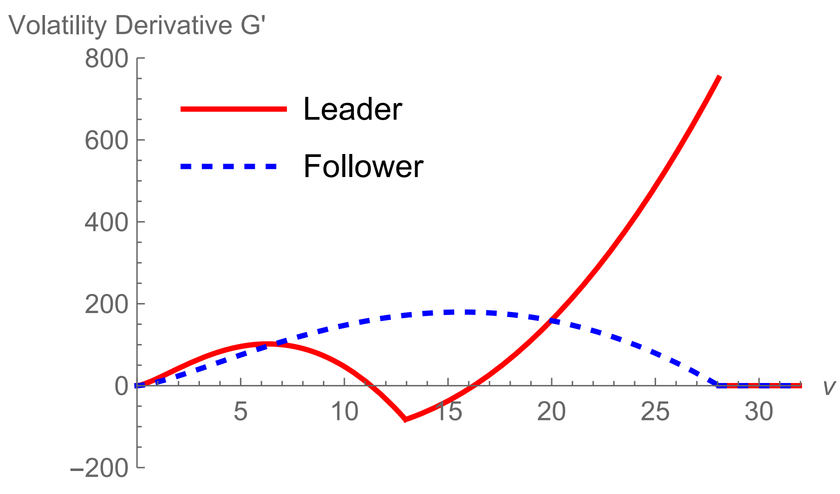


Figure 3 corroborates the predicted properties for the follower's and the leader's value functions. The effect of volatility changes on the follower's value $G_F{}'$ behaves as a continuous function but not continuously differentiable at $v = v_{F1}$, attaining a maximum at $v_{F,MAX} = 15.66367$. In contrast, the effect of volatility changes on the leader's value $G_L{}'$ behaves as **a** semi-continuous function, continuous for $v < v_{F1}$ and for $v > v_{F1}$ but discontinuously differentiable at $v = v_{L1}$ and having a down-jump discontinuity at $v = v_{F1}$. For $v \le v_{L1}$, $G_L{}'$ attains a maximum at $v_{L,MAX} = 6.30031$ but dips below zero for $v \in (11.28710, 16.24475)$. For $v_{L1} \le v < v_{F1}$, $G_L{}'$ is an increasing function. ($\partial g_{L2} / \partial \sigma$ exhibits a minimum at $v = 9.06760 < v_{L1}$.) Which of the two competitors benefits more from a volatility change? While $v \in (0, v_{L,MAX}]$, the leader gains more from positive volatility changes due to first mover advantages. For v increasingly greater than $v_{L,MAX}$, the follower gains more from positive volatility changes as the leader approaches its exercise threshold where the impact on the leader's value is negative. The follower maintains their advantage until where $\frac{\partial g_{L2}}{\partial \sigma}$ and $\frac{\partial g_{F1}}{\partial \sigma}$ intersect at $v_{L2,F1} = 19.97921$. Finally, the leader increasingly benefits more from positive volatility changes while $v \in (v_{L2,F1}, v_{F1})$, because those changes defer the market entry for the follower since $\frac{\partial v_{F1}}{\partial \sigma} > 0$ and thereby prolong the monopoly position for the leader.

Delta Partials

The leader delta (31) does not involve any change in the thresholds or option coefficients, while the other partial derivatives do.

$$\frac{\partial V_L(v)}{\partial v} = \begin{cases} \frac{\partial V_{L3}(v)}{\partial v} = m_{LF} \frac{1}{\delta} \quad \text{for } v \ge v_{F1} \\ \frac{\partial V_{L2}(v)}{\partial v} = \frac{1}{\delta} + \beta_1 A_{L11} v^{\beta_1 - 1} \text{ for } v_{L1} \le v < v_{F1}, \\ \frac{\partial V_{L1}(v)}{\partial v} = \beta_1 A_{L1} v^{\beta_1 - 1} \quad \text{for } v < v_{L1}. \end{cases} \tag{31}$$

Differentiate the follower's value function with respect to v yields:

$$
\frac{\partial V_F(v)}{\partial v} = \begin{cases} \dfrac{\partial V_{F2}(v)}{\partial v} = m_{FL}\dfrac{1}{\delta} & \text{for } v \geq v_{F1} \\ \dfrac{\partial V_{F1}(v)}{\partial v} = \beta_1 A_{F1} v^{\beta_1 - 1} & \text{for } v < v_{F1}, \end{cases} \tag{32}
$$

In line with conventional option pricing theory, it could be argued that for L1 and L2

$$
\begin{aligned}
\frac{\partial V_{L1}(v)}{\partial v} &= \beta_1 A_{L1} v^{\beta_1 - 1} = 10.5012 \quad v = 5 \\
\frac{\partial V_{L2}(v)}{\partial v} &= \frac{1}{\delta} + \beta_1 A_{L11} v^{\beta_1 - 1} = 16.1516 \quad v = 20
\end{aligned} \tag{33}
$$

a short position 10.5012 when v =5, and VF_L=30.6141 should be used to delta hedge the L value function which includes the strategic investment option $A_{L1} v^{\beta_1}$ in the initial L1 regime. A short position 16.1516 when v =20, and VF_L=286.3068 should be used to delta hedge the L value function which includes the negative value of the rival investment option $A_{L11} v^{\beta_1}$. These hedging guidelines are not well presented in the literature.

Other Partial Derivatives

See Appendix D, E, F & G for Rho, Epsilon (Yield), Phi (OpCost) and Kappa Derivatives

5 Delta Hedging

Risk hedging may be the most useful activity using these partial derivatives, especially over one regime such as L2 where there are no jumps. Table 3 is an illustration of delta hedging based on equations 32 and 33 for the middle regime L2. Suppose the leader is satisfied with maintaining the value function of 286 after investing (cost 140), with a PV of operations 627 and a rival option value of -200 when v=20. The leader seeks to maintain this value function value (in case v declines) by shorting v for each price interval (adjusting the delta at each interval), and marking-to-market (or model) at each interval. The leader's experiences an unhedged loss for each integer if v declines, which increases with the v decline because the rival option becomes less negative.

The deltas are all positive since increasing v benefits both, Δ F2<ΔL2 until just before the follower's investment threshold of 28. Table 3 shows the leader and follower gross loss (unhedged) for the value function VF as v falls from 22 to 13 in the L2 (after the leader invests). The largest component of the loss for the leader is in the PV of operations, which is constant at 50 for each interval. There is a small loss for the F investment option value at lower v. The mean hedged loss (combining the unhedged and hedging gain/loss) is sharply reduced for both the leader and follower.

Table 3

Delta Hedging over v=22 to 13, L2

v	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**				
HEDGED LOSS=+	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	NET LOSS	**MEAN**	**STDEV**	MAX	MIN
VF F		-0.21	-0.20	-0.20	-0.20	-0.19	-0.19	-0.19	-0.19	-0.18	**-0.19**	**0.01**	-0.18	-0.21
VF L		0.34	0.34	0.33	0.32	0.32	0.31	0.31	0.30	0.30	**0.32**	**0.01**	0.34	0.30
UNHEDGED LOSS=+		LOSS 14->13						LOSS 20->19	LOSS 21->20	LOSS 22>21	**MEAN**	**STDEV**	MAX	MIN
VF F		7.90	8.32	8.72	9.12	9.51	9.90	10.28	10.65	11.02	**9.49**	**1.07**	11.02	7.90
VF L		20.36	19.68	19.02	18.36	17.72	17.08	16.46	15.85	15.24	**17.75**	**1.75**	20.36	15.24
DELTA HEDGE		GAIN 14->13						GAIN 20->19	GAIN 21->20	GAIN 22->21				
VF F GAIN=+		8.11	8.52	8.92	9.32	9.71	10.09	10.47	10.84	11.20	9.69	1.06	11.20	8.11
VF L		20.02	19.35	18.69	18.04	17.40	16.77	16.15	15.54	14.94	17.43	1.74	20.02	14.94
dVF/dv		8.11	8.52	8.92	9.32	9.71	10.09	10.47	10.84	11.20				
dVL/dv		20.02	19.35	18.69	18.04	17.40	16.77	16.15	15.54	14.94				

For the leader, the mean loss (mostly due to the PV operations) and variability is significant unhedged, but sharply reduced with this academic hedging based on the delta partial derivatives, and choice of hedging intervals over these limited intervals. By hedging, the standard deviation of the leader's unhedged losses of 1.74 is reduced to .01. However, trying to delta hedge over the investment thresholds is likely to be problematic.

6 Summary and Conclusions

We provide several possibly unique contributions for the real option solutions and derivatives for basic once-off investment opportunities in a non-pre-emptive duopoly with operating costs (adjusted for market share): analytical solutions for the thresholds and option coefficients, and for the partial derivatives for all of the inputs; confirming all of these solutions with numerical solutions, and that all of the conventional conditions are satisfied; and simulations of the solutions and partial derivatives over a range of input parameter values. We show how the delta partial derivative can be used for delta hedging to sharply reduce risk of this portfolio of real options.

We propose three measures of the risk exposure of the real option portfolio of duopoly investment opportunities: sensitivities, partials, and value functions across a range of input parameter values[5].

(i) Sensitivities show the change in each threshold, option coefficient, value function for a 1% change in the input parameter value for a single v.

(ii) Partials show the change in continuous time, which are also compared to proportionate change over an almost infinitesimal interval (.0000000001).

(iii) Value functions are shown on a single chart (using these analytical results) over a wide range of input parameter values, including across regimes, illustrating L jumps at the F threshold. An advantage of the analytical solutions for the thresholds and option coefficients (rather than a numerical solution for the leader's threshold as in other papers, and all thresholds as in Adkins et al., 2022) is that all of these calculations can be done immediately, changing other variables as well.

What is the relationship between Delta, Vega, Rho, Epsilon, Kappa, & Alpha? How should one use volatility swaps to hedge Vega, interest-rate futures to hedge Rho, and arrangements with third parties and marketing experts (or collusion through industry associations) to hedge Alpha risk? Is there a simple measure like VaR which can be constructed out of these analytical formulae to assess risk for this basic model? Some intriguing research issues are:

1. Can these real options be replicated through dynamic trading in v?
2. Can these real options be separated and recombined into new competitive industry configurations?
3. What are the option and physical/future positions for obtaining strategic objections to protect on the downside at a cost equal to giving up some of the upside (an effective real option collar), or other desired exposures?

[5] Of course, each of these formats can be replicated for volatility changes for instance using the Appendix Table C1 (CROR Num PD Vol), using in C2=1.0000000001 for (ii), C2=1.01 for (i), and C2=.17 for the σ interval .16->.17 (iii).

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APPENDICIES

A: Determining the Partial Derivatives

B: Model PD with respect to Alpha (Market Share)

C: Model PD with respect to Vega

D: Model PD with respect to Rho

E: Model PD with respect to Epsilon (Yield)

F: Model PD with respect to Phi (OpCost)

G: Model PD with respect to Kappa