AP REAL LAYERED COLLAR OPTIONS 5 Feb 2022

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Roger Adkins*

Dean Paxson**

Submitted to The Real Options Conference, Oporto 7 Feb 2022

Acknowledgements: we thank Alcino Azevedo, Paulo Pereira, and Artur Rodrigues for valuable comments on earlier versions.

*School of Management, University of Bradford, Bradford BD9 4JL, UK.

r.adkins@bradford.ac.uk

+44 (0)1274233466.

**Alliance Manchester Business School, University of Manchester, Manchester, M15 6PB, UK. dean.paxson@manchester.ac.uk

+44(0)161-306-6456. Corresponding author.

REAL LAYERED COLLAR OPTIONS

Abstract

We derive the optimal investment timing and real option value for an investment opportunity, and separate values for each of the options with price uncertainty, where there are layers specifying the proportions of the price (at different layers) that are shared (with a third party). The general model is for collars consisting of two upside and two downside risk-sharing layers, thus eight option coefficients, from which several other models can be easily derived, where the variables other than price are constant or deterministic. Analytical solutions are derived for the separate embedded option values. Sensitivities of the real option values and the thresholds justifying immediate investment to changes in the important parameter values are examined. Notable findings are the real option value of a layered collar arrangement is *lower* with increased high volatility at high prices, that layered values as function of the sharing proportion depend on the level of prices, and the layered collar investment threshold "vegas" depend on the risk sharing proportions. Lower thresholds justifying immediate investment are obtained through reducing the investment cost and the layers, and increasing the floors, with layer and floor adjustments possibly economical for the third party (government).

JEL Classifications: D81, G31, Q42, Q48

Keywords: risk sharing, investment incentives, layered collars, real option values, thresholds.

1 Introduction

Is a government subsidy the least costly method for encouraging early investment in a socially (or politically) desirable infrastructure or renewable energy facility? Are layered collars less/more sensitive to changes in prices than basic collars and no collars? Are layered collar "vegas" (sensitivity to volatility) always positive? Are the separate option values in layered collars significant compared to other risk-sharing elements? These are the critical questions we seek to answer while developing analytical solutions for the embedded options in layered collar arrangements.

We assume that in evaluating a perpetual opportunity to invest in an infrastructure, an investor uses modern investment criteria, allowing for volatility and drift over time of the expected net price for a unit quantity ("P"). She may then consider what proportions of risk sharing (layered options), volatility and drift characteristics of a proposed arrangement justify commencing an investment expenditure, given the physical characteristics of the infrastructure.

Our approach is consistent with some other real option models, where valuing matching and smooth pasting conditions hold. Analytical solutions for perpetual collars were first introduced in Adkins and Paxson (2016, 2017). Takashima et al. (2010) design a private-public partnership deal involving government debt participation that incorporates a floor on the future maximum loss level where the investor has the right to sell back the project whenever adverse conditions emerge. Armada et al. (2012) make an analytical comparison of various subsidy policies including minimum revenue guarantees. Barbosa et al. (2018) and Barbosa et al. (2020) develop models for a feed-in tariffs contract with a minimum price guarantee (price-floor regime) with regulatory uncertainty. Adkins and Paxson (2019) provide analytical solutions for perpetual collars, floors and ceilings, plus partial floors and ceilings, and show the sensitivity of these collars to changes in most of the parameter values. Adkins et al. (2019) contain solutions for the investment criteria involving basic collars. To our knowledge, the specific models herein for active and investment layered collars, and complete decomposition of value for each regime, are novel contributions.

Shaoul et al. (2012) report that for a U.K. rail franchise agreement, investors are reimbursed for 50% of any revenue shortfall below 98% of forecast and 80% below 96%, but suffer a claw-back of 50% of revenue exceeding 102%, equivalent to partial puts and calls. The Hinkley Point C arrangement specifies that if the project IRR exceeds 11.4% in nominal terms, the gain is shared 30:70 between the GOV and the OWN, and 60:40 if the IRR exceeds 13.5% in nominal terms and 11.5% in real terms.

Section 2 develops the analytical solution for the eight options embedded in a layered collar. Section 3 extends these solutions to derive the optimal investment timing and real option value pre-investment, Section 4 shows the basic spreadsheet models for pre and post investment value with these embedded options and sharing proportions. Section 5 summarizes the interesting findings and suggests future research.

2 ACTIVE Layered Collars

We consider a number of regimes and formulate the shared price for the outer regimes of the collar to depend on a proportion (less than 100%) of the price under the floors and over the ceilings. Analytical solutions are obtainable despite the increase in complexity. Some of the sensitivities to changes in parameter values are similar to the basic collar model, but some are surprising.

For a firm in a monopolistic situation confronting a sole source of uncertainty due to output price (P) variability, and ignoring operating costs and taxes, the revenue of the firm depends on the price evolution, which is specified by the geometric Brownian motion process:

$$\mathrm{d}P = \alpha P \mathrm{d}t + \sigma P \mathrm{d}W \qquad (1)$$

where α denotes the expected price risk-neutral drift, σ the price volatility, and $\mathrm{d}W$ an increment of the standard Wiener process. Using contingent claims analysis, a project subject to a layered collar arrangement $V(P)$ follows the risk-neutral valuation relationship: For an active project, the revenue accruing to the firm subject to a four-layer collar is given by a conditional net price $\pi_C(P)$ and its value V_C is described by the risk-neutral valuation relationship:

$$\tfrac{1}{2}\sigma^2 P^2 \frac{\partial^2 V_C}{\partial P^2} + (r-\delta) P \frac{\partial V_C}{\partial P} - rV_C + \pi_C(P) = 0 \,. \qquad (2)$$

where $r > \alpha$ denotes the risk-free interest rate and $\delta = r - \alpha$ the convenience yield or the rate of return shortfall.

We suppose there is a symmetrical arrangement with two downside risk sharing and two upside risk sharing arrangements. For the purpose of determining the price to be received by the infrastructure investor-owner OWN, the agreement with the government GOV divides the price schedule into 5 distinct mutually-exhaustive regimes. The four junctions for neighboring regimes occur at $P = P_{LL}$, where P_{LL} represents the lowest limit, at $P = P_L$ where P_L is the lower limit, at $P = P_H$ where P_H is the higher limit, and at $P = P_{HH}$ where P_{HH} is the highest limit. Under Regime I with $P < P_{LL}$, the "price received" by the OWN is the actual price P plus a proportion $1 - w_{LL}$ of the shortfall below P_{LL} and a proportion $1 - w_L$ of the difference (P_L-P_{LL}). Under Regime II with $P_{LL} \le P < P_L$, the price received is P plus a proportion $1 - w_L$ of the shortfall from P_L, where $0 \le w_{LL} \le w_L \le 1$. Under Regime III with $P_L \le P < P_H$, the price received is P, and under Regime IV with $P_H \le P < P_{HH}$, the price received is P less a proportion $1 - w_H$ of $(P_H - P)$. Under Regime V with $P \ge P_{HH}$, the price received is P less a proportion $1 - w_{HH}$ of (P-P_{HH}) and less a proportion $1 - w_H$ of (P_{HH}-P_H), where $0 \le w_{HH} < w_H \le 1$. In the absence of any fixed costs and taxation, the regime value is determined not only from the price schedule but also from the presence of any switch options. The conditional net price is:

$$\begin{aligned}
&IF(P < P_{LL}, P + (1 - w_{LL})*(P_{LL} - P) + (1 - w_L)*P_L + (1 - w_L)*(P_L - P_{LL}),\\
&IF(AND(P_L > P, P >= P_{LL}, P + (1 - w_L)*(P_L - P),\\
&IF(AND(P_H > P, P >= P_L, P,\\
&IF(AND(P_{HH} > P, P >= P_H, P + (1 - w_H)*(P_H - P),\\
&P + (1 - w_{HH})*(P_{HH} - P) + (1 - w_H)*(P_H - P_{HH}).
\end{aligned} \tag{3}$$

For each regime, opportunities for switching to a higher or lower neighboring regime are represented by options, a call-style option for upward switching and a put-style option for downward switching, so both Regime II, III and IV are characterized by both call and put options, while Regime I by a call and Regime V by a put. Also, a switch producing a price advantage is represented by a positive option value coefficient, while that for a price disadvantage by a negative coefficient. The specifications for each of the five regimes are listed in Table 1.

Table 1 Regime Specification and Price Schedule

Regime	Specification	Value

I EQ 4	$P < P_{LL}$	$V_I(P) = A_{11}P^{\beta_1} + \frac{w_{LL}P}{\delta} + \frac{(1-w_L)(P_L - P_{LL})}{r} + \frac{(1-w_L)P_{LL}}{r}$
II EQ 5	$P_{LL} \le P < P_L$	$V_{II}(P) = A_{21}P^{\beta_1} + A_{22}P^{\beta_2} + \frac{w_L P}{\delta} + \frac{(1-w_L)P_L}{r}$
III EQ 6	$P_L \le P < P_H$	$V_{III}(P) = A_{31}P^{\beta_1} + A_{32}P^{\beta_2} + \frac{P}{\delta}$
IVEQ 7	$P_H \le P < P_{HH}$	$V_{IV}(P) = A_{41}P^{\beta_1} + A_{42}P^{\beta_2} + \frac{w_H P}{\delta} + \frac{(1-w_H)P_H}{r}$
V EQ 8	$P_{HH} \le P$	$V_V(P) = A_{52}P^{\beta_2} + \frac{w_{HH}P}{\delta} + \frac{(1-w_H)(P_H - P_{HH})}{r} + \frac{(1-w_{HH})P_{HH}}{r}$

The power parameters are β_1, β_2, respectively, the positive and negative roots of the fundamental equation, which are given by:

$$\beta_1, \beta_2 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (9)$$

The eight unknown switch option coefficients, $A_{11}, A_{21}, A_{22}, A_{31}, A_{32}, A_{41}, A_{42}, A_{52}$, are determined from the value matching relationships and associated smooth pasting condition s. The value matching relationships, defined at each of the 4 junctions of neighboring regimes are:

$$\left[V_{II}(P) - V_I(P)\right]\Big|_{P=P_{LL}} = 0 \qquad (10)$$

$$\left[V_{III}(P) - V_{II}(P)\right]\Big|_{P=P_L} = 0 \qquad (11)$$

$$\left[V_{IV}(P) - V_{III}(P)\right]\Big|_{P=P_H} = 0 \qquad (12)$$

$$\left[V_V(P) - V_{IV}(P)\right]\Big|_{P=P_{HH}} = 0 \qquad (13)$$

Equations (10)-(13) together with the 4 associated smooth pasting conditions are sufficient to solve for the eight unknowns. The resulting solutions[1] together with their signs are in Table 2. The coefficients having a positive value (involving the lower layers) indicate that the corresponding switch options are held by the OWN and contribute to their value, whilst those having a negative sign (involving the higher layers) are sold or written.

Table 2

Solutions and Conditions for the Option Coefficients of a Layered Collar

Coefficient Solution		**Condition**
$A_{11} = A_{21} - \dfrac{\left(w_L - w_{LL}\right) P_{LL}^{(1-\beta_2)} \left(r\beta_2 - r - \delta\beta_2\right)}{(\beta_1 - \beta_2) r\delta}$	EQ 14	$A_{22} \geq 0$
$A_{21} = A_{31} - \dfrac{\left(1 - w_L\right) P_L^{(1-\beta_1)} \left(r\beta_2 - r - \delta\beta_2\right)}{\left(\beta_1 - \beta_2\right) r\delta}$	EQ 15	$A_{21} \geq 0$
$A_{22} = \dfrac{\left(w_L - w_{LL}\right) P_{LL}^{(1-\beta_2)} \left(r\beta_1 - r - \delta\beta_1\right)}{\left(\beta_1 - \beta_2\right) r\delta}$	EQ 16	$A_{22} \geq 0$
$A_{32} = A_{22} - \dfrac{\left(1 - w_L\right) P_L^{(1-\beta_2)} \left(r\beta_1 - r - \delta\beta_1\right)}{\left(\beta_1 - \beta_2\right) r\delta}$	EQ 17	$A_{32} \geq 0$
$A_{31} = A_{41} + \dfrac{\left(1 - w_H\right) P_H^{(1-\beta_2)} \left(r\beta_2 - r - \beta_2\right)}{\left(\beta_1 - \beta_2\right) r\delta}$	EQ 18	<0
$A_{41} = \dfrac{\left(w_H - w_{HH}\right) P_{HH}^{(1-\beta_1)} \left(r\beta_2 - r - \delta\beta_2\right)}{(\beta_1 - \beta_2) r\delta}$	EQ 19	
$A_{42} = A_{32} + \dfrac{\left(1 - w_H\right) P_H^{(1-\beta_2)} \left(r\beta_1 - r - \delta\beta_1\right)}{(\beta_1 - \beta_2) r\delta}$	EQ 20	
$A_{52} = A_{42} + \dfrac{\left(w_H - w_{HH}\right) P_{HH}^{(1-\beta_2)} \left(r\beta_1 - r - \delta\beta_1\right)}{(\beta_1 - \beta_2) r\delta}$	EQ 21	

[1] Derivations are shown in Appendix A.

The first subscript 1,2,3,4,5 refers to the regime, while the second subscript indicates 1=call or 2=put. Note that most of these options are expressed by adding other option values, due to the method of deriving the analytical solutions. Possibly there are other simplified expressions. It is convenient that the denominators are the same $[(\beta_1 - \beta_2) r\delta)]$ with repeating expressions in the numerators $(r\beta_1 - r - \delta\beta_1)$, $\{r\beta_2 - r - \delta\beta_2\}$ which are stochastic adjustment functions.

3 Investment Criteria for Layered Collars

The optimal exercise of an investment opportunity held by an owner (OWN) is characterized by the unknown price threshold denoted by $\hat{P}_0$, which is derived from the value matching relationship and optimality condition. At $P = \hat{P}_0$, the opportunity value, $A_0 \hat{P}_0^{\beta_1}$ with unknown coefficient $A_0 > 0$, is sufficient to compensate the value of the net price per unit, less the investment cost K, plus the values of any available switch options. For the purpose of analysis, we presume that exercise occurs for $P_L \le P_0 < P_H$. The value matching relationship is:

$$A_0 \hat{P}_0^{\beta_1} = \frac{\hat{P}_0}{\delta} - K + A_{31} \hat{P}_0^{\beta_1} + A_{32} \hat{P}_0^{\beta_2} \tag{22}$$

The smooth pasting condition is:

$$\beta_1 A_0 \hat{P}_0^{\beta_1 - 1} = \frac{1}{\delta} + \beta_1 A_{31} \hat{P}_0^{\beta_1 - 1} + \beta_2 A_{32} \hat{P}_0^{\beta_2 - 1} \tag{23}$$

It is straightforward to deduce that $\hat{P}_0$ and $A_0 > 0$ are given by, respectively:

$$\frac{\hat{P}_0}{\delta} = \frac{\beta_1}{\beta_1 - 1} K - \frac{\beta_1 - \beta_2}{\beta_1 - 1} A_{32} \hat{P}_0^{\beta_2} \tag{24}$$

$$A_0 = \frac{1}{\beta_1 - \beta_2} \left[(1 - \beta_2) \frac{\hat{P}_0}{\delta} + \beta_2 K \right] \hat{P}_0^{-\beta_1} + A_{31}. \tag{25}$$

(24) shows that the investment threshold is determined by A_{32}, which depends on the floor-like attributes P_L and w_L, and on A_{22}, which depends on the floor-like attributes P_{LL} and w_{LL}. (25)

shows that the real option value depends not only on A_{32} but also A_{31}, which depends on the cap-like attributes P_H and w_H, so the investment option value is determined by both floor- and cap-like attributes. (26) shows the expansion of (24) with clear and distinct identification of the threshold drivers, $K, P_{LL}, P_L, w_{LL}, w_L$.

$$\hat{P}_0 = \delta[\frac{\beta_1}{\beta_1 - 1} K - \frac{\beta_1 - \beta_2}{\beta_1 - 1}\{\frac{((w_L - w_{LL})P_{LL}^{(1-\beta_2)}) - (1 - w_L)P_L^{(1-\beta_2)}))*(r\beta_1 - r - \delta\beta_1)}{(\beta_1 - \beta_2)r\delta}\}\hat{P}_0^{(\beta_2)}] \tag{26}$$

A systematic approach for a government (GOV) with the objective of motivating early (indeed instantaneous) investment[2] is to identify the threshold level $\hat{P}_0 = P$, close or just less than the prevailing P level, and determine level of K (probably the actual physical investment cost less cash subsidies, or equivalent tax credits) which results in that threshold. The direct subsidy is transparent, and in many countries enters into the fiscal budget. For comparison, with such a subsidy, it is appropriate to determine the floor-like attributes, which reduce the threshold to the same level, and then to evaluate the (less transparent) immediate value of that policy. Although these findings are based on assuming that $P_L \le \hat{P}_0 < P_H$, there are similar results when assuming that $P_{LL} \le \hat{P}_0 < P_L$. There are some alternative criteria given in Adkins et al. (2019) for investment opportunities with basic collars, even assuming possible retraction of the collar arrangements.

4. NUMERICAL RESULTS

ACTIVE layered collars have somewhat different sensitivities to changes in P and P volatility than basic collars without layers[3]. Layered collars with the specified parameter values are in Table 3: floors are 3.5,4, ceilings 10, 10.5, and the risk sharing is .25, .5 on the downside and .5, .25 on the upside.

Table 3

[2] Governments may have alternative objectives, such as maximizing the real option investment value, possibly if concessions with collar arrangements are sold to private investors at or above that value.

[3] No collars are defined by $P_{LL}=P_L=0$, $P_H=P_{HH}=\infty$. Basic collars are defined by $w_{LL}=w_{HH}=0$, $w_L=w_H=1.0$.

	A	B	C	D	E	F
1			**ACTIVE LAYERED COLLAR**			
2	**INPUT**					
3	P	6.00				
4	σ	0.25				
5	r	0.04				
6	δ	0.04				
7	PLL	3.50				
8	PL	4.00				
9	PH	10.00				
10	PHH	10.50				
11	wLL	0.25	75 % from GOV below PLL			
12	wL	0.50				
13	wH	0.50				
14	wHH	0.25	75 % to GOV over PHH			
15	**OUTPUT**			Eqs		
16	β_1	1.7369	0.5-(B5-B6)/(B4^2)+SQRT(((B5-B6)/(B4^2)-0.5)^2 + 2*B5/(B4^2))	9		
17	β2	-0.7369	0.5-(B5-B6)/(B4^2)-SQRT(((B5-B6)/(B4^2)-0.5)^2 + 2*B5/(B4^2))	9		
18	A11	**1.4501**	**B19-((B12-B11)*(B7^(1-B16))*-B28)/B26**	14	+	Hold Call
19	A21	**0.4465**	**B21-((1-B12)*B8^(1-B16)*-B27)/B26**	15	+	Hold Call
20	A22	**22.2594**	**(-((B12-B11))*B7^(1-B17)*-B27)/B26**	16	+	Hold Put
21	A31	**-1.3726**	**B23+((1-B13)*(B9^(1-B16)*-B28)/B26)**	17	-	Write Call
22	A32	**78.3993**	**B20-((1-B12)*B8^(1-B17)*-B27)/B26**	18	+	Hold Put
23	A41	**-0.4466**	**((B13-B14)*(B10^(1-B16))*-B28)/B26**	19	-	Write Call
24	A42	**-197.3193**	**B22+((1-B13)*B9^(1-B17)*-B27)/B26**	20	-	Write Put
25	A52	**-347.3708**	**B24+((B13-B14)*(B10^(1-B17))*-B27)/B26**	21	-	Write Put
26	[ ]	0.0040	(B5*(B16-B17)*B6)			
27	( )	0.0400	(B5*(1-B16)+B6*B16)			
28	{ }	0.0400	(B5*(1-B17)+B6*B17)			
29						
30	ACTIVE OWN	**140.0925**	IF(B3<B7,B31,IF(AND(B8>B3,B3>=B7),B32,IF(AND(B9>B3,B3>=B8),B33,IF(AND(B10>B3,B3>=B9),B34,B35))))			
31	Regime I	141.9579	B18*(B3^B16)+B11*B3/B6+(1-B12)*(B8-B7)/B5+(1-B11)*B7/B5	4		
32	Regime II	140.9761	B19*(B3^B16)+B20*(B3^B17)+B12*B3/B6+(1-B12)*B8/B5	5		
33	Regime III	140.0925	B21*(B3^B16)+B22*(B3^B17)+B3/B6	6		
34	Regime IV	137.2745	B23*(B3^B16)+B24*(B3^B17)+B13*B3/B6+(1-B13)*B9/B5	7		
35	Regime V	135.3675	B25*(B3^B17)+B14*B3/B6+(1-B13)*(B9-B10)/B5+(1-B14)*B10/B5	8		

Appendix B shows, with the first derivatives Δ and second derivatives Γ of the regime values, that the ODE (2) is solved for all regimes.

Figure 1 shows a static diagram of the Net Price going to the OWN as P increases.

Figure 1

Price	0	1	2	3	4	5	6	7	8	9	10	11	12
Net Price	**2.8750**	**3.1250**	**3.3750**	**3.6250**	**4.0000**	**5.0000**	**6.0000**	**7.0000**	**8.0000**	**9.0000**	**10.0000**	**10.3750**	**10.6250**
Regime I	2.8750	3.1250	3.3750	3.6250	3.8750	4.1250	4.3750	4.6250	4.8750	5.1250	5.3750	5.6250	5.8750
Regime II	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000	5.0000	5.5000	6.0000	6.5000	7.0000	7.5000	8.0000
Regime III	0.0000	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000	10.0000	11.0000	12.0000
Regime IV	5.0000	5.5000	6.0000	6.5000	7.0000	7.5000	8.0000	8.5000	9.0000	9.5000	10.0000	10.5000	11.0000
Regime V	7.6250	7.8750	8.1250	8.3750	8.6250	8.8750	9.1250	9.3750	9.6250	9.8750	10.1250	10.3750	10.6250

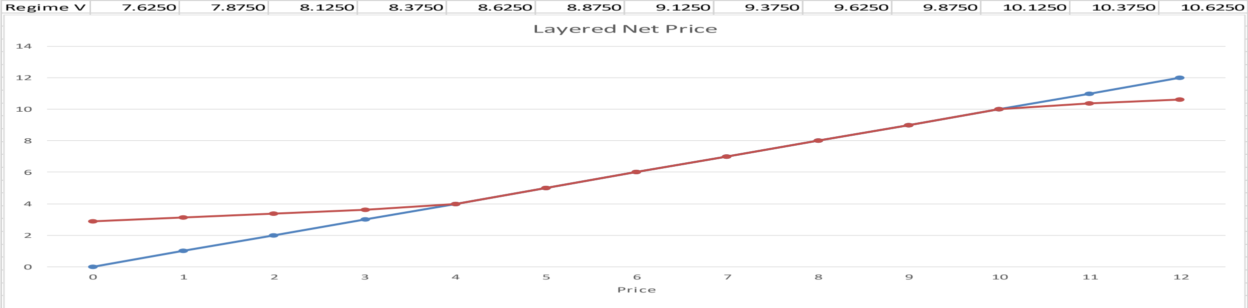


Figure 2 shows some sensitivity to increases in P, which is intuitive compared to 100% risk sharing below the floor and above the ceiling, but naturally less than for No Collar.

Figure 2

P	2	4	6	8	10	12	14	16	18
No Collar	49.79	99.31	148.60	197.70	246.61	295.34	343.91	392.32	440.58
Basic Collar	94.93	112.21	133.63	153.07	168.83	180.60	189.40	196.25	201.75
Layered Collar	89.21	112.97	140.09	166.10	189.47	209.97	228.45	245.60	261.84
Regime I	**89.2085**	112.9863	141.9579	175.5781	213.5022	255.4821	301.3265	350.8812	404.0176
Regime II	89.8442	**112.9744**	140.9761	171.3435	203.4425	237.0064	271.8902	308.0010	345.2730
Regime III	92.4654	112.9744	**140.0925**	**166.1012**	**189.4673**	209.7559	226.8439	240.7179	251.4076
Regime IV	30.1169	98.9993	137.2745	165.8350	189.4673	209.9335	228.0580	244.2895	258.8993
Regime V	-5.3024	90.5656	135.3675	165.5876	189.4657	**209.9692**	**228.4457**	**245.6015**	**261.8447**

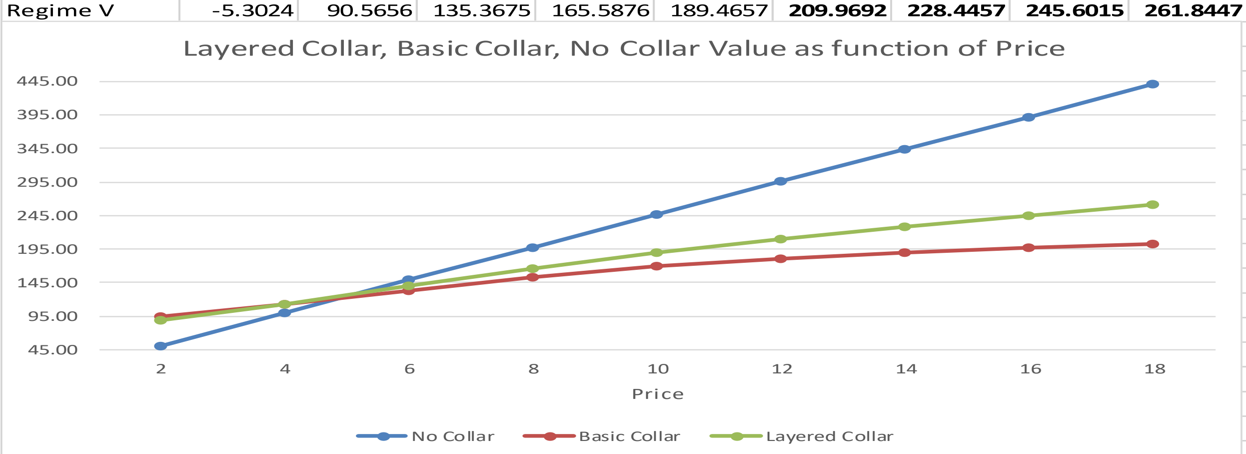
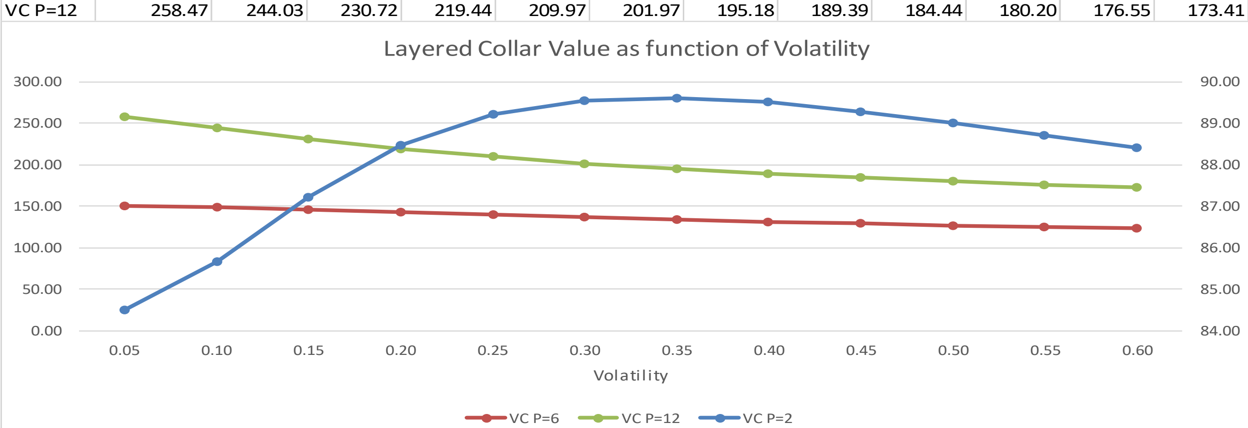


Figure 3

σ	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60
VC P=2	84.50	85.65	87.23	88.46	89.21	89.56	89.62	89.51	89.29	89.02	88.72	88.42
VC P=6	150.01	148.77	146.25	143.22	140.09	137.07	134.26	131.70	129.40	127.35	125.54	123.94
VC P=12	258.47	244.03	230.72	219.44	209.97	201.97	195.18	189.39	184.44	180.20	176.55	173.41



The sensitivity of a layered collar to changes in P volatility ("vegas") when P=2 (right vertical axis) first increases then decreases as volatility increases as shown in Figure 3, but this is dependent on the particular other parameter values in addition to the P level.

How does the particular collar setting affect the exposure of an OWN to changes in prices, which is a critical risk consideration? Without a collar, the price exposure (delta) is of course simply

1/δ=25. With a collar, the price exposure is shown in Figure 4, changing regimes as P increases. G has a U-shape possibly due to the various combinations of puts and calls across the regimes.

Figure 4

ACTIVE Δ	10.45	13.18	13.50	12.40	10.93	9.67	8.87	8.32	7.94
ACTIVE Γ	1.55	1.04	-0.35	-0.68	-0.77	-0.49	-0.32	-0.23	-0.16

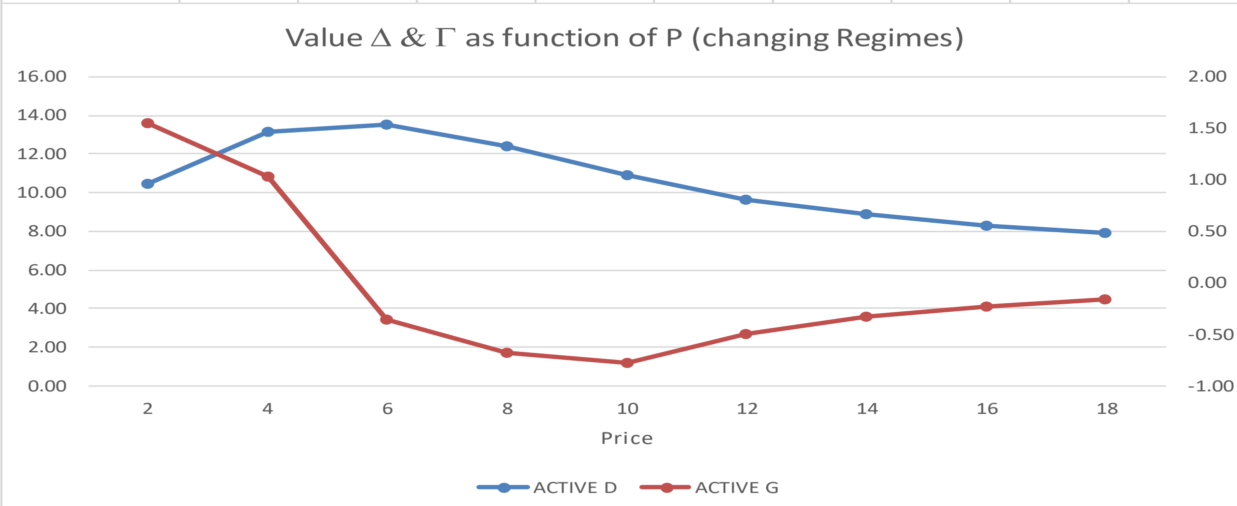
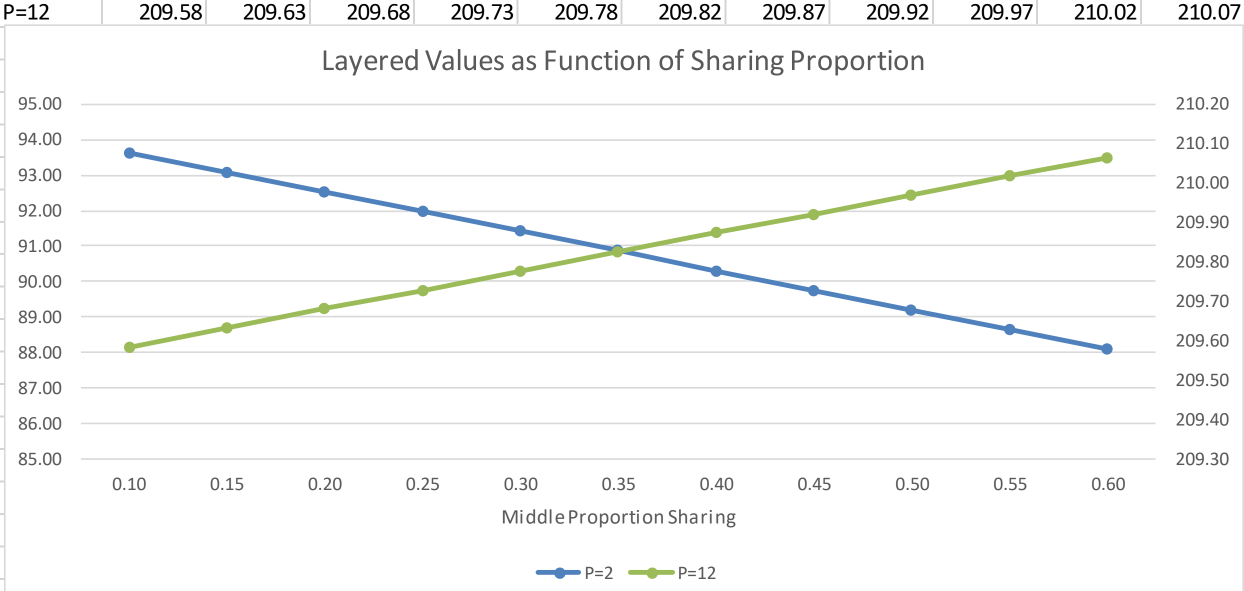


Figure 5

P=2	93.62	93.07	92.52	91.97	91.41	90.86	90.31	89.76	89.21	88.66	88.11
P=6	141.99	141.75	141.51	141.28	141.04	140.80	140.57	140.33	140.09	139.86	139.62
P=12	209.58	209.63	209.68	209.73	209.78	209.82	209.87	209.92	209.97	210.02	210.07

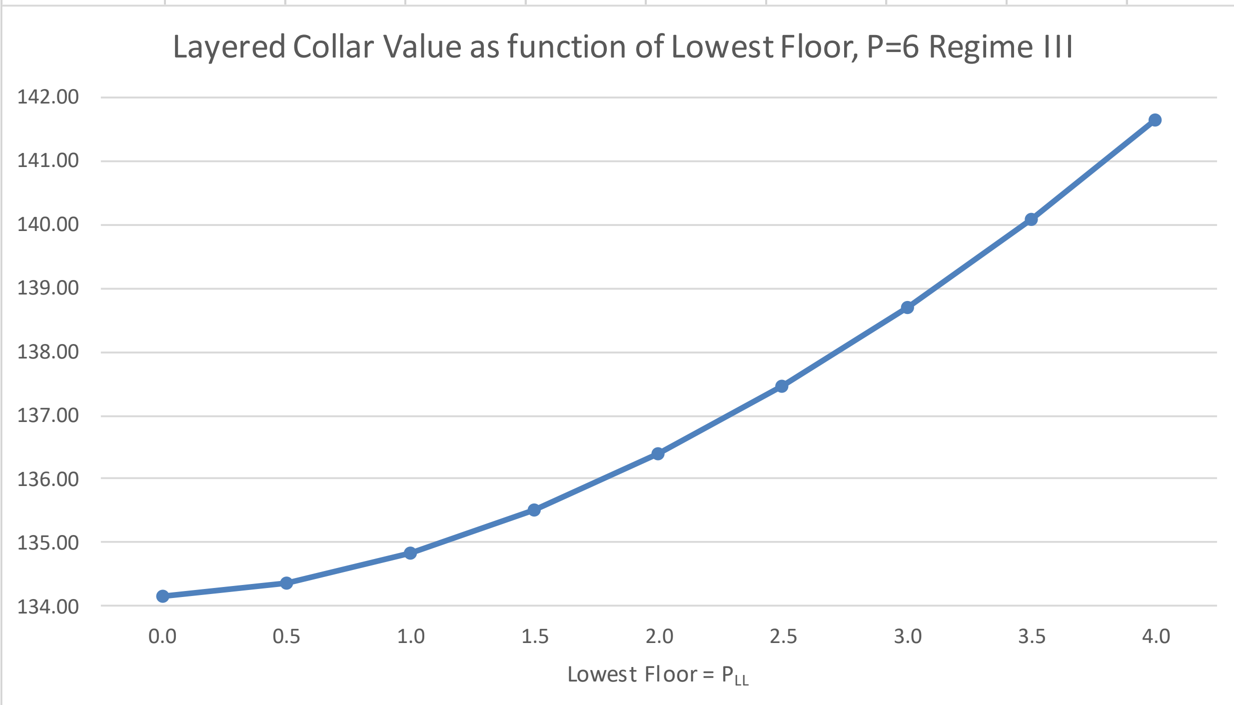


The layered collar value sensitivity to the middle risk-sharing proportion w_L (when $w_L=w_H$, $w_{LL}=w_{HH}=.5w_L$) is dependent on the price level P, shown in Figure 5. When $P<P_L$, the layered collar option value is given by A11, A21, (including A31 and A41), involving both w_{LL} and w_L in equations 14-15-18-19.

Naturally, the layered collar value will be affected by changes in the floor and ceiling levels. Figure 6 shows that increasing the lowest floor P_{LL} does not increase the overall layered collar value by much.

Figure 6

ACTIVE OWN	**134.15**	**134.35**	**134.82**	**135.51**	**136.40**	**137.46**	**138.70**	**140.09**	**141.64**

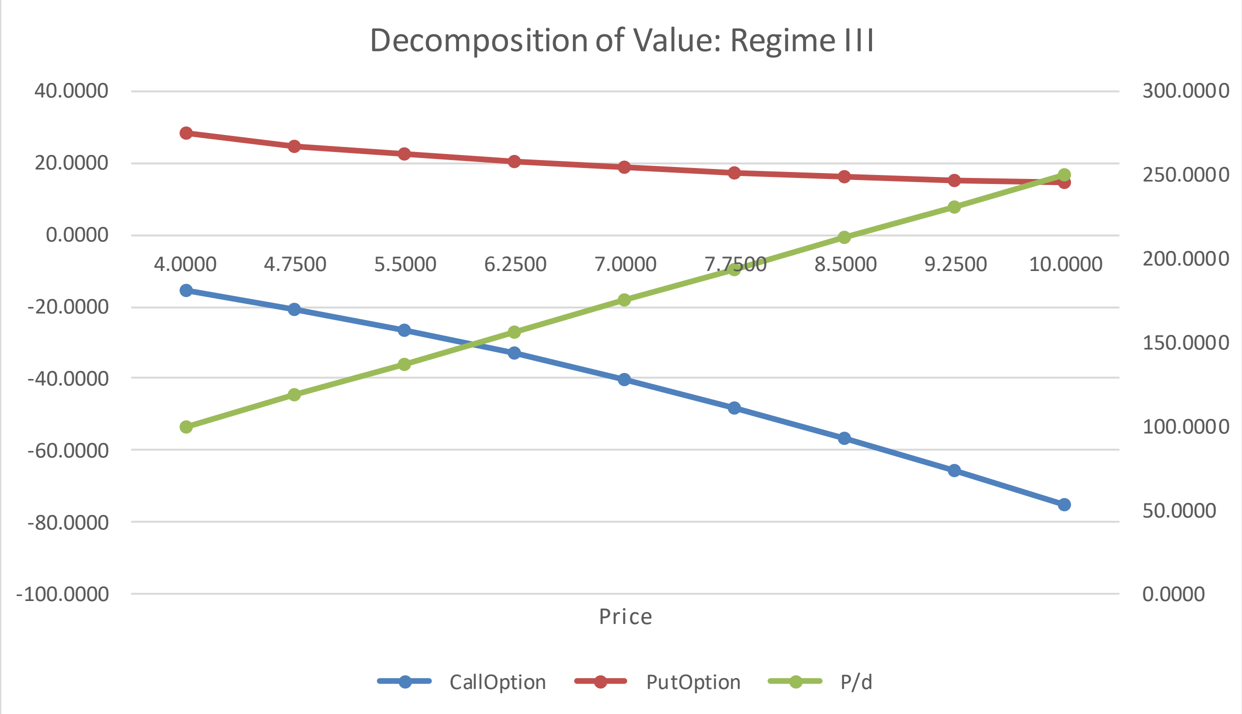


DECOMPOSITION of REGIME VALUES

Most of the regime values are dominated by the present values of stochastic prices and floors/ceiling but at different risk sharing proportions. Then each regime has a collection of switching option values.

Figure 7

Price	4.0000	4.7500	5.5000	6.2500	7.0000	7.7500	8.5000	9.2500	10.0000
Net Price	4.0000	4.7500	5.5000	6.2500	7.0000	7.7500	8.5000	9.2500	10.0000
Value	112.9744	123.0625	133.3049	143.4558	153.3752	162.9794	172.2163	181.0527	189.4673
CallOption	-15.2506	-20.5552	-26.5161	-33.1086	-40.3115	-48.1069	-56.4792	-65.4144	-74.9001
PutOption	28.2251	24.8677	22.3211	20.3143	18.6867	17.3363	16.1955	15.2171	14.3675
P/δ	100.0000	118.7500	137.5000	156.2500	175.0000	193.7500	212.5000	231.2500	250.0000



In Regime III shown in Figure 7, the aggregate ACTIVE value increases as P increases substantially due the 100% proportion of P/δ received, but decreases as the negative value of the written call option $A_{31}P^{\beta_1}$ to give up the stochastic price P to a ceiling P_H increases, while the put option $A_{32}P^{\beta_2}$ to receive the floor P_L should P fall decreases. Generally, in this middle Regime, real option values are relatively less important than in other regimes.

Whether any of these separate layered collar options could be detached, monetarized, or hedged is an interesting research (and practical) topic.

INVESTMENT NUMERICAL ILLUSTRATIONS

Table 4 shows the analytical solution for the real option value of the without collar (ROV CALL) and the with layered collar (ROV L COLLAR) with the same general parameter values as Table 3 for the ACTIVE operation. At these parameter values, the ROV CALL is 50% more valuable than the ROV L COLLAR, and the threshold that justifies immediate investment $\hat{P}$ is one-third higher than threshold $\hat{P}_0$ with a layered collar.

Table 4

	A	B	C	D
1	**INVESTMENT OPPORTUNITY WITH A LAYERED COLLAR OPTION**			
2	**INPUT**			
3	P	6.00	if PL<P<PH	
4	K	100.00		
5	σ	0.25		
6	r	0.04		
7	δ	0.04		
8	PLL	**3.5**		
9	P_L	**4.0**		
10	P_H	**10.0**		
11	PHH	**10.5**		
12	wLL	**0.25**		
13	wL	**0.50**		
14	wH	**0.50**		
15	wHH	**0.25**		
16	**OUTPUT**			EQ
17	β_1	1.7369	0.5-(B6-B7)/(B5^2)+SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2))	9
18	$\beta 2$	-0.7369	0.5-(B6-B7)/(B5^2)-SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2))	9
19	ROV CALL	61.8978	IF(B3<B21,((B4/(B17-1))*(B3/B21)^B17),B20)	
20	P/δ-K	50.0000	MAX(B3/B7-B4,0)	
21	P^	9.4279	(B17/(B17-1))*B4*B7	
22	A0	2.7547	(B4*(B21^-B17))/(B17-1)	
23				
24	ROV L COLLAR	**40.8466**	IF(B3<B26,B27*(B3^B17),B3/B7-B4+B29*(B3^B17)+B30*(B3^B18))	22
25	FIND P^C	0.0000	B26/B7-(B17/(B17-1))*B4+((B17-B18)/(B17-1))*B30*(B26^B18)	24
26	P^0	**6.8889**	**Set B25=0, changing B26**	
27	AC0	1.8179	(1/(B17-B18))*((1-B18)*(B26/B7)+B18*B4)*(B26^-B17)+B30	25
28	A22	22.2594	(-((B13-B12))*B8^(1-B18)*-B33)/B32	16
29	A31	-1.3726	B31+((1-B14)*B10^(1-B17)*-B34)/B32	18
30	A32	78.3993	B28-((1-B13)*B9^(1-B18)*-B33)/B32	17
31	A41	-0.4466	((B14-B15)*(B11^(1-B17))*-B34)/B32	19
32	[ ]	0.0040	(B6*(B17-B18)*B7)	
33	( )	0.0400	-(B6*(B17-1)-B7*B17)	
34	{ }	0.0400	-(B6*(B18-1)-B7*B18)	
35	Value Matching at P^0			
36	Pre	51.9245	B27*(B26^B17)	22
37	Post	51.9245	B26/B7-B4+B29*(B26^B17)+B30*(B26^B18))	22
38	AC31 P0^$\beta 1$	-39.2070	B29*(B26^B17)	
39	AC32 P0^$\beta 2$	18.9083	B30*(B26^B18)	

Figure 8 shows that the investment threshold “vega” is significantly different for the risk-sharing proportions w_L

Figure 8

P^	4.77	5.21	5.69	6.20	6.76	7.36	8.00	8.69	9.43
P^0 wL=.05	4.19	4.27	4.35	4.44	4.53	4.61	4.70	4.79	4.88
P^0 wL=.25	4.39	4.57	4.76	4.95	5.15	5.36	5.57	5.79	6.02
P^0 wL=.50	4.52	4.78	5.05	5.32	5.61	5.91	6.22	6.55	6.89
P^0 wL=.75	4.62	4.93	5.25	5.59	5.95	6.33	6.72	7.14	7.57
P^0 wL=1	4.70	5.05	5.43	5.82	6.24	6.68	7.14	7.64	8.15

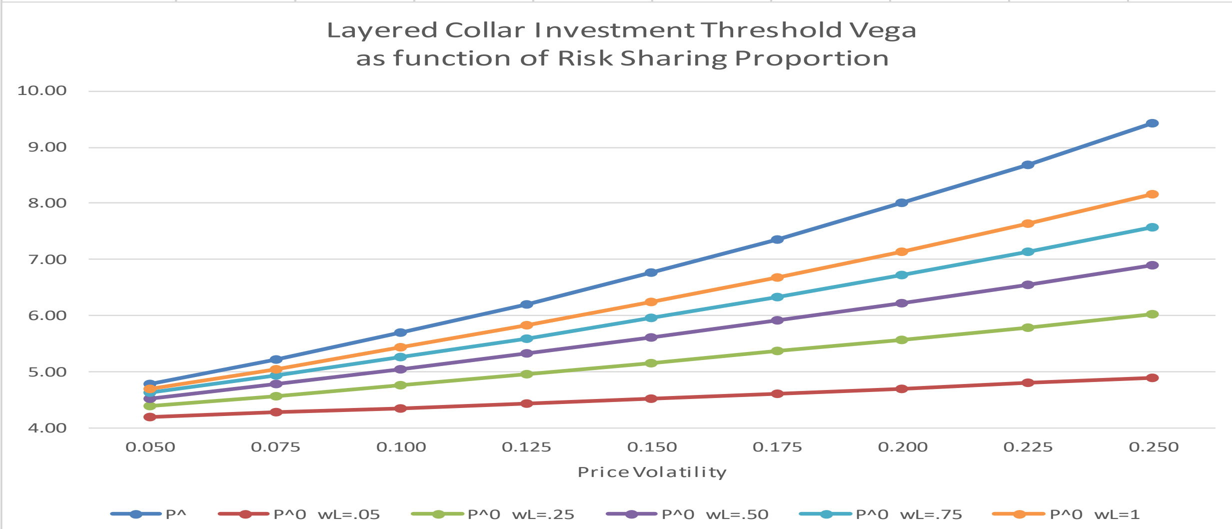
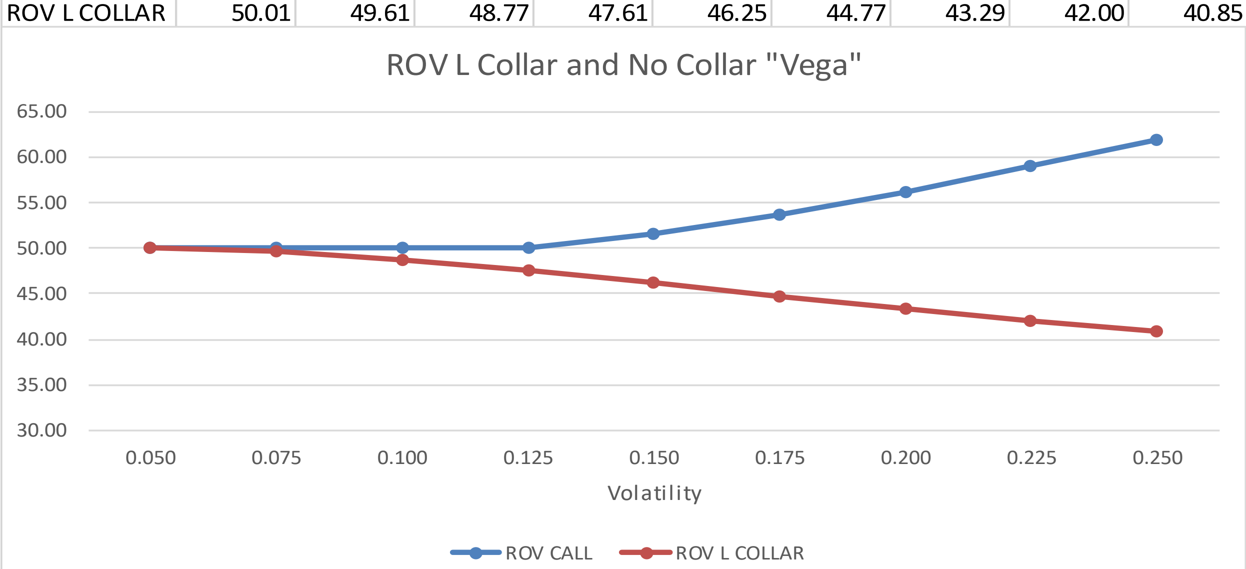


Figure 9

ROV CALL	50.00	50.00	50.00	50.15	51.51	53.68	56.25	59.03	61.90
ROV L COLLAR	50.01	49.61	48.77	47.61	46.25	44.77	43.29	42.00	40.85



In Figure 9, the ROV for the layered collar decreases with increases in volatility, due to the constrained variability imposed by the collar, which possibly has significant implications for optimal collar arrangement design.

As discussed in Section 3, the investment threshold $\hat{P}_0$ depends on A_{32}, which involves w_L and P_L, but also A_{22} (w_L, w_{LL}, P_{LL}). Figures 10-11-12 show that the threshold is nearly a linear function of changing K, but an upward convex function of decreasing the risk sharing proportion w_L (when w_{LL}=.5w_L), and a downward convex function of P_L (similarly for w_{LL} and P_{LL}).

Figure 10

P^	8.4851	8.7208	8.9565	9.1922	9.4279	9.6636	9.8993	10.1350	10.3707
P^0	5.4797	5.8606	6.2186	6.5600	6.8889	7.2079	7.5189	7.8232	8.1218

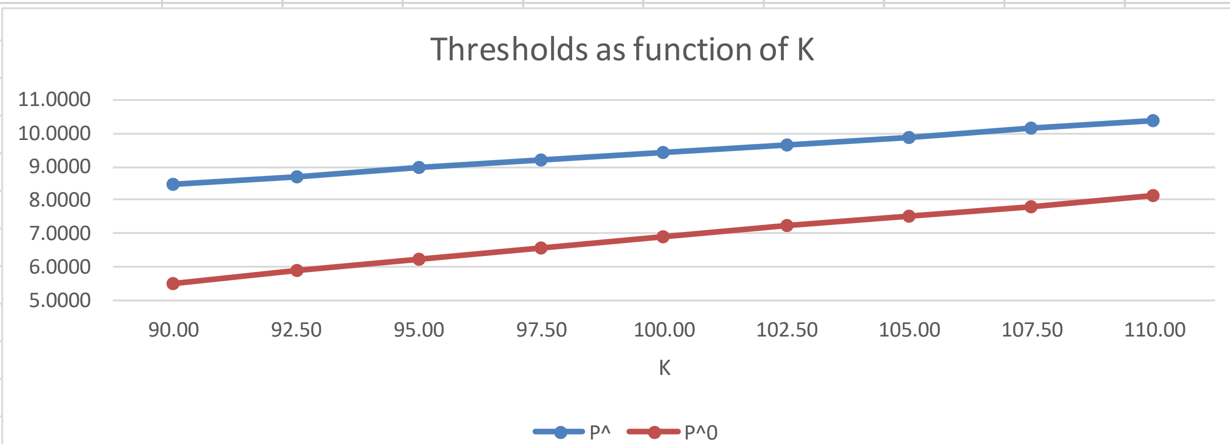


Figure 11

wL	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80
P^0	4.00	5.26	5.80	6.22	6.57	6.89	7.18	7.44	7.69

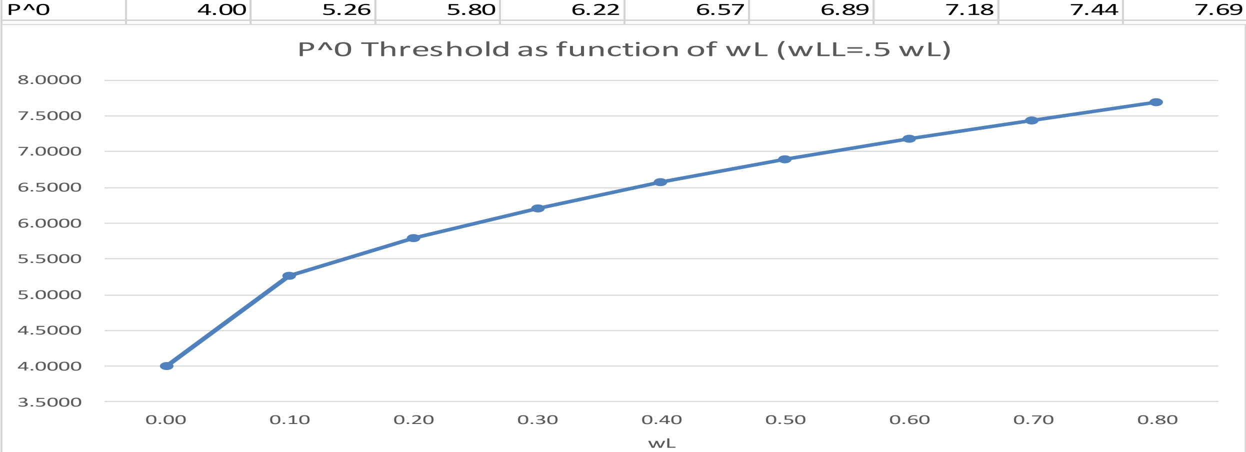
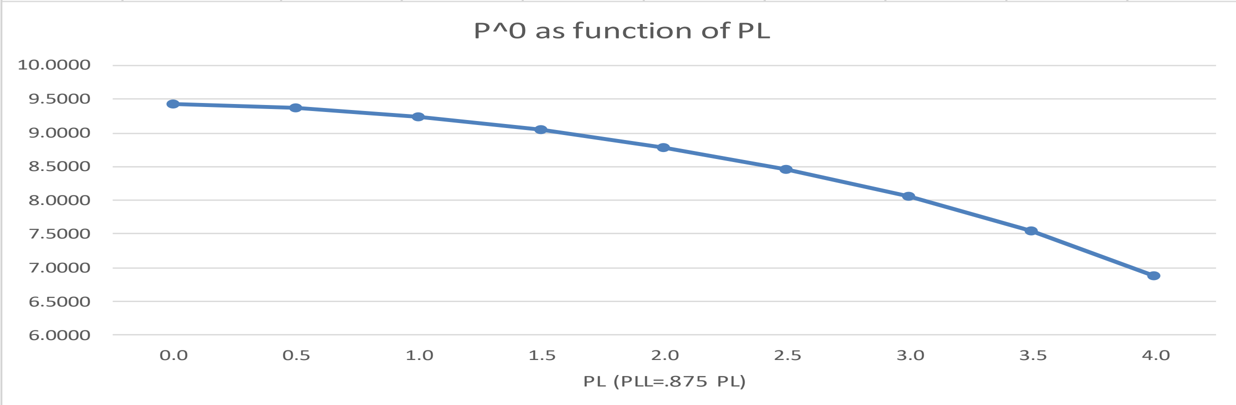


Figure 11

PL	0.00	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
P^0	9.43	9.37	9.24	9.05	8.79	8.46	8.06	7.55	6.89



This offers possibilities for designing a layered collar with a low threshold that has a lower effective cost (the combined option values, the opposite signs for the OWN and the GOV) than reducing K. An example is in Table 5, where a reduction in K (through a \$1 direct subsidy) reducing $\hat{P}_0$ from 6.8889 to 6.7587 is compared to the reduction of w_{LL}, $w_{L,}$ or increase of P_{LL},P_L resulting in the same threshold.

Table 5

EQUAL THRESHOLDS THROUGH CHANGING PARAMETER VALUES						
CASH COST FOR GOV		$1 GOV				
P	6.00	6.00	6.00	6.00	6.00	6.00
K	100.00	**99.00**	100.00	100.00	100.00	100.00
σ	0.25	0.25	0.25	0.25	0.25	0.25
r	0.04	0.04	0.04	0.04	0.04	0.04
δ	0.04	0.04	0.04	0.04	0.04	0.04
PLL	3.50	3.5	**3.7531**	3.5	3.5	3.5
PL	4.00	4.0	4.0	**4.1165**	4.0	4.0
PH	10.00	10.0	10.0	10.0	10.0	10.0
PHH	10.50	10.5	10.5	10.5	10.5	10.5
wLL	0.25	0.25	0.25	0.25	**0.2178**	0.23
wL	0.50	0.50	0.50	0.50	0.50	**0.4576**
wH	0.50	0.50	0.50	0.50	0.50	0.50
wHH	0.25	0.25	0.25	0.25	0.25	0.25
OUTPUT						
ROV L COLLAR	40.8466	**41.6464**	**41.4041**	**41.4041**	**41.4041**	**41.4041**
FIND P^C	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
P^0	**6.8889**	**6.7587**	**6.7587**	**6.7587**	**6.7587**	**6.7587**
COMBINED RO'	**-20.2987**	**-18.7523**	**-18.0502**	**-18.0502**	**-18.0502**	**-18.0502**
Pre	51.9245	51.2148	50.9169	50.9169	50.9169	50.9169
Post	51.9245	51.2148	50.9169	50.9169	50.9169	50.9169
AC31 P0^β1	-39.2070	-37.9284	-37.9284	-37.9284	-37.9284	-37.9284
AC32 P0^β2	18.9083	19.1761	19.8782	19.8782	19.8782	19.8782

In the base case OWN holds a perpetual option to invest in a perpetual project with a perpetual collar arrangement worth 40.8466 when P=6, and 51.9245 when $P = \hat{P}_0 = 6.8889$. At the threshold the OWN has written a call option worth -39.2070 and holds a put option worth 18.9082, for a combined value of -20.2987. Naturally GOV holds a call option and has written a put option for a combined value of 20.2987. Pre-investment GOV could reduce the threshold by spending \$1 subsidy to reduce K to 99, reducing $\hat{P}_0 = 6.7587$ but also reducing the GOV combined option values to 18.7523. Total cost is 1+(20.2987-18.7523) =**2.5454** for the GOV. Alternatively, the GOV could reduce the risk-sharing proportion w_L to 45.76% from 50% (keeping w_{LL}=.5w_L) thereby reducing $\hat{P}_0 = 6.7587$ but reducing the GOV combined option values to 18.0502. Total cost is (20.2987-18.0502) =**2.2485**, so this alternative is economical for the GOV (and perhaps sadly less transparent). The results will be somewhat different for other regimes, and changes in the levels of other parameter values (r, δ, σ). The real option values change after the collar arrangement is established as other parameter values, especially volatility, change, as illustrated in Adkins and Paxson (2019).

5. Conclusion

Analytical solutions are suggested for the eight separate embedded option values in an operation subject to a layered collar consisting of two upside and two downside risk-sharing arrangements. Such layered collars are less sensitive to increases in prices than no collar, but more than basic collars with 100% risk sharing below the floor and above the ceiling. Layered collar value "vegas" (sensitivity to volatility) are not always positive, or monotonic, sometimes increasing, then decreasing with increasing volatilities. The optimal investment layered collar threshold "vegas" depend on the risk-sharing proportions. The separate option values in layered collars are not always large compared to other risk-sharing elements, but the increase/decrease with increased prices can be significant. We provide some examples of floor/ceiling (collar) arrangements that may have significant effects on reducing the threshold that justifies immediate investment, compared to reducing the thresholds through direct subsidies. Alternative configurations (and for lower price regimes) are intriguing.

Future research might focus on such alternative optimal arrangements incentivizing early investment timing, along with the possibility of replicating these real option values through dynamic trading in prices, along with monetarizing and/or hedging embedded options. Extension to other multi-layer arrangements including finite, retractable collars, and competition, with stochastic floors, ceilings, proportions and price volatility is challenging.

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APPENDIX A Derivations of Analytical Solutions for Layered Collar

Let x described by the geometric Brownian motion process:

$$\mathrm{d}\, x_t = \alpha x_t \, \mathrm{d}t + \sigma x_t \, \mathrm{d}W, \tag{1}$$

where α denotes the expected drift, σ the volatility, and $\mathrm{d}W$ an increment of the standard Wiener process. At $t = 0$, $x_t = x_0 > 0$. Using Ito's Lemma, the valuation relationship for the opportunity, $F(x)$, has the form:

$$\tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} + (r-\delta) x \frac{\partial F}{\partial x} - r = 0, \tag{2}$$

where r is the risk-free rate, and $\delta = r - \alpha > 0$ the return shortfall. The generic solution to (2) is:

$$F(x) = A_{01} x^{\beta_1} + A_{02} x^{\beta_2} \tag{3}$$

where A_{01}, A_{02} are to-be-determined constants, and $\beta_1 \geq 1, \beta_2 < 0$ are the roots of:

$$Q(\beta) = \tfrac{1}{2}\sigma^2 \beta(\beta - 1) + (r-\delta)\beta - r = 0. \tag{4}$$

Since $F(0) = 0$, $A_{02} = 0$. From (4), $r\beta - r - \delta\beta < 0.$

The periodic net revenue flow $g(\cdot)$ is formulated to be continuous but not smooth and is specified by:

$$g(x) = \left\{ \begin{array}{l} g_1(x) = x + (1-w_{LL})(x_{LL} - x) + (1-w_L)(x_L - x_{LL}) \quad \text{for } \theta \in R_1, \\ g_2(x) = x + (1-w_L)(x_L - x) \quad \text{for } \theta \in R_2, \\ g_3(x) = x \quad \text{for } \theta \in R_3, \\ g_4(x) = x + (1-w_H)(x_H - x) \quad \text{for } \theta \in R_4, \\ g_5(x) = x + (1-w_{HH})(x_{HH} - x) + (1-w_H)(x_H - x_{HH}) \quad \text{for } \theta \in R_5, \end{array} \right\} \tag{5}$$

where $R_1 = [0, x_{LL})$, $R_2 = [x_{LL}, x_L)$, $R_3 = [x_L, x_H)$, $R_4 = [x_H, x_{HH})$, and $R_5 = [x_{HH}, \infty)$.

The valuation relationship for the installed opportunity is:

$$\tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + (r-\delta) x \frac{\partial V}{\partial x} - rV + g(x) = 0, \tag{6}$$

where V denotes the value of an installed opportunity. The generic solution to (6) is:

$$V\left(x\middle|\theta\in R_j\right)=V_j\left(x\right)=A_{j1}x^{\beta_1}+A_{j2}x^{\beta_2}+G_j\left(x\middle|\theta\in R_j\right),\quad j=1,2,3,4,5,\tag{7}$$

where the coefficients A_{j1},A_{j2} depend upon x_{LL},x_L,x_H,x_{HH} and w_{LL},w_L,w_H,w_{HH}, and:

$$G\left(x\right)=\left\{\begin{array}{l} G_1\left(x\right)=\dfrac{\left(1-w_L\right)\left(x_L-x_{LL}\right)}{r}+\dfrac{\left(1-w_{LL}\right)x_{LL}}{r}+\dfrac{w_{LL}x}{\delta}\quad\text{for }\theta\in R_1,\\ G_2\left(x\right)=\dfrac{\left(1-w_L\right)x_L}{r}+\dfrac{w_Lx}{\delta}\quad\text{for }\theta\in R_2,\\ G_3\left(x\right)=\dfrac{x}{\delta}\quad\text{for }\theta\in R_3,\\ G_4\left(x\right)=\dfrac{\left(1-w_H\right)x_H}{r}+\dfrac{w_Hx}{\delta}\quad\text{for }\theta\in R_4,\\ G_5\left(x\right)=\dfrac{\left(1-w_H\right)\left(x_H-x_{HH}\right)}{r}+\dfrac{\left(1-w_{HH}\right)x_{HH}}{r}+\dfrac{w_{HH}x}{\delta}\quad\text{for }\theta\in R_5.\end{array}\right\}$$

In (7), $A_{12}=0$ since $V\left(0\right)$ is finite and $A_{51}=0$ since $V\left(\infty\right)$ is not explosive. The expressions $A_{11}x^{\beta_1},A_{22}x^{\beta_2}$ represent the expected present values accruing to the installed opportunity as x approaches the bound $x\to x_{LL}$ from below and above, $A_{21}x^{\beta_1},A_{32}x^{\beta_2}$ as x approaches the bound $x\to x_L$ from below and above, $A_{31}x^{\beta_1},A_{42}x^{\beta_2}$ as x approaches the bound $x\to x_H$ from below and above, and $A_{41}x^{\beta_1},A_{52}x^{\beta_2}$ as x approaches the bound $x\to x_{HH}$ from below and above. Dependent upon whether it is economically advantageous or disadvantageous when x equals these bounds, the respective coefficient is either positive or negative, respectively.

The value-matching relationship at $x=x_{LL}$ is $V_1\left(x\right)-V_2\left(x\right)=0$:

$$\begin{aligned} A_{11}x^{\beta_1}-A_{21}x^{\beta_1}-A_{22}x^{\beta_2}-\frac{\left(1-w_L\right)x_L}{r}+\frac{\left(1-w_L\right)\left(x_L-x_{LL}\right)}{r}\\ +\frac{\left(1-w_{LL}\right)x_{LL}}{r}-\frac{w_Lx}{\delta}+\frac{w_{LL}x}{\delta}=0.\end{aligned}\tag{8}$$

The smooth-pasting condition associated with (8) is:

$$\beta_1A_{11}x^{\beta_1-1}-\beta_1A_{21}x^{\beta_1-1}-\beta_2A_{22}x^{\beta_2-1}-\frac{w_L}{\delta}+\frac{w_{LL}}{\delta}=0.\tag{9}$$

Multiplying (9) by x/β_1, adding to (8) and setting $x=x_{LL}$ yields:

$$A_{22} = -\frac{\left(w_L - w_{LL}\right) x_{LL}{}^{1-\beta_2}\left(r\beta_1 - r - \beta_1\delta\right)}{r\left(\beta_1 - \beta_2\right)\delta} > 0. \qquad (10)=(\text{EQ } 16)$$

The value-matching relationship at $x = x_L$ is $V_2\left(x\right) - V_3\left(x\right) = 0$:

$$A_{21}x^{\beta_1} - A_{31}x^{\beta_1} + A_{22}x^{\beta_2} - A_{32}x^{\beta_2} + \frac{\left(1 - w_L\right)x_L}{r} - \frac{x}{\delta} + \frac{w_L x}{\delta} = 0. \qquad (11)$$

The smooth-pasting condition associated with (11) is:

$$\beta_1 A_{21}x^{\beta_1 - 1} - \beta_1 A_{31}x^{\beta_1 - 1} + \beta_2 A_{22}x^{\beta_2 - 1} - \beta_2 A_{32}x^{\beta_2 - 1} - \frac{1}{\delta} + \frac{w_L}{\delta} = 0. \qquad (12)$$

Multiplying (12) by x/β_1 , adding to (11) and setting $x = x_L$ yields:

$$A_{32} = A_{22} - \frac{\left(1 - w_L\right)x_L^{1-\beta_2}\left(r\beta_1 - r - \delta\beta_1\right)}{r\left(\beta_1 - \beta_2\right)\delta} > 0. \qquad (13)=(\text{EQ } 18)$$

The value-matching relationship at $x = x_H$ is $V_3\left(x\right) - V_4\left(x\right) = 0$:

$$A_{31}x^{\beta_1} - A_{41}x^{\beta_1} + A_{32}x^{\beta_2} - A_{42}x^{\beta_2} - \frac{\left(1 - w_H\right)x_H}{r} + \frac{x}{\delta} - \frac{w_H x}{\delta} = 0. \qquad (14)$$

The smooth-pasting condition associated with (14) is:

$$\beta_1 A_{31}x^{\beta_1 - 1} - \beta_1 A_{41}x^{\beta_1 - 1} + \beta_2 A_{32}x^{\beta_2 - 1} - \beta_2 A_{42}x^{\beta_2 - 1} + \frac{1}{\delta} - \frac{w_H}{\delta} = 0. \qquad (15)$$

Multiplying (15) by x/β_1 , adding to (14) and setting $x = x_H$ yields:

$$A_{42} = A_{32} + \frac{\left(1 - w_H\right)x_H^{1-\beta_2}\left(r\beta_1 - r - \delta\beta_1\right)}{r\left(\beta_1 - \beta_2\right)\delta}. \qquad (16)=(\text{EQ } 20)$$

The value-matching relationship at $x = x_{HH}$ is $V_4\left(x\right) - V_5\left(x\right) = 0$:

$$\begin{aligned} &A_{41}x^{\beta_1} + A_{42}x^{\beta_2} - A_{52}x^{\beta_2} + \frac{\left(1 - w_H\right)x_H}{r} \\ &\quad - \frac{\left(1 - w_H\right)\left(x_H - x_{HH}\right)}{r} - \frac{\left(1 - w_{HH}\right)x_{HH}}{r} + \frac{w_H x}{\delta} - \frac{w_{HH} x}{\delta} = 0. \end{aligned} \qquad (17)$$

The smooth-pasting condition associated with (17) is:

$$\beta_1 A_{41}x^{\beta_1 - 1} + \beta_2 A_{42}x^{\beta_2 - 1} - \beta_2 A_{52}x^{\beta_2 - 1} + \frac{w_H}{\delta} - \frac{w_{HH}}{\delta} = 0. \qquad (18)$$

Multiplying (18) by x/β_2 , adding to (17) and setting $x = x_{HH}$ yields:

$$A_{41} = \frac{\left(w_H - w_{HH}\right) x_{HH}^{1-\beta_1} \left(r\beta_2 - r - \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta} < 0. \qquad (19)=(EQ\ 19)$$

Substituting (19) in (18) and setting $x = x_{HH}$ yields:

$$A_{52} = A_{42} + \frac{\left(w_H - w_{HH}\right) x_{HH}^{1-\beta_2} \left(r\beta_1 - r - \delta\beta_1\right)}{r\left(\beta_1 - \beta_2\right)\delta}. \qquad (20)=(EQ\ 21)$$

Substituting (16) in (15) and setting $x = x_H$ yields:

$$A_{31} = A_{41} + \frac{\left(1 - w_H\right) x_H^{1-\beta_1} \left(r\beta_2 - r - \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta}. \qquad (21)=(EQ\ 17\ )$$

Substituting (13) in (12) and setting $x = x_L$ yields:

$$A_{21} = A_{31} - \frac{\left(1 - w_L\right) x_L^{1-\beta_1} \left(r\beta_2 - r - \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta}. \qquad (22)=(EQ\ 15)$$

Substituting (10) in (9) and setting $x = x_{LL}$ yields:

$$A_{11} = A_{21} - \frac{\left(w_L - w_{LL}\right) x_{LL}^{1-\beta_1} \left(r\beta_2 - r - \delta\beta_2\right)}{r\left(\beta_1 - \beta_2\right)\delta}. \qquad (23)=(EQ\ 14)$$

APPENDIX B Table B1 Complete Analytical Solutions for Table 3

	A	B	C	D	E	F
1			**ACTIVE LAYERED COLLAR**			
2	**INPUT**					
3	P	6.00				
4	σ	0.25				
5	r	0.04				
6	δ	0.04				
7	PLL	3.50				
8	PL	4.00				
9	PH	10.00				
10	PHH	10.50				
11	wLL	0.25	75 % from GOV below PLL			
12	wL	0.50				
13	wH	0.50				
14	wHH	0.25	75 % to GOV over PHH			
15	**OUTPUT**			Eqs		
16	β_1	1.7369	0.5-(B5-B6)/(B4^2)+SQRT(((B5-B6)/(B4^2)-0.5)^2 + 2*B5/(B4^2))	9		
17	$\beta 2$	-0.7369	0.5-(B5-B6)/(B4^2)-SQRT(((B5-B6)/(B4^2)-0.5)^2 + 2*B5/(B4^2))	9		
18	A11	**1.4501**	**B19-((B12-B11)*(B7^(1-B16))*-B28)/B26**	14	+	Hold Call
19	A21	**0.4465**	**B21-((1-B12)*B8^(1-B16)*-B27)/B26**	15	+	Hold Call
20	A22	**22.2594**	**(-((B12-B11))*B7^(1-B17)*-B27)/B26**	16	+	Hold Put
21	A31	**-1.3726**	**B23+((1-B13)*(B9^(1-B16)*-B28)/B26)**	17	-	Write Call
22	A32	**78.3993**	**B20-((1-B12)*B8^(1-B17)*-B27)/B26**	18	+	Hold Put
23	A41	**-0.4466**	**((B13-B14)*(B10^(1-B16))*-B28)/B26**	19	-	Write Call
24	A42	**-197.3193**	**B22+((1-B13)*B9^(1-B17)*-B27)/B26**	20	-	Write Put
25	A52	**-347.3708**	**B24+((B13-B14)*(B10^(1-B17))*-B27)/B26**	21	-	Write Put
26	[ ]	0.0040	(B5*(B16-B17)*B6)			
27	( )	0.0400	(B5*(1-B16)+B6*B16)			
28	{ }	0.0400	(B5*(1-B17)+B6*B17)			
29						
30	ACTIVE OWN	**140.0925**	IF(B3<B7,B31,IF(AND(B8>B3,B3>=B7),B32,IF(AND(B9>B3,B3>=B8),B33,IF(AND(B10>B3,B3>=B9),B34,B35))))			
31	Regime I	141.9579	B18*(B3^B16)+B11*B3/B6+(1-B12)*(B8-B7)/B5+(1-B11)*B7/B5	4		
32	Regime II	140.9761	B19*(B3^B16)+B20*(B3^B17)+B12*B3/B6+(1-B12)*B8/B5	5		
33	Regime III	140.0925	B21*(B3^B16)+B22*(B3^B17)+B3/B6	6		
34	Regime IV	137.2745	B23*(B3^B16)+B24*(B3^B17)+B13*B3/B6+(1-B13)*B9/B5	7		
35	Regime V	135.3675	B25*(B3^B17)+B14*B3/B6+(1-B13)*(B9-B10)/B5+(1-B14)*B10/B5	8		
36	OPTION	**-9.9075**	IF(B3<B7,B37,IF(AND(B8>B3,B3>=B7),B38,IF(AND(B9>B3,B3>=B8),B39,IF(AND(B10>B3,B3>=B9),B40,B41))))			
37	Regime I	32.5829	B18*(B3^B16)			
38	Regime II	15.9761	B19*(B3^B16)+B20*(B3^B17)			
39	Regime III	-9.9075	B21*(B3^B16)+B22*(B3^B17)			
40	Regime IV	-62.7255	B23*(B3^B16)+B24*(B3^B17)			
41	Regime V	-92.7575	B25*(B3^B17)			
42	Net Price	**6.0000**	IF(B3<B7,B43,IF(AND(B8>B3,B3>=B7),B44,IF(AND(B9>B3,B3>=B8),B45,IF(AND(B10>B3,B3>=B9),B46,B47))))	3		
43	Regime I	4.3750	B11*B3+(1-B12)*B8+(B12-B11)*B7			
44	Regime II	5.0000	B12*B3+(1-B12)*B8			
45	Regime III	6.0000	B3			
46	Regime IV	8.0000	B13*B3+(1-B13)*B9			
47	Regime V	9.1250	B14*B3+(1-B13)*B9+(B13-B14)*B10			
48	ODE	**0.0000**	IF(B3<B7,B49,IF(AND(B8>B3,B3>=B7),B50,IF(AND(B9>B3,B3>=B8),B51,IF(AND(B10>B3,B3>=B9),B52,B53))))	2		
49	Regime I	0.0000	0.5*(B4^2)*(B3^2)*B61+(B5-B6)*B3*B55-B5*B31+B43			
50	Regime II	0.0000	0.5*(B4^2)*(B3^2)*B62+(B5-B6)*B3*B56-B5*B32+B44			
51	Regime III	0.0000	0.5*(B4^2)*(B3^2)*B63+(B5-B6)*B3*B57-B5*B33+B45			
52	Regime IV	0.0000	0.5*(B4^2)*(B3^2)*B64+(B5-B6)*B3*B58-B5*B34+B46			
53	Regime V	0.0000	0.5*(B4^2)*(B3^2)*B65+(B5-B6)*B3*B59-B5*B35+B47			
54	ACTIVE Δ	**13.5003**	IF(B3<B7,B55,IF(AND(B8>B3,B3>=B7),B56,IF(AND(B9>B3,B3>=B8),B57,IF(AND(B10>B3,B3>=B9),B58,B59))))			
55	Regime I	15.6824	B16*B18*(B3^(B16-1))+B11/B6			
56	Regime II	14.6742	B16*B19*(B3^(B16-1))+B17*B20*(B3^(B17-1))+B12/B6			
57	Regime III	13.5003	B16*B21*(B3^(B16-1))+B17*B22*(B3^(B17-1))+1/B6			
58	Regime IV	16.0662	B16*B23*(B3^(B16-1))+B17*B24*(B3^(B17-1))+B13/B6			
59	Regime V	17.6427	B17*B25*(B3^(B17-1))+B14/B6			
60	ACTIVE Γ	**-0.3523**	IF(B3<B7,B61,IF(AND(B8>B3,B3>=B7),B62,IF(AND(B9>B3,B3>=B8),B63,IF(AND(B10>B3,B3>=B9),B64,B65))))			
61	Regime I	1.1585	B16*(B16-1)*B18*(B3^(B16-2))			
62	Regime II	0.5680	B16*(B16-1)*B19*(B3^(B16-2))+B17*(B17-1)*B20*(B3^(B17-2))			
63	Regime III	-0.3523	B16*(B16-1)*B21*(B3^(B16-2))+B17*(B17-1)*B22*(B3^(B17-2))			
64	Regime IV	-2.2302	B16*(B16-1)*B23*(B3^(B16-2))+B17*(B17-1)*B24*(B3^(B17-2))			
65	Regime V	-3.2980	B17*(B17-1)*B25*(B3^(B17-2))			

APPENDIX C Table C1 Complete Analytical Solutions for Table 4

	A	B	C	D
1	**INVESTMENT OPPORTUNITY WITH A LAYERED COLLAR OPTION**			
2	**INPUT**			
3	P	6.00	if PL<P<PH	
4	K	100.00		
5	σ	0.25		
6	r	0.04		
7	δ	0.04		
8	PLL	**3.5**		
9	P_L	**4.0**		
10	P_H	**10.0**		
11	PHH	**10.5**		
12	wLL	**0.25**		
13	wL	**0.50**		
14	wH	**0.50**		
15	wHH	**0.25**		
16	**OUTPUT**			EQ
17	β_1	1.7369	0.5-(B6-B7)/(B5^2)+SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2))	22
18	β2	-0.7369	0.5-(B6-B7)/(B5^2)-SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2))	24
19	ROV CALL	61.8978	IF(B3<B21,((B4/(B17-1))*(B3/B21)^B17),B20)	
20	P/δ-K	50.0000	MAX(B3/B7-B4,0)	
21	P^	9.4279	(B17/(B17-1))*B4*B7	
22	A0	2.7547	(B4*(B21^-B17))/(B17-1)	
23				
24	ROV L COLLAR	**40.8466**	IF(B3<B26,B27*(B3^B17),B3/B7-B4+B29*(B3^B17)+B30*(B3^B18))	
25	FIND P^C	0.0000	B26/B7-(B17/(B17-1))*B4+((B17-B18)/(B17-1))*B30*(B26^B18)	
26	P^0	**6.8889**	**Set B25=0, changing B26**	
27	AC0	1.8179	(1/(B17-B18))*((1-B18)*(B26/B7)+B18*B4)*(B26^-B17)+B30	25
28	A22	22.2594	(-((B13-B12))*B8^(1-B18)*-B33)/B32	16
29	A31	-1.3726	B31+((1-B14)*B10^(1-B17)*-B34)/B32	18
30	A32	78.3993	B28-((1-B13)*B9^(1-B18)*-B33)/B32	17
31	A41	-0.4466	((B14-B15)*(B11^(1-B17))*-B34)/B32	19
32	[ ]	0.0040	(B6*(B17-B18)*B7)	
33	( )	0.0400	-(B6*(B17-1)-B7*B17)	
34	{ }	0.0400	-(B6*(B18-1)-B7*B18)	
35	Value Matching at P^0			
36	Pre	51.9245	B27*(B26^B17)	22
37	Post	51.9245	B26/B7-B4+B29*(B26^B17)+B30*(B26^B18))	22
38	AC31 P0^$^{\beta1}$	-39.2070	B29*(B26^B17)	
39	AC32 P0^$^{\beta2}$	18.9083	B30*(B26^B18)	
40	PV	172.2232	B26/B7	
41	ODE	0.0000	0.5*(B5^2)*(B3^2)*B43+(B6-B7)*B3*B42-B6*B24	
42	VC Δ	11.8246	B17*B27*(B3^(B17-1))	
43	VC Γ	1.4523	B17*(B17-1)*B27*(B3^(B17-2))	
44	ROV	40.8466	B27*(B3^B17)	
45	P0^	6.8889	B7*((B17/(B17-1))*B4-((B17-B18)/(B17-1))*B30*(B26^B18))	
46	Part I	235.6978	(B17/(B17-1))*B4	
47	Part II	3.3570	((B17-B18)/(B17-1))	
48	Part III	-7.7580	((-((B13-B12))*B8^(1-B18)-((1-B13)*B9^(1-B18)))*-B33/B32)	
49	Part IV	0.0400	(-(B6*(B17-1)-B7*B17))	
50	Part V	0.0040	(B6*(B17-B18)*B7)	
51	P0^	6.8889	B7*(B46-(B47*B48*(-B33/B32)*(B26^B18)))	