Not just the option to exclude: Valuing patents as a portfolio of options*

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Abstract

We use a compound real options model to investigate the impact of product market characteristics on patent value by considering Imperfect patent protection. Patent enforcement can be regarded as a portfolio of options. Once the patent lawsuit is filed, an alleged infringer firm (challenger) and an infringed firm (incumbent) pay for their ongoing litigation cost using operating cash flows from product market profits. We consider the challenger's strategy to exit the market during litigation due to shortage of funds, the incumbent's strategy to withdraw from value-reducing litigation or to force the challenger to exit the market by a threat to litigate, and firms' strategies to set up royalty payments to avoid a lawsuit, or to settle with each other after a lawsuit is filed. We distinguish between the effects of litigation and settlement on patent values and show first that settlement options raise patent values. By focusing on each firm's ability and willingness to pay for litigation costs, we find that product market characteristics such as the challenger's profit relative to the incumbent's loss of profits due to the alleged infringement (gain-to-loss ratio) has to be high enough for settlements to be possible. Settlements are also more likely in less volatile product markets, with more questionable patent validity, and when litigation costs are similar for the two firms. Our model generates new testable implications regarding patent values in a rigorous and comprehensive way.

Keywords: *Patent value, patent litigation, settlement, real options*

JEL Classification: *O34; L10; D81; C73*

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Extended abstract

Patents are important component of firm value, particularly for innovation intensive firms (Hall and Ziedonis, 2001; Nagaoka et al., 2010). Yet theoretical methods for valuing patents are not well developed. The existing literature has focussed on the right patents give the holder to litigate against infringing firms and thereby exclude other firms from the market [1] and have largely ignored the additional options conferred by a patent.

A patent gives its holder the right, defendable in law, not only to exclude unwanted firms from using the protected intellectual property (IP), but also, should this prove more profitable, to allow firms to use the IP on payment of royalty or licensing fees. Due to the high cost, firms tends to settle either before or after the lawsuit is filed and less than 5% of the cases go through final judgment (Lemley and Shapiro, 2005). Both the exclusion of unwanted competitors and the credibility to require royalty payments rely on the right given by the patent to litigate and thereby potentially exclude competing firms. In the absence of this right to litigate, there would be little reason for a user of IP to pay royalties (in the absence of any signalling effects, which we do not consider).

We thus view a patent as giving a number of inter-related options or rights - to decide who is legally allowed to utilise the IP as well as the right to enforce that decision by litigation. The economics of litigation and settlements were investigated in the early literature (e.g., Landes, 1971; Ordover et al., 1983; Bebchuk, 1984; Bebchuk, 1996; Choi, 1998). More recently, patent litigation has caught the attention of the economics literature, especially after the Leahy-Smith America Invents Act was enacted in 2012 (e.g., Bessen and Meurer, 2006; Choi and Spier, 2018; Lee et al., 2019). Nevertheless, little is known regarding the patent value by considering all possible options in the patent enforcement process. Marco (2005) is the first study that treats patent as an option to bring a lawsuit against an alleged infringer, but he does not model any other strategies. Our goal is to fill that gap from a corporate finance perspective by considering patent values based on firms' various options in patent enforcement procedure. We aim to provide a baseline understanding of the following questions: How to quantify patent value by considering patent enforcement (i.e., firms' settlement and litigation decisions)? How does the relationship between the infringed and the infringing products, in terms of market sizes and shares, affect the patent value?

To answer these questions, we build a compound real options model to value a patent both before and after infringement of the patent by a competing firm. Our model incorporates not only the right of the patent holder to litigate, taking account of the associated litigation costs for both the patent-holder or incumbent and the challenger, but also the resulting options the two parties have to settle, either before or after litigation has commenced. The option to settle will be exercised only if settlement (weakly) increases value for both parties via the reduction in associated costs. We separate out the components of value arising from the option to litigate in order to exclude and the additional value of the option to settle and investigate the drivers of value in each case.

[1] Indeed in many patent race models, the patent is assumed to confer monopoly rights (Denicolò, 2000; Meng, 2008).

Our model shows the value of a patent to its holder is significant, even after infringement by a competitor and that product market characteristics, i.e. the firm's competitive environment, is a leading order influence on the magnitude of patent value. Infringement causes a loss in profitability for the patent-holder and it is the expected reinstatement of the original higher profit level after the court's judgement which provides the incentive for patent-holders to litigate. Naturally, the magnitude of this reduction in profitability and the probability of winning at trial have significant positive impacts on patent value, whereas, as in Marco (2005), increases in the incumbent's litigation cost reduce the value of this option to litigate and hence the patent value. However, the competitor's gain in profits and costs also have an impact on the overall value of the patent to the patent-holder. This arises partly because the higher the competitor's profits, the greater their ability to pay royalties. This increases the likelihood that settlement will be feasible. There is also a more subtle effect due to the differences in the two parties' willingness to continue to finance the litigation, which impacts the value of the option to litigate and exclude but also the likelihood and value of settlement.

Overall we show that patent values vary depending on the relative magnitudes of the challenger's gain and patent-holder's loss in profits (the "gain-to-loss ratio"), and also on the challenger's and patent-holder's relative saving of costs from settlement through their willingness to continue to finance litigation. The value of the option to litigate and exclude is particularly valuable when the incumbent's loss in profits is large and significantly greater than the challenger's gain. In contrast, settlement is only feasible when the challenger's gain in profits is sufficiently large relative to the incumbent's loss. Our initial findings also suggest both the option value of litigation and of settlement depend on the relative willingness of each firm to continue financing litigation.

We then extend our analysis to value a newly-granted patent, i.e. before any infringement, continuing to define the patent value as the increase in firm value as a result of the existence of the patent. We recognise that the existence of a patent does not guarantee monopoly profits. Instead we explicitly model any challenger's infringement decision and show that the existence of the patent not only delays infringement (relative to equivalent unpatented IP) but also increases the patent-holder's firm value after infringement because of the options to litigate (and potentially exclude the challenger from the market, recovering monopoly profits) and to settle (and receive agreed royalty payments). Nevertheless, by recognising that infringement may occur, we show the impact of the costs the firm is forced to incur if it needs to enforce the legal rights conferred by the patent. These reduce patent value relative to the pure monopoly case but produce a more realistic patent value.

Finally, we model a firm's R&D incentives given this more realistic characterisation of patent value. We show that, relative to unpatented IP, the additional value associated with holding a patent on IP feeds back into a greater commitment to R&D at the research stage, i.e. a lower R&D abandonment threshold, so firms are less likely to abandon research before completion.

This paper takes a further theoretical step towards quantifying patent value as a portfolio of options by considering patent enforcement process. We build on Marco (2005), but recognise that patent value is not only the value of the option to litigate but represents a portfolo of options to settle, litigate or force out[2]. This provides a link to the most common way used to value patents in industry, which is to use the royalty payments. We model firms strategies in patent lawsuit including royalty payment in a rigorous way and can present patent value and the impact of product market characteristics on patent value in various cases.

This paper is one of the first studies to examine the effect of product market characteristics on firms' strategies in litigation and the litigation outcomes, as well as the impact on patent value and each of its different components. We contribute to this literature by establishing the importance of the relationship between the plaintiff's and the defendant's product markets and re-examining the role of market volatility in the context of patent litigation. We believe that the gain-to-loss ratio, i.e., the challenger's gain in profitability as a proportion of the incumbent's loss in profitability as a result of infirngement, is a first-order factor in determining firms' litigation strategies, similar to the previously argued judgment amount, litigation cost, and information asymmetry (e.g. Spier (2007); Hughes and Snyder (1995); Bebchuk (1984)).

This paper also adds to the recent discussion of how financing considerations affect litigation (e.g., Cohen et al. (2016); Choi and Spier (2018)). A number of papers have modeled both generic litigation and patent enforcement using litigation in real options models (e.g., Grundfest and Huang, 2005; Marco, 2005; Jeon, 2015). New to this literature, our model incorporates the possibility that the defendant may exit during litigation due to its inability to pay for the litigation cost. As Lee et al. (2019) find, defendants in patent litigation become much more financially constrained.

As in Jeon (2015), we model two risk-neutral all-equity firms operating in the same product market and their decision makers maximize the firm values. The incumbent ("I") owns a patent, and the challenger ("C") has allegedly infringed the patent[3]. Based on the patented technology, each firm generates a net operating income linear of the stochastic market demand x_t in each period. The market demand follows a geometric Brownian motion (GBM):

$$dx_t = \mu x_t dt + \sigma x_t dW_t, \tag{1}$$

where the growth rate of the demand $\mu < r$ (the risk-free rate), $\sigma > 0$, and W_t is standard Brownian motion.

Figure 1 presents the game tree of the model, which begins with an alleged infringement. Before the alleged infringement, the incumbent earns a monopoly flow profit $\pi_1 x$. After the alleged infringement, the incumbent's flow profit drops to $\pi_2^I x$, with $\pi_2^I < \pi_1$, whilst the challenger receives a duopoly flow profit of $\pi_2^C x$. Denote the total size or profits of the duopoly as $\pi_2 = \pi_2^C + \pi_2^I$. The model is flexible to study the situations in which the total

[2] Jeon (2015) considers the possibility of settlement as well as litigation as the outcome of patent infringement, but focuses on the potential outcome of infringement rather than the patent value.

[3] Bessen and Meurer (2005) document the opportunistic patent litigation occurs substantially, which refers to patent lawsuits that rely on weak patents to induce licensing without observing the infringement.

market profit increases (i.e., $\pi_2 > \pi_1$) or not (i.e., $\pi_2 \leq \pi_1$) as a result of the alleged infringement. In describing the game, we separate it into the part before and the part after a potential litigation. Firms' strategies take the form of optimal timing decisions, which are equivalent to threshold strategies under standard assumptions.

[Insert Figure 1 here.]

We solve the model using backward induction:

During litigation Once the incumbent litigates, both firms incur a flow litigation cost (C_l^I for the incumbent and C_l^C for the challenger) until the lawsuit ends. Any firm strategy after litigation starts features a put option, as the firm abandons litigation upon exercising the option/taking the action of the strategy. The game proceeds with one of the four possible ways if the court is not ruled before the action threshold is reached: (1) "I-withdraw": the incumbent withdraws from the litigation when the market demand drops to or below his withdraw threshold (represented by x_w) for the first time after litigation starts, i.e., $x_t \leq x_w$. After his withdrawal, the incumbent keeps sharing the duopoly profits with the challenger. (2) "C-exit": the challenger exits the market once the market demand drops to her exit threshold x_e, i.e., she ceases to sell any products using the technology related to the patent once $x_t \leq x_e$. After her exit, the challenger stops getting any operating income whilst the incumbent restores his monopoly profit. (3) "ex-post settlement": the two firms agree to settle after the litigation starts. Similar to ex-ante settlement, we follow Lukas et al. (2012) in modelling the ex-post settlement. The royalty rate, settlement threshold, and the one-time settlement costs in ex-post settlement are represented by $(\theta_p, x_p; C_s^I, C_s^C)$. Ex-post settlement may happen after litigation starts for a while ("later ex-post settlement" with $x_p < x_l$) , or right at the litigation ("immediate ex-post settlement" with $x_p = x_l$). After ex-post settlement, both firms earn duopoly profits whilst the challenger pays royalty fees to the incumbent in each period thereafter. (4) "court ruling": the court reaches judgement regarding the alleged infringement, which is equivalent to the court deciding which firms wins at the trial in this model.[4]

Among the aforementioned four possibilities that terminate a patent litigation after it starts, court ruling is modelled as exogenous with two parameters λ and p (Jeon, 2015). For model tractability, we assume the court ruling happens at a random time τ, and follows a Poisson process with a mean arrival rate λ. The expected time of the court ruling is therefore $E(\tau) = \frac{1}{\lambda}$. With probability p, the incumbent wins at the ruling, and restores its monopoly profit $\pi_1 x$ whilst the challenger is ordered to leave the market and gets nothing afterwards. With probability $1 - p$, the challenger wins at the court ruling, and the two firms keep sharing the duopoly profits $(\pi_2^I x, \pi_2^C x)$. The probability parameter p is assumed to be common knowledge (Lemley and Shapiro, 2005) and remains unchanged during the course of litigation. The assumption that court ruling happens at a random time

[4] In a patent litigation case, a counterclaim can be filed against the plaintiff (i.e., the incumbent in the model) in the form of an invalidity case. For simplicity, we assume the result of court ruling/judgement is either confirming the infringement ("the incumbent wins") or invalidating the patent ("the challenger wins").

makes it impossible for us to investigate the definite outcome during litigation. Therefore, we focus on examining the *likely outcome* during litigation, which refers to the outcome if the court has not yet ruled by the time the firms take actions. The likely outcomes during litigation include non-settlement (either I-withdraw or C-exit) and ex-post settlement.

Proposition 1. *During litigation, the firm values for the incumbent and the challenger are*

$$V_{dl}^{I}(x) = \Big(\frac{\pi_2^I}{r-\mu} + p\delta(\pi_1 - \pi_2^I)\Big)x - H_l^I + B_d^I x^{\beta_\lambda}, \tag{2}$$

$$V_{dl}^{C}(x) = \Big(\frac{\pi_2^C}{r-\mu} - p\delta\pi_2^C\Big)x - H_l^C + B_d^C x^{\beta_\lambda}, \tag{3}$$

where $\delta = \frac{1}{r-\mu} - \frac{1}{r+\lambda-\mu}, \beta_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}} < 0, H_l^I = \frac{C_l^I}{r+\lambda}, H_l^C = \frac{C_l^C}{r+\lambda}$*, and* B_d^I *and* B_d^C*,* $d \in \{w, e, p\}$*, are the arbitrary constants to be determined by real options during litigation.*

Before litigation Facing the alleged infringement, the two firms can sign licensing agreement to avoid the costly litigation (“ex-ante settlement”), or the incumbent can file a patent infringement lawsuit against the challenger if they fail to reach ex-ante settlement, or the incumbent may make a threat of litigation to force the challenger out of the market before litigation starts if the ongoing litigation cost for the challenger would be sufficiently high that she would rather exit the market than continuing in the market when litigation starts (”forcing-out”), or no firm takes any actions.

In modelling ex-ante settlement between the two firms, we follow Lukas et al. (2012): the incumbent proposes a royalty rate θ_a to the challenger, which is the fraction of the challenger’s future profit payable to the incumbent. Upon receiving this settlement offer, the challenger decides whether and when to accept the offer, i.e., the settlement threshold x_a at which ex-ante settlement happens. By reaching an ex-ante settlement agreement, both firms agree to settle with the royalty rate proposed by the incumbent at the settlement threshold determined by the challenger (i.e., θ_a and x_a), and that terminates the game. Meanwhile, both firms pay a one-time cost in settlement, C_s^I for the incumbent and C_s^C for the challenger. If the firms fail to settle, either because the incumbent refuses to offer settlement, or the challenger rejects the incumbent’s offer, then the incumbent can start a patent infringement lawsuit against the challenger. Because the incumbent’s litigation decision features a call option, he optimally starts the litigation once the market demand rises to or above a relatively high level (denoted as x_l). That is, the incumbent litigates whenever the demand $x_t \geq x_l$. The game proceeds to the next stage if the litigation happens. Alternatively, the incumbent may force the challenger out of the market without starting a litigation, and ends the game. However, whether the incumbent’s forcing-out strategy is relevant or not depends on the how market demand fluctuates after the alleged infringement (i.e., whether x_t drops to certain level first or goes up to certain level first), as well as what would be the likely non-settlement outcome during litigation (i.e., I-withdraw or C-exit defined in the next part of the game below). Due to the nature of this dynamic game, firms take no actions only during the period of time that the demand level has not yet reached any action threshold.

Proposition 2. *After the alleged infringement and before firms take any action(s), firms' value functions depend on the likely non-settlement outcome during litigation (i.e., I-withdraw or C-exit):*

$$(V_{bl}^I, V_{bl}^C) = \begin{cases} (\dfrac{\pi_2^I x}{r-\mu} + A_{bl}^I x^\alpha, \dfrac{\pi_2^C x}{r-\mu} + A_{bl}^C x^\alpha), & \textit{I-withdraw} \\ (\dfrac{\pi_2^I x}{r-\mu} + a_{bl}^I x^\alpha + b_{bl}^I x^\beta, \dfrac{\pi_2^C x}{r-\mu} + a_{bl}^C x^\alpha + b_{bl}^C x^\beta). & \textit{C-exit} \end{cases} \tag{4}$$

where $\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$, $\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$ *and the arbitrary constants* A_{bl}^I, A_{bl}^C, a_{bl}^I, a_{bl}^C, b_{bl}^I *and* b_{bl}^C *are determined by the incumbent's litigation option or the firms' option to settle ex-ante (i.e., license).*

Before infringement Based on our main model, we can further investigate the incumbent's innovation decision and the challenger's market entry decision in which she uses technologies similar to the patented one. The challenger is subject to litigation risk for the potential infringement.

For simplicity, we assume the incumbent can pay a one-time cost of C_r to obtain the patented innovation, and he gets the monopoly profit flow after he exercises the innovation option but gets no profits or any cash flows before his innovation. After the incumbent's innovation and his market entry, the challenger can pay a one-time cost of C_g to enter the market with a suspected infringement, after which the two firms receive duopoly profits. We use V_r to denote firm values before the incumbent's innovation, and V_g for firm values after the incumbent's innovation but before the challenger's market entry.

To obtain the action threshold on the market demand for alleged infringement, note that the challenger decides on the market entry with suspected infringement and the incumbent is the recipient of that decision. Thus we apply the value-matching condition and the smooth-pasting conditions for the challenger and only the value-matching condition for the incumbent at the relevant threshold x_g on their firm values. for both firms and the smooth-pasting condition for the challenger for the firm value with the option to infringe and the value before litigation at the infringement threshold x_g.

Firm values before the challenger's alleged infringement for $x < x_g$ are

$$V_g^C(x) = A_g^C x^\alpha, \quad V_g^I(x) = \frac{\pi_1}{r-\mu} x + A_g^I x^\alpha, \tag{5}$$

where $A_g^i, i = \{I, C\}$ are defined in Corollary 1.

Corollary 1. *The arbitary constants in value functions with the infringement option V_g can be expressed as.*

$$(A_g^C, A_g^I) = \begin{cases} \left(\left[\frac{\pi_2^C}{r-\mu} x_g + A_{bl}^C x_g^\alpha - C_g\right] x_g^{-\alpha}, \left[\frac{\pi_2^I - \pi_1}{r-\mu} x_g + A_{bl}^I x_g^\alpha\right] x_g^{-\alpha} \right) & \textit{I-withdraw}, \\ \left(\left[\frac{\pi_2^C}{r-\mu} x_g + a_{bl}^C x_g^\alpha + b_{bl}^C x_g^\beta - C_g\right] x_g^{-\alpha}, \left[\frac{\pi_2^I - \pi_1}{r-\mu} x_g + a_{bl}^I x_g^\alpha + b_{bl}^I x_g^\beta\right] x_g^{-\alpha} \right) & \textit{C-exit}. \end{cases} \tag{6}$$

The infringement threshold is $x_g = \frac{\alpha(r-\mu)C_g}{(\alpha-1)\pi_2^C}$ in I-withdraw and it satisfies $(\alpha-1)\frac{\pi_2^C x_g}{r-\mu} + (\alpha-\beta) b_{bl}^C x_g^\beta - \alpha C_g = 0$ in C-exit. where A_{bl}^I, a_{bl}^C and b_{bl}^C are defined in Proposition 2.

Before innovation We seperate the incumbent's innovation in two stages. The first stage is the research stage where the incument pays flow cost C_r per period on R&D and obtian breakthrough that modelled as possion process with hazard rate ϵ. The value function with the option to abandon is represented by V_r^I. The second stage is the develoment stage. Once R&D succeed, the incumbent has the option to produce with cost C_d and it is represented by V_d^I.

Using backward induction, we first consider the second stage of development. The value function for the incumbent can be expressed as

$$V_d^I = A_d^I x^\alpha. \tag{7}$$

Applying smooth-pasting and value-matching condition between V_d^I and V_g^I, we obtain Corollary 2.

Corollary 2. *In the incumbent's value function with his option to develop the invention V_d^I, we have:The arbitrary constant $A_d^I = \left[\frac{\pi_1}{r-\mu} x_d + A_g^I x_d^\alpha - C_d\right] x_d^{-\alpha}$ in I-withdraw, and $A_d^I = \left[\frac{\pi_1}{r-\mu} x_d + A_g^I x_d^\alpha - C_d\right] x_d^{-\alpha}$ in C-exit, where A_g^I is defined in Corollary 1 for I-withdraw and C-exit respectively. The innovation threshold $x_d = \frac{\alpha(r-\mu)C_d}{(\alpha-1)\pi_1}$.*

In the first stage of research, the incumbent's value function with the option to research for $x \geq x_r$ can be written as

$$V_r^I = B_r^I x^{\beta_\epsilon} - \frac{C_r}{r+\epsilon} + A_d^I x^\alpha, \tag{8}$$

where $\beta_\epsilon = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\epsilon)}{\sigma^2}}$, A_d^I is defined in Corollary 2 and B_r^I is the abandonment option to be determined.

We can obtain the innovation abandonment threshold and arbitrary constants in the first stage,

$$x_r = \left[\frac{\beta_\epsilon}{(\beta_\epsilon - \alpha) A_d^I} \frac{C_r}{r+\epsilon}\right]^{\frac{1}{\alpha}}, \quad B_r^I = -\frac{\alpha}{\beta_\epsilon} A_d^I x_r^{\alpha - \beta_\epsilon}. \tag{9}$$

Firms' value without patents Without patent system, the model can be simplified as a duopoly setting where there is a leader who innovates first and develops a product and a follower who enters the market without the risk of patent litigation by paying the lump-sum cost C_g. If there are two firms in the market, they share the market with duopoly profit $\frac{\pi_2^I}{r-\mu}x$ for the incumbent and $\frac{\pi_2^C}{r-\mu}x$ for the challenger. Therefore, the post-infringement patent value for the incumbent V_p can be quantified as the difference between the patent value before litigation V_{bl}^I defined in Proposition 2 or the payoff in settlement or litigation depending on the market demand and the duopoly profit $\frac{\pi_2^I}{r-\mu}x$.

Proposition 3. *If $x < x_{l/a}$, the post-infringement patent value can be expressed as*

$$V_p = V_{bl}^I - \frac{\pi_2^I}{r-\mu}x = \begin{cases} A_{bl}^I x^\alpha, & \textit{I-withdraw,} \\ a_{bl}^I x^\alpha + b_{bl}^I x^\alpha, & \textit{C-exit and } x > x_e, \\ \frac{\pi_1 - \pi_2^I}{r-\mu}x. & \textit{Force out when C-exit and } x \leq x_e. \end{cases} \tag{10}$$

where A_{bl}^I, a_{bl}^I and b_{bl}^I are defined in Proposition 2.

If $x \geq x_{l/a}$, the post-infringement patent value can be expressed as

$$V_p = V_{payoff}^I - \frac{\pi_2^I}{r-\mu}x = \begin{cases} \frac{\theta_a \pi_2^C}{r-\mu}x - C_s^I, & \textit{Ex-ante settlement,} \\ p\delta(\pi_1 - \pi_2^I)x - H_l^I + B_d^I x^{\beta_\lambda}. & \textit{During litigation.} \end{cases} \tag{11}$$

Before the challenger enters, the incumbent earns monopoly profit $\frac{\pi_1}{r-\mu}x$. Therefore, their firm values before the challenger enters/infringes can be expressed as

$$V_g^{I,NP} = \frac{\pi_1}{r-\mu}x + A_g^{I,NP}x^\alpha, \qquad V_g^{C,NP} = A_g^{C,NP}x^\alpha. \tag{12}$$

Applying the value-matching and smoothing pasting at the challenger's entry threshold x_g^{NP}, we obtain

Corollary 3. *Without patent system, firms' value functions before infringement follows Eq. (12). The arbitrary constants and infringement thresholds in the functions are*

$$A_g^{C,NP} = (\frac{\pi_2^C}{r-\mu}x_g^{NP} - C_g)(x_g^{NP})^{-\alpha}, \qquad A_g^{I,NP} = \frac{\pi_2^I - \pi_1}{r-\mu}(x_g^{NP})^{1-\alpha}, \qquad x_g^{NP} = \frac{\alpha C_g(r-\mu)}{(\alpha-1)\pi_2^C}, \tag{13}$$

Specifically, the entry threshold x_g^{NP} is the same with the entry threshold with patents x_g in the I-withdraw case, whereas in the C-exit case, the non-patent infringement threshold x_g^{NP} is lower than infringement threshold with patent x_g. This shows that with patent system, the infringement threshold is delayed in the case when the challenger's unwillingness to continue to finance litigation is stronger.

As a result, the patent value can be quantified as the difference between the incumbent's value with patents and without patents, i.e., $V_g^I - V_g^{I,NP}$. Therefore, the pre-infringement patent value $\hat{V}_p$ that can be expressed as

$$\hat{V}_p = \begin{cases} A_{bl}^I x^\alpha, & \text{I-withdraw,} \\ \frac{\pi_2^I - \pi_1}{r-\mu} x^\alpha [x_g^{1-\alpha} - (x_g^{NP})^{1-\alpha}] + a_{bl}^I x^\alpha + b_{bl}^I x_g^{\beta-\alpha} x^\alpha. & \text{C-exit.} \end{cases} \tag{14}$$

where A_{bl}^I, a_{bl}^I and b_{bl}^I are defined in Proposition 2. The infringement threshold x_g and x_g^{NP} are defined in Corollary 1 and 3.

We then move to the earlier stage, i.e., before innovation by the incumbent. In this case, we can obtian the development threshold $x_d^{NP} = \frac{\alpha(r-\mu)C_d}{(\alpha-1)\pi_1}$ and innovation abandonment threshold $x_r^{NP} = [\frac{\beta_\epsilon}{(\beta_\epsilon - \alpha)A_d^{I,NP}} \frac{C_r}{r+\epsilon}]^{\frac{1}{\alpha}}$. We then have $x_r^{NP} > x_r$ because $A_d^{I,NP} < A_d^I$.

Theorem 1. *The innovation abandonment threshold with patent is lower than without patent (i.e., $x_r^{\text{no patent}} > x_r^{\text{with patent}}$).*

We use numerical methods to analyse and list our baseline parameter values in Table 1, which is similar in Jeon (2015). The risk-free rate is set at $r = 0.05$, the growth rate of the demand shock is $\mu = 0.02$, and the volatility of the demand condition is $\sigma = 0.3$. The average duration of litigation is 2.5 years, as suggested by the empirical evidence on patent litigation in the US. The probability of patent validity p is 0.5 at the baseline. For simplicity, we assume the costs of ex-ante settlement and ex-post settlement are the same and the costs of litigation and settlement are assumed equal for the two parties (i.e., $\Gamma = 1$). Competition reduces the overall market profit, i.e. $\pi_1 (= 1.2) > \pi_2 (= 1)$. This represents an infringement where the challenger's product is a close substitute. The duopoly profits are such that the incumbent earns more profit than the challenger after the infringement, $\pi_2^C = 0.3$ and $\pi_2^I = 0.7$ (i.e., $\Phi = 0.6$).

[Insert Table 1 here.]

In Figure 2, we plot the different possible outcome regions if the market demand reaches the litigation threshold, as we vary the gain-to-loss ratio Φ and relative cost saving Γ. The green area is where ex-post settlement is possible, i.e., the incumbent does not litigate. The blue area is where ex-ante settlement is possible, i.e., firms reach an agreement to settle ex-post rather than continuing litigation. We also use the lighter blue coloured region to represent the area where firms are likely to settle immediately. The blank region is where the settlement does not occur. Specifically, above the dashed line, the challenger exits first. Below the dashed line, the incumbent withdraws first.

[Insert Figure 2 here.]

Figure 2 shows that Φ has to be high enough to make ex-ante or ex-post settlement possible. From the incumbent's perspective, when Φ is high enough, he does not suffer a significant loss due to the challenger's entry. This reduces the potential gain from continuing costly litigation (i.e. $\pi_1 - \pi_2^I \downarrow$), so the incumbent is willing to settle. When Φ is low, the incumbent suffers a significant loss due to the challenger's infringement (i.e. $\pi_1 - \pi_2^I \uparrow$), increasing his incentives to litigate. Increasing Φ, on the other hand, raises the challenger's profit π_2^C, increasing her financial capability to pay the settlement royalty that the incumbent required.

The graph also depicts that firms in the withdrawal region (bottom) are more likely to settle immediately when compared to firms in the exit region (top). In the withdrawal region, when the market demand first reaches the immediate settlement threshold, the incumbent's value of settling immediately exceeds his value of waiting to litigate as the incumbent is in a disadvantageous position and is less likely to wait. In the exit region, the incumbent in patent litigation is less financially constrained than the challenger and can force the challenger to exit with the threat of litigation, and thus is willing to wait to litigate and then settle ex-post.

The figure shows that settlement is more likely when the gain-to-loss ratio and relative cost saving are close to the boundary between the withdrawal region. In general, this boundary represents the two firms' equal willingness to continue to finance litigation. The further away from this boundary, the less likely that firms agree to settle because one of the parties knows the other party is disadvantageous to fight until the court ruling. It is thus obvious that firms' willingness to continue to finance litigation relates to both the gain-to-loss ratio and the relative cost saving. In the exit region (i.e. above the dashed line), the incumbent is the main party who decides whether a settlement can be reached because the challenger prefers settlement. Therefore, the greater the incumbent's cost saving from settlement (the lower the relative cost saving) in this region, the wider the range of gain-to-loss ratios for which the incumbent is willing to offer ex-ante settlement instead of litigating. In the withdrawal region, the higher the incumbent's litigation cost, the greater is the likelihood that he will be unable to continue to pay the costs and thus will withdraw from the litigation. This increases the challenger's value of continuing instead of settling, and hence decreases the likelihood of settlement.

In Table 2, we show how post-infringement patent value defined in Proposition 3 changes with different litigation outcomes that are affected by product market characteristics. These tables list the exmaple patent values when the market demand x is set to be 1. Panel (a) depicts the patent value in our model with options to litigate, settle and force out, whereas panel (b) depicts patent values when ex-ante and ex-post settlement options are not included. In panel (c), we also show the difference in patent values between these two cases (i.e. (a)-(b)). The black dashed line denotes the boundary between regions where the challenger exits first and where the incumbent withdraws first.

[Insert Table 2 here.]

Two major observations can be found in Table 2. First, we discover that the gain-to-loss ratio has a negative impact on patent value since the patent value drops as the gain-to-loss ratio rises, as shown in panel (a). The gain-to-loss ratio, according to Crampes and Langinier (2002), measures the extent to which the two litigants' products are substitutes for one other. Intuitively, when the gain-to-loss ratio is low, two products in the market are substituted for each other; as a result, the market competition between them is more intensive, and the right to exclude the other party has a higher value, which is the patent value. Second, we are able to separate the impact of litigation and settlement on patent value because we model settlement as compound options, which has not been studied in previous literature. We demonstrate the rise in patent value by considering settlement options in panel (c) by comparing the patent value with settlement options as in panel (a) with the patent value without settlement options as in panel (b). On the one hand, when the outcome is non-settlement, the table demonstrates that there is no rise in patent value, i.e. the difference is zero. On the other hand, settlement occurs if the difference is positive (i.e., the gain-to-loss ratio is high enough), implying that it is critical to consider settlement options when quantifying patent value.

Our detailed model allows us to incorporate a richer set of components of value associated with holding a patent into patent valuation, and show the impact of these additional components: the option to force out a competitor because of the threat of litigation and the option to settle, have an overall patent value. Furthermore, by modelling the full portfolio of inter-related options associated with a patent, we are better able to identify the drivers of patent value at all stages in a firm's product development and life cycle. In further work, we plan to investigate the comparative statics with repect to other product market characteristics such as volatility σ and the probability that the incumbent wins in court ruling p. We will also quantify the value of the patent at earlier stage: before infringement, which can be thought of as the value of a newly-granted patent, as well as quantifying the impact of patent value on research incentives and the key determinants in this stage.

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Figure 1: Strategies of the incumbent ("I") and the challenger ("C") in the main model

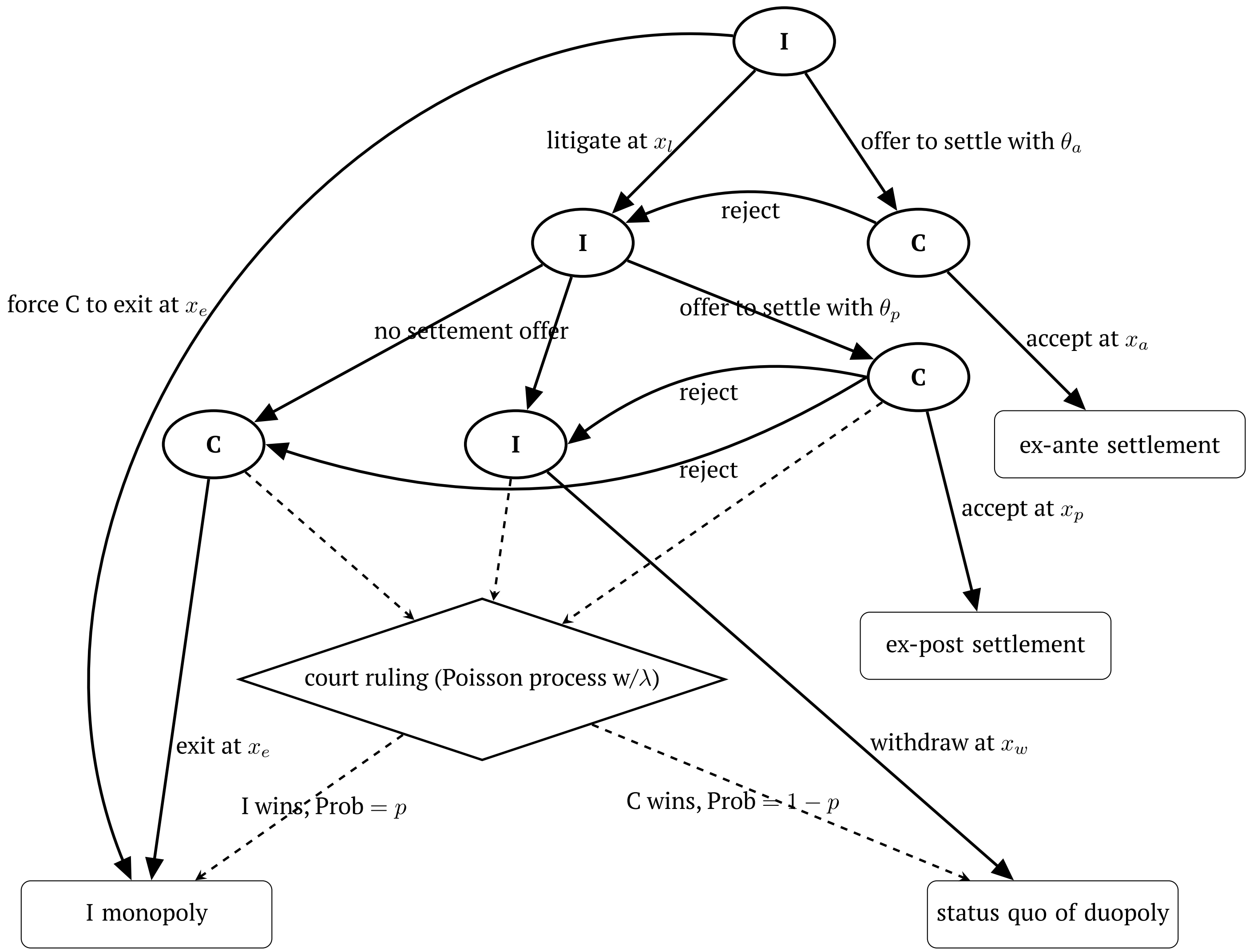

Figure 2: Possible outcomes in patent infringement

The green area is the possible region for ex-post settlement. The blue area is the possible region for ex-ante settlement , while the immediate settlement region is represented by light blue regions. The black dashed line represents the boundary between regions where the challenger exits first and where the incumbent withdraws first. The relative cost saving is defined as $\Gamma = \frac{H_l^C - C_s^C}{H_l^I - C_s^I}$, and the gain-to-loss ratio is defined as $\Phi = \dfrac{\pi_2^C}{\pi_1 - \pi_2^I}$. Other parameter values are given in table 1.

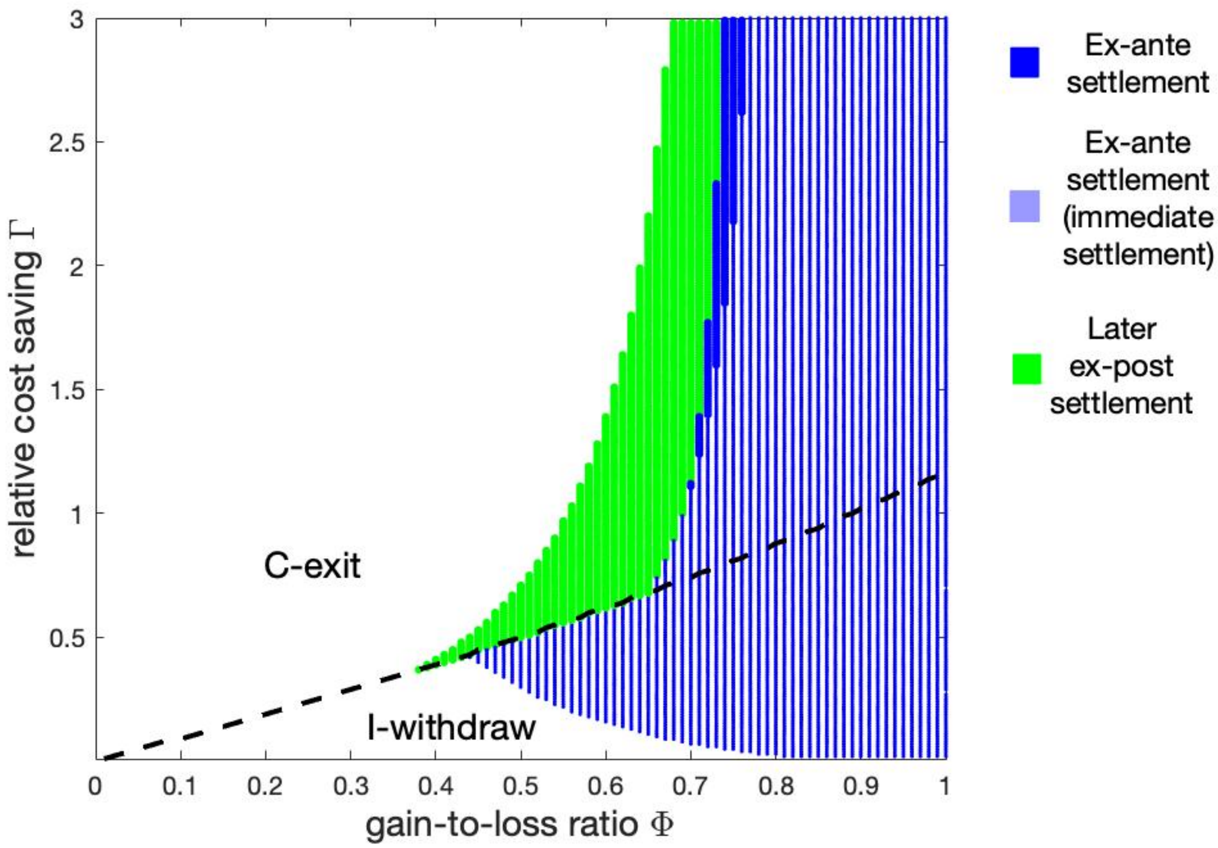

Table 1: Baseline model parameter values

Parameter	Value
Basics	
Risk free rate	$r = 0.05$
Arrival rate of court ruling ($\frac{1}{\text{year}}$)	$\lambda = \frac{1}{2.5}$
Probability of patent validity	$p = 0.5$
Growth rate/volatility of the demand shock	$\mu = 0.02, \sigma = 0.3$
I's monopoly profit multiplier (profit = $\pi_1 x$)	$\pi_1 = 1.2$
Duopoly profit multipliers	$\pi_2^I = 0.7, \pi_2^C = 0.3$
Flow litigation costs	$C_l^I = 1, C_l^C = 1$
One-time settlement costs for ex-post and ex-ante	$C_s^I = 0.5, C_s^C = 0.5$
Ratios	
gain-to-loss ratio	$\Phi = \frac{\pi_2^C}{\pi_1 - \pi_2^I} = 0.6$
relative cost saving	$\Gamma = \frac{H_l^C - C_s^C}{H_l^I - C_s^I} = 1$
Other greeks or expressions	
discount rate of x_t from court ruling onwards	$\delta = \frac{1}{r-\mu} - \frac{1}{r+\lambda-\mu} = 31.01$
constant in value functions of during litigation	$\beta_\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}} = -2.90$
constant in value functions of before litigation	$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}} = 1.37$
constant in value functions of before litigation	$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}} = -0.81$

Table 2: Post-infringement patent values in possible litigation outcomes

These tables list the exmaple post-infringement patent values defined in Proposition 3 when the market demand x is set to be 1. Panel (a) depicts the patent value in our model with options to litigate, settle and force out, whereas panel (b) depicts patent values when ex-ante and ex-post settlement options are not included as in prvious studies. In panel (c), we also show the difference in patent values between these two cases (i.e. (a)-(b)). The black dashed line denotes the boundary between regions where the challenger exits first and where the incumbent withdraws first.

(a)

Patent value

relative cost saving \ gain-to-loss ratio	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
3	46.07	22.58	14.77	10.88	8.55	7.01	6.62	5.37	4.91	4.32
2.8	46.03	22.54	14.74	10.85	8.53	6.99	6.58	5.36	4.89	4.29
2.6	45.99	22.5	14.7	10.82	8.5	6.96	6.54	5.35	4.87	4.26
2.4	45.94	22.46	14.66	10.78	8.46	6.93	6.48	5.34	4.85	4.23
2.2	45.88	22.41	14.61	10.73	8.42	6.89	6.42	5.33	4.82	4.19
2	45.82	22.34	14.55	10.68	8.37	6.85	6.35	5.32	4.79	4.14
1.8	45.73	22.27	14.49	10.62	8.32	6.79	6.25	5.3	4.75	4.08
1.6	45.63	22.18	14.4	10.54	8.25	6.73	6.14	5.28	4.7	4.01
1.4	45.5	22.06	14.3	10.45	8.16	6.65	5.99	5.25	4.63	3.92
1.2	45.33	21.91	14.16	10.33	8.05	6.58	5.8	5.21	4.54	3.8
1	45.09	21.7	13.98	10.17	7.91	6.76	6.14	5.44	6.79	6.35
0.8	44.74	21.41	13.73	9.95	7.72	6.76	6.1	6.51	6.04	5.66
0.6	44.19	20.95	13.35	9.62	7.54	6.71	6.09	5.61	5.23	4.92
0.4	43.15	20.15	12.7	9.37	6.02	5.39	4.92	4.56	4.27	4.03
0.2	40.56	18.36	8.65	5.74	4.23	3.76	3.46	3.23	3.04	2.88

(b)

Patent value without settlement options

relative cost saving \ gain-to-loss ratio	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
3	46.07	22.58	14.77	10.88	8.555	7.014	5.92	5.105	4.474	3.973
2.8	46.03	22.54	14.74	10.85	8.528	6.989	5.897	5.083	4.454	3.954
2.6	45.99	22.5	14.7	10.82	8.497	6.961	5.87	5.058	4.43	3.932
2.4	45.94	22.46	14.66	10.78	8.462	6.928	5.839	5.029	4.403	3.907
2.2	45.88	22.41	14.61	10.73	8.421	6.89	5.804	4.996	4.373	3.878
2	45.82	22.34	14.55	10.68	8.373	6.846	5.763	4.958	4.337	3.844
1.8	45.73	22.27	14.49	10.62	8.316	6.793	5.715	4.913	4.295	3.805
1.6	45.63	22.18	14.4	10.54	8.247	6.73	5.656	4.859	4.245	3.759
1.4	45.5	22.06	14.3	10.45	8.161	6.652	5.585	4.794	4.185	3.703
1.2	45.33	21.91	14.16	10.33	8.053	6.554	5.496	4.712	4.11	3.634
1	45.09	21.7	13.98	10.17	7.911	6.427	5.381	4.608	3.182	2.755
0.8	44.74	21.41	13.73	9.949	7.718	6.255	5.228	3.502	2.981	2.581
0.6	44.19	20.95	13.35	9.625	7.438	4.754	3.85	3.207	2.73	2.364
0.4	43.15	20.15	12.7	9.096	5.354	4.172	3.379	2.815	2.396	2.074
0.2	40.56	18.36	8.655	5.741	4.231	3.297	2.67	2.225	1.894	1.639

(c)

Difference in patent values with and without settlement options

relative cost saving \ gain-to-loss ratio	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
3	0	0	0	0	0	0	0.7	0.27	0.44	0.35
2.8	0	0	0	0	0	0	0.68	0.28	0.44	0.34
2.6	0	0	0	0	0	0	0.67	0.29	0.44	0.33
2.4	0	0	0	0	0	0	0.64	0.31	0.45	0.32
2.2	0	0	0	0	0	0	0.62	0.33	0.45	0.31
2	0	0	0	0	0	0	0.59	0.36	0.45	0.3
1.8	0	0	0	0	0	0	0.54	0.39	0.45	0.27
1.6	0	0	0	0	0	0	0.48	0.42	0.45	0.25
1.4	0	0	0	0	0	0	0.41	0.46	0.45	0.22
1.2	0	0	0	0	0	0.03	0.3	0.5	0.43	0.17
1	0	0	0	0	0	0.33	0.76	0.83	3.61	3.6
0.8	0	0	0	0	0	0.5	0.87	3.01	3.06	3.08
0.6	0	0	0	0	0.1	1.96	2.24	2.4	2.5	2.56
0.4	0	0	0	0.27	0.67	1.22	1.54	1.75	1.87	1.96
0.2	0	0	0	0	0	0.46	0.79	1.01	1.15	1.24