# Valuing Real Options with Endogenous Payoff

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**Abstract**

This study investigates irreversible investment decisions when the exercise payoff is scale-dependent; thus, it is endogenously determined by the firm's risk management. We find that the scale-dependency gives rise to a speculative risk management strategy: a positive relationship between the firm's derivatives position and unhedged cash flow. Moreover, investment can be hastened or delayed as the underlying uncertainty increases depending on the economic conditions due to the speculative strategy. The main force driving these results, different from those known in the existing literature, is that the firm's risk management is designed to optimize the risk-return trade-off of the endogenous payoff.

JEL classification: G11; G31; G32
Keyword: real options; endogenous payoff; scale-dependency; risk management; speculative usage of derivatives; risk-return trade-off

## 1 Introduction

This study investigates a mixed problem of a firm's optimal risk management and real option exercise decision under uncertainty. Specifically, we consider the model in which the firm's self-financing risk management strategy endogenously determines the payoff due to its scale-dependency. More precisely, the real option payoff depends on the underlying process (e.g., output price) and the amount of investment at the exercise time. We show that the scale-dependency gives rise to a speculative risk management strategy and the investment timing can be delayed or hastened depending on the economic conditions. We find that these results are driven by that the firm's risk management is designed to optimize the risk-return trade-off of the real option payoff.

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It is worth noting that in standard real option models, the firm only chooses the exercise time since the amount of investment is exogenously given.[1] This assumption is innocuous since it represents the lumpy investment (e.g., the total cost of building a plant or a factory). The technical advantage of this assumption is that the underlying payoff is separate from the amount of investment (the exercise price). Hence, a usual real option problem has the same structure as a perpetual American option problem. However, there are many cases where the payoff is also scale-dependent. The output increases as a firm invests more, which is also true for irreversible investments (see Example 1 in Section 2). In line of this feature, so-called capacity choice problems also have been considered in the literature (e.g., Dangl (1999), Huisman and Kort (2015), and Bensoussan et al. (2021)).

A novel and important departure from the literature is that we consider the case in which the real option payoff is endogenously determined by the firm's risk management strategy. Note also that in our model the firm's risk management strategy is *self-financing* due to two types of capital market frictions. The first one is a credit constraint: borrowing is limited and so is the size of investment when the firm exercise the real option. Second, a firm (or a decision maker in the firm) cannot freely sell the real option to others (or the firm does not want to sell it since selling incurs a huge discount). These frictions often arise because while the firm views that the real investment opportunity is fairly valuable, other parties, such as banks and lenders, significantly underestimate the value due to moral hazard, adverse selection, limited commitment, or other contractual restrictions. Then, with a given limited amount of capital, the firm needs to accumulate capital by using a self-financing risk management strategy before exercising the real option. We assume that there may exist a derivative or an insurance asset that can be used to hedge the idiosyncratic risk of the underlying process of the real option.[2] Therefore, the firm's problem is to maximize the value of investment, consisting of two components: an optimal real option exercise problem and an optimal control problem of allocating capital over time between insurance contracts and safe (or risk-free) assets. The problem is transformed into a stochastic control problem of a nonlinear Hamilton-Jacobi-Bellman (HJB) equation with a free boundary. We obtain explicit solutions and present the implications for the firm's risk management and investment strategy.

Our model provides two novel implications. First, the optimal risk management strategy turns out to be speculative, which seems risk-seeking. In our model, the firm's position of insurance holdings takes a positive value even if the insurance (risky) asset is positively correlated with the underlying price process and its risk premium is zero. This result is in sharp contrast with that of the usual portfolio selection and conventional risk management wisdom in which the hedge position

[1] There are models in which the investment cost is given as a stochastic process. However, the cost is still an exogenous variable unaffected by a firm's decision.

[2] We first focus on the case in which the insurance asset exists and is perfectly correlated with the underlying process of the real option, and then extend the model to more general cases: (i) the case in which no such asset exists and (ii) the case in which the correlation is imperfect.

takes a negative value if the hedge asset is positively correlated with the underlying asset process. Considering the real-estate development example, a typical risk management is to trade housing futures contracts. It is often taught in MBA classes that producers are recommended to *short* futures contracts on the commodity that they are selling for hedging (see, for example, Chapter 6 of McDonald (2013)). However, our result indicates that the real-estate development firm should take a *long* position for the housing futures contracts. Thus, the firm looks like utilizing derivatives not for hedging but for speculating. This result is even more counter-intuitive, considering that the firm's concave objective function essentially generates risk aversion. Moreover, the firm not only takes such a long position, but also extends the long position as the volatility of the underlying price increases. Although this risk-seeking behavior basically originates from the nature of real investment, it does not appear without the scale-dependency of the exercise payoff. It means that the fundamental driving force of the result is that the insurance strategy is used to optimize the *risk-return trade-off* of the payoff process, as explained below.

The second contribution of this study is that the optimal exercise threshold can increase or decrease with risk, i.e., the volatility of the underlying price process ($\sigma_x$). More precisely, the threshold increases (decreases, respectively) with $\sigma_x$ when the expected growth rate of the underlying process ($\alpha_x$) is small (large, respectively). This result also sounds counter-intuitive at first because the value of waiting decreases with $\sigma_x$ when the underlying economic environment is favorable ($\alpha_x$ is large). However, recall that in our model, the total payoff is the underlying price multiplied by the size of the output. An increase in $\sigma_x$ increases the absolute value of the proportion of insurance asset holdings; thus, it increases the expected return of the payoff process and its volatility (even when the insurance asset has no risk premium). However, a change in $\sigma_x$ affects the payoff risk and return by a different magnitude. To quantify their relative magnitude, we characterize the risk-return trade-off by investigating the Sharpe ratio of the total payoff process. When the underlying economic environment is unfavorable ($\alpha_x$ is small), the optimal risk management increases the expected return of the total payoff, which is substantial. Since the original expected return ($\alpha_x$) is small, the extra return added by the risk management outweighs the added extra risk, thus, increasing the Sharpe ratio. Therefore, the firm will have more incentives to delay investment as $\sigma_x$ increases for low $\alpha_x$. In contrast, the threshold decreases with $\sigma_x$ when the underlying economic environment is favorable (high $\alpha_x$). Here, the additional return is small relative to the size of the added risk added due to the risk management; thus, the Shape ratio decreases with $\sigma_x$, which deteriorates the value of waiting. We also present the impact of the parameter values different from $\sigma_x$ in the main body of the paper.

According to the conventional risk management theory, non-financial firms should use derivatives mostly for hedging, indicating a negative relationship between a firm's derivatives position and its unhedged cash flow. Compared with the theoretical clarity, the empirical literature is largely inconclusive. Guay (1999), Allayannis and Ofek (2001), and Bartram et al. (2011) obtained a reduction in risk of derivatives users, which is consistent with the hedging argument. However, speculative usage of derivatives has also been widely reported (e.g., Adama and Fernado (2006), Chernenko and Faulkender (2011), Faulkender (2005)), and Geczy et al. (2007)). Furthermore, Hentschel and Kothari (2001) obtained no difference in risk between derivatives users and non-users. Bartram (2019) reported that commodity price exposure of firms using commodity derivatives is marginally high. Guay and Kothari (2003) obtained that the magnitude of the derivatives position relative to firms' risk is quite small. In our theory, firms with scale-dependent real investment opportunities optimally take a positive derivatives position against the unhedged cash flow. Although firms as commodity producers generally have hedging motives, the existence of firms with scale-dependent investment opportunities in the sample can offset the hedging effect, which can lead to no relationship or even a slightly positive relationship between the firm's derivatives position and its risk. The above literature attributes firms' speculative usage of derivatives to that managerial incentives are designed in such a way or to that hedging with derivatives contributes to a small fraction of firms' overall risk management. We suggest another channel to potentially explain the positive relationship caused by firms' real investment decisions.

We hope that our theory can help shed light on the empirical debate by suggesting researchers to design more detailed empirical specifications. Existing empirical studies mostly compare derivatives users and non-users. Based on our theory, one can further compare firms with different degrees of scale-dependency in investment opportunities. Our theory predicts that among derivatives users, firms with high scale-dependency can use derivatives more speculatively. Additionally, among firms with high scale-dependency, derivatives users have a higher firm value but higher return volatility than non-users.

This study is broadly related to the real option literature; however, we do not summarize it since it is too vast and is quite well developed. We note that the main insight from the real option literature since Brennan and Schwartz (1985) and McDonald and Siegel (1986) is that an increase in risk increases the value of waiting; thus, delaying investment. This result is preserved for many extensions, including investment under uncertainty (see, e.g., Abel and Eberly (1994, 1996), Dixit and Pindyck (1994), and Guo et al. (2005)) and sequential expansion choice (see, e.g., Bensoussan and Chevalier-Roignant (2019) and references therein). A recent body of literature has pointed out that when the underlying process has idiosyncratic risk (the market is incomplete) and the agent is risk-averse, the result can be overturned depending on the level of risk aversion (Henderson (2007), Miao and Wang (2007), and Evans et al. (2008)) and according to the degree of time-discounting (Choi et al. (2017)). A similar feature is also found in American option exercise and optimal stopping problems when the market is incomplete (e.g., Carpenter (2000), Henderson and Hobson (2013), and Carpenter et al. (2010)). Additionally, the option value may not globally increase with its volatility if there is incomplete information; thus, the decision-maker learns the drift of the underlying process over time (Décamps et al. (2005)). However, our study deviates from the literature and proposes a different channel that affects investment timing. Our model considers the case in which the firm's risk management endogenously determines the payoff process, while the aforementioned literature has considered the case with the exogenous payoff.

Among the real investment studies, our paper is more closely related to the capacity choice literature such as Dangl (1999), Huisman and Kort (2015), and Bensoussan et al. (2021) in the sense that these papers study a firm's optimal investment timing decision when the payoff is dependent on its scale choice. Dangl (1999) studies the case in which a firm needs to choose the maximum production capacity when it invests, considering the future demand.[3] Huisman and Kort (2015) study the duopoly (leader-follower) capacity choice problem, in which the total demand is unchanged when a new producer enters the market. Bensoussan et al. (2021) consider the expansion timing problem of a levered firm that optimally chooses its debt capacity. Our paper is different from these papers in that the capacity choice is a one-time decision in these models. In our model a firm continuously monitors its derivatives position by choosing the optimal portfolio of insurance and risk-free assets over time until it exercises the real option, and this risk management decision eventually affects the exercise payoff. The main insight from our model is that an increase in the underlying uncertainty can increase or decrease the Sharpe ratio of the payoff process, depending on the economic environment, which can increase or decrease the value of waiting. We hope that our setup can shed light on understanding different aspects of the role of risk management when a firm faces irreversible investment decisions.

Finally, the firm's risk-seeking behavior in our model is similar to the result of Henderson and Hobson (2013) where an agent optimally chooses to participate in a risk-increasing gamble when the risk-averse agent decides on the optimal timing of selling an indivisible asset. However, the fundamental reason behind our result differs from that suggested by Henderson and Hobson (2013). The risk-averse agent indeed engages in a pure gamble in Henderson and Hobson (2013): the agent is better off by buying a gambling asset with a zero risk premium uncorrelated with the underlying asset since it leads to a local convexity in the agent's value function. However, the market incompleteness in our model is not necessary for deriving the risk-seeking result as in Henderson and Hobson (2013). Moreover, buying or selling such an asset does not generate any value or lead to a local convexity. Thus, in terms of risk management, the firm optimally chooses never to trade such a gambling asset (see Proposition 8). Instead, the firm's optimal choice is to buy a correlated asset that provides rewards when there is good news on the underlying price and damages when there is bad news on the price, which seemingly amplifies only risk. Furthermore, the firm increases its bet on the correlated asset as the underlying uncertainty increases and even chooses to buy an asset with a negative risk premium. The fundamental force driving these results in our model is that the firm's risk management strategy is designed to optimize the risk-return trade-off. Thus, a firm's risk-seeking insurance strategy seems at odds, but it is in fact optimal.

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3

[3] For example, when a dam is built, the maximum capacity for electricity generation is set.

explains our solution analysis. We investigate the main implications in Section 4. Section 5 provides a discussion on the case where the insurance asset has an imperfect correlation with the underlying process. It also shows that the firm never chooses a pure gamble in our context. Finally, Section 6 presents the conclusion. All proofs are given in appendices.

## 2 Model

To help understanding of the model we present an example first.

**Example 1** (Real Estate Development Company). *Consider a real estate development company having a parcel of vacant land and plans to build condos on the land.*[4] *The value of the land can be considered the value of developing the real estate (Titman (1985) and Grenadier (1996)). The firm will consider when to build condos, as well as the number of condos to build and variety of luxuries to add to a condo unit. If the firm has a tight budget at the investment time, it can build only a two-story building with simple facilities (e.g., four units of condos). If its budget is sufficiently large, it can build a five-story building with luxury options (e.g., ten units of condos). The fundamental uncertainty originates from the condo price; thus, the price is the underlying variable. However, the firm's total payoff is not the price but the condo price multiplied by the total number of condos built at the exercise time. Additionally, the unit price increases if the firm attaches luxury options into each condo by investing more. Therefore, as soon as the firm creates the condo-building project, it naturally must engage in self-financing risk management to maximize its profits. More precisely, the firm decides the building time, as well as optimally accumulates capital to invest by continuously buying or selling the proper insurance contracts to hedge the condo price risk or allocating capital between insurance assets and risk-free assets.*

Now we introduce the baseline model.

**The Firm's Production Technology and Real Option**: Time is continuous. Consider a firm holding the following real investment option. Let $K_t$ be the dollar amount of capital firm holds at time $t$ with initial capital $K_0 = k$. If the firm exercises the real option at $\tau$ by investing $K_{\tau-}$, the firm will obtain output $f(K_{\tau-})$ , where $f$ is an increasing and concave production function. The exercise payoff is $f(K_\tau)X_\tau$, where the underlying process $X_t$ is the firm's productivity or the price of the output (netting out the cost) at time $t$. Considering the condo-building example, $X_t$ can be interpreted as the net profit from selling a unit of condo (i.e., the price of a condo minus the building cost per unit), and $f(K)$ can be interpreted as the number of condo units that the firm builds by investing $K$ or the multiplier effect in the price by attaching more luxuries. For simplicity, we assume that $f(K) = K$ without loss of generality.[5]

[4] This example is an extension of the example suggested by Miao and Wang (2007).

[5] The main result will not change if we assume $f(K) = C_k K^\delta$ with $C_k > 0$ and $\delta \in (0, 1]$.

The underlying process $X_t$ follows a geometric Brownian motion (GBM):

$$dX_t = \alpha_x X_t dt + \sigma_x X_t dB_t, \quad X_0 = x > 0, \tag{1}$$

where $\alpha_x$ and $\sigma_x$ are positive constants, and $B_t$ is a standard Brownian motion (BM) on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that in general, $B_t$ represents the idiosyncratic risk instead of the systemic risk.

**Risk Management**: We start with the assumption that the market is complete in the sense that there exists a risky asset $S_t$ that can fully replicate the risk in $X_t$. In our model, the risky asset $S_t$ is not the market index nor an asset in the efficient frontier. Instead, $S_t$ is any asset that can be used for the firm's risk management for hedging the risk in the underlying process $X_t$, such as a derivative or an *insurance asset* (e.g., the housing futures contracts). Assume that $S_t$ follows a GBM as follows:

$$dS_t/S_t = \mu_s dt + \sigma_s dB_t, \tag{2}$$

where $\mu_s \in \mathbb{R}$ and $\sigma_s \neq 0$. We assume $\sigma_x > 0$ without loss of generality; however, there is no restriction on the sign of $\sigma_s$. We call the case where $\mu_s = r$ the *baseline* case, where $r > 0$ is the risk-free rate. It means that the baseline case refers to the case where the firm can freely buy and sell futures contracts on $X_t$ provided by a financial intermediary. If insurance companies offer no such contracts, the problem the firm faces is the incomplete market problem. First, we focus on the perfectly correlated case for simplicity of exposition. The extension to the general imperfectly correlated case is not difficult once we understand the solutions to the perfect correlation and the other extreme case of zero-correlation. Note that the zero-correlation case is interpreted as that of pure gambling (Henderson and Hobson, 2013). The pure gambling and the general case with an imperfect correlation are considered in Section 5.

The firm's output increases with the capital that the firm holds at the investment time. At the same time, the firm can hedge the risk in the underlying process using the insurance contract $S_t$. Therefore, the firm is engaged with risk management over time. Then, the firm needs to allocate its capital between the insurance asset and risk-free savings account over time. Let $\pi_t$ be the proportion of the firm's capital invested in $S_t$ at time $t \in [0, \tau)$. Then, the dynamics of the firm's capital before the exercise of the real option is given by

$$\begin{aligned} dK_t &= r(1-\pi_t)K_t dt + \pi_t K_t \frac{dS_t}{S_t} \\ &= (r + (\mu_s - r)\pi_t)K_t dt + \sigma_s \pi_t K_t dB_t, \quad K_0 = k > 0. \end{aligned} \tag{3}$$

**Optimization Problem**: The optimization problem before the exercise of the real option is as follows:

$$J(k,x) = \sup_{\pi_t,\tau} \mathbb{E}\left[e^{-(r+\lambda)\tau} V(K_\tau X_\tau) | K_0 = k, X_0 = x\right], \tag{4}$$

subject to (1) and (3). Here, $V(\cdot)$ is the utility function of the entrepreneur or the CEO of the firm. That is, the firm is private and our problem is a real exercise problem of a firm's key decision-maker, such as Carpenter (2000), Chen et al. (2010), Choi et al. (2017), Henderson (2007), and Miao and Wang (2007). The agent's subject discount rate is given as $r + \lambda$, where $\lambda$ is the excess rate relative to the risk-free rate. We assume that the agent's discount rate is greater than or equal to the risk-free rate, i.e., $\lambda \geqslant 0$. To obtain an explicit solution, we assume a constant absolute risk aversion (CARA) setup, i.e.,

$$V(k) = 1 - e^{-\gamma k}. \tag{5}$$

Our basic setup is a private firm's problem for simplicity (i.e., $V(\cdot)$ is the utility function of the firm's decision-maker). However, the problem can also be interpreted as a problem of a public firm if $V(\cdot)$ is the firm value after the exercise of the real option (see Example 2). When we interpret the problem as a public firm's problem, $\lambda = 0$.

**Example 2.** *Consider the case where the firm only distributes the payoff to the shareholders as a dividend after harvesting the exercise payoff. Here, we define $V(\cdot)$ by the firm's value function from distributing the dividend. The firm chooses dividend rate $d_t$ to maximize the value of the dividend net the cost incurred by the dividend payment as follows:*

$$V(k) = \sup_{d_t} \mathbb{E}\left[\int_0^\infty e^{-rt}(d_t - C(d_t))dt \middle| K_0 = k\right], \tag{6}$$

*where $K_t$ follows $dK_t = (rK_t - d_t)dt$, and $C(\cdot)$ is the cost of providing the dividend $C(d) = \frac{\gamma}{2}d^2$, where $\gamma > 0$. All results are the same under this specification (see Appendix B).*

However, we are only interested in the real option exercise decision and not in the firm's activities after it. On the modeling choice of $V(\cdot)$, i.e., the value function of a firm after exercising the real option, the key for deriving the main results is to introduce the concavity of $V(\cdot)$. Moreover, $V(\cdot)$ only affects the boundary conditions, not the risk-return trade-off that will be discussed in later sections if $V(\cdot)$ is increasing and concave. It means the entire results and intuitions are preserved as long as $V(\cdot)$ is concave.

To obtain the concavity, the example presented in Example 2 assumes the existence of the dividend distribution cost. However, concavity can also be achieved by different types of market frictions. To obtain the concavity, one can introduce alternative assumptions, such as the existence of the liquidation cost, cost of providing the fund's liquidity, and bankruptcy cost, with a proper setup for the firm's operating revenue generated by the real option exercise (see, for example, Bolton et al. (2013), Décamps et al. (2011), Leland (1998), and Della Seta et al. (2020)). Additionally, $V(w)$ in Example 2 turns out to be quadratic (see Lemma 3 in the Appendix); thus, it can also be interpreted as the case when the decision-maker of a firm has a quadratic utility function.

# 3 Solution Analysis

We have two state variables in our setup. Problem (4) can be reduced as a one-dimensional problem by introducing $W_t = K_t X_t$. $W_t$ is the *endogenous* payoff process since $W_\tau = K_\tau X_\tau$ is the exercise payoff at $\tau$. $W_t$ satisfies the following dynamics:

$$dW_t = W_t\Big[\{r + \alpha_x + \sigma_s(\theta + \sigma_x)\pi_t\}dt + (\sigma_x + \sigma_s\pi_t)dB_t\Big], \tag{7}$$

where $\theta \triangleq (\mu_s - r)/\sigma_s$. We redefine the value function as $J(w) = J(k, x)$ with a slight abuse of notation. We will use both $J(w)$ and $J(k, x)$ interchangeably later in our analysis since there is little confusion. Then, the value function can be rewritten as follows:

$$J(w) = \sup_{\pi_t,\tau} \mathbb{E}\left[e^{-(r+\lambda)\tau}V(W_\tau)|W_0 = w = kx\right], \tag{8}$$

subject to (7) and $W_0 = w = K_0X_0 = kx$.

There are two remarks on Problem (8). First, at first sight, the redefined problem looks similar to a standard portfolio choice problem of maximizing utility from terminal wealth. However, the stopping time dimension generates implications different from those of a standard portfolio choice problem, as will be explained in the next sections. Second and conceptually more important, the drift term of $dW_t/W_t$ is $r + \alpha_x + \pi_t(\mu_s + \sigma_x\sigma_x - r)$, which is greater than $\alpha_x$, the drift of the original underlying process $X_t$ even when $\pi_t = 0$. This is because the total value gains derive from the combination of capital gains and price appreciation. Moreover, the expected return associated with investments in $S_t$ is $\mu_s + \sigma_x\sigma_s$, not $\mu_s$. This implies that the total payoff $W$ is effectively *convex* in $X$ when $\pi_t > 0$. However, risk in the volatility part of $dW_t/W_t$ also arises if $\pi_t > 0$. This risk-return trade-off in the total payoff process $W_t$ is essential for understanding the main results explained in the following sections.

Before the real option exercise, the Hamilton-Jacobi-Bellman (HJB) equation for the value function $J(w)$ in (8) is given by

$$(r + \lambda)J = \max_{\pi}\left[\{\alpha_x + r + \sigma_s(\theta + \sigma_x)\pi\}wJ' + \frac{1}{2}(\sigma_x + \sigma_s\pi)^2w^2J''\right]. \tag{9}$$

From the first-order condition

$$\pi^* = -\frac{\sigma_x}{\sigma_s} - \frac{(\theta + \sigma_x)J'}{\sigma_s w J''}, \tag{10}$$

we can obtain the following ordinary differential equation (ODE):

$$(r+\lambda)J = (\alpha_x + r - \theta\sigma_x - \sigma_x^2)wJ' - \frac{1}{2}\frac{(\theta+\sigma_x)^2(J')^2}{J''}. \tag{11}$$

Based on the ODE in (11), we prove the Proposition 1 that the value function $J(w)$ satisfies the following Variational Inequality 1, where the optimal exercise timing of the real option is

$$\tau^* = \inf\{t > 0 \mid W_t \geqslant \bar{w}\}, \tag{12}$$

with the free boundary $\bar{w}$ of Variational Inequality 1 as the optimal exercise threshold.

**Variational Inequality 1.** *Find a function $J(w) \in C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+ \backslash \{\bar{w}\})$ and the free boundary values $\bar{w} > 0$ satisfying*

$$\mathcal{L}J(w) = 0, \quad \textit{if} \quad 0 < w < \bar{w}, \tag{13}$$

$$\mathcal{L}J(w) \leqslant 0, \quad \textit{if} \quad w \geqslant \bar{w}, \tag{14}$$

$$J(w) > V(w), \quad \textit{if} \quad 0 < w < \bar{w}, \tag{15}$$

$$J(w) = V(w), \quad \textit{if} \quad w \geqslant \bar{w}, \tag{16}$$

*where*

$$\mathcal{L}J(w) = (\alpha_x + r - \theta\sigma_x - \sigma_x^2)wJ'(w) - \frac{1}{2}\frac{(\theta+\sigma_x)^2 J'(w)^2}{J''(w)} - (r+\lambda)J(w) \tag{17}$$

*with $V(\cdot)$in* (5)*.*

The solutions to the value function $J(w)$ satisfying Variational Inequality 1 and the optimal strategies $\pi^*$ and $\tau^*$ are presented in Proposition 1. Before that, we assume the following and introduce Lemma 1 used for the proof of Proposition 1.

**Assumption 1.**

$$\theta > -\sigma_x, \quad \textit{and} \tag{18}$$

$$\alpha_x > \lambda. \tag{19}$$

**Lemma 1.** *There exists a unique $\xi \in (0,1)$ that satisfies the following equation:*

$$(\alpha_x + r - \theta\sigma_x - \sigma_x^2)\xi - \frac{1}{2}(\theta + \sigma_x)^2 \frac{\xi}{\xi - 1} - (r + \lambda) = 0. \tag{20}$$

Assumption 1 is directly used for proving Lemma 1. Regarding Condition (18) in Assumption 1, note that a weaker condition, such as $\theta \neq -\sigma_x$, guarantees the existence of a unique solution. We assume a stronger condition (18) to simplify the exposition and analysis. Since we assume $\sigma_x > 0$, (18) is automatically satisfied for $\theta = 0$, the baseline case, where $S_t$ is the futures contracts on $X_t$. However, even if $\theta < 0$, the solution exists provided (18) is satisfied. This implies that even when the financial intermediary offers a contract more disadvantageous to the firm than a simple futures contract, the firm is willing to make the contract for risk management purposes.

Condition (19) in Assumption 1 is required only for an incomplete market case. The solution exists without (19) if the market is complete. In an incomplete market, if the inequality in (19) is reversed, the optimal exercise timing is $\tau^* = 0$, for any $K_0$ and $X_0$, making the problem trivial. As $\lambda$ increases, the firm (or the agent) further discount future cash flows. Here, Condition (19) implies that the productivity or price grows sufficiently on average, compensating for the opportunity cost from waiting. If $\lambda$ is very high (i.e., the agent is fairly impatient) so that (19) fails, there is no value of waiting.

To better understand the characteristic equation (20), we rearrange the terms in (20) as follows:

$$\frac{\xi}{2(\xi - 1)}(\theta + \sigma_x\xi)^2 = \frac{1}{2}\sigma_x^2\xi(\xi - 1) + (\alpha_x + r)\xi - (r + \lambda). \tag{21}$$

Note that the right-hand side of (21) is the quadratic equation that often appears in standard real option problems when the underlying asset is $X_t$ without scale-dependency. That is, the left-hand side is generated by the risk management effect.

**Proposition 1.** *Let $\xi \in (0,1)$ be the root of the equation (20). Then, the solution to the value function (8) is given as follows:*

$$J(w) = \begin{cases} Aw^{\xi}, & \text{if } 0 \leqslant w < \bar{w}, \\ V(w), & \text{if } w \geqslant \bar{w}, \end{cases} \tag{22}$$

*where $\bar{w}$ is the unique positive root of the following equation*

$$\bar{w} = \frac{\xi}{\gamma}(e^{\gamma\bar{w}} - 1), \tag{23}$$

*and $A$ is given by*

$$A = \frac{1 - e^{-\gamma \bar{w}}}{\bar{w}^{\xi}}. \tag{24}$$

*Moreover, the optimal hedging $\pi^*$ is given by*

$$\pi^* = \frac{1}{\sigma_s(1-\xi)}(\theta + \sigma_x \xi), \tag{25}$$

*and the optimal exercise timing of the real option $\tau^*$ is* (12) *with $\bar{w}$ satisfying* (23).

We will investigate the implications of the optimal risk management strategy (25) and optimal exercise strategy (12) in the next section.

**Remark 1.** *The value function in Example 2 has the same form with* (22)*, where $\bar{w}$ and $A$ are given by* (71) *and* (72)*. See Appendix B for the detail.*

**Remark 2.** *Note that $\xi \in (0, 1)$ in our model while the solution to a characteristic equation is usually greater than 1 in standard real option models. This difference results from the concavity of $V(\cdot)$.*

# 4 Implications

## 4.1 Risk Management Effect

It is necessary to consider the following two properties to understand the risk management effects. First, the optimal risk management strategy $\pi^*$ in (25) turns out to be positive even if $\theta = 0$ and $\sigma_x$ and $\sigma_s$ have the same sign. This result is in contrast to the result from the standard portfolio selection or risk management theory, i.e., the optimal demand of the hedge asset with zero risk premium is negative when the asset positively correlates with the underlying risky asset (or the market) since the agent is risk averse (or the agent's utility function is concave). However, in our model, the optimal insurance portfolio is positive even if the objective function (value function after exercising the option) is concave. This enhanced risk-taking or speculative behavior is fundamentally driven since the payoff is endogenously determined as it is scale-dependent.

Second and more importantly, if $\sigma_x$ and $\sigma_s$ have the same sign (positive correlation), a decrease in $\alpha_x$ increases the optimal risk management portfolio and an increase in $\sigma_x$ increases the optimal risk management portfolio, as in Proposition 2.

**Proposition 2.** *$\sigma_s \pi^*$ decreases with $\alpha_x$ and increases with $\sigma_x$, i.e., we have*

$$\frac{\partial(\sigma_s \pi^*)}{\partial \alpha_x} < 0 \quad \textit{and} \quad \frac{\partial(\sigma_s \pi^*)}{\partial \sigma_x} > 0.$$

To further investigate how payoff dynamics are related to the optimal exercise timing, let us compute the drift and volatility part of the optimal payoff process $W_t = K_t X_t$ with $\pi_t = \pi^*$:

$$\text{drift part of } \frac{dW_t}{W_t} = (r + \alpha_x) + \underbrace{\frac{(\theta + \sigma_x)(\theta + \sigma_x \xi)}{1 - \xi}}_{\text{risk management effect}} \tag{26}$$

$$\text{volatility part of } \frac{dW_t}{W_t} = \sigma_x + \underbrace{\frac{\theta + \sigma_x \xi}{1 - \xi}}_{\text{risk management effect}} = \frac{\theta + \sigma_x}{1 - \xi}. \tag{27}$$

The first component in (26) and (27) are from the risk-free rate and the underlying process exogenously given. However, the second components are endogenously generated by the optimal choice of risk management. We will call the effect from the second term the *risk management effect* on the expected return and volatility of the optimal payoff process. Note that the second components (risk management effect) in drift (26) and volatility (27) of $K_t X_t$ have the same sign with $\sigma_s \pi^*$. Since $\sigma_s \pi^* > 0$, provided $\theta + \sigma_x \xi > 0$, including the case $\theta = 0$, risk management increases both the expected return and risk of the payoff even if $\theta = 0$.

In most cases, the risk management effect increases with $\sigma_x$, as in Proposition 3.

**Proposition 3.** *The following are true:*

*(i) An increase in $\sigma_x$ increases the volatility of the payoff process $K_t X_t$.*

*(ii) If $\theta \geqslant 0$ or $\theta$ is not too small when $\theta < 0$, an increase in $\sigma_x$ increases the drift of the payoff process $K_t X_t$.*

An increase in the volatility of the underlying process increases the volatility of the payoff and the expected return of the payoff process under reasonable conditions. However, the size of each increase is not the same, and it depends on the fundamental parameters. In the next section, we will investigate how much the return increases relative to the volatility of the payoff process. Thus, the intuition underlying the main result of the study will be explained.

## 4.2 Impact of $\sigma_x$ on the Optimal Exercise Strategy

In standard real option models, when the market is complete, the exercise threshold increases with the risk of the underlying process. This is driven by the convexity or irreversible nature of the real option exercise. However, in our case, it is not always true. The real option exercise can be hastened or delayed depending on the market (or insurance asset) and the nature of the underlying process. Proposition 4 provides the general and exact condition for each case.

**Proposition 4.** *The threshold level* $\bar{w}$ *decreases as* $\sigma_x$ *increases, i.e.,* $\frac{\partial \bar{w}}{\partial \sigma_x} < 0$ *if and only if*

$$\frac{1}{2}\theta^2 + \Big\{\frac{1}{2}\sigma_x - \frac{1}{\sigma_x}(r+\lambda)\Big\}\theta + (r+\alpha_x) - 2(r+\lambda) > 0. \tag{28}$$

Investment is hastened as the volatility of the underlying process increases if and only if (28) is satisfied. To understand the intuition behind the result, we need the following lemma.

**Lemma 2.** *The following are satisfied:*

*(i)* $\frac{\theta+\sigma_x}{1-\xi}$ *increases with* $\sigma_x$*, i.e.,* $\frac{\partial\left(\frac{\theta+\sigma_x}{1-\xi}\right)}{\partial \sigma_x} > 0$.

*(ii)* $\sigma_x \xi$ *increases with* $\sigma_x$*, i.e.,* $\frac{\partial(\sigma_x\xi)}{\partial \sigma_x} > 0$.

Let us define $\bar{\alpha}$ by

$$\bar{\alpha} \triangleq r + 2\lambda - \frac{1}{2}\theta^2 - \Big\{\frac{1}{2}\sigma_x - \frac{1}{\sigma_x}(r+\lambda)\Big\}\theta.$$

Then, (28) is equivalent to $\alpha_x > \bar{\alpha}$. We divide the parameter set into two cases: (i) $\alpha_x < \bar{\alpha}$, and (ii) $\alpha_x > \bar{\alpha}$. In Case (i), the firm delays the exercise as $\sigma_x$ increases. However, if $\alpha_x$ is sufficiently high (Case (ii)), investment is hastened as volatility increases. Why does the size of $\alpha_x$ matter? To understand this, let us define $SR$, the Sharpe ratio of the optimal payoff process $K_t X_t$, as follows:

$$SR \triangleq \frac{\alpha_x + \frac{(\theta+\sigma_x)(\theta+\sigma_x\xi)}{1-\xi}}{\frac{\theta+\sigma_x}{1-\xi}} = \alpha_x\left(\frac{1-\xi}{\theta+\sigma_x}\right) + (\theta + \sigma_x\xi), \tag{29}$$

using (26) and (27). By Lemma 2, $\left(\frac{1-\xi}{\theta+\sigma_x}\right)$ in the first term of (29) decreases with $\sigma_x$. However, the second term $(\theta+\sigma_x\xi)$ in (29) increases with $\sigma_x$. Therefore, an increase in $\sigma_x$ can increase or decrease the Sharpe ratio depending on the size of $\alpha_x$. If $\alpha_x$ is sufficiently small (or close to zero), the second term dominates the first term in (29); thus, the Sharpe ratio of the payoff increases with $\sigma_x$. In other words, the extra return added by risk management is relatively high since the original expected return ($\alpha_x$) is small; thus, increasing Sharpe ratio. Therefore, the firm will have more incentives to delay investment as $\sigma_x$ increases when $\alpha_x$ is low.

However, the threshold decreases with $\sigma_x$ when $\alpha_x$ is sufficiently high since the Sharpe ratio also decreases with $\sigma_x$. Here, the additional return is small relative to the size of the risk added due to the risk management when the value of the exogenously given $\alpha_x$ is high.

### 4.3 Impact of Other Parameters on the Optimal Exercise Strategy

**Proposition 5.** *The impacts of* $\alpha_x$*,* $\lambda$*, and* $r$ *on the exercise threshold* $\bar{w}$ *are as follows:*

$$\frac{\partial \bar{w}}{\partial \alpha_x} > 0, \quad \frac{\partial \bar{w}}{\partial \lambda} < 0, \quad \textit{and} \quad \frac{\partial \bar{w}}{\partial r} < 0.$$

It is intuitive that an increase in $\alpha_x$ increases the threshold: if the output's productivity or price grows more quickly, the firm has more incentive to wait. This result is true for our model and universally true for any real option model. However, an increase in the interest rate and subject discounting decrease the value of waiting; thus, decrease the threshold level.

### 4.4 Incomplete Market

What if there is no insurance asset or contract available? If we solve the problem without having such an asset, it turns out that the solution is the same as the case where $\pi_t^* = 0$. Specifically, if we choose $\theta$ so that the optimal choice of $\pi_t^*$ is zero for all $t \geqslant 0$, it covers the case of the incomplete market. In this case, it is equivalent to introducing a fictitious asset in solving portfolio selection problems in an incomplete market, as shown by Karatzas et al. (1991) and Cvitanić and Karatzas (1992). Therefore, we do not have to re-analyze the incomplete market case independently. This result is summarized in Corollary 1.

**Corollary 1** (When $\pi^* = 0$)**.** *Consider $\theta$ that satisfies $\theta + \sigma_x \xi = 0$, where $\xi$ is the solution to* (20) *for given $\theta$. Here, we have $\pi^* = 0$. Moreover, the following inequality holds*

$$\frac{1}{2}\theta^2 + \left\{\frac{1}{2}\sigma_x - \frac{1}{\sigma_x}(r+\lambda)\right\}\theta + (r + \alpha_x) - 2(r+\lambda) > 0,$$

*so that we always have $\frac{\partial \bar{w}}{\partial \sigma_x} < 0$.*

There is no such case in the incomplete market, where the firm delays investment as volatility increases. There are two cases: Case (i), where the firm exercises the option immediately, regardless of the current productivity level; Case (ii), where the firm will wait, but an increase in volatility decreases the threshold. Specifically, Case (i) is the one where the inequality is reversed in condition (19) in Assumption 1, i.e., if $\alpha_x$ is sufficiently small or $\lambda$ is sufficiently high, then there is no value of waiting. If condition (19) in Assumption 1 is satisfied, the firm will wait until $K_t X_t \geqslant \bar{w}$. However, neither the expected return nor the risk of the exercise payoff changes by the risk management since $\pi^* = 0$. Here, only the concavity of the value function matters so that the threshold level decreases with risk.

### 4.5 Volatility of $J(K_t X_t)$

Here, we consider the impact of the fundamental parameters on the volatility of the firm value $J(K_t X_t)$.

**Proposition 6.** *Let $J_t = J(K_t X_t)$. Then, we have*

$$dJ_t/J_t = (r+\lambda)dt + \Sigma dB_t,$$

*where*

$$\Sigma = \frac{\xi}{1-\xi}(\theta + \sigma_x), \tag{30}$$

*where $\xi$ is the root of the equation* (20)*. Moreover, for a given $\theta$, the following hold*

$$\frac{\partial \Sigma}{\partial \sigma_x} > 0, \quad \frac{\partial \Sigma}{\partial \lambda} > 0, \quad \frac{\partial \Sigma}{\partial r} > 0, \quad \textit{and} \quad \frac{\partial \Sigma}{\partial \alpha_x} < 0. \tag{31}$$

The expected growth rate of the firm value process is not affected by the underlying process ($\alpha_x$ and $\sigma_x$) or risk management. However, the expected growth rate and volatility of the underlying process, as well as the risk-free rate and subjective discounting, impact the volatility of the firm value process. First, an increase in the volatility of the underlying process increases the volatility of the firm value. Second, an increase in $r$ and $\lambda$ increase the volatility. Finally, an increase in the growth rate of the underlying process decreases the firm value volatility.

### 4.6 Risk of Derivatives Users versus Non-users

In this section, we investigate the impact of risk management on risk in detail. More precisely, we compare the return volatilities between firms using derivatives and non-users. The return volatility of a user is given by (30) with $\theta = 0$ since the risk premium of financial derivatives is zero. To characterize the volatility of a non-user, we need to consider the case when there is no insurance asset available or the firm does not use derivatives by any reason. Then, other things being equal, a non-user's volatility is given by

$$\Sigma_{\text{no hedge}} = \sigma_x \xi_{\text{no hedge}}, \tag{32}$$

where $\xi_{\text{no hedge}}$ is the positive root of the following quadratic equation:

$$q(\xi) = \frac{1}{2}\sigma_x^2 \xi^2 + (r + \alpha_x - \frac{1}{2}\sigma_x^2)\xi - (r + \lambda) = 0. \tag{33}$$

The following proposition provides the comparison between the two volatilities when $\theta = 0$.

**Proposition 7.** *Let $\theta = 0$. Other things being equal, the volatility of the firm value of the derivatives user is greater than that of the non-user, i.e.,*

$$\Sigma > \Sigma_{\text{no hedge}}.$$

First, although we only consider the case when $\theta = 0$ in Proposition 7 for the simplicity of exposition, the result is true for more general cases when $\theta$ is non-zero (see Appendix C).

Second and more importantly, this result can help design a more detailed empirical study regarding the usage of derivatives by firms. Recall that the optimal risk management strategy $\pi^*$ is positive even if the hedge asset does not provide an excess return ($\theta = 0$). In line with this, Proposition 7 shows that derivatives users with scale-dependent investment opportunities have a more volatile firm value than non-users. However, it is worth noting that the relationship should be the opposite among firms whose investment opportunities have little scale-dependency: among firms with little scale-dependency, derivatives users have lower return volatility than non-users, other things being equal.

## 5 Pure Gambling and Imperfect Correlation

The optimal insurance strategy (25) is risk-seeking. So far, we have assumed, for simplicity, that the asset has a perfect correlation with the underlying price or productivity process. Any asset that correlates with the firm's underlying process can be used for risk management. In this section, we first consider the case with no correlation. Then, we consider a general case.

Let us first consider an extreme case where there is no insurance contract, such as (2) provided by the financial intermediary. Instead, there is a contract or an asset such that

$$dG_t/G_t = \mu_g dt + \sigma_g dB_t^G, \tag{34}$$

where $B_t^G$ is the BM independent of $B_t$. Then, does the firm still buy or sell $G_t$? The answer is yes if $\mu_g > r$ or $\mu_g < r$. The firm can take advantage of the non-zero risk premium of $G_t$. In this case, the firm trades $G_t$ for the investment motive, not for the risk management. However, if $\mu_g = r$ and $\sigma_g \neq 0$, the firm never trades such an asset. This result is summarized in Proposition 8.

**Proposition 8** (Pure Gambling)**.** *Suppose $S_t$ in* (2) *is not available, but the firm is allowed to buy and sell $G_t$ in* (34) *over time. If $\mu_g = r$, the optimal portfolio choice $\pi_G$ of asset $G_t$ is zero.*

Note that when the asset does not correlate with the underlying process and its risk premium is zero, buying such a zero-risk premium asset is equivalent to a pure gamble. Proposition 8 implies that the firm does not engage in a pure gamble. However, it happens in Henderson and Hobson (2013) since the risk-averse agent is better off by doing. In our case, the firm trades the correlated assets to optimize the risk-return trade-off in the payoff process. Trading a gambling asset only increases the risk, which decreases the Sharpe ratio of the payoff process. As mentioned in our introduction, this is the key difference between the motivation for the risk-seeking behavior suggested by Henderson and Hobson (2013) and that for the risk-seeking insurance strategy in our model.

With the above result, we consider the case where the underlying process $X_t$ and the insurance asset $S_t$ are imperfectly correlated as follows:

$$dX_t = \alpha_x X_t dt + \sigma_x X_t \left( \rho dB_t + \sqrt{1-\rho^2} d\tilde{B}_t \right), \tag{35}$$

where $S_t$ still follows (2). In (35), $\rho \in (-1, 1)$ and $W_t$ is a standard BM independent of $B_t$. Let $\pi_I$ be the portfolio choice for this case with $\theta = 0$ (or $\mu_s = r$). Then, from the proof of Proposition 8, it can be easily inferred that the optimal portfolio choice $\pi_I$ is

$$\pi_I = \rho \pi^*, \tag{36}$$

where $\pi^*$ is in (25) with $\theta = 0$. That is, it follows from Proposition 8 that there is no additional demand on the insurance asset regarding the risk factor $\tilde{B}_t$, and the demand on the insurance asset only comes from the risk factor $B_t$. Since the volatility associated with $B_t$ of the imperfect correlation case in (35) is $\rho$ times the one with a perfect correlation, the corresponding optimal portfolio choice $\pi_I$ is also $\rho$ times the optimal portfolio choice for the perfectly correlated case $\pi^*$.

# 6 Concluding Remarks

We investigated the firm's real option exercise problem when the payoff is scale-dependent; thus, it is endogenously determined by the firm's capital investment at the exercise time. The firm not only selects the optimal exercise time of real option, but also efficiently manages the risk and accumulates capital for investment. We obtained the explicit solution and investigated the implications of risk management. We found that the firm's risk management strategy exhibits speculative (seemingly risk-seeking), not hedging since the speculative risk management optimizes the risk-return trade-off of the payoff process. We also showed that, due to the risk management strategy, an increase in the risk in the underlying price could increase or decrease the optimal exercise threshold depending on the economic environment. We hope that our setup can shed light on understanding different aspects of the role of risk management when a firm faces irreversible investment decisions.

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# Appendix

## A Proofs

### A.1 Proof of Lemma 1

*Proof.* First, consider the case in which $\theta + \sigma_x \xi = 0$, or equivalently $\xi = -\frac{\theta}{\sigma_x}$. This happens only when $\theta$ satisfies

$$\frac{1}{2}\theta^2 + \left(\frac{1}{2}\sigma_x - \frac{r+\alpha_x}{\sigma_x}\right)\theta - (r+\lambda) = 0, \tag{37}$$

which can be obtained by substituting $\xi = -\frac{\theta}{\sigma_x}$ into equation (20). Since $r + \lambda > 0$ and inequality (19) is imposed, it is guaranteed that $\theta$ satisfying (37) is in between $-\sigma_x$ and 0. Thus, $\xi = -\frac{\theta}{\sigma_x}$ for this case is in between 0 and 1.[6]

Now, we consider the remaining case such that $\theta + \sigma_x \xi \neq 0$. Let us define two functions of a dummy variable $u \in [0, 1)$ as follows:

$$L(u) \triangleq (r+\lambda) - (r+\alpha_x)u, \quad R(u) \triangleq -\frac{1}{2}(\theta+\sigma_x)^2 \frac{u}{u-1} - (\theta\sigma_x + \sigma_x^2)u.$$

Then, it is straightforward that equation (20) is equivalent to $L(\xi) = R(\xi)$.

Note that $R(0) = 0$ and $\lim_{u \to 1-} R(u) = \infty$ when $\theta \neq -\sigma_x$, which is guaranteed by the assumption (18). Moreover, we can show that $R(u)$ is a strictly increasing function on $[0, 1)$ when $\theta > \sigma_x$, or $R(u)$ is decreasing in $u$ for $0 \leqslant u \leqslant u^*$, and increasing in $u$ for $u^* < u < 1$, where $u^* \triangleq 1 - \sqrt{(1 - \frac{1}{2}(1 - \frac{\theta}{\sigma_x}))} \in [0, 1)$ when $-\sigma_x < \theta \leqslant \sigma_x$. In either case, there exists a unique $\xi \in (0, 1)$ such that $L(\xi) = R(\xi)$ because the graph of $L(u)$ is a straight line with positive $y$-intercept and negative slope. □

### A.2 Proof of Proposition 1

*Proof.* We can easily obtain $\pi^*$ in (25) by substituting the form of $J(w)$ in (22) into the first-order condition (10), and $\tau^*$ is already given in (12). Thus, we focus on showing that $J(w)$ in (22) satisfies Variational Inequality 1.

It is obvious that $J(w) \in \mathbb{C}^2(\mathbb{R}^+ \backslash \{\bar{w}\})$. In order to satisfy $J(w) \in \mathbb{C}^1(\mathbb{R}^+)$, we need the following value matching and smooth-pasting conditions at $w = \bar{w}$:

$$J(\bar{w}) = V(\bar{w}) \quad \text{and} \quad J'(\bar{w}) = V'(\bar{w}),$$

[6] Note that, for this case, assumption (18) is not utilized but induced by applying assumption (19) to equation (37).

which are equivalent to

$$A\bar{w}^{\xi} = 1 - e^{-\gamma\bar{w}}, \tag{38}$$

$$A\xi\bar{w}^{\xi-1} = \gamma e^{-\gamma\bar{w}}. \tag{39}$$

Combining (38) and (39), we derive $A$ in (24) and equation (23) for $\bar{w}$. Here, we have to show that $\bar{w}$ is the unique positive root of equation (23), which is equivalent to showing that $\bar{w}$ is the unique positive root of $g(w) = 0$ in which the function $g(w)$ is defined as

$$g(w) \triangleq \frac{\xi}{\gamma}(e^{\gamma w} - 1) - w. \tag{40}$$

Since $g'(w) = \xi e^{\gamma w} - 1$, it follows that $g'(w) > 0$ ($g(w)$ is strictly increasing) for $w > \hat{w}$ and $g'(w) < 0$ ($g(w)$ is strictly decreasing) for $0 \leqslant w < \hat{w}$, where $\hat{w} \triangleq \frac{1}{\gamma}\ln(\frac{1}{\xi})$ is the unique root of $g'(w) = 0$. Note that $0 < \xi < 1$ (by Lemma 1) guarantees that $\hat{w} > 0$. Since $g(0) = 0$ and $g(w)$ is strictly decreasing for $0 \leqslant w < \hat{w}$, it is obvious that $g(w) < 0$ for $0 < w \leqslant \hat{w}$. In addition, since $\lim_{w\to\infty} g(w) = \infty$ and $g(w)$ is strictly increasing for $w > \hat{w}$ and $g(w)$ is continuous, there exists a unique root $\bar{w}$ of $g(w) = 0$ such that $\hat{w} < \bar{w}$ (thus $\bar{w} > 0$), and

$$g(w) < 0 \text{ for } 0 < w < \bar{w}, \text{ whereas } g(w) > 0 \text{ for } w > \bar{w}. \tag{41}$$

Moreover, since $\hat{w} < \bar{w}$, it follows that $g'(\bar{w}) > 0$, i.e.,

$$\xi e^{\gamma\bar{w}} > 1, \tag{42}$$

which is useful in the remaining part of the proof.

Now we verify the equations and inequalities (13)–(16) of Variational inequality 1. In (22), $J(w) = V(w)$ for $w \geqslant \bar{w}$. Moreover, the equation (20) satisfied by $\xi$ guarantees that

$$\mathcal{L}J(w) = \left[(\alpha_x + r - \theta\sigma_x - \sigma_x^2)\xi - \frac{1}{2}(\theta + \sigma_x)^2\frac{\xi}{\xi - 1} - (r + \lambda)\right] J(w) = 0$$

for $0 < w < \bar{w}$. Thus, two equations (13) and (16) hold with $J(w)$ in (22).

We are left to show that two inequalities (14) and (15) hold. To show (14), by substituting $V(w)$ in (5) into (17) we have

$$\mathcal{L}V(w) = \left[(\alpha_x + r - \theta\sigma_x - \sigma_x^2)\gamma w + \frac{1}{2}(\theta + \sigma_x)^2 + (r + \lambda)\right] e^{-\gamma w} - (r + \lambda). \tag{43}$$

By using equation (23) satisfied by $\bar{w}$, it can be verified that

$$\mathcal{L}V(\bar{w}) = -\frac{1}{2}(\theta + \sigma_x)^2 \frac{1}{1-\xi}(\xi - e^{-\gamma \bar{w}}) < 0, \tag{44}$$

where the inequality (44) comes from $\xi \in (0,1)$ and $\xi - e^{-\gamma \bar{w}} > 0$ by (42). Note that

$$\frac{d\mathcal{L}V(w)}{dw} = l(w)\gamma e^{-\gamma w}, \tag{45}$$

where the function $l(w)$ is defined as

$$l(w) \triangleq (\alpha_x + r - \theta\sigma_x - \sigma_x^2)(1 - \gamma w) - \frac{1}{2}(\theta + \sigma_x)^2 - (r + \lambda). \tag{46}$$

Thus, $l(w)$ and $\frac{d\mathcal{L}V(w)}{dw}$ always have same sign.

- (Case I) If $\alpha_x + r - \theta\sigma_x - \sigma_x^2 \geqslant 0$, using (20) and (23), we can show that

$$l(\bar{w}) = (\alpha_x + r - \theta\sigma_x - \sigma_x^2)(1 - \xi e^{\gamma \bar{w}}) - \frac{1}{2}(\theta + \sigma_x)^2 \frac{1}{1-\xi} < 0, \tag{47}$$

  where inequality (47) holds since $\xi \in (0,1)$ and $1 - \xi e^{\gamma \bar{w}} < 0$ by (42). Moreover, $l(w)$ is decreasing in $w$. Thus, $l(w) \leqslant l(\bar{w}) < 0$ for $w \geqslant \bar{w}$, i.e., $\mathcal{L}V'(w) < 0$ ($\mathcal{L}V(w)$ is decreasing in $w$) for $w \geqslant \bar{w}$. Since $\mathcal{L}V(\bar{w}) < 0$ in (44), we can conclude that $\mathcal{L}V(w) < 0$ for $w \geqslant \bar{w}$.

- (Case II) If $\alpha_x + r - \theta\sigma_x - \sigma_x^2 < 0$, $l(w)$ is an increasing function of $w$. More precisely, $l(w) < 0$ (or $\mathcal{L}V(w)$ is decreasing) in $w$ for $0 < w < \tilde{w}$, where $\tilde{w} \triangleq \frac{(\alpha_x + r - \theta\sigma_x - \sigma_x^2) - \frac{1}{2}(\theta+\sigma_x)^2 - (r+\lambda)}{\gamma(\alpha_x + r - \theta\sigma_x - \sigma_x^2)} > 0$ is the unique root of $l(w) = 0$. On the other hand, $l(w) > 0$ for $w > \tilde{w}$, i.e., $\mathcal{L}V(w)$ is increasing in $w$ for $w > \tilde{w}$. Then, regardless of the relationship between $\tilde{w}$ and $\bar{w}$, we can deduce that

$$\mathcal{L}V(w) \leqslant \max\left(\mathcal{L}V(\bar{w}), \lim_{w\to\infty} \mathcal{L}V(w)\right)$$

  for $w \geqslant \bar{w}$. Note that $\mathcal{L}V(\bar{w}) < 0$ in (44) and $\lim_{w\to\infty} \mathcal{L}V(w) = -(r + \lambda) < 0$. Thus, it follows that $\mathcal{L}V(w) < 0$ for $w \geqslant \bar{w}$.

In summary, regardless of the sign of $\alpha_x + r - \theta\sigma_x - \sigma_x^2$, we can show that $\mathcal{L}V(w) < 0$ for $w \geqslant \bar{w}$.

In order to show that inequality (15) holds, let us define a function

$$h(w) \triangleq \frac{1 - e^{-\gamma w}}{w^\xi}.$$

Then, it follows from the form of $A$ in (24) that

$$J(w) = \frac{1 - e^{-\gamma \bar{w}}}{\bar{w}^\xi} w^\xi > V(w) = 1 - e^{-\gamma w}$$

is equivalent to $h(\bar{w}) > h(w)$ for $0 < w < \bar{w}$. Thus, it is enough to show that $h(w)$ is increasing for $0 < w < \bar{w}$ to verify inequality (15). Direct computations gives us

$$h'(w) = -\gamma e^{-\gamma w} e^{-\xi-1} g(w) \tag{48}$$

with $g(w)$ in (40). From (41), it follows that $h'(w) > 0$ for $0 < w < \bar{w}$ as desired.

□

### A.3 Proof of Proposition 2

*Proof.* From (25), we have $\sigma_s \pi^* = \frac{\theta+\sigma_x\xi}{1-\xi}$, and it follows that

$$\frac{\partial(\sigma_s\pi^*)}{\partial\alpha_x} = \frac{\theta+\sigma_x}{(1-\xi)^2} \cdot \frac{\partial\xi}{\partial\alpha_x}, \tag{49}$$

$$\frac{\partial(\sigma_s\pi^*)}{\partial\sigma_x} = \frac{\theta+\sigma_x}{(1-\xi)^2} \cdot \frac{\partial\xi}{\partial\sigma_x} + \frac{\xi}{1-\xi}. \tag{50}$$

Since $0 < \xi < 1$, the sign of $\frac{\partial(\sigma_s\pi^*)}{\partial\alpha_x}$ coincides with that of $\frac{\partial\xi}{\partial\alpha_x}$. Since $\xi \neq 1$, we can obtain

$$Q(\xi) = 0,$$

which is equivalent to equation (20), where

$$Q(u) = \left[\sigma_x^2 + \sigma_x\theta - (r+\alpha_x)\right] u^2 + \left[-\frac{1}{2}\sigma_x^2 + \frac{1}{2}\theta^2 + (r+\lambda) + (r+\alpha_x)\right] u - (r+\lambda), \tag{51}$$

by multiplying $(1-\xi)$ to the both sides of equation (20) and rearranging the terms. By differentiating both sides of $Q(\xi) = 0$ with respect to $\alpha_x$, we have

$$\frac{\partial\xi}{\partial\alpha_x} = -\frac{\xi(1-\xi)}{Q'(\xi)}.$$

Note that $Q(0) = -(r+\lambda) < 0$ and $Q(1) = \frac{1}{2}(\sigma_x+\theta)^2 > 0$. Since $Q(u)$ is a quadratic function and $\xi$ is the root of $Q(u)$ between 0 and 1, it follows that $Q'(\xi) > 0$. Moreover, we have $0 < \xi < 1$. Therefore, we have

$$\frac{\partial\xi}{\partial\alpha_x} < 0$$

and consequently,

$$\frac{\partial(\sigma_s\pi^*)}{\partial\alpha_x} < 0.$$

By differentiating both sides of $Q(\xi) = 0$ with respect to $\sigma_x$, we have

$$\frac{\partial \xi}{\partial \sigma_x} = \frac{-(2\sigma_x + \theta)\xi + \sigma_x}{Q'(\xi)}\xi. \tag{52}$$

Substituting (52) into (50),

$$\frac{\partial(\sigma_s \pi^*)}{\partial \sigma_x} = \frac{H(\xi)}{(1-\xi)^2 Q'(\xi)}, \tag{53}$$

where

$$H(\xi) = \left[-2\sigma_x^2 - 2\sigma_x\theta + 2(r+\alpha_x)\right]\xi^3 + \left[\frac{1}{2}\sigma_x^2 - \sigma_x\theta - \frac{3}{2}\theta^2 - (r+\lambda) - 3(r+\alpha_x)\right]\xi^2$$
$$+ \left[\frac{1}{2}\sigma_x^2 + \sigma_x\theta + \frac{1}{2}\theta^2 + (r+\lambda) + (r+\alpha_x)\right]\xi.$$

Since $Q(\xi) = 0$, we can show that

$$H(\xi) = H(\xi) + 2\xi Q(\xi) = \xi(1-\xi)\left[\frac{1}{2}(\sigma_x+\theta)^2 + (r+\alpha_x) - (r+\lambda)\right] > 0,$$

where the inequality comes from $0 < \xi < 1$ and Condition (19) in Assumption 1. Since $H(\xi) > 0$, $\xi \neq 1$, and $Q'(\xi)$ in (53), we have

$$\frac{\partial(\sigma_s \pi^*)}{\partial \sigma_x} > 0.$$

□

## A.4 Proof of Proposition 3

*Proof.* Note that the volatility of $K_t X_t$ is

$$\sigma_x + \frac{\theta + \sigma_x \xi}{1-\xi} = \sigma_x + \sigma_s \pi^*.$$

Since it is shown that $\frac{\partial(\sigma_s \pi^*)}{\partial \sigma_x} > 0$ in Proposition 2, it is obvious that

$$\frac{\partial}{\partial \sigma_x}\Big(\sigma_x + \sigma_s \pi^*\Big) = 1 + \frac{\partial(\sigma_s \pi^*)}{\partial \sigma_x} > 0.$$

Note that the drift of $K_t X_t$ is

$$(r+\alpha_x) + \frac{(\theta+\sigma_x)(\theta+\sigma_x\xi)}{1-\xi} = (r+\alpha_x) + (\theta+\sigma_x)\sigma_s\pi^*.$$

By differentiating with respect to $\sigma_x$, we have

$$\begin{aligned}\frac{\partial}{\partial\sigma_x}\left[(r+\alpha_x)+(\theta+\sigma_x)\sigma_s\pi^*\right] &= \sigma_s\pi^* + (\theta+\sigma_x)\frac{\partial(\sigma_s\pi^*)}{\partial\sigma_x}\\ &= \frac{\theta+\sigma_x\xi}{1-\xi} + (\theta+\sigma_x)\frac{\partial(\sigma_s\pi^*)}{\partial\sigma_x}\end{aligned}$$

Thus, if $\theta$ satisfies the following condition

$$\theta > \underbrace{-\sigma_x\xi - (1-\xi)(\theta+\sigma_x)\frac{\partial(\sigma_s\pi^*)}{\partial\sigma_x}}_{<0} \tag{54}$$

we have

$$\frac{\partial}{\partial\sigma_x}\left[(r+\alpha_x)+(\theta+\sigma_x)\sigma_s\pi^*\right] > 0.$$

Although the right-hand side of (54) varies depending on $\theta$, it is guaranteed that it is negative because $\sigma_x > 0$, $0 < \xi < 1$, $(\theta+\sigma_x) > 0$ by Condition (18) in Assumption 1, and $\frac{\partial(\sigma_s\pi^*)}{\partial\sigma_x} > 0$ by Proposition 2. In the cases with $\theta \geqslant 0$, condition (54) is satisfied and the drift of $K_tX_t$ increases as $\sigma_x$ increases. □

### A.5 Proof of Proposition 4

*Proof.* Recall that $\frac{\partial\xi}{\partial\sigma_x}$ is given as (52) and it is shown that $Q'(\xi) > 0$ in the proof of Proposition 2. Since $\xi > 0$, we can deduce that $\frac{\partial\xi}{\partial\sigma_x}$ and $-(2\sigma_x+\theta)\xi+\sigma_x$ have same sign. Moreover, from (23) we can show that

$$\frac{\partial\bar{w}}{\partial\sigma_x} = -\frac{e^{\gamma\bar{w}}-1}{\gamma(\xi e^{\gamma\bar{w}}-1)}\cdot\frac{\partial\xi}{\partial\sigma_x},$$

and thus $\frac{\partial\bar{w}}{\partial\sigma_x}$ and $\frac{\partial\xi}{\partial\sigma_x}$ have the opposite signs because $e^{\gamma\bar{w}}-1 > 0$ and $\xi e^{\gamma\bar{w}}-1 > 0$ as shown by (42) in the proof of Proposition 1. Hence, we can deduce that

$$\begin{aligned}\frac{\partial\bar{w}}{\partial\sigma_x} < 0 \quad \text{if} \quad \xi < \frac{1}{2+\frac{\theta}{\sigma_x}},\\ \frac{\partial\bar{w}}{\partial\sigma_x} > 0 \quad \text{if} \quad \xi > \frac{1}{2+\frac{\theta}{\sigma_x}}.\end{aligned}$$

By assumption (18), we have $0 < \frac{1}{2+\frac{\theta}{\sigma_x}} < 1$. Thus, $\xi < \frac{1}{2+\frac{\theta}{\sigma_x}}$ is equivalent to $Q(\frac{1}{2+\frac{\theta}{\sigma_x}}) > 0$. Note that

$$Q(\frac{1}{2+\frac{\theta}{\sigma_x}}) = \frac{(1+\frac{\theta}{\sigma_x})}{(2+\frac{\theta}{\sigma_x})^2}\left[\frac{1}{2}\theta^2 + \left\{\frac{1}{2}\sigma_x - \frac{1}{\sigma_x}(r+\lambda)\right\}\theta + (r+\alpha_x) - 2(r+\lambda)\right] \tag{55}$$

and $\frac{(1+\frac{\theta}{\sigma_x})}{(2+\frac{\theta}{\sigma_x})^2} > 0$ provided (18). Thus, we can conclude that

$$\frac{\partial \bar{w}}{\partial \sigma_x} < 0 \quad \text{if} \quad \frac{1}{2}\theta^2 + \Big\{\frac{1}{2}\sigma_x - \frac{1}{\sigma_x}(r+\lambda)\Big\}\theta + (r+\alpha_x) - 2(r+\lambda) > 0,$$
$$\frac{\partial \bar{w}}{\partial \sigma_x} > 0 \quad \text{if} \quad \frac{1}{2}\theta^2 + \Big\{\frac{1}{2}\sigma_x - \frac{1}{\sigma_x}(r+\lambda)\Big\}\theta + (r+\alpha_x) - 2(r+\lambda) < 0,$$

which completes the proof. □

### A.6 Proof of Lemma 2

*Proof.* Since $\sigma_s\pi^* = \frac{\theta+\sigma_x\xi}{1-\xi}$, we have

$$\frac{\theta+\sigma_x}{1-\xi} = \sigma_x + \sigma_s\pi^*.$$

We have already shown in Proposition 2 that $\frac{\partial(\sigma_s\pi^*)}{\partial\sigma_x} > 0$. Thus, it is straightforward to see

$$\frac{\partial\left(\frac{\theta+\sigma_x}{1-\xi}\right)}{\partial\sigma_x} = 1 + \frac{\partial(\sigma_s\pi^*)}{\partial\sigma_x} > 0.$$

Using (52), we have

$$\begin{aligned}\frac{\partial\left(\sigma_x\xi\right)}{\partial\sigma_x} &= \xi + \sigma_x\frac{\partial\xi}{\partial\sigma_x} = \xi + \sigma_x\frac{-(2\sigma_x+\theta)\xi+\sigma_x}{Q'(\xi)}\xi \\ &= \frac{\xi}{Q'(\xi)}\left[Q'(\xi) - (2\sigma_x^2+\sigma_x\theta)\xi + \sigma_x^2\right].\end{aligned}$$

Since

$$Q'(\xi) = \left[2\sigma_x^2 + 2\sigma_x\theta - 2(r+\alpha_x)\right]\xi + \left[-\frac{1}{2}\sigma_x^2 + \frac{1}{2}\theta^2 + (r+\lambda) + (r+\alpha_x)\right],$$

we have

$$\frac{\partial\left(\sigma_x\xi\right)}{\partial\sigma_x} = \frac{\xi}{Q'(\xi)}\left[(r+\alpha_x)(1-\xi) + (r+\lambda) - (r+\alpha_x)\xi + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\theta^2 + \sigma_x\theta\xi\right]. \tag{56}$$

Note that $L(\xi) = R(\xi)$ in the proof of Lemma 1 is

$$\begin{aligned}(r+\lambda) - (r+\alpha_x)\xi &= \frac{1}{2}\sigma_x^2\left[\xi(\xi-1) - \frac{\xi}{2(\xi-1)}(\theta+\sigma_x\xi)^2\right] \\ &= \frac{1}{2}\sigma_x^2\frac{2\xi^2-\xi}{1-\xi} + \frac{1}{2}\theta^2\left(\frac{\xi}{1-\xi}\right) + \sigma_x\theta\left(\frac{\xi^2}{1-\xi}\right).\end{aligned} \tag{57}$$

By substituting (57) into (56),

$$\frac{\partial(\sigma_x\xi)}{\partial\sigma_x} = \frac{\xi}{Q'(\xi)}\left[(r+\alpha_x)(1-\xi)+\frac{1}{2}\sigma_x^2\left(\frac{\xi^2+(1-\xi)^2}{1-\xi}\right)+\frac{1}{2}\theta^2\left(\frac{1}{1-\xi}\right)+\sigma_x\theta\left(\frac{\xi}{1-\xi}\right)\right]$$
$$= \frac{\xi}{Q'(\xi)}\left[(r+\alpha_x)(1-\xi)+\frac{1}{2}\sigma_x^2(1-\xi)+\frac{1}{2(1-\xi)}(\theta+\sigma_x\xi)^2\right] > 0, \quad (58)$$

where the last inequality comes from $Q'(\xi) > 0$ (shown in the proof of Proposition 2) and $0 < \xi < 1$. □

### A.7 Proof of Proposition 5

*Proof.* Recall that equation (20) is equivalent to $Q(\xi) = 0$, where $Q(\cdot)$ is given as (51). Moreover, it is shown in the proof of Proposition 2 that $\frac{\partial\xi}{\partial\alpha_x} < 0$. Thus, it follow that

$$\frac{\partial\bar{w}}{\partial\alpha_x} = -\frac{e^{\gamma\bar{w}}-1}{\gamma(\xi e^{\gamma\bar{w}}-1)}\cdot\frac{\partial\xi}{\partial\alpha_x} > 0.$$

By differentiating equation $Q(\xi) = 0$ with respect to $\lambda$ or $r$, we have

$$\frac{\partial\xi}{\partial\lambda} = \frac{(1-\xi)}{Q'(\xi)} > 0, \quad \frac{\partial\xi}{\partial r} = \frac{(1-\xi)}{Q'(\xi)} > 0,$$

where the inequalities come from $0 < \xi < 1$ and $Q'(\xi) > 0$. Then, it follows that

$$\frac{\partial\bar{w}}{\partial\lambda} = -\frac{e^{\gamma\bar{w}}-1}{\gamma(\xi e^{\gamma\bar{w}}-1)}\cdot\frac{\partial\xi}{\partial\lambda} < 0, \quad (59)$$
$$\frac{\partial\bar{w}}{\partial r} = -\frac{e^{\gamma\bar{w}}-1}{\gamma(\xi e^{\gamma\bar{w}}-1)}\cdot\frac{\partial\xi}{\partial r} < 0. \quad (60)$$

□

### A.8 Proof of Corollary 1

*Proof.* If $\theta$ satisfies $\theta + \sigma_x\xi = 0$, where $\xi$ is the solution to (20) for given $\theta$, it is obvious that corresponding $\pi^*$ is

$$\pi^* = \frac{1}{\sigma_x(1-\xi)}(\theta+\sigma_x\xi) = 0.$$

Note that, as mentioned in the proof of Lemma 1, such $\theta$ satisfies equation (37). By using equation (37), we have

$$
\begin{aligned}
&\frac{1}{2}\theta^2 + \Big\{\frac{1}{2}\sigma_x - \frac{1}{\sigma_x}(r+\lambda)\Big\}\theta + (r+\alpha_x) - 2(r+\lambda) \\
&= \left[\frac{1}{2}\theta^2 + \Big\{\frac{1}{2}\sigma_x - \frac{1}{\sigma_x}(r+\lambda)\Big\}\theta + (r+\alpha_x) - 2(r+\lambda)\right] - \underbrace{\left[\frac{1}{2}\theta^2 + \left(\frac{1}{2}\sigma_x - \frac{r+\alpha_x}{\sigma_x}\right)\theta - (r+\lambda)\right]}_{=0} \\
&= (1+\frac{\theta}{\sigma_x})(\alpha_x - \lambda) > 0.
\end{aligned}
$$

where the last inequality comes from the assumption (19) (see the proof of Lemma 1 and footnote 6 for more details). By applying Proposition 4, we always have $\frac{\partial \bar{w}}{\partial \sigma_x} < 0$ for this case. □

## A.9 Proof of Proposition 6

*Proof.* By applying Itô's formula to $J_t = J(K_t X_t) = A(K_t X_t)^{\xi}$ with the optimal $\pi^*$ in (25) and using (20), we can show that

$$dJ_t/J_t = (r+\lambda)dt + \Sigma dB_t,$$

where

$$\Sigma = \xi\left[\frac{\theta + \sigma_x \xi}{1-\xi} + \sigma_x\right] = \frac{\xi}{1-\xi}(\theta + \sigma_x).$$

Note that $\Sigma = \sigma_s \pi^* - \theta$. From Proposition 2, it follow that

$$\frac{\partial \Sigma}{\partial \sigma_x} > 0, \quad \frac{\partial \Sigma}{\partial \alpha_x} < 0.$$

For given $\theta$, by differentiating $\Sigma$ with respect to $r$ or $\lambda$, we have

$$\frac{\partial \Sigma}{\partial r} = \frac{(\theta+\sigma_x)}{(1-\xi)^2}\cdot\frac{\partial \xi}{\partial r}, \quad \frac{\partial \Sigma}{\partial \lambda} = \frac{(\theta+\sigma_x)}{(1-\xi)^2}\cdot\frac{\partial \xi}{\partial \lambda}$$

By Condition (18) in Assumption 1, it is obvious that $\frac{(\theta+\sigma_x)}{(1-\xi)^2} > 0$. Thus the signs of $\frac{\partial \Sigma}{\partial r}$ and $\frac{\partial \Sigma}{\partial \lambda}$ coincide with $\frac{\partial \xi}{\partial r}$ and $\frac{\partial \xi}{\partial \lambda}$, respectively. Since it is shown in the proof of Proposition 5 that $\frac{\partial \xi}{\partial r} > 0$ and $\frac{\partial \xi}{\partial \lambda} > 0$, it follows that

$$\frac{\partial \Sigma}{\partial r} > 0, \quad \frac{\partial \Sigma}{\partial \lambda} > 0.$$

□

## A.10 Proof of Proposition 7

*Proof.* Since $\xi$ satisfying (20) and $\Sigma$ the volatility of the firm value are dependent on $\theta$, we use notations $\xi(\theta)$ and $\Sigma(\theta)$ in this proof if necessary to indicate that $\xi$ and $\Sigma$ are functions of $\theta$.

When $\theta = 0$, $Q(u)$ in (51) can be rewritten as

$$Q(u) = -u^2 \hat{Q}(\frac{1}{u}),$$

where

$$\hat{Q}(x) \triangleq (r+\lambda)x^2 - \left[-\frac{1}{2}\sigma_x^2 + (r+\lambda) + (r+\alpha_x)\right] x - \left[\sigma_x^2 - (r+\alpha_x)\right], \tag{61}$$

and $\xi(0)$ satisfies $\hat{Q}(\frac{1}{\xi(0)}) = 0$. Note that

$$\Sigma(0) = \sigma_x \frac{1}{\frac{1}{\xi(0)} - 1}, \quad \Sigma_{\text{no hedge}} = \sigma_x \frac{1}{\frac{1}{\xi_{\text{no hedge}}}}. \tag{62}$$

Thus, we have the following equivalences:

$$\Sigma(0) > \Sigma_{\text{no hedge}} \quad \Longleftrightarrow \quad \frac{1}{\xi(0)} < 1 + \frac{1}{\xi_{\text{no hedge}}} \quad \Longleftrightarrow \quad \hat{Q}(1 + \frac{1}{\xi_{\text{no hedge}}}) > 0. \tag{63}$$

Indeed, it can be verified that

$$\hat{Q}(1 + \frac{1}{\xi_{\text{no hedge}}}) = -\frac{1}{\xi^2_{\text{no hedge}}} \left[ \underbrace{q(\xi_{\text{no hedge}})}_{=0} - (r+\lambda)\xi_{\text{no hedge}} \right] \tag{64}$$

$$= \frac{(r+\lambda)}{\xi_{\text{no hedge}}} > 0, \tag{65}$$

where $q(\xi)$ is given in (33). Consequently, we can conclude that $\Sigma(0) > \Sigma_{\text{no hedge}}$. □

## A.11 Proof of Proposition 8

*Proof.* If we replace (2) by (34), the corresponding HJB equation becomes

$$(r+\lambda)J = \max_{\pi_G} \Big[ \{r + \alpha_x + \sigma_g \theta_g \pi_G\} w J' + \frac{1}{2}(\sigma_x^2 + \sigma_g^2 {\pi_G}^2) w^2 J'' \Big]. \tag{66}$$

Then, we can show that the optimal portfolio in this case becomes

$$\pi_G = -\frac{\theta_g J'}{\sigma_g w J''},$$

and $\pi_G = 0$ because $\theta_g = (\mu_g - r)/\sigma_g = 0$ when $\mu_g = r$.

$\square$

## B Analysis of Example 2

We first summarize $V(k)$ and the optimal dividend policy after the exercise of the real option in Example 2 as in the following lemma:

**Lemma 3.** *The value function after the real option exercise in Example 2 is*

$$V(k) = \begin{cases} k - \frac{\gamma r}{2}k^2, & \text{if } 0 \leqslant k < \frac{1}{\gamma r}, \\ \frac{1}{2\gamma r}, & \text{if } k \geqslant \frac{1}{\gamma r}, \end{cases} \tag{67}$$

*and the optimal dividend policy $d^*(k)$ is*

$$d^*(k) = \begin{cases} rk, & \text{if } 0 \leqslant k < \frac{1}{\gamma r}, \\ \frac{1}{\gamma}, & \text{if } k \geqslant \frac{1}{\gamma r}. \end{cases} \tag{68}$$

*Proof.* The HJB equation that corresponds to $V(k)$ is

$$rV(k) = \max_d \left[ d - \frac{\gamma}{2}d^2 + (rk - d)V'(k) \right]. \tag{69}$$

In the definition of $V(k)$ in (6), $d - C(d)$ is maximized when $d = \frac{1}{\gamma}$ if $C(d)$ is given as (2). Note that the minimum capital required to keep the dividend rate $d = \frac{1}{\gamma}$ permanently is $\frac{1}{\gamma r}$. Thus, if $k \geqslant \frac{1}{\gamma r}$, the optimal dividend policy is $d^* = \frac{1}{\gamma}$ and the corresponding value function is

$$V(k) = \int_0^\infty e^{-rt} \left[ \frac{1}{\gamma} - \frac{\gamma}{2}\left(\frac{1}{\gamma}\right)^2 \right] dt = \frac{1}{2\gamma r}.$$

If $k < \frac{1}{\gamma r}$, from HJB equation (69) for $V(k)$, the candidate for the optimal dividend is $d = \frac{1 - V'(k)}{\gamma}$, and the corresponding ordinary differential equation (ODE) for $V(k)$ is

$$rV(k) = rkV'(k) + \frac{(1 - V'(k))^2}{2\gamma}, \tag{70}$$

with

$$V(0) = 0 \quad \text{and} \quad \lim_{k \uparrow \frac{1}{\gamma r}} V(k) = \frac{1}{2\gamma r}.$$

Then the solution is $V(k) = k - \frac{\gamma r}{2}k^2$ and the corresponding optimal dividend policy is $d^*(k) = rk$. $\square$

It is notable that, even if we consider quadratic $V(\cdot)$ in Example 2 instead of exponential utility in (5), the solution in Proposition 1 does not change except the explicit forms of $A$ and $\bar{w}$, and the solution to $A$ and $\bar{w}$ for Example 2 are as follows:

$$A = \frac{\gamma r}{2(1-\xi)}\bar{w}^{2-\xi}, \tag{71}$$

$$\bar{w} = \frac{1-\xi}{\gamma r(1-\frac{\xi}{2})}. \tag{72}$$

Moreover, all of the results and implications in Sections 4 and 5 still hold with quadratic $V(\cdot)$. Only the proofs of propositions about the exercise threshold $\bar{w}$ require slight modification.

# C Relationship between $\Sigma$ and $\Sigma_{\textbf{no hedge}}$ for Non-zero $\theta$

Proposition 7 provides the size comparison between $\Sigma$ and $\Sigma_{\text{no hedge}}$ for the case where $\theta = 0$. We can obtain more general result on the relationship between $\Sigma$ and $\Sigma_{\text{no hedge}}$ as in the following proposition:

**Proposition 9.** *There exist $\bar{\theta}$ and $\underline{\theta}$ ($\underline{\theta} < 0 < \bar{\theta}$) such that*

$$\Sigma(\theta) > \Sigma_{\text{no hedge}} \quad \textit{for } \underline{\theta} < \theta < \bar{\theta},$$

*and $\Sigma(\theta)$ is maximized at $\tilde{\theta} \triangleq -\sigma_x + \sqrt{2(\alpha_x - \lambda)} \in (\underline{\theta}, \bar{\theta})$, and $\Sigma(\underline{\theta}) = \Sigma(\bar{\theta}) = \Sigma_{\textit{no hedge}}$.*

*Proof.* Consider $\underline{\theta}$ satisfying $\underline{\theta} + \sigma_x \xi(\underline{\theta}) = 0$, then $\underline{\theta} < 0$ because $\xi(\underline{\theta}) > 0$ by Lemma 1 and $\sigma_x > 0$ and $\pi_t^* = 0$ for this case. As explained in Section 4.4, the solution to the case without trading the insurance asset is identical to the solution when $\pi_t^* = 0$. In line with this, we can show that

$$\Sigma(\underline{\theta}) = \Sigma_{\text{no hedge}}.$$

By differentiating $\Sigma(\theta)$ with respect to $\theta$, we have

$$\frac{\partial \Sigma}{\partial \theta} = \frac{\xi G(\xi)}{(1-\xi)^2 Q'(\xi)}, \tag{73}$$

where

$$G(\xi) = (1-\xi)Q'(\xi) - (\theta + \sigma_x)(\theta + \sigma_x \xi).$$

Using $Q(\xi) = 0$, we have

$$G(\xi) = G(\xi) + 2Q(\xi) = \left[(\alpha_x - \lambda) - \frac{1}{2}(\sigma_x + \theta)^2\right](1-\xi).$$

Since $\xi \in (0,1)$ and $Q'(\xi) > 0$ (see the proof of Proposition 4), $\frac{\partial \Sigma}{\partial \theta}$ and $\left[(\alpha_x - \lambda) - \frac{1}{2}(\sigma_x + \theta)^2\right]$

have same sign. Note that $\left[(\alpha_x-\lambda)-\frac{1}{2}(\sigma_x+\theta)^2\right]>0$ is equivalent to

$$-\sigma_x-\sqrt{2(\alpha_x-\lambda)}<\theta<-\sigma_x+\sqrt{2(\alpha_x-\lambda)}\triangleq\tilde{\theta}.$$

Since $-\sigma_x-\sqrt{2(\alpha_x-\lambda)}<-\sigma_x<\tilde{\theta}$, we can conclude that

- $\frac{\partial\Sigma}{\partial\theta}>0$ ($\Sigma(\theta)$ is strictly increasing in $\theta$) for $\theta\in(-\sigma_x,\tilde{\theta})$
- $\frac{\partial\Sigma}{\partial\theta}<0$ ($\Sigma(\theta)$ is strictly decreasing in $\theta$) for $\theta>\tilde{\theta}$.

It is obvious that $\underline{\theta}=-\sigma_x\xi(\underline{\theta})\in(-\sigma_x,0)$ because $\xi(\theta)\in(0,1)$. Indeed, we can compute $\underline{\theta}$ satisfying (37) as follows

$$\begin{aligned}\underline{\theta}&=-\left[\frac{\sigma_x}{2}-\frac{(r+\alpha_x)}{\sigma_x}\right]-\sqrt{\left[\frac{\sigma_x}{2}-\frac{(r+\alpha_x)}{\sigma_x}\right]^2+2(r+\lambda)}\\&=-\left[\frac{\sigma_x}{2}-\frac{(r+\alpha_x)}{\sigma_x}\right]-\sqrt{\left[\frac{\sigma_x}{2}+\frac{(r+\alpha_x)}{\sigma_x}\right]^2-2(\alpha_x-\lambda)},\end{aligned}$$

and direct computation gives us

$$\underline{\theta}-\tilde{\theta}=B_1-B_2-\sqrt{B_1^2-B_2^2}, \tag{74}$$

where

$$B_1\triangleq\frac{\sigma_x}{2}+\frac{(r+\alpha_x)}{\sigma_x}>0,\quad B_2\triangleq\sqrt{2(\alpha_x-\lambda)}>0.$$

Since $B_1^2-B_2^2>0$, we have $B_1>B_2>0$, and it follows that

$$(B_1-B_2)^2-(B_1^2-B_2^2)=2B_2(B_2-B_1)<0,$$

or equivalently,

$$\underline{\theta}-\tilde{\theta}=(B_1-B_2)-\sqrt{B_1^2-B_2^2}<0.$$

In summary, we have $-\sigma_x<\underline{\theta}<\tilde{\theta}$, and $\Sigma(\theta)$ is strictly increasing in $\theta$ for $\underline{\theta}\leqslant\theta<\tilde{\theta}$. This implies that

$$\Sigma(\theta)>\Sigma_{\text{no hedge}}\quad\text{for }\underline{\theta}<\theta\leqslant\tilde{\theta}$$

because $\Sigma(\underline{\theta})=\Sigma_{\text{no hedge}}$.

Moreover, since we have

$$\lim_{\theta\to\infty}\Sigma(\theta)=\lim_{\theta\to\infty}\frac{1}{-\frac{1}{\xi^2}\frac{\partial\xi}{\partial\theta}}=0, \tag{75}$$

there exists $\bar{\theta}>\tilde{\theta}$ such that $\Sigma(\bar{\theta})=\Sigma_{\text{no hedge}}$ because $\Sigma(\theta)$ is continuous in $\theta$, $\Sigma(\theta)$ is strictly decreasing in $\theta$ for $\theta > \tilde{\theta}$, and $\Sigma(\tilde{\theta}) > \Sigma_{\text{no hedge}}$. Therefore, we have

$$\Sigma(\theta) > \Sigma_{\text{no hedge}} \quad \text{for } \theta \in (\underline{\theta}, \bar{\theta})$$

and $\Sigma(\theta)$ is maximized when $\theta = \tilde{\theta}$. Moreover, since it is proven in Proposition 7 that $\Sigma(0) > \Sigma_{\text{no hedge}}$, we have $\bar{\theta} > 0$, which completes the proof. □