Investment under Uncertainty: the Positive Impact of Economic Depreciation on Size and Timing *

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Abstract

This paper identifies and analyzes the effects of the rate of economic deprecation of capital stock on a monopolist's investment problem in a dynamic and uncertain market environment, where continuous economic depreciation cannot be fully offset. We find that a higher rate of depreciation increases the investment trigger but can have mixed effects on the scale of investment. When investment is undertaken immediately, the monopolist has an incentive, relative to a zero-depreciation scenario, to preemptively increase its capital stock and counter losses in productive capacity for low (positive) rates of depreciation. For high rates, however, the reduced return on investment overtakes this incentive, leading to the firm investing less. Furthermore, the analysis on the interplay between the rate of depreciation and the level of uncertainty reveals that only high rates of depreciation can mitigate the impact of uncertainty on the real option's value, partially lifting the irreversibility constraint. The fact that the impact of economic depreciation on the firm's timing option and capacity decisions is level dependent demonstrates that its consideration is not trivial.

Keywords: Economic Depreciation, Capital Investment, Investment under Uncertainty, Monopoly.

JEL classification: D42, D81, L12

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1 Introduction

The inevitable physical and productive deterioration of assets, due to the passage of time or recurrent use, imposes constraints on firms' production capabilities that cannot be overcome in real-life. These constraints are often dismissed in the study of the optimal investment behavior, with the introduction of strong assumptions, such as the possibility of continuous investment or constant maintenance. Not only are these assumptions hard to observe for individual firms, given any financial attrition or the need to stop production to carry on maintenance, but they also dissociate the value of the asset from its usage. We show that, in the absence of such assumptions, economic depreciation of the capital stock (henceforth simply "depreciation") can play a crucial role in the investment decision, even in the absence of tax considerations.

In fact, this paper shows that the impact of depreciation on the firm's decision considering the timing and scale of investment is not trivial nor typically monotonic.

In the spirit of Abel and Eberly (1996), Bertola and Caballero (1994), and Bar-Ilan and Strange (1999), this paper proposes a simple, yet effective, framework where we consider a monopolist that has an American-style perpetual option to undertake a one-off lump-sum irreversible investment. By acquiring capital stock, the firm can immediately start up production in a market with a downward sloping demand curve, where the willingness-to-pay of consumers evolves stochastically over time. Capital stock is assumed to depreciate at a constant geometric rate. By allowing the firm's flexibility in terms of capacity size, we can show that an increase in the depreciation rate may have two competing effects on the firm's decisions: on the one hand, the firm may wish to install a higher level of capital stock to account for future falls in productivity. On the other hand, the total future aggregate revenue generated by each unit of capital stock decreases as a result of depreciation, which makes investment relatively more expensive and makes the firm invest less. We refer to the first effect as the *buffer effect* and the second effect as the *relative cost effect*.

While depreciation can have an ambiguous impact on the scale of investment, it also partially relaxes the irreversibility constraint and thereby interacts with uncertainty, meriting an analysis to the net impact it has on the firm's timing and option value.

Our main results can be summarized as follows.

(i) If the firm decides to invest immediately, under not too strict conditions, we find that the buffer effect is dominant for small depreciation rates, whereas the relative cost effect is dominant for higher rates. Thus, we say that the firm *overinvests* for small depreciation rates, relative to a zero-depreciation benchmark, and the firm *underinvests* for large depreciation rates.

It may seem surprising that depreciation can have a mixed effect on the size of investment. Intuitively, however, economic depreciation creates a need to preemptively replace productive capacity. However, whereas the costs of acquiring more capital scale proportionally in the investment size, the benefits from such investment have diminishing returns on the amount of capital acquired. Therefore, the benefits of acquiring capital stock may be typically large but are dominated by the relative costs of investing at sufficiently high depreciation rates, i.e. when the need for capital replenishment is greater.

(*ii*) Depreciation unambiguously increases the threshold for undertaking investment.

This means that if the consumers' initial willingness-to-pay is not sufficiently high, then depreciation leads to a later exercise of the option, in expectation. Consequently, depreciation also increases the size of investment and decreases the present value of capital.

The increase in the scale of investment found due to a later exercise of the option is a common result in the literature, and in line with, e.g., Manne (1961), Bar-Ilan and Strange (1999), and Dangl (1999), who show that the scale of investment is increasing in the consumers' willingness-to-pay. We show that our result stands, even when the rate of depreciation is so high that the firm would typically underinvest if it were to invest immediately.

(*iii*) Depreciation mitigates the impact of uncertainty on investment, thereby partially lifting the irreversibility constraint, only for sufficiently high rates of depreciation. For low rates, depreciation compounds on the positive effect of uncertainty on the option value.

Neoclassical literature on capital investment has made widespread use of the assumption that economic depreciation can be fully neutralized by continuous investment (see, e.g., pioneering studies like Jorgenson (1963), Hartman (1972), and Abel (1983)), if not fully ignored for any non-tax related purposes. Such practice overlooks that, if not for some exogenous salvage value, a productive asset's worth is intrinsically connected to its productive capacity, and therefore by the loss of this capacity, as well as heavily influenced by the fluctuations of market prices for the end good produced.

The flexibility in investment size is a crucial element in our framework. Conventional dynamic investment set-ups, starting with the seminal work by, e.g., McDonald and Siegel (1986) and Dixit and Pindyck (1994), typically assume that the scale of the investment is fixed and exogenously determined. Dangl (1999) and Bar-Ilan and Strange (1999) were among the firsts to study capacity choice for single firm set-ups. More recent contributions studying capacity choice in various dynamic lumpy investment monopoly settings are Della Seta *et al.* (2012), Wen *et al.* (2017), Azevedo *et al.* (2020), Sarkar (2021), and Jeon (2021) (also see Huberts *et al.* (2015) for a general survey for contributions prior to 2015). The work in this paper extends on the single firm case by including and studying economic depreciation.

Surprisingly, considering the literature on investment under uncertainty in a dynamic framework, only few studies incorporate the effects of economic depreciation on the timing of capital investments. Those studies that do, in turn, do not simultaneously allow for a decision on the size of investment. For some, depreciation is present, but not studied (e.g., Abel (1983), Bertola and Caballero (1994), Bloom (2000), Gryglewicz and Hartman-Glaser (2019), Mauer and Ott (1995), Cooper (2006), and Lyandres *et al.* (2018)). For others, depreciation plays a more prominent role, but the scale of investment is, as mentioned, not considered (e.g. Arkin and Slastnikov (2007), Jou and Lee (2011), Ruffino and Treussard (2006), and Adkins and Paxson

(2017)).[1] To the best of our knowledge, there are currently no other studies that analyze both timing and scale of capital investments, simultaneously, under the presence of positive economic depreciation in a dynamic and uncertain environment.

Popular alternative ways of modeling depreciation include assuming finite life-time of capital or complete capital depreciation after production, as studied by, e.g., Gryglewicz *et al.* (2008), Dixit and Pindyck (1994), and Nakamura (1999, 2002). Although some situations are very well modeled by this "light bulb" model of depreciation, or similar approaches, we find that this choice of treatment obscures the identification of the opposing and level-dependent effects we uncover in this paper, which are present under a more general setting. Nakamura, in a discrete time setting without optimal timing, also considers the relationship between uncertainty and depreciation and finds that market uncertainty has a negative impact on investment. Femminis (2008) and Saltari and Ticchi (2005) challenge his findings: Femminis shows that the negative relationship between investment and uncertainty fully relies on the assumption that capital fully depreciates after production. He also shows that this is not always the case when capital depreciates at a constant geometric rate in a model with risk-aversion. Section 4.2 discusses the implications from using this alternative formulation.

This paper is organized as follows. First, the model is introduced in Section 2. Section 3 studies the firm's investment strategy and particularly focuses on the impact of depreciation on scale and timing. We consider two alternative versions of the main model in Section 4 and Section 5 offers concluding remarks and comments on further research extensions this paper.

2 Model

Consider a monopolist that holds an American style perpetual (real) option to undertake investment and acquire some capital stock. Capital stock is denoted by $K(t)$, where $t \geq 0$ denotes time, and can be obtained by a lump-sum irreversible investment. The firm is assumed to be risk-neutral, rational, financially unconstrained, and value-maximizing. After investment, capital stock is assumed to change over time according to the dynamics

$$\mathrm{d}K(t) = I(t) - \delta \cdot K(t)\mathrm{d}t, \tag{1}$$

where $I(t)$ denotes the instantaneous investment at time t and where $\delta > 0$ denotes the depreciation rate. Although we do not allow for zero-depreciation cases in the problem specification, we will include $\delta = 0$ in

[1] In fact, Arkin and Slastnikov and Jou and Lee find that depreciation accelerates investment, which is a crucially different finding from the work in this paper. However their results are strongly driven by financial (dis)advantages. In the duopoly game by Ruffino and Treussard with time-to-build and technology adoption, capital depreciation is considered as a necessary requirement for a capital-replacement option to be exercised. Adkins and Paxson explore the role of depreciation on optimal investment, with stochastic deterioration of the salvage value and operating cost, in a capital replacement model. However, their use of a price-taking assumption limits the application of their set-up to the study of capacity choice problems, since the direct effect of output on prices as a part of the firm's instantaneous profits, a crucial property in these problems, is lost.

our analysis in order to be able to reflect on the effect of depreciation.

For our main model, we assume that the instantaneous investment takes a positive value at most once.[2] Let $\bar{K}$ denote the initial capital stock at the time of investment, i.e. for some time $s \geq 0$, $I(s) = \bar{K} > 0$ and $I(t \neq s) = 0$. It then follows from (1) that the capital stock held by the firm is given by $K(t) = \bar{K}e^{-\delta(t-s)}, t \geq s$.

To finance this project, the firm has a sunk (adjustment) cost of $\kappa > 0$, proportional to $I(t)$, which is incurred at the moment of investment.

Output $Q(t)$ is determined by the production function

$$Q(t) = \frac{a}{\gamma} K^{\gamma}(t), \tag{2}$$

where $a > 0$ is the production technology parameter and $\gamma \in (0, 1)$ denotes an output elasticity that ensures diminishing returns to capital.[3] This type of production function is in line with, e.g., Bertola (1988), Bertola and Caballero (1994), Nakamura (2002) and Lyandres *et al.* (2018), where labor is assumed to be flexible.

The market the firm operates in, is characterized by the following inverse demand function:

$$p(t) = x(t)(1 - Q(t)), \tag{3}$$

so that prices clear markets. Here, $x(t)$ captures uncertainty and noise and it follows a geometric Brownian motion with trend μ and volatility parameter $\sigma > 0$, i.e.

$$\mathrm{d}x(t) = \mu x(t)\mathrm{d}t + \sigma x(t)\mathrm{d}z(t). \tag{4}$$

The first term on the right-hand side represents the trend of the process. The second term on the right-hand side contains the Wiener process $z(t)$ through which exogenous shocks are brought in. The Wiener process has a normal distribution with expected value 0, standard deviation $\sqrt{t}$, and has the property that $(\mathrm{d}z)^2 = \mathrm{d}t$. Let us denote the initial value of the shock process $(x(t))_{t \geq 0}$ by $X = x(0)$. We will assume $2\mu > \sigma^2$ to ensure finite expected hitting times. Discounting is done under rate ρ, with $\rho > \mu$.

This type of inverse demand function follows, e.g., Pindyck (1988), He and Pindyck (1992), Aguerrevere (2003), Wu (2007), and Huisman and Kort (2015). Firms are assumed to be committed to produce the amount dictated by their capacity allowance. In the literature on capacity constrained firms, this so-called capacity clearing assumption is used on a large scale (e.g. Deneckere *et al.* (1997), Chod and Rudi (2005), Anand and Girotra (2007), Goyal and Netessine (2007), and Huisman and Kort (2015)). For example, Goyal and Netessine (2007) argue that producing below capacity may be found to be difficult for firms as a result of fixed costs associated with commitments to suppliers, labor, and production ramp-up.

[2]Section 4.1 considers an extension where the firm can replenish its capital stock any number of times and it confirms the results we obtain from our main model.

[3]We assume that $a > \gamma e^{-1}$ to ensure a monotonic relationship between Q and γ. We want to note that our results will still apply for other values of a, however, some of the intuitive properties of the production function are lost in this simple formulation if a is chosen to be too small.

The firm's strategy comprises two decisions: the timing of investment and the size of the initial capital stock ($\bar{K}$). We base ourselves on the work of McDonald and Siegel (1986), Smets (1991), and Dixit and Pindyck (1994) to find the firm's optimal investment (stopping) behavior under uncertainty. To formally write down the firm's optimization problem, denote the filtered probability space of $(x(t))_{t\geq 0}$ by $(\Omega, \mathcal{F}^x, \mathbf{F}, \mathbb{P})$, so that the filtration associated with the process $x(\cdot)$ is denoted by $\mathbf{F} = (\mathcal{F}_t^x)_{t\geq 0}$, with natural filtrations $\mathcal{F}_t^x$, collecting the available information at time $t \geq 0$. Conditional expectation operator $\mathbb{E}_X$ is taken with respect to measure $\mathbb{P}$, i.e. $\mathbb{E}\{\,\cdot\,|\mathcal{F}_0^x\}$, where $X = x(0)$.

Let τ be a stopping time and let $\mathcal{M}$ consist of all finite $\mathcal{F}_t^x$-stopping times. Given production function (2) and inverse demand (3), the firm then faces the following optimization problem, at time $t = 0$, over the initial capital stock $\bar{K}$ and timing τ,

$$V(X) = \sup_{\tau\in\mathcal{M},\bar{K}\geq 0} \mathbb{E}_X \left\{ \int_0^\infty \left(p(t)Q(t) - \kappa I(t)\right) e^{-\rho t} \mathrm{d}t \right\} \tag{5}$$

subject to $I(t) = \bar{K}\chi\{t = \tau\}$, $\mathrm{d}K(t) = I(t) - \delta \cdot K(t)\mathrm{d}t$. The function $\chi\{v\}$ equals 1 if v is true and 0 otherwise.[4]

In line with the literature, we will write the optimal stopping moment in terms of X. This means that we will determine a threshold X^* such that, if $x(0) < X^*$ the firm invests when $x(t)$ hits the investment threshold X^* for the first time.[5] If $x(0) \geq X^*$, the firm invests immediately and investment takes place at $t = 0$. It follows that the (stochastic) investment time is given by hitting time $\tau^* = \inf\{t \geq 0 \mid x(t) \geq X^*\}$. The set of all values of X such that investment takes place immediately is called the *stopping region* and the complementary region is called the *continuation region*. For typical scenarios like ours, the stopping region equals $\mathcal{S} = \{X \in \mathbb{R}_+ \mid X \geq X^*\}$ and the continuation region equals $\mathcal{C} = \{X \in \mathbb{R}_+ \mid X < X^*\}$.

The problem in (5) can then be rewritten as

$$V(X) = \sup_{\tau\in\mathcal{M},\bar{K}>0} \mathbb{E}_X \left\{ \int_0^\infty x(t)(1 - Q(t))Q(t)e^{-\rho t}\mathrm{d}t - e^{-\rho\tau}\kappa\bar{K} \right\}, \tag{6}$$

with

$$Q(t) = \begin{cases} 0 & \text{if } t < \tau, \\ \frac{a}{\gamma}\left(\bar{K}e^{-\delta(t-\tau)}\right)^\gamma, & \text{if } t \geq \tau. \end{cases}$$

For our results we will denote by $\bar{K}^*(X)$ the optimal level for the initial stock if $X \geq X^*$, i.e. if $X \in \mathcal{S}$. For $X < X^*$, i.e. for $X \in \mathcal{C}$, the firm delays investment and will set, upon investment, $\bar{K}^*(X^*)$, which we will denote as $\bar{K}^{opt}$. Note that these represent the level of capital stock upon investment.

[4]Notice that for all finite stopping times it holds that $\lim_{t\to\infty} K(t) = 0$, which implies that there are no issues with the transversality.

[5]The proof for this result in our set-up is based on Dixit and Pindyck (1994), who show that the state space can be divided into two consecutive regions for standard real options problems without capacity choice, and whose results are extended by Huberts *et al.* (2019) for models where capacity choice is explicitly modeled. Optimality can be shown using a verification theorem based on, e.g., Gozzi and Russo (2006).

Proposition 1 *Let $\gamma < \frac{\beta-1}{\beta}$, where β is given by*

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}. \tag{7}$$

Then, $\mathcal{C}$ is empty. The firm's capital stock in the stopping region $\bar{K}^*(X)$ is the solution of

$$\frac{X}{\rho+\gamma\delta-\mu}\left(1 - 2\frac{a}{\gamma}\bar{K}^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) = \frac{\kappa}{a}(\bar{K})^{1-\gamma}. \tag{8}$$

Let $\gamma \geq \frac{\beta-1}{\beta}$. Then $\mathcal{C}$ is non-empty. For $X \in \mathcal{S}$, as in the previous case, the firm invests immediately and acquires $\bar{K}^*(X)$, given by the solution of (8). For $X \in \mathcal{C}$ the firm delays investment and waits until the process $x(t)$ reaches the investment threshold X^* to acquire $\bar{K}^{opt}$, given by

$$X^* = \left(\frac{\gamma}{a}\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}\right)^{\frac{1-\gamma}{\gamma}} \frac{\beta(2\gamma-1)+1}{\beta-1}\frac{\kappa}{a}(\rho+\delta\gamma-\mu), \tag{9}$$

$$\bar{K}^{opt} = \left(\frac{\gamma}{a}\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}\right)^{\frac{1}{\gamma}}, \tag{10}$$

respectively. As a result, the firm's value function is given by

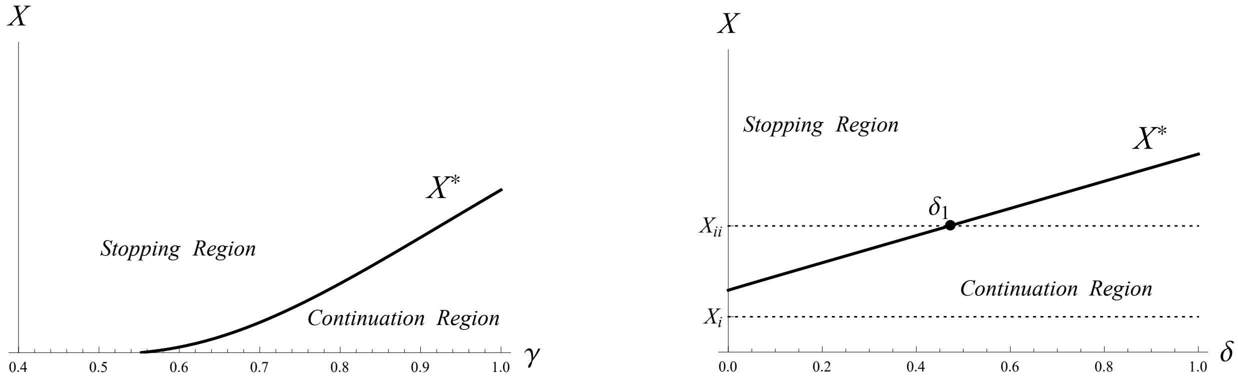
$$V(X) = \begin{cases} \left(\frac{X}{X^*}\right)^\beta \frac{\kappa}{\beta-1}\bar{K}^{opt} & \text{if } X < X^*, \\ \frac{X}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}(\bar{K}^*(X))^\gamma\left(1 - \frac{a}{\gamma}(\bar{K}^*(X))^\gamma\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) - \kappa\bar{K}^*(X) & \text{if } X \geq X^*. \end{cases} \tag{11}$$

All proofs can be found in Appendix A. Notice that, as a result of the assumption that $\rho > \mu$, we have $\beta > 1$. Additionally, as it's been shown extensively in the literature, β has the property that it is decreasing in σ. Also note that the first case of (11) gives the *option value.*

Proposition 1 shows that only for $\gamma > \frac{\beta-1}{\beta}$ we have a non-empty continuation region, i.e., there exists a trigger as given by (9) and it is positive. Bar-Ilan and Strange (1999) also established a relationship between the *marginal productivity of capital* parameter γ and the trigger. As shown by Lyandres *et al.* (2018), assumptions on γ are required. In order for the value of the waiting option to exceed the value of immediate investment, γ must be sufficiently large. From the formulation of our production function, it follows that capital gets more productive as the output elasticity γ goes down. This means that for each unit of output, less capital is required when γ is smaller, so that investment in each unit of output becomes relatively cheaper, which ultimately accelerates investment. Figure 1a illustrates the regions for different γ.

We would like to note here that we distinguish two ways in which the optimal capital stock is affected by the rate of depreciation. First, assuming the firm invests immediately, i.e. X is fixed, δ has an effect on $\bar{K}^*(X)$ as can be noted from (8). We will refer to this as a *direct effect.*

The proof of Proposition 1 shows that $\frac{\partial}{\partial X}K^*(X) > 0$, i.e. the firm's optimal capital stock in the stopping region is increasing in X. Then, in case the firm delays investment, from (9), δ has a direct impact on X^*, which in turn has an impact on $\bar{K}^{opt} = \bar{K}^*(X^*)$. We will refer to the effect of δ on the acquired capital stock through a change in the threshold of investment as the *indirect effect.*



(a) Stopping region and continuation region for different γ with $\delta = 0.1$.

(b) Stopping region and continuation region for different δ with $\gamma = 0.8$.

Figure 1: Stopping region and continuation region for different parametrizations.
$\mu = 0.02,\ \rho = 0.1,\ \sigma = 0.2,\ a = 0.6$, and $\kappa = 0.3$.

3 Analysis

With our main model in place, we now address the question of how economic depreciation, which cannot be fully offset, affects the firm's optimal investment behavior.

Section 3.1 details how depreciation affects the investment threshold, i.e. the boundary between the continuation region and stopping region.

The case where the monopolist remains in the stopping region when studying the effect of different rates of δ, i.e. in the region where the timing of investment is unaffected, offers an opportunity to look at the direct effect of depreciation on the optimal capital stock, which is analyzed in Section 3.2. Then, Section 3.3 extends the analysis to the case where the monopolist remains in the continuation region for an infinitesimal change in δ, now considering indirect effects due to the simultaneous adjustments of size and timing of investment, and expands on how depreciation affects, in addition, the impact of uncertainty on the firm's investment problem.

Section 3.4 then consolidates these findings and allows for cases where the initial state can potentially switch regions for a change in δ.

3.1 Timing of Investment

Intuition tells us that, since a higher rate of depreciation leads to each unit of capital stock generating less output, total revenue streams would be negatively impacted by depreciation. The firm would then have an incentive to delay investment until the state process reaches a higher level of the consumers' willingness-to-pay. That way, the firm could allow for the gains from the higher prices to compensate the losses in productivity, minimizing any expected revenue loss.

Indeed, one can check directly from equation (9) that, since $\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}$ is an increasing function of δ, so is the investment threshold X^* increasing in δ. This confirms that, comparatively, depreciation delays investment in expectation for all states X below the modified threshold.

Figure 1b illustrates the investment threshold X^* for different values of δ and distinguishes the regions where the firm delays investment (continuation) and where the firm invests immediately (stopping). As illustrated, for a given parametrization, there exist initial states, like, e.g., X_i, such that they are in the continuation region for any δ. Others, like X_{ii}, can fall in either region depending on the range of δ values being analyzed.

By first restricting our analysis to the range where initial states remain in the stopping region, like X_{ii} for $\delta \in [0, \delta_1)$, we can provide a characterization of the firm's optimal investment scale even as the threshold increases in δ, because it remains optimal to invest immediately.[6] For our analysis on the continuation region we, equivalently, first implicitly assume that we restrict our analysis to the range where initial states remain in the stopping region.

3.2 Investment in the Stopping Region

Let $X \in \mathcal{S}$, i.e. the firm undertakes immediate investment. The parameter δ appears in two terms in equation (8), each having a different effect on capital stock $\bar{K}^*(X)$. Restructuring equation (8) gives

$$\overset{\text{“Relative Cost Effect”}}{\underbrace{\frac{\kappa}{a}(\bar{K})^{1-\gamma}\frac{\rho+\gamma\delta-\mu}{X}}_{\uparrow \text{ as } \delta\uparrow \;\Rightarrow\; \bar{K}\downarrow}} = \overset{\text{“Buffer Effect”}}{\underbrace{\left(1-2\frac{a}{\gamma}\bar{K}^{\gamma}\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right)}_{\uparrow \text{ as } \delta\uparrow \;\Rightarrow\; \bar{K}\uparrow}}. \tag{12}$$

Depending on which of the two effects is dominant, $\bar{K}^*(X)$ is either pushed downwards or upwards as δ increases. To understand why this happens, for the first effect, notice that when depreciation is stronger, each unit of capital stock will produce less units of output in the future, which means that the marginal revenue of capital is negatively affected by depreciation. Although the cost of investment, i.e., the cost of acquiring capital, is unaffected by a change in δ, it becomes relatively less rewarding, or more expensive, to invest when δ is higher. As a result, the optimal quantity goes down. We will call this the *relative cost effect.*

The second effect, where depreciation pushes the quantity up, follows from anticipating the changes in capacity that will restrict output. This gives the firm the incentive to set an initially higher output and to therefore acquire a higher level of capital stock upon investment. We will call this the *buffer effect.*

To illustrate the buffer effect, consider the situation of a firm with a capital stock that does not depreciate, represented by the solid lines in Figure 2, where Panel (a) represents the capital stock and Panel (b) represents the instantaneous cash flows $\pi(t) = X(t)(1-Q(t))Q(t)$ in expectation. If the firm now faces a scenario where

[6] In other words, the timing decision is the same for $X > X^*$ and thus plays no direct role in the choice of $\bar{K}$. Nevertheless, we don't explicitly exclude X^* from the analysis in the stopping region, but simply note that the sensitivity of the investment scale will differ on this point if taken from the left and from the right of δ.

its capital depreciates, its capital stock and the corresponding instantaneous cash-inflows will erode over time as illustrated by the dotted curves, assuming the firm has the same capital stock at $t = 0$. This gives the firm an incentive to invest in a higher initial capital stock (and move from the dotted to the dashed curve) and thus increase the area under $\mathbb{E}_X \pi(t)$ (from dotted to dashed) to recapture some of the lost total cash-inflows, which, in essence, is the buffer effect.

The degree to which it is optimal increase the capital stock, however, will depend on the cost associated with acquiring additional capital stock. Note that the dashed line in Panel (b) was built from the optimal level of capital stock, taking the cost of acquisition into consideration.

The next proposition formally shows that the buffer effect is dominant for small values of δ whereas the relative cost effect is dominant for larger values. Figure 3a illustrates the typical shape of $\bar{K}^*(X)$ as a function of depreciation parameter δ. Figure 3b illustrates how $V(X)$ is overall affected by the depreciation rate δ.

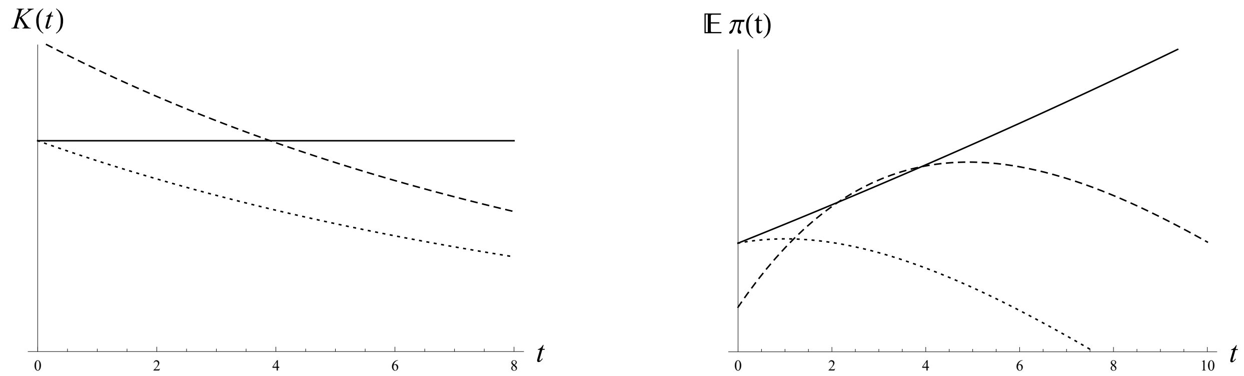
Proposition 2 *Let $\hat{\delta}(X)$ be the (unique) solution to*

$$\frac{X(\rho-\mu)}{2(\rho+\gamma\delta-\mu)^2} = \frac{\kappa}{a}\left(\frac{\gamma}{4a}\left(\frac{\rho+2\gamma\delta-\mu}{\rho+\gamma\delta-\mu}\right)^2\right)^{\frac{1-\gamma}{\gamma}}. \tag{13}$$

Let $X \in \mathcal{S}$.

(i) If $X > (\rho-\mu)\frac{8\kappa}{\gamma}\left(\frac{\gamma}{4a}\right)^{\frac{1}{\gamma}}$, then $\hat{\delta}(X) > 0$ and

- *for $\delta \in (0, \hat{\delta}(X))$ the capital stock $\bar{K}^*(X)$ is increasing in δ;*
- *for $\delta > \hat{\delta}(X)$ the capital stock $\bar{K}^*(X)$ is decreasing in δ.*

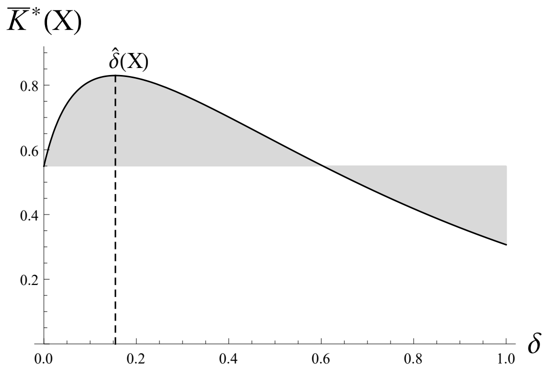


(a) Level of capital stock as a function of t (stopping region).

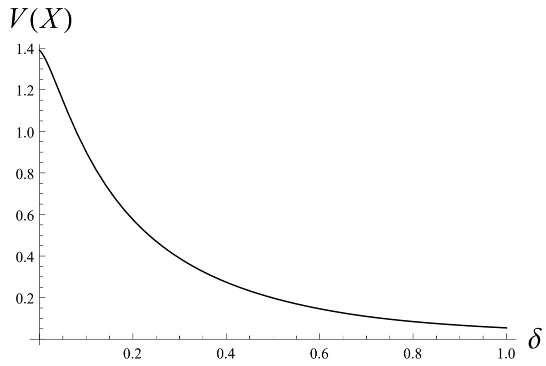
(b) Instantaneous cash-inflows (stopping region).

Figure 2: Illustration of buffer effect. Investment when $\delta = 0$ (solid), when $\delta = 0.1$ but setting capital stock as if $\delta = 0$ (dotted), and optimal investment for $\delta = 0.1$ (dashed).

$\mu = 0.02,\ \rho = 0.1,\ \gamma = 0.8,\ X = 0.5$, and $a = 0.6$.



(a) Optimal level of $\bar{K}^*(X)$ as a function of δ.



(b) Firm's value function $V(X)$ as a function of δ.

Figure 3: The optimal investment size $\bar{K}^*(X)$ and the resulting value to the firm $V(X)$ as a function of depreciation parameter δ.

$$\mu = 0.02,\ \rho = 0.1,\ \sigma = 0.2,\ \gamma = 0.8,\ X = 0.5,\ a = 0.6, \text{ and } \kappa = 0.3.$$

(ii) If $X \leq (\rho - \mu)\frac{8\kappa}{\gamma}\left(\frac{\gamma}{4a}\right)^{\frac{1}{\gamma}}$, *then* $\hat{\delta}(X) \leq 0$ *and*

- *for all* $\delta > 0$ *the capital stock* $\bar{K}^*(X)$ *is decreasing in* δ.

Section 3.4 will show that the condition in Case (*ii*) of Proposition 2 is not typically met, since these values of X are often part of the continuation region.

Case (*i*) of Proposition 2 shows that, in the stopping region, the buffer effect dominates for low rates of depreciation, i.e. the capital stock is increasing in δ. The firm adjusts for the erosion in future productive capacity by *overinvesting*, that is acquiring a higher level of capital stock, relative to the zero-depreciation case, and thus minimizing the loss in total revenue.[7] As illustrated by Figure 3a, for higher rates of depreciation, investing in each unit of capital becomes relatively too expensive, since at these rates any additional capital translates very poorly into extra production, while remaining just as costly, and the *net* effect on the size of investment reverses for some $\delta > \hat{\delta}(X)$, which we refer to as *underinvestment* relative to the zero-depreciation case.

For Case (*ii*) of Proposition 2, consumers' willingness-to-pay is too low for the firm to be able to invest in a large stock and therefore the buffer effect cannot dominate for any δ. This case arises in scenarios where, relatively, κ is high, a is small, and/or X is low relative to $(\rho - \mu)$. In other words, when each unit of capital stock is relatively too expensive.

Since the model accommodates mixed effects of depreciation on investments, these results feed into the wider discussion taking place in the literature on determinants of industry growth in the presence of depreciation (see, e.g., Samaniego and Sun (2019)). For the cases where the initial willingness-to-pay is sufficiently

[7] It is worth noting that this result is not driven by the capacity clearing constraint. In fact, relaxing this assumption will amplify the buffer effect, and therefore the non-monotonicity observed in this case is preserved.

high, so that the firm finds itself in the stopping region, these results illustrate the case for moderate depreciation as a growth driver.

The next section studies how depreciation can influence investment in more detail, by including the continuation region in the analysis.

The overall effect of δ on $V(X)$ in the stopping region, as illustrated in Figure 3b, is unambiguously negative. Depreciation leads to lower expected total cash flows which, in turn, translates into a lower value for the firm.

Lemma 1 *Let $X \in \mathcal{S}$. Then $\frac{\partial}{\partial \delta} V(X) < 0$.*

3.3 Investment in the Continuation Region

Assume that $\mathcal{C}$ is nonempty and let $X \in \mathcal{C}$, i.e. the firm always delays investment. Section 3.3.1 builds on the stopping region case to introduce the indirect effect of δ on the scale and value of capital. Section 3.3.2 looks at the effect of uncertainty on the firm's investment and focuses on the interplay between depreciation and uncertainty.

3.3.1 The Effect of δ on Capital Stock

Capital stock $\bar{K}^{opt} = K^*(X^*)$ is affected by δ in two ways: indirectly through X^* as studied in Section 3.1 and directly as studied in Section 3.2. The indirect effect is positive: the scale of investment, as established in the real options literature, increases when investment is undertaken at a higher level of the state process, which we find as well. This follows from $\bar{K}^*(X)$ being an increasing function of X (see proof of Proposition 1). The direct effect can be mixed as established in Section 3.2.

The overall effect of δ on the scale of investment $\bar{K}^{opt}$, however, is unambiguously positive as depreciation gets stronger and thus the indirect effect always dominates any negative direct effects. Similar to what we did to investigate the effect of depreciation on the investment trigger, one can easily verify that, since $\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}$ is an increasing function of δ, $\bar{K}^{opt}$ must also be increasing in δ, which follows directly from equation (10).

However, one must keep in mind that $\bar{K}^{opt}$ represents the size of investment (at the time of investment). An increase in the size as a result of more rapid depreciation does not necessarily mean that the value of investment, i.e. the present value of capital investment, in expectation, at time zero is increasing as well. In fact, as investment is delayed for higher rates of depreciation, a change in δ has opposing effects on the present value of investment,

$$\begin{aligned} \mathbb{E}_X\left\{e^{-\rho\tau^*}\kappa I(\tau^*)\right\} &= \left(\frac{X}{X^*}\right)^{\beta} \kappa \bar{K}^{opt} \\ &= \kappa \underbrace{\left(\frac{\gamma}{a}\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}\right)^{\frac{1-\beta(1-\gamma)}{\gamma}}}_{\uparrow \text{ as } \delta\uparrow} \underbrace{\left(\frac{\beta-1}{\beta(2\gamma-1)+1}\frac{a}{\kappa}\frac{X}{\rho+\delta\gamma-\mu}\right)^{\beta}}_{\downarrow \text{ as } \delta\uparrow}, \end{aligned} \tag{14}$$

for all $X \in \mathcal{C}$. Notice that $\beta(1-\gamma) < 1$, which follows from the condition in Proposition 1 that allows for non-empty $\mathcal{C}$.

The following lemma shows that the negative effect is always dominant for the present value of the acquired capital stock, even when the level of capital stock upon investment is increasing in δ. Hence, the decrease in present value of investment due to the delay outweighs the increase in the optimal level of capital stock at the time of investment, in expectation.

Lemma 2 *Let $X \in \mathcal{C}$. Then $\frac{\partial}{\partial \delta}\left(\frac{X}{X^*}\right)^{\beta} \kappa \bar{K}^{opt} < 0$.*

3.3.2 Uncertainty and Depreciation

In the early real options literature (see, e.g., Bertola (1988), Pindyck (1988), Dixit (1989)), for models with fixed capacity, a direct relationship was established between the volatility of the underlying process and the firm's real option (also see Dixit and Pindyck (1994)).[8] Uncertainty pushes the associated investment threshold up and leads to a "late" exercise of the options so that uncertainty is bad for investments. However, since the increased uncertainty also directly affects the distribution of the underlying stochastic process and, therefore, the probability of reaching a higher threshold within a certain amount of time (see, e.g., Lund (2005), Sarkar (2021)), an appropriate way to measure the effect of uncertainty on the investment option is to look at the value of the option. For these models, the relationship is unambiguous: more uncertainty increases the value of the option and thus uncertainty is "bad" for investment. When including capacity choice, the same relationship can be established (see, e.g., Bar-Ilan and Strange (1999), Huberts *et al.* (2015)).

However, allowing economic depreciation to take place could call into question this unambiguously positive impact of uncertainty on the option value, since this increased value stems in part from the irreversible nature of the investment and considering the fact that depreciation erodes such irreversibility. Intuitively, the firm should attach less value to flexibility when depreciation is stronger and, as such, the present value of the option should go down.

One can check directly from (9) and (10) that, also in our set-up, higher uncertainty increases the investment threshold and the scale of investment. As for the option value, the next lemma shows that uncertainty also increases the option value unambiguously in our set-up, while depreciation decreases the option value.

Lemma 3 *Let $X \in \mathcal{C}$. Then*

(i) $\frac{\partial}{\partial \sigma} V(X) > 0$, *and*

(ii) $\frac{\partial}{\partial \delta} V(X) < 0$.

[8]See, e.g., Gryglewicz *et al.* (2008) and Sarkar (2021) for a more contemporary discussion on the uncertainty-investment relationship in the theoretical and empirical literature.

Case (ii) of Lemma 3 confirms the intuition that the option value should decrease in δ, but makes no distinction between a loss in value due to extra flexibility and losses due to reduced expected total revenues. To investigate the effect of δ on the value attached to flexibility, we look at how depreciation affects the impact of uncertainty, by evaluating how $\frac{\partial}{\partial\sigma}V(X)$ is affected by a change in the rate of depreciation. In the literature, there is mixed evidence as to how an increase in uncertainty affects capital investment when depreciation is stronger (see, e.g., Samaniego and Sun (2019) for a discussion). Thereto, the next proposition discusses the relationship between depreciation and uncertainty by studying the sensitivity of the option to uncertainty for changes in δ. To that end, the cross derivative is studied for both the investment trigger and the option value.

Proposition 3 *Let $X \in \mathcal{C}$. Then*

(i) $\frac{\partial^2}{\partial\delta\partial\sigma}X^* > 0$, *and*

(ii) $\frac{\partial^2}{\partial\delta\partial\sigma}V(X) > 0$ *if and only if*

$$\mathrm{Ln}\left(X^*\right) - \mathrm{Ln}\left(X\right) < \left(\beta - \frac{\rho-\mu}{\rho+2\gamma^2\delta-\mu}\right)^{-1}. \tag{15}$$

The cross-derivative in case (i) is positive: more uncertainty and stronger depreciation push the trigger further up. In other words, higher depreciation compounds on the delay associated with more uncertainty. It is worth noting that a higher threshold does not necessarily mean that, in expectation, more time will pass before investment is undertaken, when this is caused by a larger σ, but indeed that the firm strictly requires a higher willingness-to-pay before acquiring the capital stock.

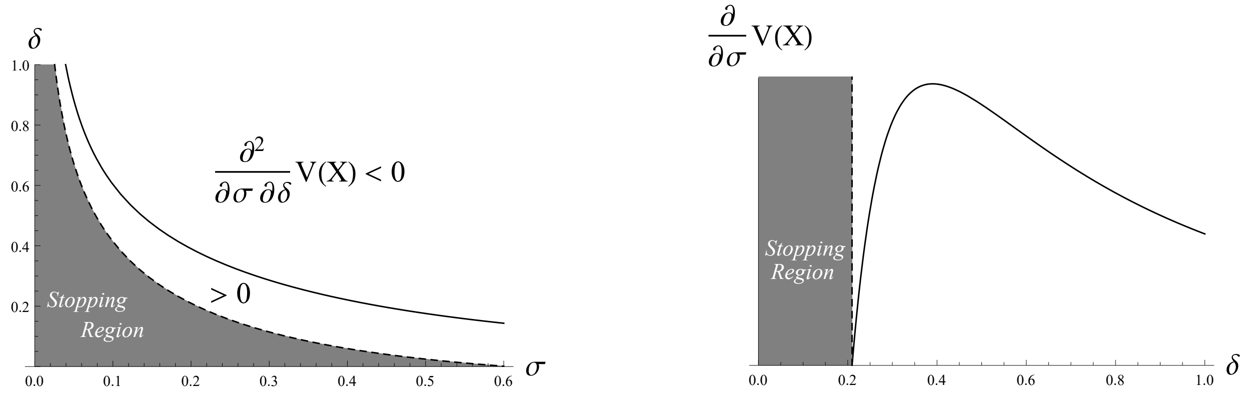
As for the option value, since X^* on the left-hand side of (15) is increasing in δ and since the right-hand side is decreasing in δ, the inequality in (15) only holds for small values of δ. This is illustrated by Figure 4b. This also implies that for sufficiently high δ condition (15) never holds. As a result, one can distinguish two regions with respect to δ and σ, as illustrated by Figure 4a. For small δ or σ it holds that $X^* < X$ so that for these parameter values X falls in the stopping region.

To interpret this condition, notice that (15) holds if X^* and X are sufficiently close. Then, irreversibility is less relevant to the firm, since in that case investment is expected to be undertaken soon. When investment is not expected to be undertaken soon, depreciation mitigates the (positive) impact of uncertainty, which is reflected by $\frac{\partial^2}{\partial\delta\partial\sigma}V(X) < 0$.

Hence, depreciation can indeed mitigate the impact of uncertainty and therefore partially lifts the irreversibility constraint, but only if depreciation is sufficiently strong.

3.4 Optimal Investment across Regions

We now bring the analyses on the stopping and continuation regions together, allowing for cases where the monopolist may switch between these regions for a change in δ, and identify the overall effect of depreciation



(a) Regions distinguishing sign of cross derivative.

(b) Example of derivative $\frac{\partial}{\partial\sigma}V(X)$ for different δ with $\sigma = 0.2$.

Figure 4: Compound effect of uncertainty and depreciation.
$\mu = 0.02,\ \rho = 0.1,\ \gamma = 0.8,\ X = 0.2,\ a = 0.6$, and $\kappa = 0.3$.

on the firm's investment behavior. In order to establish how the size of capital investment is affected by the depreciation rate for all X, let us denote the inverse of the $\hat{\delta}(X)$ in Proposition 2 by $\hat{X}(\delta)$, i.e.

$$\hat{X}(\delta) = \frac{2(\rho + \gamma\delta - \mu)^2}{\rho - \mu}\frac{\kappa}{a}\left(\frac{\gamma}{4a}\left(\frac{\rho + 2\gamma\delta - \mu}{\rho + \gamma\delta - \mu}\right)^2\right)^{\frac{1-\gamma}{\gamma}}. \tag{16}$$

Notice that $\hat{X}(\delta)$ is increasing in δ and, following Proposition 2, it divides the stopping region into a region with net positive and a region with net negative *direct* effects of an increase in the value of δ on $\bar{K}^*(X)$, i.e. for $X < \hat{X}(\delta)$ and $X \geq \hat{X}(\delta)$, respectively.

Section 3.1 established that X^* is increasing in δ, and that X^* divides the space into a region where the firm delays investment (continuation) with the *indirect* effect of an increase in the value of δ on $\bar{K}^{opt}$ dominating positively over all others (for $X < X^*$), and a region where the firm invests immediately (stopping) with the indirect effect of δ on capital stock not playing any part (for $X \geq X^*$).

In summary, we can compare both boundaries to distinguish cases where $\hat{X} < X^*$ and where $\hat{X} > X^*$, which leads to the following two cases.

Case 1:

- For $X \in (0, X^*)$ the firm delays investment until the state process hits X^* for the first time. The scale of investment $\bar{K}^{opt} = \bar{K}^*(X^*)$ is increasing in δ, while the present value of capital is decreasing in δ.
- For $X \in [X^*, \hat{X})$ the firm undertakes investment immediately and $\bar{K}^*(X)$ is decreasing in δ.
- For $X \in (\hat{X}, \infty)$ the firm undertakes investment immediately and $\bar{K}^*(X)$ is increasing in δ.

Case 2:

– For $X \in (0, X^*)$ the firm delays investment until the state process hits X^* for the first time. The scale of investment $\bar{K}^{opt} = \bar{K}^*(X^*)$ is increasing in δ, while the present value of capital is decreasing in δ;

– For $X \in [X^*, \infty)$ the firm undertakes investment immediately and $\bar{K}^*(X)$ is increasing in δ.

Since X^* is increasing in σ and since $\hat{X}(\delta)$ is not affected by σ, the condition for the stopping region to consist only of points where $\bar{K}^*(X)$ is positively affected by depreciation is that σ be sufficiently large. For lower levels on uncertainty, capital is negatively impacted by depreciation for $X \in [X^*, \hat{X}(\delta))$ and positively otherwise.

Proposition 4 *Let $\tilde{\sigma}$ be the (unique) solution to*

$$\left(\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\right)^{1-\gamma}\left(\frac{\beta(2\gamma-1)+1}{\beta-1}\right)^{\gamma} = \frac{(8)^{\gamma}}{4}. \tag{17}$$

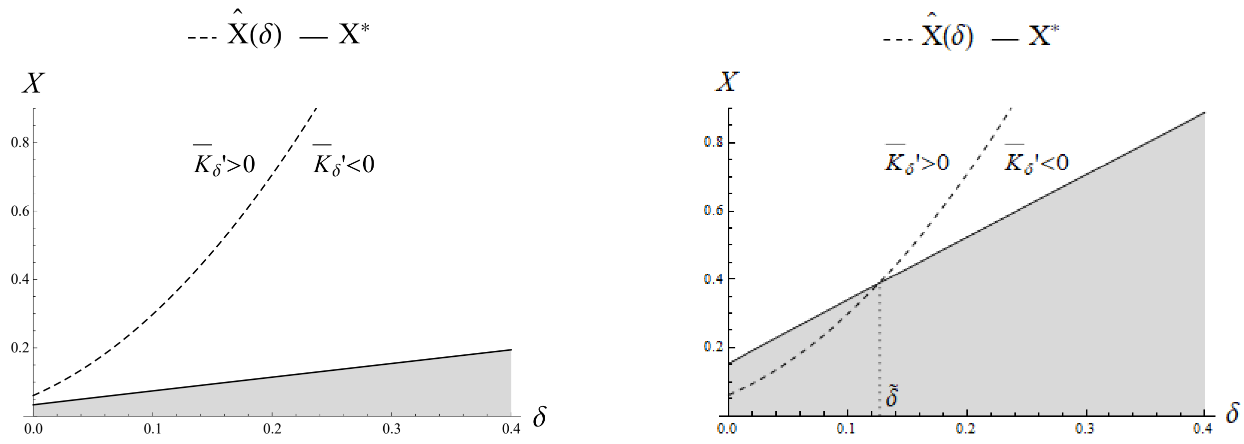
(i) If $\sigma < \tilde{\sigma}$, then Case 1 applies, for all δ.

(ii) If $\sigma \geq \tilde{\sigma}$, then Case 1 applies if and only if δ is larger than some $\tilde{\delta}$, which is the solution of

$$\frac{8^{\gamma}}{4}\left(\frac{\rho+\gamma\delta-\mu}{\rho-\mu}\right)^{\gamma}\left(\frac{\rho+2\gamma\delta-\mu}{\rho+\gamma\delta-\mu}\right)^{1-\gamma} = \left(\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\right)^{1-\gamma}\left(\frac{\beta(2\gamma-1)+1}{\beta-1}\right)^{\gamma}. \tag{18}$$

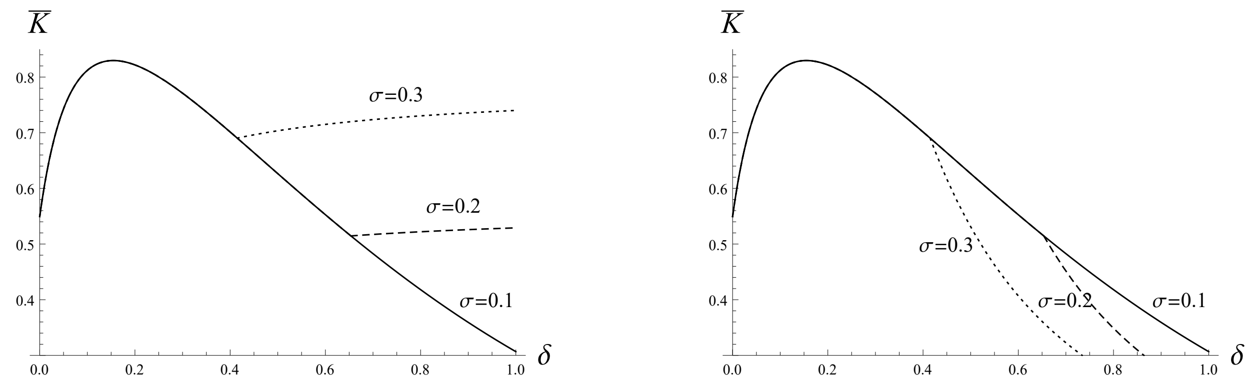
Moreover, Case 2 applies if and only if δ is smaller than some $\tilde{\delta}$, which is the solution of (18).

The proposition is illustrated by Figure 5 where $\tilde{\sigma} = 0.22$. Notice that (17) only depends on σ, μ, ρ, and γ.



(a) $\sigma = 0.1 < 0.22 = \tilde{\sigma}$. (b) $\sigma = 0.5 > 0.22 = \tilde{\sigma}$.

Figure 5: Curves corresponding to $\hat{X}$ and X^*. Solid curve separates regions where $\bar{K}^*(X)$ is increasing and decreasing in δ for $X \in \mathcal{S}$ and dashed line separates stopping region and continuation region (dotted area). $\mu = 0.02$, $\rho = 0.1$, $\gamma = 0.8$, $a = 0.6$, and $\kappa = 0.3$.



(a) Optimal level of $\bar{K}$ as a function of δ for different values of σ.

(b) Present value of capital as a function of δ for different values of σ.

Figure 6: The effect of σ on $\bar{K}$ for $\sigma = 0.1$ (solid), $\sigma = 0.2$ (dashed), and $\sigma = 0.3$ (dotted). $\mu = 0.02,\ \rho = 0.1,\ \gamma = 0.8,\ X = 0.5,\ a = 0.6$, and $\kappa = 0.3$.

The proposition shows that, for depreciation to have an unambiguously positive effect on (the level of) capital stock for all X, market uncertainty needs to be sufficiently high, which comes a result of the expansion of the continuation region.

Figure 6 illustrates the optimal capital stock for different values of δ at the moment of investment (Panel (a)) and in present terms (Panel (b)). Small values of δ correspond to $X \in \mathcal{S}$ and high values of δ correspond to $X \in \mathcal{C}$. A higher value of σ pushes the trigger up, so that $X \in \mathcal{C}$ for a larger range of δ values, as is illustrated in Panel (a). Panel (b) illustrates what was shown in Section 3.3, namely that the level of capital stock is increasing in σ and the present value of capital is decreasing in σ, for all $X \in \mathcal{C}$.

4 Robustness

In this section the robustness of our results is tested when considering two alternative versions of the main model. In Section 4.1 the main model is extended using a (s, S)-inspired type of (inventory investment) model to offer the firm the possibility to replenish its capital stock.[9] This section shows that our results do not depend, qualitatively, on the assumption that the firm has a one-off opportunity to acquire capital.

In Section 4.2 an alternative way of modeling economic depreciation is considered: capital is assumed to not depreciate at a constant geometric rate, but instead is assumed to remain fully productive whilst having a finite life-time. This section illustrates that this formulation leads to a qualitatively different investment strategy: some of the intuitive dynamics are lost when eliminating constant depreciation.

[9]This is inspired by the abundant literature on inventory problems, also known as lot sizing problems, emerged after the seminal work by Scarf (1960). This type of models can commonly be found in the Economics literature on irreversible investments with a stochastic state process, see, e.g., Federico *et al.* (2019). Federico *et al.* also provide an extensive summary of this literature stream and show optimality for problems very similar to ours.

Finally, Section 4.3 comments on alternative specifications of the inverse demand function and shows that overinvestment and underinvestment can be found for convex or concave functional forms.

4.1 Option to Replenish

It could be argued that the effect of depreciation on the scale of investment is partially driven by the assumption that the firm can only acquire capital once. Therefore, in this section, we introduce a simple extension to our model where the firm, in principle, can replenish its capital stock an infinite number of times and we will argue that, qualitatively, this assumption has no impact on what was already established for the main model.

We do not aim to fully analyze the outcomes of this extension, but rather to illustrate that both the buffer effect and the relative cost effect are still present, and that the investment behavior in the stopping region is not qualitatively different from our main model.

Consider the scenario where the monopolist, after undertaking investment, can replenish its capital stock. We assume that the firm, upon investment, will choose a $\bar{K}$ and a $\underline{K}$, with $\underline{K} < \bar{K}$, such that when the capital stock reaches the level $\underline{K}$, the firm acquires additional capital to reset its capital stock to a level equal to $\bar{K}$. Since depreciation is deterministic, this is equivalent to saying that the firm places an order every T periods, where T follows from $\underline{K} = \bar{K}e^{-\delta T}$. As a result, the period between replenishing is fixed, despite the stochastic nature of the state process. This could arise from contractual reasons as, e.g., commitments to the supplier (see, e.g., Dural-Selcuk *et al.* (2016) for an overview of the literature on (s, S)-type policies with stochastic demand).

Purchasing K units of capital stock is associated with costs $\kappa_0 K + \kappa_1$. This formulation introduces a fixed cost, which was not present in our main model. This term ensures that the optimal *replenishment time* T is strictly positive. When $T \to 0$ the model collapses into a continuous investment model, which contradicts our original assumption that this manner of investment cannot actually take place in real life.

In the stopping region, the firm then considers the following optimization problem

$$\sup_{\bar{K},T\geq 0} \mathbb{E}_X \left\{ \sum_{i=0}^{\infty} \left(\int_{t=iT}^{(i+1)T} X(t)(1-Q_i(t))Q_i(t)e^{-\rho t}\mathrm{d}t - e^{-\rho iT}\left[\kappa_0\left(\bar{K}-\bar{K}e^{-\delta T}\right)+\kappa_1\right] \right) - \kappa_0\bar{K}e^{-\delta T} \right\}, \tag{19}$$

where $Q_i(t) = \frac{a}{\gamma}(K_i(t))^\gamma$ and $K_i(t) = \bar{K}e^{-\delta(t-iT)}$ are the output and capital stock, respectively, for the $(i+1)$-th cycle, for all $i = 0, 1, 2, \ldots,$ and $iT \leq t < (i+1)T$. Equation (19) contains of three terms: two terms inside the integral and one term at the end. The first terms represents the firm's discounted instantaneous cash-inflows. The total discounted cost for each replenishment is captured by the second term for $i \geq 1$. The total cost involved with the initial investment is captured by the second term for $i = 0$, $\kappa_0(\bar{K} - \underline{K}) + \kappa_1$, plus the final term, which can also be written as $\kappa_0\underline{K}$.

The firm's optimization problem in (19) can be rewritten as

$$
\begin{aligned}
&\sup_{\bar{K},T\geq 0} \sum_{i=0}^{\infty} e^{-(\rho-\mu)iT}\left[\frac{X}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}\bar{K}^{\gamma}\left(1-e^{-(\rho+\gamma\delta-\mu)T}\right)-\frac{X}{\rho+2\gamma\delta-\mu}\left(\frac{a}{\gamma}\bar{K}^{\gamma}\right)^2\left(1-e^{-(\rho+2\gamma\delta-\mu)T}\right)\right]\\
&\qquad -\sum_{i=0}^{\infty} e^{-\rho iT}\left[\kappa_0\left(\bar{K}-\bar{K}e^{-\delta T}\right)+\kappa_1\right]-\kappa_0\bar{K}e^{-\delta T}\\
&=\sup_{\bar{K},T\geq 0}\frac{X}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}\bar{K}^{\gamma}\frac{1-e^{-(\rho+\gamma\delta-\mu)T}}{1-e^{-(\rho-\mu)T}}\left[1-\frac{a}{\gamma}\bar{K}^{\gamma}\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\frac{1-e^{-(\rho+2\gamma\delta-\mu)T}}{1-e^{-(\rho+\gamma\delta-\mu)T}}\right]\\
&\qquad -\kappa_0\bar{K}\frac{1-e^{-(\rho+\delta)T}}{1-e^{-\rho T}}+\frac{\kappa_1}{1-e^{-\rho T}}.
\end{aligned}
$$

First order conditions give the optimal capital stock $\bar{K}^*(X)$ and period $T^*(X)$.[10] The optimal capital stock is given as the solution of

$$
\begin{aligned}
&\frac{\kappa}{a}(\bar{K})^{1-\gamma}\frac{\rho+\gamma\delta-\mu}{X}\frac{1-e^{-(\rho-\mu)T^*(X)}}{1-e^{-(\rho+\gamma\delta-\mu)T^*(X)}}\frac{1-e^{-(\rho+\delta)T^*(X)}}{1-e^{-\rho T^*(X)}}\\
&\qquad=\left(1-2\frac{a}{\gamma}\bar{K}^{\gamma}\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\frac{1-e^{-(\rho+2\gamma\delta-\mu)T^*(X)}}{1-e^{-(\rho+\gamma\delta-\mu)T^*(X)}}\right).
\end{aligned}
\tag{20}
$$

Notice that, for sufficiently large T, equation (20) gives a (nearly) identical solution to $\bar{K}^*(X)$ as (12), where the buffer effect and relative cost effect can also be observed in a model where the firm can top-up its capital stock. In fact, for $\delta \to 0$, they are identical for any T.

Figure 7 illustrates the optimal strategy of the firm in the stopping region. Panel (a), equivalent to Figure 3, illustrates the optimal capital stock $\bar{K}^*(X)$. The solid curve represents the optimal capital stock as found for the main model. The dashed, dotted, and dash-dotted curves correspond to scenarios where the firm is faced with fixed replenishment cost $\kappa_1 = 0.5$, $\kappa_1 = 1$, and $\kappa_1 = 1.5$, respectively. The panel illustrates that the curves are qualitatively equivalent and that for κ_1 sufficiently large, the acquired capital stock is nearly identical to the capital stock the firm would set when it was only allowed to invest once.

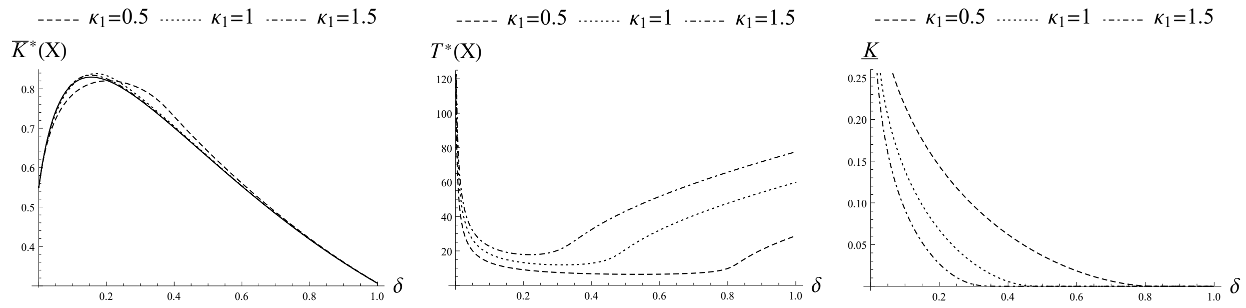
Panel (b) and Panel (c) illustrate the optimal replenishment time and the resulting replenishment trigger $\underline{K}$, respectively. For small values for δ, capital depreciates slowly so that the cycle can be long, i.e. $T^*(X)$ is large, whilst not letting capital completely depreciate before replenishment, i.e. $\underline{K}> 0$. For higher values of δ it is optimal to choose a lower $\underline{K}$.

Due to the presence of the fixed cost in this model, investment will be delayed compared to the main model. Nonetheless, we conclude that the direct effects of depreciation on the firm's capital stock, as described in the main model, are preserved when the investment strategy is modeled as a (s, S)-type of policy.

4.2 Full Depreciation in Finite Time

To emphasize the relevance of the way this paper models depreciation, in contrast to some existing work (e.g., Dixit and Pindyck (1994), Gryglewicz *et al.* (2008), Nakamura (2002)), this section briefly discusses what happens if capital retains its productive capabilities, but fully depreciates after a fixed moment in time.

[10] The Hessian confirms that this is a local maximum. Numerical analysis shows that the maximum is global.



(a) Optimal level of $\bar{K}(X)$ as a function of δ.

(b) Optimal $T(X)$ as a function of δ

(c) Optimal level of $\underline{K}$ as a function of δ

Figure 7: Optimal investment strategy for the firm with different κ_1. The solid curve represents the investment strategy for the main model (Section 2).

$\mu = 0.02,\ \rho = 0.1,\ \sigma = 0.2,\ \gamma = 0.8,\ a = 0.6$, and $\kappa_0 = 0.3$.

Denote by $\lambda > 0$ the lifetime of capital. The firm's optimization problem can then be written as

$$V(X) = \sup_{\tau \geq 0, \bar{K} > 0} \mathbb{E}_X \left\{ \int_0^\infty p(t) Q(t) e^{-\rho t} \mathrm{d}t - e^{-\rho \tau} \kappa \bar{K} \right\},$$

with

$$Q(t) = \begin{cases} 0 & \text{if } t < \tau \text{ or } t > \tau + \lambda, \\ \frac{a}{\gamma} \left(\bar{K}\right)^\gamma, & \text{if } t \in [\tau, \tau + \lambda]. \end{cases}$$

One can show that the optimal capital stock in the stopping region the solution of

$$\frac{\kappa}{a} (\bar{K})^{1-\gamma} \frac{\rho - \mu}{X} \quad = \quad \underbrace{\overbrace{\left(1 - 2\frac{a}{\gamma} \bar{K}^\gamma\right) \left(1 - e^{-\rho\lambda}\right)}^{\text{“Relative Cost Effect”}}}_{\uparrow \text{ as } \lambda\downarrow \ \Rightarrow \ \bar{K}\downarrow}. \tag{21}$$

With λ only appearing in the right-hand side of (21), stronger depreciation is only associated with a lower value of $\bar{K}$, meaning that we lose the buffer effect. A lower value for the lifespan of capital, λ, leads to a smaller period over which revenues are accumulated and, as such, investment is relatively more expensive. Figure 8 illustrates this effect.

This characterization also illustrates that the buffer effect originates from compensating for losses caused by gradual erosion in productivity. Since this type of depreciation does not impose such losses, the firm does not compensate by increasing the capital stock when depreciation is stronger.

The trigger is given by

$$X^* \quad = \quad \left(\frac{\gamma}{a} \frac{\beta(\gamma - 1) + 1}{\beta(2\gamma - 1) + 1}\right)^{\frac{1-\gamma}{\gamma}} \frac{\beta(2\gamma - 1) + 1}{\beta - 1} \frac{\kappa}{a} \frac{\rho - \mu}{1 - e^{-\rho\lambda}}, \tag{22}$$

which is decreasing in λ, i.e. investment is delayed when λ goes down.

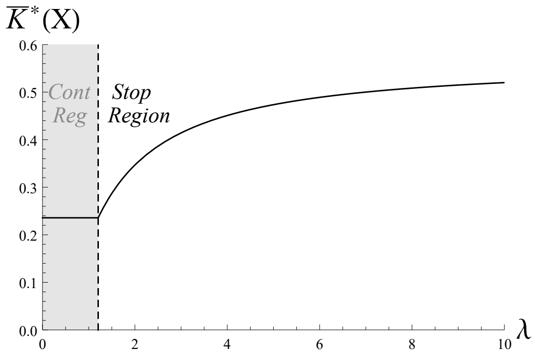


Figure 8: Optimal capital stock $\bar{K}^*(X)$ as a function of λ.

$\mu = 0.02,\ \rho = 0.1,\ \sigma = 0.2,\ \gamma = 0.8,\ X = 0.5,\ a = 0.6,$ and $\kappa = 0.3$.

It is interesting to note that, in contrast to our main model, the capital stock upon investment, when starting in the continuation region, $\bar{K}^{opt}$ is not affected by λ,

$$\bar{K}^{opt} = \left(\frac{\gamma}{a}\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\right)^{\frac{1}{\gamma}}.$$

This means that, in the absence of the buffer effect, we find that the effect of delaying on the scale of investment is matched by the relative cost effect, with the firm waiting for higher prices to exactly compensate for the loss in expected revenue.

Furthermore, for $X \in \mathcal{C}$,

$$\frac{\partial}{\partial\lambda}V(X) = V(X)\frac{\rho}{e^{\rho\lambda}-1}(\beta-1) > 0 \tag{23}$$

and

$$\frac{\partial^2}{\partial\sigma\partial\lambda}V(X) < 0 \quad \text{if and only if} \quad \mathrm{Ln}(X^*) - \mathrm{Ln}(X) < 2(\beta-1)^{-1}, \tag{24}$$

which are equivalent results to those found in Proposition (3).

4.3 Alternative Inverse Demand Specifications

Consider a slight generalization of our model by assuming the inverse demand to be

$$p(t) = x(t)(1 - Q^\eta(t)),$$

where $\eta > 0$. Notice that for $\eta < 1$ the inverse demand is convex and for $\eta > 1$ the inverse demand is concave. One can verify that for this specification the result as described by Proposition 2, i.e., the buffer effect dominates for small rates δ for sufficiently high X, still applies unless η is sufficiently close to 0. Therefore, our results are not contingent on the linearity of the inverse demand function.

One could, however, identify some functional forms that lock in the relationship of the two effects we described, such that the relative cost effect always dominates for all δ; incidentally, this drastic characterization of the functional form can be achieved by imposing constant elasticity on the demand function.

5 Conclusions

The framework introduced is admittedly simple, yet sufficiently robust to characterize the effects of eroding productivity in the optimal investment behavior of a monopolist under uncertainty.

By not assuming away economic depreciation, and without resorting to any tax implications, we reveal competing direct (buffer and relative cost) and indirect (investment threshold) effects on the optimal scale and timing of investment.

We find that more depreciation increases the investment trigger, i.e. depreciation can lead to a later exercise of the option and, upon a delayed investment, this increases the level of capital stock acquired, while decreasing the present value of capital.

Additionally, we find that, under not too strict conditions, an increase in depreciation can lead to overinvestment, when comparing to the standard zero-depreciation case, but always leads to underinvestment for sufficiently high rates.

These findings showcase how the net effect of depreciation on the size of investment cannot be simply reduced to a positive/negative offset of the choice under no-depreciation (or under a continuous investment/maintenance assumption) and that this effect is, in fact, dependent on the level of economic depreciation.

Furthermore, we find that depreciation can either have a compounding or a mitigating effect on the impact of uncertainty on the firm's option value, again depending on the level of depreciation.

Our findings illustrate that the treatment of economic depreciation is not trivial when addressing a monopolist's investment problem in a dynamic and uncertain market environment. In fact, alternative modeling choices, as illustrated in Section 4.2 and Section 4.3, may actively hide or dismiss the effects we have identified and, with that, hide or dismiss delays, over- and underinvestment that would otherwise take place.

We are able to verify that our findings are robust to set-ups with multiple sequential replenishment options and with non-linear demand specifications.

For future research, with this framework at hand, more complex cases can be investigated, such as (i) the decision to replace capital with the same or with superior productivity, (ii) the interplay of the tax benefits and the productivity losses of depreciation, and (iii) optimal investment behavior on Incumbent-Entrant games and/or other competitive setups.

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Appendix: Proofs

Proof of Proposition 1 In the stopping region, i.e. for $X \in \mathcal{S}$, we get

$$\begin{aligned}
V(X) &= \mathbb{E}_X \left\{ \int_0^\infty e^{-\rho t} \frac{a}{\gamma} (\bar{K})^\gamma e^{-\delta\gamma t} x(t) \left(1 - \frac{a}{\gamma}(\bar{K})^\gamma e^{-\delta\gamma t}\right) \mathrm{d}t - \kappa \bar{K} \right\} \\
&= \mathbb{E}_X \left\{ \int_0^\infty \frac{a}{\gamma} (\bar{K})^\gamma e^{-t(\rho+\delta\gamma)} x(t) \mathrm{d}t - \int_0^\infty \left(\frac{a}{\gamma}(\bar{K})^\gamma\right)^2 x(t) e^{-t(\rho+2\delta\gamma)} \mathrm{d}t - \kappa \bar{K} \right\} \\
&= \frac{a}{\gamma}(\bar{K})^\gamma \left(\frac{X}{\rho+\gamma\delta-\mu} - \frac{a}{\gamma}(\bar{K})^\gamma \frac{X}{\rho+2\gamma\delta-\mu} \right) - \kappa \bar{K}. \qquad (25)
\end{aligned}$$

The optimal capital stock at investment follows from the first order condition where marginal revenue (left-hand side) is equal to marginal cost (right-hand side):

$$\frac{X}{\rho+\gamma\delta-\mu} \left(1 - 2\frac{a}{\gamma}\bar{K}^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) = \frac{\kappa}{a}(\bar{K})^{1-\gamma}. \qquad (26)$$

The second order condition shows that this is indeed a maximum:

$$\frac{\partial^2}{\partial \bar{K}^2} V(X) = -\frac{X}{\rho+\gamma\delta-\mu} a(\bar{K})^{\gamma-2} \left[(1-\gamma)\left(1 - 2\frac{a}{\gamma}\bar{K}^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) + a\bar{K}^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu} \right] < 0.$$

Notice that it follows directly from (26) that (26) only gives solutions where $\bar{K} \geq 0$ so that we always have interior solutions. In addition, we find that $\frac{\partial}{\partial X}\bar{K}^*(X)$ is always positive by applying the implicit function theorem, which gives

$$\frac{\partial \bar{K}^*(X)}{\partial X} \left[\frac{\kappa}{a}(1-\gamma)(\bar{K}^*(X))^\gamma + \frac{X}{\rho+\delta\gamma-\mu} 2a \frac{\rho+\delta\gamma-\mu}{\rho+2\delta\gamma-\mu} \right] = \gamma \frac{1 - 2\frac{\rho+\delta\gamma-\mu}{\rho+2\delta\gamma-\mu}\frac{a}{\gamma}(\bar{K}^*(X))^\gamma}{\rho+\delta\gamma-\mu} > 0.$$

Following Dixit and Pindyck (1994), the value before investment (i.e. in the continuation region) is $V = \phi$ where ϕ is the solution of $\mathcal{L}\phi = \rho\phi$, where the infinitesimal generator is equal to $\mathcal{L} = \mu X \frac{\partial}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2}{\partial X^2}$. In other words,

$$\frac{1}{2}\sigma^2 X^2 \phi''(X) + \mu X \phi'(X) - \rho\phi(X) = 0.$$

They show that the unique solution to this equation is $\phi(X) = AX^\beta$ where β is the positive root of

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - \rho = 0.$$

The value of $A \in \mathbb{R}$ as well as the investment trigger X^* follow as a solution of the so called *value matching* and *smooth pasting* conditions:

$$AX^\beta = \frac{X}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}(\bar{K})^\gamma \left(1 - \frac{a}{\gamma}(\bar{K})^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) - \kappa\bar{K}, \text{ and} \qquad (27)$$

$$A\beta X^{\beta-1} = \frac{1}{\rho+\gamma\delta-\mu}\frac{a}{\gamma}(\bar{K})^\gamma \left(1 - \frac{a}{\gamma}(\bar{K})^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right), \qquad (28)$$

respectively. To find A, $\bar{K}^{opt}$, and X^*, (26), (27), and (28) are solved simultaneously,

$$\begin{aligned}
X(\beta-1)\left(1 - \frac{a}{\gamma}(\bar{K})^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) &= \beta\kappa(\rho+\delta\gamma-\mu)\frac{\gamma}{a}(\bar{K})^{1-\gamma} \\
&= X\gamma\beta\left(1 - 2\frac{a}{\gamma}(\bar{K})^\gamma \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right).
\end{aligned}$$

The first equality follows from the smooth pasting and value matching conditions and the the second equality follows from plugging in the first order condition. Rewriting leads to

$$\frac{a}{\gamma}(\bar{K})^{\gamma} = \frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\frac{\rho+2\delta\gamma-\mu}{\rho+\delta\gamma-\mu}$$

which leads to (10). The solution to (27) and (28) gives X^* and A. □

Proof of Proposition 2 First notice that, since $\bar{K} = \left(\frac{\gamma}{a}Q(0)\right)^{\frac{1}{\gamma}}$ and $\frac{\partial}{\partial\delta}\bar{K} = \frac{1}{a}\left(\frac{\gamma}{a}Q(0)\right)^{\frac{1-\gamma}{\gamma}}\frac{\partial}{\partial\delta}Q(0)$, the signs of $\frac{\partial}{\partial\delta}\bar{K}$ and $\frac{\partial}{\partial\delta}Q(0)$ are the same. Plugging $\bar{K} = \left(\frac{\gamma}{a}Q(0)\right)^{\frac{1}{\gamma}}$ into (8) gives

$$\frac{X}{\rho+\gamma\delta-\mu}\left(1-2Q(0)\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) = \frac{\kappa}{a}\left(\frac{\gamma}{a}Q(0)\right)^{\frac{1-\gamma}{\gamma}}. \tag{29}$$

Define $\xi = \frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}$. Applying the implicit function theorem to (29) leads to

$$\frac{\partial Q(0)}{\partial\delta}\left[\frac{\kappa}{a}\frac{1-\gamma}{a}\left(\frac{\gamma}{a}Q(0)\right)^{\frac{1-2\gamma}{\gamma}} + \frac{2X}{\rho+2\delta\gamma-\mu}\right] = -\gamma\frac{X}{(\rho+\delta\gamma-\mu)^2}(1-Q(0)(2\xi)^2). \tag{30}$$

Therefore $\frac{\partial Q(0)}{\partial\delta} > 0 \Leftrightarrow Q(0) > \frac{1}{4\xi^2}$. As the left-hand side of (29) is decreasing in $Q(0)$ and the right-hand side is increasing in $Q(0)$ it is sufficient to evaluate both sides at $Q(0) = \frac{1}{4\xi^2}$ to establish a condition for $\frac{\partial\bar{K}^*(X)}{\partial\delta} > 0$, i.e. plugging $Q(0) = \frac{1}{4\xi^2}$ into

$$\frac{X}{\rho+\gamma\delta-\mu}\left(1-2Q(0)\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) > \frac{\kappa}{a}\left(\frac{\gamma}{a}Q(0)\right)^{\frac{1-\gamma}{\gamma}} \tag{31}$$

gives a sufficient condition.

Next is to show that $\hat{\delta}(X)$ exists and is unique. Notice that the left-hand side of (29) is a function of δ but the right-hand side is not. Therefore, studying the left-hand side of (29) when δ changes is sufficient. As such,

$$\frac{\partial}{\partial\delta}\frac{X}{\rho+\gamma\delta-\mu}\left(1-2Q(0)\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) = \frac{X}{(\rho+\gamma\delta-\mu)^2}(4\xi^2Q(0)-1),$$

which equals 0 for $Q(0) = \frac{1}{4\xi^2}$. As $\frac{1}{\xi^2}$ is increasing in δ we have that there is a unique value of δ, $\hat{\delta}$, such that (31) holds if and only if $\delta < \hat{\delta}(X)$.

For $\delta = 0$ with $Q(0) = \frac{1}{4\xi^2}$, (31) becomes

$$\frac{X}{\rho-\mu} > \frac{2r}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}},$$

which gives a unique value of X such that $\frac{X}{\rho-\mu} = \frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$.

Finally, notice that $(\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}} > 0$, so that $(\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ is part of the state space. □

Proof of Lemma 1 Notice that $\frac{\partial}{\partial\bar{K}}V(X) = 0$ as $\bar{K}(X)$ is chosen to maximize $V(X)$. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}V(X) &= \frac{\partial}{\partial\delta}V(X) + \frac{\partial}{\partial\bar{K}}V(X)\frac{\partial\bar{K}}{\partial\delta} \\ &= -\frac{\gamma X}{(\rho+\gamma\delta-\mu)^2}\left(1-2\frac{a}{\gamma}\bar{K}^{\gamma}(X)\left(\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right)^2\right)+0 \\ &< -\frac{\gamma X}{(\rho+\gamma\delta-\mu)^2}\left(1-2\frac{a}{\gamma}\bar{K}^{\gamma}(X)\frac{\rho+\gamma\delta-\mu}{\rho+2\gamma\delta-\mu}\right) < 0.\end{aligned}$$

For the last inequality, we make use of the fact that the left-hand side of equation (8) is positive. $\square$

Proof of Lemma 2 One can show that the sign of the derivative of (14) with respect to δ is equal to the sign of

$$(\rho-\mu)(1-\beta-\beta(1-\gamma))-2\beta\gamma\delta<0,$$

which is negative since $1-\beta<0$ and all other terms are negative. $\square$

Proof of Lemma 3 Taking the derivatives gives

$$\begin{aligned}
\frac{\partial}{\partial\sigma}V(X) &= \frac{\partial}{\partial\sigma}\left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\\
&= \left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\left(\mathrm{Ln}\,\frac{X}{X^*}-\frac{\beta}{X^*}\frac{\partial X^*}{\partial\beta}+\frac{1}{\bar{K}^{opt}}\frac{\partial\bar{K}^{opt}}{\partial\beta}-\frac{1}{\beta-1}\right)\frac{\partial\beta}{\partial\sigma}\\
&= \left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\,\mathrm{Ln}\left(\frac{X}{X^*}\right)\frac{\partial\beta}{\partial\sigma}>0,\\
\frac{\partial}{\partial\delta}V(X) &= \frac{\partial}{\partial\delta}\left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\\
&= \left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\left(-\frac{\beta}{X^*}\frac{\partial X^*}{\partial\delta}+\frac{1}{\bar{K}^{opt}}\frac{\partial\bar{K}^{opt}}{\partial\delta}\right)\\
&= -\left(\frac{X}{X^*}\right)^{\beta}\frac{\kappa\bar{K}^{opt}}{\beta-1}\left(\beta-\frac{\rho-\mu}{\rho+2\gamma^2\delta-\mu}\right)\frac{\partial X^*}{\partial\delta}\frac{1}{X^*}<0,
\end{aligned}$$

so that (i) and (ii) follow directly. $\square$

Proof of Proposition 3 For (i), one can directly check that

$$\frac{\partial^2}{\partial\delta\partial\sigma}X^*=-\left(\frac{\gamma}{a}\frac{(\rho+2\gamma\delta-\mu)(1-\beta(1-\gamma))}{(\rho+\gamma\delta-\mu)(1+\beta(1-2\gamma))}\right)^{\frac{1-\gamma}{\gamma}}\frac{\kappa}{a}\frac{\rho+2\gamma^2\delta-\mu}{\rho+2\gamma\delta-\mu}\frac{\beta(1-\gamma)^2+\gamma(1-\beta(1-\gamma))}{(\beta-1)^2(1-\beta(1-\gamma))}\frac{\partial\beta}{\partial\sigma}>0.$$

For (ii), one can check the following for any function $h:\mathbb{R}^2\to\mathbb{R}$: if $h(x,y)$ can be written as $h(x,y)=f(x)g(y)\left(l(y)\right)^{k(x)}$ with functions $f,g,k,l:\mathbb{R}\to\mathbb{R}$, then

$$\frac{\partial^2}{\partial x\partial y}h(x,y)=\frac{1}{h(x,y)}\frac{\partial}{\partial y}h(x,y)\frac{\partial}{\partial x}h(x,y)+h(x,y)\frac{1}{l(y)}\frac{\partial l(y)}{\partial y}\frac{\partial k(x)}{\partial x}.$$

This can be used to obtain

$$\begin{aligned}
\frac{\partial^2}{\partial\sigma\partial\delta}\left(\frac{X}{X^*}\right)^{\beta}\frac{K^{opt}}{\beta-1} &= \underbrace{\left(\frac{X}{X^*}\right)^{-\beta}\frac{\beta-1}{K^{opt}}}_{>0}\underbrace{\frac{\partial}{\partial\delta}\left(\frac{X}{X^*}\right)^{\beta}\frac{K^{opt}}{\beta-1}}_{<0}\underbrace{\frac{\partial}{\partial\sigma}\left(\frac{X}{X^*}\right)^{\beta}\frac{K^{opt}}{\beta-1}}_{>0}\\
&\qquad -\underbrace{\left(\frac{X}{X^*}\right)^{\beta}\frac{K^{opt}}{X^*(\beta-1)}\frac{\partial X^*}{\partial\delta}\frac{\partial\beta}{\partial\sigma}}_{<0}\\
&= \underbrace{\left(\frac{X}{X^*}\right)^{\beta}\frac{\bar{K}^{opt}}{\beta-1}\frac{\partial X^*}{\partial\delta}\frac{1}{X^*}}_{>0}\underbrace{\frac{\partial\beta}{\partial\sigma}}_{<0}\left\{\left(\frac{\rho-\mu}{\rho+2\gamma^2\delta-\mu}-\beta\right)\left(\mathrm{Ln}\,\frac{X}{X^*}\right)-1\right\}.
\end{aligned}$$

Rewriting the last term gives (15). $\square$

Proof of Proposition 4 A large part of the proof follows from the main text.

Rewriting $\hat{X}(\delta) = X^*$ gives (18). Since the left-hand side of (18) is (strictly) increasing in δ and the right-hand side does not depend on δ, the intersection is unique. This can be used to show that $\frac{\partial}{\partial \delta}\hat{X} > \frac{\partial}{\partial \delta}X^*$ for all δ. Substituting the left-hand side of (18) into $\hat{X}(\delta)$ gives

$$\tilde{X} = \left(\frac{\gamma}{a}\right)^{\frac{1-\gamma}{\gamma}} \frac{\kappa}{a} \left(\frac{\beta(\gamma-1)+1}{\beta(2\gamma-1)+1}\right)^{2\frac{1-\gamma}{\gamma}} \left(\frac{\beta(2\gamma-1)+1}{\beta-1}\right)^{2} (\rho-\mu)\frac{4^{\frac{1}{\gamma}}}{8}.$$

We now need to check conditions such that $\tilde{X} > (\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ where $(\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ follows from Proposition 2. Rewriting $\tilde{X} = (\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ gives (17). Uniqueness of $\tilde{\sigma}$ follows from the fact that $\tilde{X}$ is monotone in σ and that $(\rho-\mu)\frac{2\kappa}{a}\left(\frac{\gamma}{4a}\right)^{\frac{1-\gamma}{\gamma}}$ does not depend on σ. $\square$