Optimal Capacity with a Production Cap under Uncertain Demand

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Abstract

What is the optimal capacity and investment timing when there is a ceiling (cap) on capacity due to physical or economic constraints? What is the impact of the production cap on the capacity choice? We show the optimal timing with an inverse demand function with production cap model under unbreached, and breached conditions. There are novel results, with negative sensitivity to increases in demand uncertainty, and others that are not always intuitive.

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Introduction

There are many infrastructure choices with a limited physical or economic capacity, where the timing of the infrastructure investment is optional. Bridges, pipelines, buildings, farmlands, media and sporting facilities may be constrained because of available space or technical reasons, even if potential demand is (potentially almost) unlimited. What is the optimal capacity and investment timing when there is a ceiling (cap) on capacity due to physical or economic constraints? What is the impact of the production cap on the capacity choice? Eventually unsatisfied demand may lead to congestion, crowding and constraints on economic growth within certain spaces, or local economies.

In a world of demand uncertainty, a monopoly intends to establish a facility having an optimally selected capacity to produce a product, or accommodate a demand such as tolled traffic, where the rates are priced according to a downward inverse demand function. An investment commitment is made when the market demand attains an optimal threshold. This project opportunity is subject to a single source of uncertainty due to demand volume variability. The productive capacity of the plant and the resulting investment cost incurred by installing such a capacity are positively related, so it is more expensive to develop a plant having a greater capacity, even assuming the marginal investment cost decreases with increasing capacity. Also, at the time of (instantaneous) installation, the firm selects the demand threshold aware of the bounds of productive capacity that optimises the rendered value. The model does not rule out the possibility of unmet demand arising when market demand exceeds the capacity as well as the latent cost incurred from having a cap on the output level. Besides the upfront investment cost, the firm faces a periodic cost structure in the form of both a variable and fixed cost of production. The periodic fixed cost of production depends positively on the installed capacity size, so greater capacity levels are not only more expensive to install but are also more expensive to operate. After the plant is established, there are no options to abandon and temporary suspend operations, or make any changes in capacity.

Many authors have considered the real option aspects of capacity choice. Dangl (1999) considers choosing both the timing and upper boundary for capacity with uncertain demand, with an investment cost function similar to (3). Huiman and Kort (2015) extend the Dangl approach using an inverse demand function for a duopoly. Huberts et al. (2015) summarize some capacity model developments, allowing for production suspension, and also for bounded capacity. Hagspiel et al. (2016) consider holding costs per unit of capacity, and also a linear demand structure, but apparently do not look at breaches in capacity. De Giovanni and Massabò (2018) focus on volume flexibility, but follow the Dangl approach with utilization considerations. Balter et al. (2019) extend Huisman and Kort (2015) to finite project life, with different terminations for the follower and leader, and possible deferral/accommodation for the follower. Wen et al. (2019) look at a dedicated leader, and a possibly flexible follower, who could be useful for the leader. Paxson et al. (2020) consider the effect of a price collar on optimal capacity.

Our model is for a monopoly, and so ignores some of the advances in the literature, in order to focus on upper capacity, and the possibly that it is breached by eventual demand. While we examine the effect of this possibility on optimal capacity and timing from the viewpoint of an investing firm, unsatisfied demand may lead to congestion, and other social problems, with significant public policy issues.

We offer novel solutions for optimal timing of capacity choices, considering both unbreached and breached circumstances. The next section outlines the basic economic model. Section 3 provides numerical illustrations, including a separate analysis of without-breach and with breached capacity, contrasted with more complex joint solutions for these considerations. Section 4 offers some practical examples of such infrastructure choices where these models might be usefully applied. The last section concludes with interpretation of the unique contributions and also limitations of this study, suggesting further research.

2 The Basic Model

A monopoly considers a project opportunity that is subject to a single source of uncertainty due to market demand variations which is described by the geometric Brownian motion process:

$$\mathrm{d}q = \alpha q\,\mathrm{d}t + \sigma q\,\mathrm{d}W\,, \tag{1}$$

where q denotes the periodic demand volume, α the expected drift, σ the volatility, and $\mathrm{d}W$ an increment of the standard Wiener process. Demand is fulfilled provided the installed capacity denoted by q_U is not exceeded, so the project q_O is given by $q_O = \min\{q, q_U\}$. The market price commanded by this output is obtained from the inverse demand function taking the form:

$$p = \nu\, q_O^{\chi} + \phi \tag{2}$$

where $\nu, \phi \geq 0$ and $\chi < 0$ since it is downward sloping. Clearly $p \geq \phi$ and ϕ is interpreted as the resulting price for an infinite demand. The price elasticity of demand is $p/\left(\chi\left(p-\phi\right)\right) < 0$. Because of (2), revenue can be formulated as a function of output. The unit variable cost is denoted by c and the unit contribution by $\left(p-c\right)$. The positive relationship between investment cost denoted by K and capacity q_U adopts the form, Luss (1982), Dangl (1999):

$$K = K\left(q_U\right) = a\, q_U^{\lambda}, \tag{3}$$

where $a > 0$ denotes the unit capacity multiplier and $\lambda \leq 1$ its power parameter. (3) implies that the marginal investment cost is decreasing with increasing capacity. The periodic operating cost f depends on the installed capacity $f = f\left(q_U\right)$ and is set to be proportional so $f = b\, q_U$, $b \geq 0$. The plant capacity is instrumental to determining the overall cost structure through the investment cost and the periodic operating cost. At full utilization of capacity, the active plant is expected to be viable so $\left(p-c-b\right) > 0$, suggesting possibly that $\left(\phi - c - b\right) > 0$.

Because of the capacity constraint, there are three possible distinct states. During the idle pre-investment state-0, the firm is waiting for more propitious information to emerge before committing to the investment. An investment commitment results in the post-investment active state-1, when the firm is actively producing output if there is no breach of capacity, $q \leq q_U$, but in active state-11 when the capacity is breached, $q > q_U$. This notation is adopted elsewhere. The plant value, denoted generically by $V\left(q\right)$, is denoted by $V_0\left(q\right)$, $V_1\left(q\right)$, $V_{11}\left(q\right)$ for state-0, -1, -11, respectively. Similarly, we denote the optimal market volume threshold for installing the project by $\hat{q}_{01} \leq q_U$ if the cap is not exceeded, and by $\hat{q}_{011} > q_U$ if otherwise.

In real-option models, the investment threshold for an uncertain factor is determined by maximising the net value created from transiting from the pre- to a post-investment state and setting the net created value to zero. In a similar way, for the current problem we can obtain the optimal demand threshold signalling when to make an investment commitment. However, since the productive capacity is treated as being optimally selected by management, we need a single rule for identifying the optimal capacity. There are two alternatives. Dangl (1999) develops a method for identifying the optimal productive capacity based on maximising the net-present-value (NPV), a method also adopted by Chronopoulos et al. (2017) amongst others, see also Shimko (1992). As soon as the capacity is identified by the optimal NPV method, then its value is used for obtaining the optimal investment threshold. This procedure incurs the possible shortcoming that two distinct rules are adopted to obtain the solution and the real-option approach eschews the NPV method.

The alternative proposed by Dixit (1993) and Kort et al. (2010) claims that the optimal choice decision amongst alternative investment strategies should be based on the magnitude of the investment option value. Dixit (1993) identifies the optimal capacity choice as the capacity maximising the investment option coefficient. This involves evaluating the optimal investment thresholds and the corresponding option values for a plausible set of productive capacity levels, then identifying the capacity level that maximises the investment option coefficient. This has the merit of being based on a single method, although criticized by Décamps et al. (2006) and others. Our analysis focuses on evaluating the two alternatives for obtaining the optimal threshold and the optimal capacity. It demonstrates that since the two alternatives yield identical solutions, they are equivalent whether the productive cap is without-breach when the optimal threshold is no more than the optimal capacity, or is with-breach when otherwise.

2.1 Pre-Investment Value

Following Dixit and Pindyck (1994), then from (1) the value of the project opportunity V_0 as a function of demand volume q is described by:

$$\tfrac{1}{2}\sigma^2 q^2 \frac{\partial^2 V_0}{\partial q^2} + (r-\delta) q \frac{\partial V_0}{\partial q} - rV_0 = 0 \tag{4}$$

where $r > \alpha$ denotes the risk-free rate and $\delta = r - \alpha$ the rate of return shortfall. The generic solution to (4) is:

$$V_0(q) = A_{01}\, q^{\beta_1} + A_{02}\, q^{\beta_2}\,, \tag{5}$$

where $A_{01}, A_{02} \geq 0$ are to-be-determined coefficients and β_1, β_2 are the respective positive and negative roots of the characteristic equation:

$$\tfrac{1}{2}\sigma^2\, \beta(\beta-1) + (r-\delta)\beta - r = 0\,, \tag{6}$$

with solution values:

$$\beta_1, \beta_2 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}\,, \tag{7}$$

where $\beta_1 > 1, \beta_2 < 0, \beta_1 - \beta_2 > 0.$ Since $V_0(q)$ represents the option to invest in the project, then $V_0(0) = 0$ so $A_{02} = 0$. If the capacity is not breached and the investment threshold is no more than the capacity, then that investment option is given by:

$$V_0(q) = A_{01}\, q^{\beta_1}\,. \tag{8}$$

If the capacity is breached and the investment threshold is greater than the capacity, then the investment option is given by:

$$V_0(q) = A_{011}\, q^{\beta_1}\,. \tag{9}$$

where $A_{01}, A_{011} \geq 0$ are to-be-determined coefficients.

2.2 Post-Investment Value

If the productive capacity is not breached by market demand, then similarly from (1) the value $V_1(q)$ for the active project in state-1 is described by:

$$\tfrac{1}{2}\sigma^2 q^2 \frac{\partial^2 V_1}{\partial q^2} + (r-\delta) q \frac{\partial V_1}{\partial q} + (p-c)q - f - rV_1 = 0\,. \tag{10}$$

Replacing p by (2), the generic solution for (10) is:

$$V_1(q) = \frac{\nu\, q^{\chi+1}}{\omega} + \frac{(\phi-c)q}{\delta} - \frac{f}{r} + A_{11}q^{\beta_1} + A_{12}q^{\beta_2}\,, \tag{11}$$

where A_{11}, A_{12} are to-be-determined coefficients and

$$\omega = \delta - \chi\left(r - \delta + \tfrac{1}{2}\sigma^2\right) - \tfrac{1}{2}\chi^2\sigma^2 .$$

In (11), $A_{11}q^{\beta_1}, A_{12}q^{\beta_2}$ respectively represent the American perpetuity call, Samuelson (1965), and put options, Merton (1973). Generally, options may be held or written identified by the sign of the respective coefficient. A positive coefficient indicates an option that is held and a negative coefficient an option that is written. Held options are value-creating and economically advantageous, while written options are value-destroying and economically disadvantageous. Typically, held call options represent investment (expansion) opportunities and held put options divestment (contraction) opportunities, neither of which by assumption are available herein. In contrast, written options arise when the firm's output volume attains the productive limit, either from below or from above. If the volume attains the limit from below, then the consequential loss in value is due to a breached capacity limiting productive output. The written call value $A_{11}q^{\beta_1}$ increases in magnitude as the volume approaches the cap from below. If, in contrast, the volume attains the limit from above, then the loss in value is due to market demand falling short of the cap. The written put value $A_{12}q^{\beta_2}$ increases in magnitude as the volume approaches the cap from above.

In the current formulation, $A_{11}q^{\beta_1}, A_{12}q^{\beta_2}$ represent written call and put options, respectively. Since an upper limit is present for state-1 but no lower limit exists, then $A_{11} \le 0$ and $A_{12} = 0$. From (11) the without-breach project value becomes:

$$V_1(q) = \frac{\nu\, q^{\chi+1}}{\omega} + \frac{(\phi - c)q}{\delta} - \frac{f}{r} + A_{11}q^{\beta_1} . \tag{12}$$

If the productive capacity is breached by market demand, then from (1) the value $V_{11}(q)$ for the active project in state-11 is similarly described by:

$$\tfrac{1}{2}\sigma^2 q^2 \frac{\partial^2 V_{11}}{\partial q^2} + (r - \delta) q \frac{\partial V_{11}}{\partial q} + (p - c) q_U - f - rV_{11} = 0 . \tag{13}$$

Replacing p by (2), the generic solution for (13) is:

$$V_{11}(q) = \frac{\nu\, q_U^{\chi+1}}{r} + \frac{(\phi - c) q_U}{r} - \frac{f}{r} + A_{111}q^{\beta_1} + A_{112}q^{\beta_2} . \tag{14}$$

As before, $A_{111}q^{\beta_1}$, $A_{112}q^{\beta_2}$ in (14) represent written call and put options, respectively. Since a lower limit is present for state-11 but no upper limit exists, then $A_{111}=0$ and $A_{112}\le 0$. From (14) the with-breach project value becomes:

$$V_{11}\left(q\right)=\frac{\nu\,q_U^{\chi+1}}{r}+\frac{\left(\phi-c\right)q_U}{r}-\frac{f}{r}+A_{112}q^{\beta_2}\ . \tag{15}$$

The values A_{11}, A_{112} are obtained from the value-matching and smooth-pasting conditions ruling at the juncture between state-1 and -11. The derivation, relegated to Appendix A, yields, (A4):

$$A_{11}=-\frac{q_U^{1-\beta_1}\left(\Omega_{11}+\Lambda_{11}q_U^{\chi}\right)}{r\left(\beta_1-\beta_2\right)\delta\omega},\quad A_{112}=-\frac{q_U^{1-\beta_2}\left(\Omega_{112}+\Lambda_{112}q_U^{\chi}\right)}{r\left(\beta_1-\beta_2\right)\delta\omega}, \tag{15 a,b}$$

where

$$\Omega_{11}=\left(r\left(\beta_2-1\right)-\delta\beta_2\right)\left(c-\phi\right)\omega,\quad \Lambda_{11}=\left(r\left(1-\beta_2+\chi\right)-\omega\beta_2\right)\delta\nu, \tag{15 c,d}$$

$$\Omega_{112}=\left(r\left(\beta_1-1\right)-\delta\beta_1\right)\left(c-\phi\right)\omega,\quad \Lambda_{112}=\left(r\left(1-\beta_1+\chi\right)-\omega\beta_1\right)\delta\nu. \tag{15 e,f}$$

The net-present-values for the without-breach and with-breach projects, $NPV_1=V_1\left(q\right)-K$ and $NPV_{11}=V_{11}\left(q\right)-K$, respectively, are:

$$NPV_1\left(q,q_U\right)=\frac{\nu\,q^{\chi+1}}{\omega}+\frac{\left(\phi-c\right)q}{\delta}-\frac{bq_U}{r}-\frac{q_U^{1-\beta_1}\left(\Omega_{11}+\Lambda_{11}q_U^{\chi}\right)}{r\left(\beta_1-\beta_2\right)\delta\omega}q^{\beta_1}-a\,q_U^{\lambda}, \tag{16}$$

$$NPV_{11}\left(q,q_U\right)=\frac{\nu\,q_U^{\chi+1}}{r}+\frac{\left(\phi-c\right)q_U}{r}-\frac{bq_U}{r}-\frac{q_U^{1-\beta_2}\left(\Omega_{112}+\Lambda_{112}q_U^{\chi}\right)}{r\left(\beta_1-\beta_2\right)\delta\omega}q^{\beta_2}-a\,q_U^{\lambda}. \tag{17}$$

2.3 Without-Breach Solution

Assuming that the productive capacity is not breached, the optimal capacity $\hat{q}_U$ and the optimal investment threshold $\hat{q}_{01}$ can be found jointly from maximising the net-present-value (16) and the optimality conditions identifying the threshold. From (A6), the optimal capacity defined by $\hat{q}_U=\arg\max_{q_U}\left\{NPV_1\left(q,q_U\right)\right\}$, which when evaluated at the optimal investment threshold $\hat{q}_{01}$ can be expressed as:

$$\hat{q}_{01}^{\beta_1} = \hat{q}_U^{\beta_1} \frac{(\beta_1 - \beta_2)\delta\omega\left(b + \lambda ra\,\hat{q}_U^{\lambda-1}\right)}{\Omega_{11}(\beta_1 - 1) + \Lambda_{11}\hat{q}_U^{\chi}(\beta_1 - 1 - \chi)}. \tag{18}$$

From (A10), the optimal investment threshold $\hat{q}_{01}$, which is obtained from the value-matching relationship and smooth-pasting condition, is given by:

$$0 = \frac{\beta_1 - 1}{\delta}(\phi - c)\hat{q}_{01} + \frac{\nu(\beta_1 - 1 - \chi)}{\omega}\hat{q}_{01}^{\chi+1} - \frac{\beta_1}{r}\left(b\,\hat{q}_U + ra\,\hat{q}_U^{\lambda}\right). \tag{19}$$

By using appropriate numerical methods, values for $\hat{q}_{01}$ and $\hat{q}_U$ can be determined from solving (18) and (19) simultaneously.

Alternatively, the solution for $\hat{q}_U$ is obtainable from maximising the option coefficient A_{01}. The first order condition is given by (A15):

$$0 = \left(\Omega_{11}(\beta_1 - 1)\hat{q}_{01}^{\beta_1}\hat{q}_U + \Lambda_{11}(\beta_1 - 1 - \chi)\hat{q}_{01}^{\beta_1}\hat{q}_U^{\chi+1} - \delta\omega(\beta_1 - \beta_2)\left(b\hat{q}_U + \lambda ra\, q_U^{\lambda}\right)\hat{q}_U^{\beta_1}\right). \tag{20}$$

Since (18) and (21) are equivalent, identical solutions for $\hat{q}_{01}$ and $\hat{q}_U$ are obtained whether the solution procedure applies the optimal NPV method or the optimal option method.

From (A13), any increase in the optimal capacity is associated with a corresponding increase in the investment threshold, since

$$\frac{\partial \hat{q}_{01}}{\partial \hat{q}_U} = \frac{\beta_1\delta\omega\left(b + \lambda ra\,\hat{q}_U^{\lambda-1}\right)}{r\omega(\beta_1 - 1)(\phi - c) + r\delta\nu(\chi + 1)(\beta_1 - 1 - \chi)\hat{q}_{01}^{\chi}} > 0 \tag{21}$$

assuming that $\beta_1 - 1 > \chi$ and $\chi > -1$. This demonstrates that a capacity increase, which has an accompanying increase in the investment cost and periodic fixed cost, engenders a rise in the investment threshold to compensate the consequential greater cost structure.

2.4 With-Breach Solution

Assuming that the productive capacity is breached, the optimal capacity $\hat{q}_U$ and the optimal investment threshold $\hat{q}_{011}$ can be found jointly from maximising the net-present-value (17) and the optimality conditions identifying the threshold. From (A17), the optimal capacity defined by

$\hat{q}_U = \arg\max_{q_U} \left\{ NPV_{11}\left(q, q_U\right) \right\}$, which when evaluated at the optimal investment threshold $\hat{q}_{011}$ can be expressed as:

$$\hat{q}_{011}^{\beta_2} = \frac{\left(\beta_1 - \beta_2\right)\delta\omega\left(\left(\phi - c - b\right)\hat{q}_U + \nu\left(\chi + 1\right)\hat{q}_U^{\chi+1} - \lambda ra\,\hat{q}_U^{\lambda}\right)}{\Omega_{112}\left(1 - \beta_2\right) + \Lambda_{112}\left(1 - \beta_2 + \chi\right)\hat{q}_U^{\chi}}\hat{q}_U^{\beta_2 - 1}. \tag{22}$$

From (A22), the optimal investment threshold $\hat{q}_{011}$, which is obtained from the value-matching and smooth-pasting conditions, is given by:

$$\hat{q}_{011}^{\beta_2} = \frac{\beta_1\delta\omega\left(\left(\phi - c - b\right)\hat{q}_U - ra\,\hat{q}_U^{\lambda} + \nu\,\hat{q}_U^{\chi+1}\right)}{\Omega_{112} + \Lambda_{112}\hat{q}_U^{\chi}}\hat{q}_U^{\beta_2 + 1}. \tag{23}$$

By using appropriate numerical methods, values for $\hat{q}_{011}$ and $\hat{q}_U$ can be determined from solving (22) and (23) simultaneously.

Alternatively, the solution for $\hat{q}_U$ is obtainable from maximising the option coefficient A_{011}. The first order condition is given by (A26):

$$\begin{aligned}&\left(\Omega_{112}\left(\beta_2 - 1\right)q_U + \Lambda_{112}\left(\beta_2 - 1 - \delta\right)q_U^{\chi+1}\right)q_{011}^{\beta_2}\\ &\quad = \left(\beta_1 - \beta_2\right)\delta\omega\left(q_U\left(\phi - c - b\right) + \nu\left(\chi + 1\right)q_U^{\chi+1} - \lambda ra q_U^{\lambda}\right)q_U^{\beta_2}.\end{aligned} \tag{24}$$

Since (22) and (23) are equivalent, identical solutions for $\hat{q}_{011}$ and $\hat{q}_U$ are obtained whether the solution procedure applies the optimal NPV method or the optimal option method.

From (A13), any increase in the optimal capacity is associated with a corresponding increase in the investment threshold, since

$$\begin{aligned}\frac{\partial\hat{q}_{011}}{\partial\hat{q}_U} &= \frac{\beta_1\delta\omega\hat{q}_U^{\beta_2}\hat{q}_{011}^{1-\beta_2}\left[\left(\phi - c - b\right)\hat{q}_U - \lambda ra\,\hat{q}_U^{\lambda} + \nu\left(\chi + 1\right)\hat{q}_U^{\chi+1}\right]}{\beta_2\left(\Omega_{112}\hat{q}_U + \Lambda_{112}\hat{q}_U^{\chi+1}\right)\hat{q}_U^2}\\ &\quad - \frac{\hat{q}_{011}\left[\Omega_{112}\left(1 - \beta_2\right)\hat{q}_U + \Lambda_{112}\left(1 - \beta_2 + \chi\right)\hat{q}_U^{\chi+1}\right.}{\beta_2\left(\Omega_{112}q_U + \Lambda_{112}q_U^{\chi+1}\right)q_U^2} > 0.\end{aligned} \tag{25}$$

2.5 Testing for a Breach

A breach occurs whenever the investment threshold exceeds the optimal capacity. It is feasible to test for the presence or otherwise of a breach prior to any without- and with-breached capacity evaluations by recognising from (22) that for $\hat{q}_{011} \geq \hat{q}_U$ then:

$$\begin{aligned} -(\beta_1-\beta_2)\delta\omega\left(b+\lambda ra\,\hat{q}_U^{\lambda-1}\right)-\omega(\beta_1-1)(-r+r\beta_2-\delta\beta_2)(\phi-c) \\ +\delta\nu(\beta_1-1-\chi)\left(r-r(\beta_2-\chi)+\omega\beta_2\right)\hat{q}_U^{\chi} \geq 0. \end{aligned} \tag{26}$$

2.6 Method Equivalence

In determining the optimal capacity, we now demonstrate under geometric Brownian motion that the optimal NPV method and the optimal option method are equivalent and yield identical solutions. If q and q_U denote the market demand and productive capacity, respectively, and the opportunity value of investing in a project with net-present-value $NPV(q,q_U)$ is specified by $A_0\,q^{\beta_1}$, then the value-matching relationship is given by $0 = NPV(q,q_U) - A_0\,q^{\beta_1}$ so:

$$A_0 = q^{-\beta_1} NPV(q,q_U). \tag{27}$$

The associated smooth-pasting condition is:

$$0 = \frac{\partial NPV(q,q_U)}{\partial q} - \beta_1 A_0\,q^{\beta_1-1} = \frac{\partial NPV(q,q_U)}{\partial q} - \beta_1 q^{-1} NPV(q,q_U). \tag{28}$$

From (27), the option coefficient A_0 and the option is maximised for variations in capacity when $\left.\frac{\mathrm{d}A_0}{\mathrm{d}q_U}\right|_{q_U=\hat{q}_U,\,q=\hat{q}} = 0$, where $\hat{q}_U$, $\hat{q}$ denote the optimal capacity and threshold, respectively, or for:

$$\left. -\beta_1\,q^{-\beta_1-1} NPV(q,q_U)\frac{\partial q}{\partial q_U} + q^{-\beta_1}\,\frac{\partial NPV(q,q_U)}{\partial q}\frac{\partial q}{\partial q_U} + q^{-\beta_1}\,\frac{\partial NPV(q,q_U)}{\partial q_U}\right|_{q_U=\hat{q}_U,\,q=\hat{q}} = 0. \tag{29}$$

Combining (28) and (29) yields:

$$\left. q^{-\beta_1}\,\frac{\partial NPV(q,q_U)}{\partial q_U}\right|_{q_U=\hat{q}_U,\,q=\hat{q}} = 0. \tag{30}$$

Identifying the optimal capacity from the total differentiation of the investment option coefficient yields the identical solution as the partial differentiation of the net-present-value. The optimal NPV and the optimal option methods for determining the optimal capacity are equivalent.

3 Numerical Illustration

We obtain further insights on model behaviour through several numerical illustrations. Initially, we separate the without-breach capacity solution from the with-breach capacity solution to investigate the extent of their differences in behaviour. Then we proceed to examine the without- and with-breach capacity solutions jointly particularly with reference to volatility variations. While the net-present values are reported, all the evaluations reported below are performed from using the optimal option solution method, since the two solution methods are shown to be equivalent.

3.1 Parameter Values

The assumed parameters values are given in Table 1, along with the formulae defined price, investment and fixed operating costs, and power parameters.

Table 1 Base Case Parameter Values

	A	B	C	D
1		**Capacity Size 15 Jan 2020**		
2	INPUT		Eqs	
3	r	**0.04**		
4	δ	**0.015**		
5	σ	**0.15**		
6	q	**0.1**		
7	c	**5**		
8	b	**2**		
9	λ	**0.7**		
10	a	**50**		
11	υ	**2**		
12	χ	**-0.7**		
13	φ	**7**		
14	p	17.0237	2	B11*(B15^B12)+B13
15	q0	0.1000		MIN(B6,B26)
16				
17	ω	0.0349		B4-B12*(B3-B4+0.5*(B5^2))-0.5*(B12^2)*(B5^2)
18	f	0.3884		B8*B26
19	K	15.8750	3	B10*(B26^B9)
20	t θ1	0.01375		B3-B4-0.5*(B5^2)
21	t θ2	0.01375		B3-B4-0.5*(B5^2)
22	OUTPUT	unbreached		
23	β1	1.3711	7	(-B20+SQRT(B20^2+2*B3*(B5^2)))/(B5^2)
24	β2	-2.5933	7	(-B20-SQRT(B20^2+2*B3*(B5^2)))/(B5^2)

Table 2 Derived Optimal Thresholds for the Unbreached Case

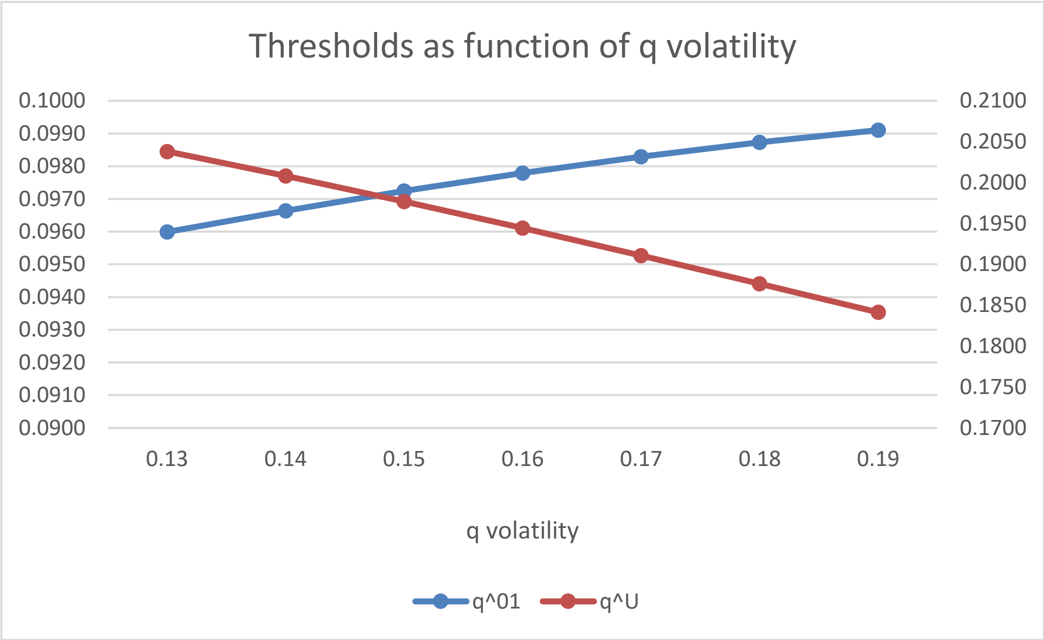
	A	B	C	D
25	q^$_{01}$	**0.0954**		Solution to B36
26	q^U	**0.1942**		Solution to B36
27	A01	1398.3503		B37*B26^(1-B23)+((B14-B7)/(B23*B4))*B25^(1-B23)
28	A11	-593.4242	15a	(-(B26^(1-B23))*(B39+B37*(B26^B12)))/(B3*(B23-B24)*B4*B17)
29	A112	0.2727	15b	(-(B26^(1-B24))*(B40+B38*(B26^B12)))/(B3*(B23-B24)*B4*B17)
30				
31				
32				
33	q^	0.0236	18	B39*(B23-1)+B37*(B26^B12)*(B23-1-B12)
34	q^	0.0000	18	(B26^B23)*(B23-B24)*B4*B17*(B8+B9*B3*B10*(B26^(B9-1)))/B33-B25^B23
35	q^	0.0000	19	((B23-1)/B4)*(B13-B7)*B25+((B11*(B23-1-B12)/B17))*(B25^(B12+1))-(B23/B3)*(B8*B26+B3*B10*(B26^B9))
36	Solver	0.0000		Solver:B36=0, changing B25:B26.
37	Δ11	0.0062	15d	(B3*(1-B24+B12)-B17*B24)*B4*B11
38	Δ112	-0.0027	15f	(B3*(1-B23+B12)-B17*B23)*B4*B11
39	Ω11	0.0073	15c	(B3*(B24-1)-B4*B24)*(B7-B13)*B17
40	Ω112	0.0004	15e	(B3*(B23-1)-B4*B23)*(B7-B13)*B17
41	V0(q)	59.5050	8	B27*(B15^B23)
42	V'(q)	815.8507		B27*B23*(B15^(B23-1))
43	V''(q)	3027.3164		B27*B23*(B23-1)*(B15^(B23-2))
44	ODE	**0.0000**	4	0.5*(B5^2)*(B15^2)*B43+(B3-B4)*B15*B42-B3*B41
45	V(q) 1	7.1241	12	(B11*(B15^(B12+1)))/B17+(B13-B7)*B15/B4-B18/B3+B28*(B15^B23)
46	V'(q) 1	-126.6363		((B11*(B12+1)*(B15^(B12)))/B17)+(B13-B7)/B4+B28*B23*(B15^(B23-1))
47	V''(q) 1	-1888.5127		((B11*(B12+1)*(B12)*(B15^(B12-1)))/B17)+B28*B23*(B23-1)*(B15^(B23-2))
48	ODE V1	**0.0000**	10	0.5*(B5^2)*(B15^2)*B47+(B3-B4)*B15*B46+(B14-B7)*B15-B18-B3*B45
49	V(q) 11	155.5426	15a	B14*B26/B3-B7*B26/B3-B18/B3+B29*(B15^B24)
50	V'(q) 11	-2771.7589		B29*B24*(B15^(B24-1))
51	V''(q) 11	99597.1916		B29*B24*(B24-1)*(B15^(B24-2))
52	ODE V11	**0.0000**	13	0.5*(B5^2)*(B15^2)*B51+(B3-B4)*B15*B50+(B14-B7)*B26-B18-B3*B49
53	ROV	7.1241		IF(B15<B25,B41,IF(AND(B15>B25,B15<B26),B45,B49))

Given the Table 1 base case parameter values, the optimal capacity is .1942, and the threshold that justifies immediate investment in that capacity is .0954, B25:B26, obtained by solving simultaneously (18) and (19). The pre-investment value is from (8), which satisfies the ODE (4) with the calculated delta and gamma B42:B43. The post-investment value is from (12), which satisfies the ODE (10). If the productive capacity is breached by market demand, the breached value is from (15a), which satisfies the ODE (13).

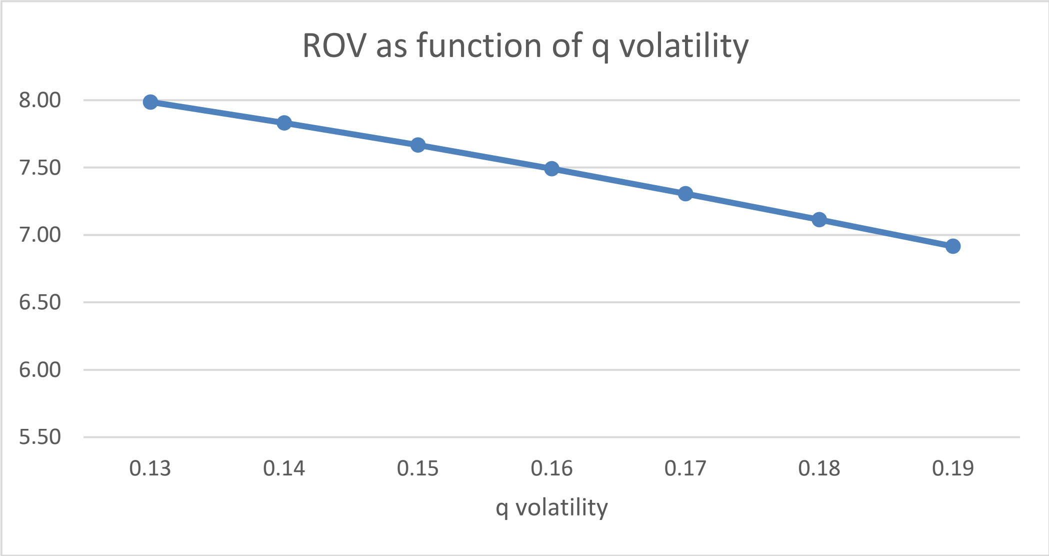
The optimal capacity and threshold are highly sensitive to changes in some of the parameter values, three (quantity volatility, operating cost, investment cost) which are illustrated below. Figure 1 shows the thresholds and ROV=$V_1(q)$, with $\hat{q}_{01} < q < \hat{q}_U$, so that this is the post-investment value, unbreached capacity. Note that $\hat{q}_{01}$ increases with q volatility, while $\hat{q}_U$ decreases, as does the ROV.

Figure 1 Base Case Parameter Values

Sensitivity of Optimal Capacity, Thresholds and ROV to Changes in Demand Volatility



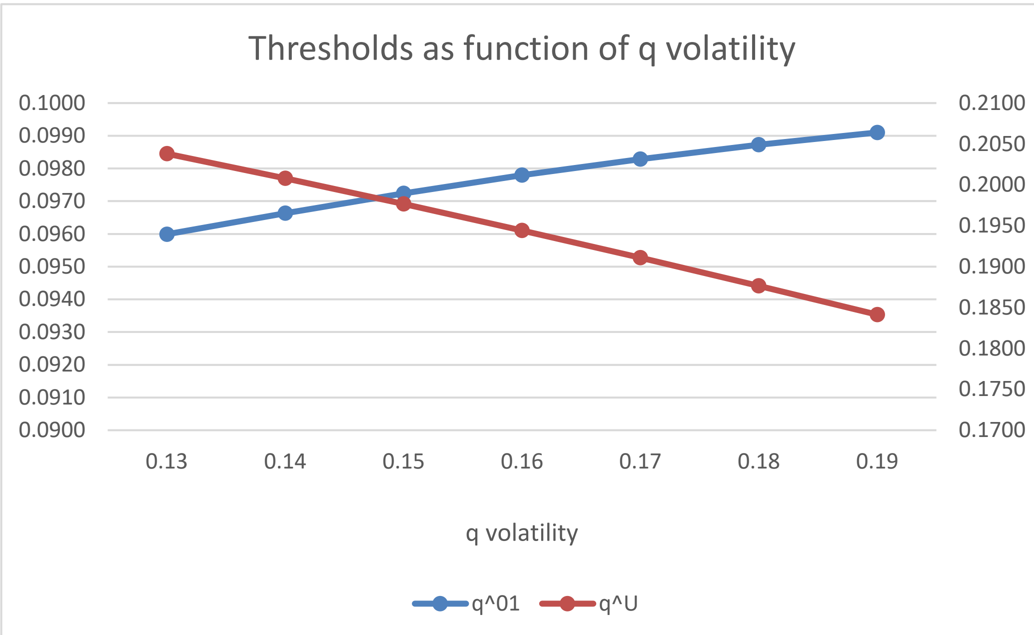
C	5
A	50



q^01	0.0940	0.0947	0.0954	0.0960	0.0965	0.0970	0.0974
q^U	0.2000	0.1972	0.1942	0.1911	0.1879	0.1846	0.1812
ROV	7.4033	7.2695	7.1241	6.9683	6.8031	6.6296	6.4487

In Figure 2, with 2% lower operating costs, the ROV naturally increase at all levels of quantity volatility, as do the thresholds. The opposite happens for 2% higher operating costs as shown in Figure 3. The spread $\hat{q}_U - \hat{q}_{01}$ increases with higher variable costs.

Figure 2 Thresholds and ROV at Lower Operating Costs



c 4.9

a 50

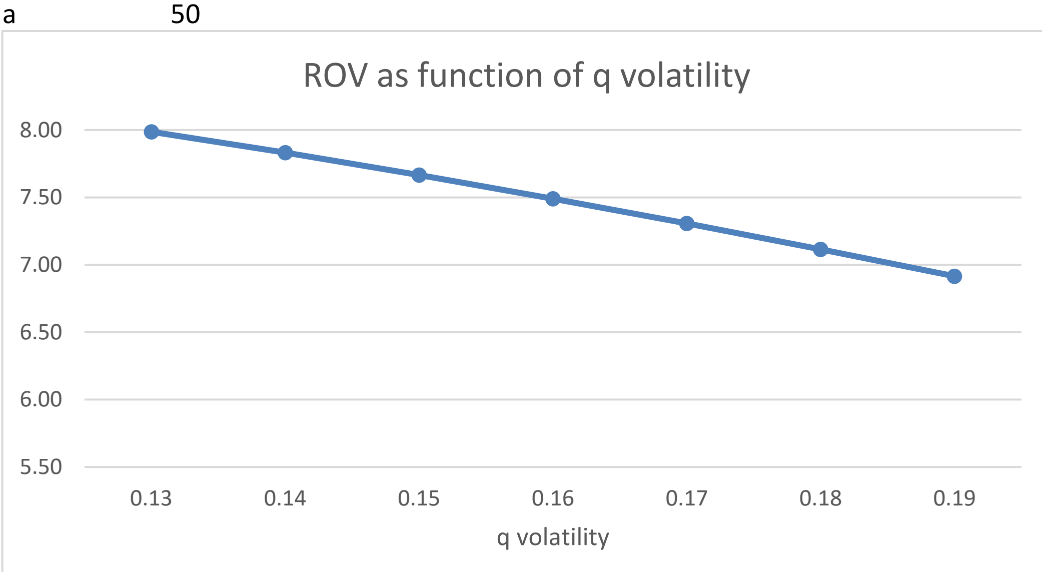
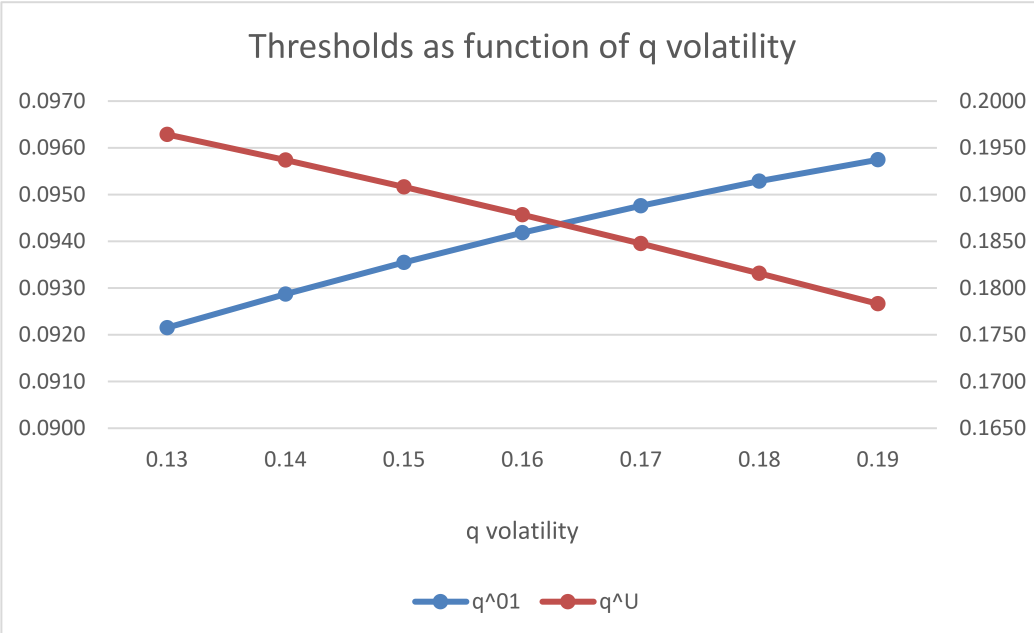
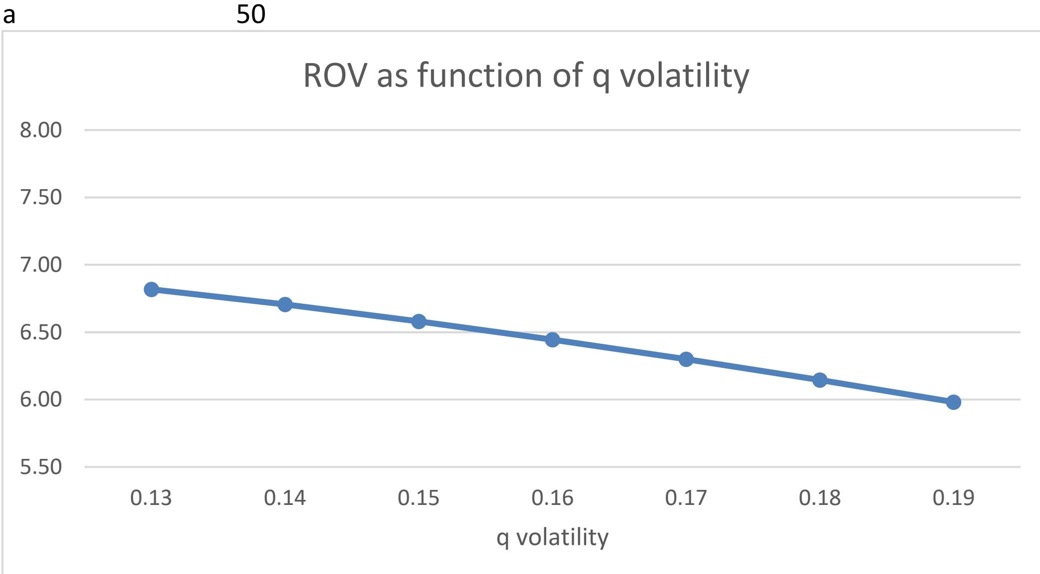


Figure 3 Higher Variable Operating Costs

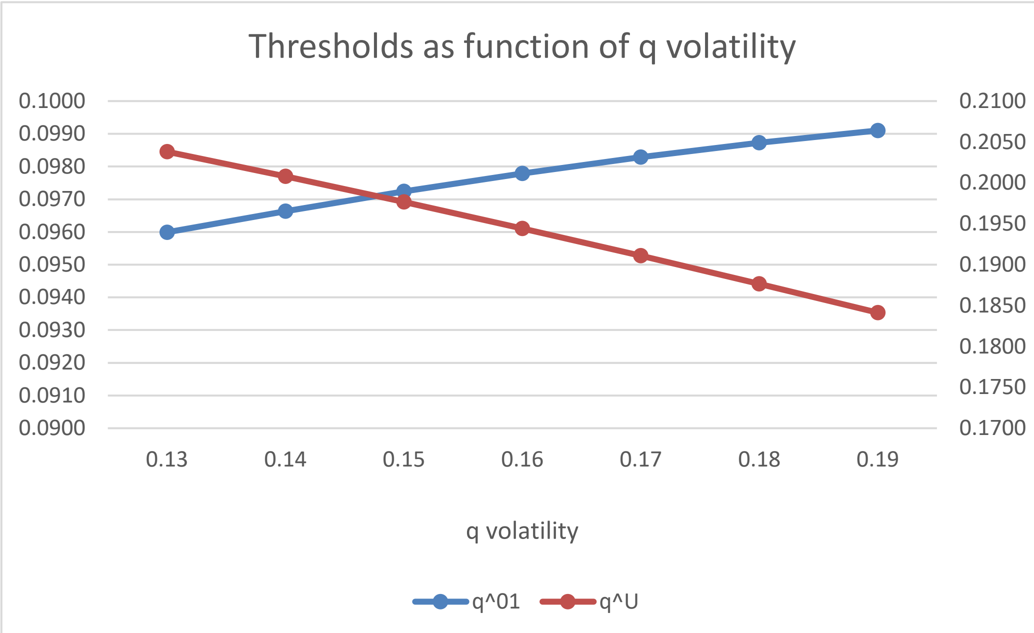


c	5.1
a	50

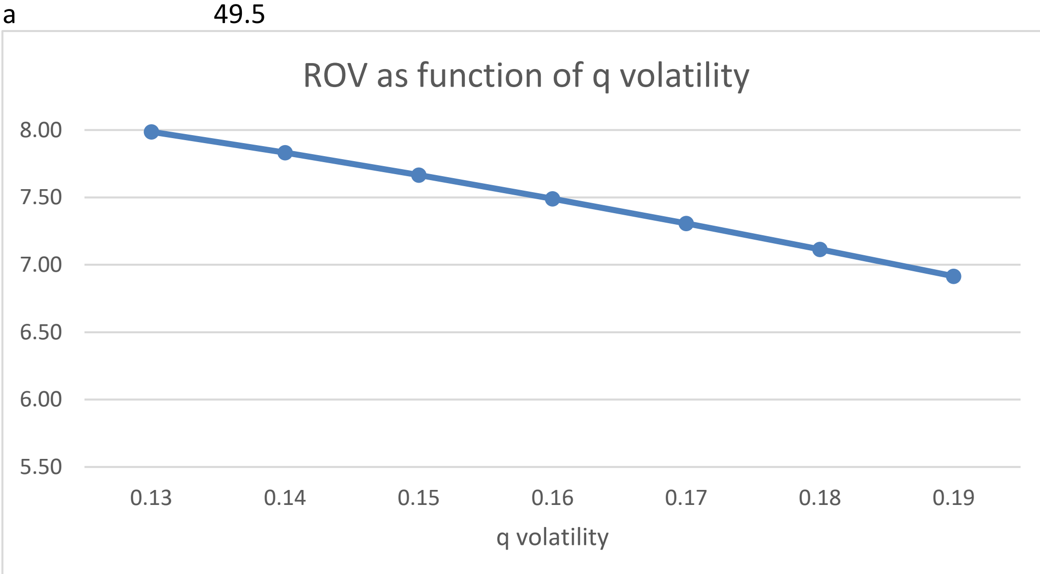


q^01	0.0922	0.0929	0.0935	0.0942	0.0948	0.0953	0.0957
q^U	0.1964	0.1937	0.1908	0.1878	0.1848	0.1816	0.1783
ROV	6.8183	6.7056	6.5808	6.4450	6.2992	6.1445	5.9816

Figure 4 Lower Unit Capacity Multiplier (K)

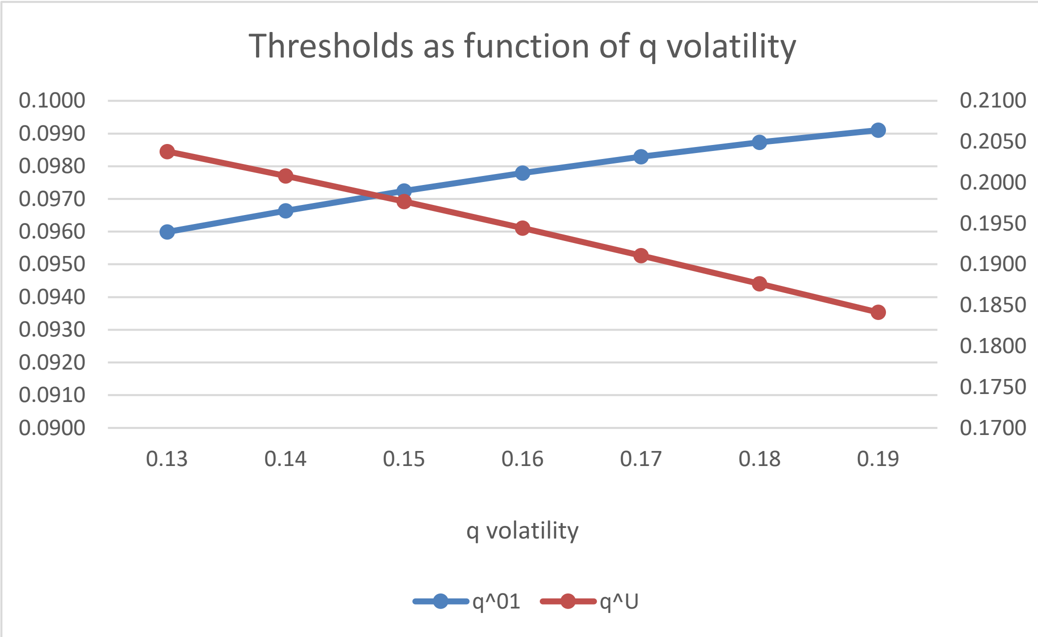


c	5
a	49.5

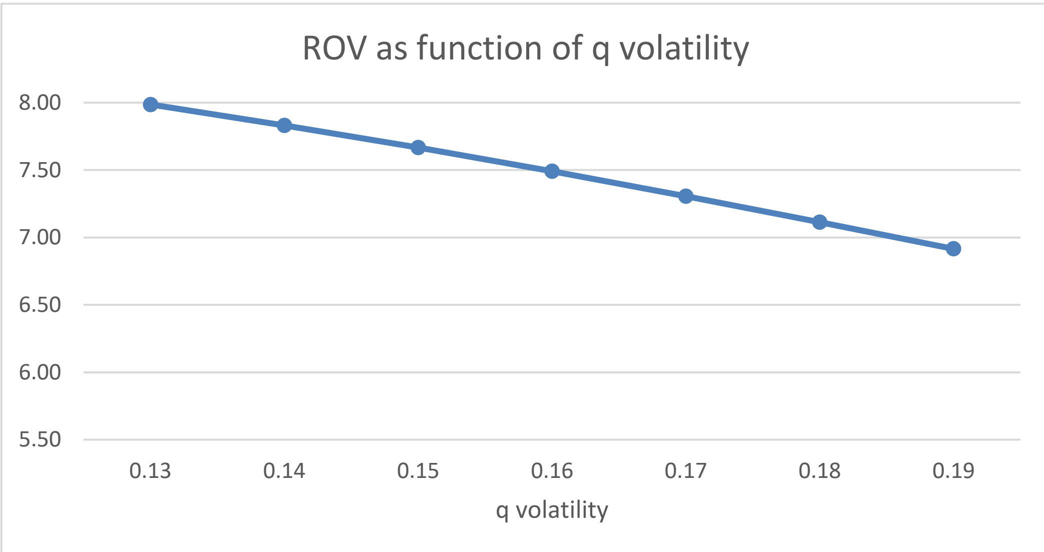


q^01	0.0957	0.0964	0.0971	0.0977	0.0982	0.0987	0.0991
q^U	0.2033	0.2004	0.1974	0.1942	0.1909	0.1876	0.1841
ROV	7.6155	7.4752	7.3237	7.1621	6.9917	6.8133	6.6278

Figure 5 Higher Unit Capacity Multiplier (K)



c	5
a	50.5



q^01	0.0924	0.0930	0.0937	0.0943	0.0948	0.0953	0.0957
q^U	0.1968	0.1940	0.1911	0.1880	0.1848	0.1816	0.1783
ROV	7.1833	7.0563	6.9173	6.7674	6.6078	6.4395	6.2633

Figure 4 shows that the thresholds and ROV increase as the K investment multiplier decreases 1%, thus increasing K per unit capacity, while the thresholds and ROV increase, which is not intuitive. Figure 5 shows the opposite happens when the K multiplier increases 1%.

Suppose a government seeks to motivate early investment in an infrastructure project undertaken by such a monopoly through lowering the threshold $\hat{q}_{01}$ but at the same time aims for high capacity $\hat{q}_U$ to avoid future congestion. First of all, from Figures 1-5 it is apparent that the sensitivity is not constant as volatility increases. But at the base case volatility, the threshold decreases with an operating cost c increase and an investment multiplier a increase, so this unusual government could motivate early investment by taxing both types of costs, surprisingly. The monopoly hoping for a high ROV, while waiting to invest, would be disappointed by such actions. Reducing demand volatility by quantity guarantees (or minimum revenue subsidies) could achieve all three objectives, reducing the investment threshold, increasing the optimal capacity, and the ROV, perhaps a "goldilocks" effect.

Table 3

Summary Sensitivity of Base Case to Parameter Value Changes				
	c decrease	c increase	a decrease	a increase
q^01	1.89%	-1.99%	1.78%	-1.78%
q^U	1.80%	-1.75%	1.65%	-1.60%
ROV	7.61%	-7.63%	2.80%	-2.90%

Breached Case illustration incomplete

4 Conclusion

The optimal capacity and investment timing when there is a ceiling (cap) on capacity due to physical or economic constraints is such in the base case that in the unbreached situation a monopoly would invest when the quantity is around half of the optimal capacity, creating lots of spare capacity initially. We show the optimal timing with an inverse demand function with production cap model under unbreached, and breached conditions. There are novel results, with negative sensitivity to increases in demand uncertainty, and others that are not always intuitive.

5 Appendix A

Derivation of A_{11}, A_{112}

The values of the coefficients A_{11}, A_{112} are obtained from the value-matching relationship and smooth-pasting condition ruling at the boundary between state-1 and -11:

$$V_1(q)\Big|_{q=q_U} = V_{11}(q)\Big|_{q=q_U}, \frac{\partial V_1(q)}{\partial q}\Bigg|_{q=q_U} = \frac{\partial V_{11}(q)}{\partial q}\Bigg|_{q=q_U} \tag{A1}$$

or respectively as:

$$\frac{(\phi-c)q_U}{\delta} + \frac{\nu q_U^{\chi+1}}{\omega} + A_{11}q_U^{\beta_1} = \frac{(\phi-c)q_U}{r} + \frac{\nu q_U^{\chi+1}}{r} + A_{112}q_U^{\beta_2}, \tag{A2}$$

$$\frac{(\phi-c)}{\delta} + \frac{\nu q_U^{\chi}(\chi+1)}{\omega} + \beta_1 A_{11}q_U^{\beta_1-1} = \beta_2 A_{112}q_U^{\beta_2-1}. \tag{A3}$$

Substituting $A_{112} = q_U^{-\beta_2}\left(\delta\nu q_U^{\chi+1}(\chi+1) + \omega q_U(\phi-c) + \beta_1\delta\omega A_{11}q_U^{\beta_1}\right)\Big/(\beta_2\delta\omega)$ from (A3) into (A2), and simplifying yields:

$$A_{11} = -\frac{q_U^{1-\beta_1}\left(\Omega_{11} + \Lambda_{11}q_U^{\chi}\right)}{r(\beta_1-\beta_2)\delta\omega}, \quad A_{112} = -\frac{q_U^{1-\beta_2}\left(\Omega_{112} + \Lambda_{112}q_U^{\chi}\right)}{r(\beta_1-\beta_2)\delta\omega}, \tag{A4}$$

where:

$$\Omega_{11} = \left(r(\beta_2-1) - \delta\beta_2\right)(c-\phi)\omega, \quad \Lambda_{11} = \left(r(1-\beta_2+\chi) - \omega\beta_2\right)\delta\nu,$$

$$\Omega_{112} = \left(r(\beta_1-1) - \delta\beta_1\right)(c-\phi)\omega, \quad \Lambda_{112} = \left(r(1-\beta_1+\chi) - \omega\beta_1\right)\delta\nu.$$

Without-Breach Optimal NPV Method

The optimal capacity is found from maximising the net-present-value. Differentiating (15) with respect to the capacity yields:

$$\frac{\partial NPV_1}{\partial q_U} = -\frac{b}{r} + \frac{(\beta_1-1)\Omega_{11}q_U^{-\beta_1} + (\beta_1-1-\chi)\Lambda_{11}q_U^{\chi-\beta_1}}{r(\beta_1-\beta_2)\delta\omega} q^{\beta_1} - \lambda a\, q_U^{\lambda-1}. \tag{A5}$$

From (A5), the first order condition evaluated at the optimal capacity $\hat{q}_U$ and investment threshold $\hat{q}_{01}$ can be expressed as:

$$\hat{q}_{01}^{\beta_1} = \hat{q}_U^{\beta_1} \frac{(\beta_1-\beta_2)\delta\omega\left(b + \lambda ra\,\hat{q}_U^{\lambda-1}\right)}{\Omega_{11}(\beta_1-1) + \Lambda_{11}\hat{q}_U^{\chi}(\beta_1-1-\chi)}. \tag{A6}$$

Without-Breach Optimal Threshold

For the optimal capacity level $\hat{q}_U$, the optimal threshold $\hat{q}_{01}$ is obtained from the value-matching relationship and smooth-pasting condition assuming the capacity is not breached:

$$0 = V_1(q) - V_0(q) - K\Big|_{q=\hat{q}_{01}, q_U=\hat{q}_U}, \quad 0 = \frac{\partial V_1(q)}{\partial q} - \frac{\partial V_0(q)}{\partial q}\Bigg|_{q=\hat{q}_{01}, q_U=\hat{q}_U}, \tag{A7}$$

or respectively as:

$$0 = \frac{\nu\,\hat{q}_{01}^{\chi+1}}{\omega} + \frac{(\phi - c)\,\hat{q}_{01}}{\delta} - \frac{f}{r} + A_{11}\hat{q}_{01}^{\beta_1} - A_{01}\hat{q}_{01}^{\beta_1} - K, \tag{A8}$$

$$0 = \frac{(\chi+1)\nu\,\hat{q}_{01}^{\chi}}{\omega} + \frac{(\phi - c)}{\delta} + \beta_1 A_{11}\hat{q}_{01}^{\beta_1 - 1} - \beta_1 A_{01}\hat{q}_{01}^{\beta_1 - 1}. \tag{A9}$$

Substituting $A_{01} = \hat{q}_{01}^{-\beta_1}\left(\nu\delta(\chi+1)\hat{q}_{01}^{\chi+1} + \omega(\phi - c)\hat{q}_{01}\right)\big/\beta_1\delta\omega + A_{11}$ from (A9) into (A8) and simplifying yields:

$$0 = \frac{\beta_1 - 1}{\delta}(\phi - c)\hat{q}_{01} + \frac{\nu(\beta_1 - 1 - \chi)}{\omega}\hat{q}_{01}^{\chi+1} - \frac{\beta_1}{r}\left(b\,\hat{q}_U + ra\,\hat{q}_U^{\lambda}\right), \tag{A10}$$

$$A_{01} = \frac{\left(\nu\delta(\chi+1)\hat{q}_{01}^{\chi+1} + \omega(\phi - c)\hat{q}_{01}\right)}{\beta_1\delta\omega}\hat{q}_{01}^{-\beta_1} - \frac{\Omega_{11} + \Lambda_{11}\hat{q}_U^{\chi}}{r(\beta_1 - \beta_2)\delta\omega}\hat{q}_U^{1-\beta_1}. \tag{A11}$$

The solutions for $\hat{q}_{01}$ and $\hat{q}_U$ are obtained simultaneously from (A6) and (A10) using an appropriate numerical method.

Without-Breach Optimal Option Method

The optimal capacity is found from maximising the option coefficient A_{01}. Differentiating (A11) with respect to the capacity has to attend to the relationship between $\hat{q}_{01}$ and $\hat{q}_U$ as specified by (A10). We define:

$$H_1(q_{01}, q_U) = \frac{\beta_1 - 1}{\delta}(\phi - c)q_{01} + \frac{\nu(\beta_1 - 1 - \chi)}{\omega}q_{01}^{\chi+1} - \frac{\beta_1}{r}\left(b\,q_U + ra\,q_U^{\lambda}\right). \tag{A12}$$

From (A12):

$$\frac{\partial H_1}{\partial q_{01}} = \frac{\beta_1 - 1}{\delta}(\phi - c) + \frac{\nu(\chi+1)(\beta_1 - 1 - \chi)}{\omega}q_{01}^{\chi},$$

$$\frac{\partial H_1}{\partial q_U} = -\frac{\beta_1}{r}\left(b + \lambda ra\, q_U^{\lambda-1}\right),$$

and since $\frac{\partial q_{01}}{\partial q_U} = -\frac{\partial H_1}{\partial q_U} \bigg/ \frac{\partial H_1}{\partial q_{01}}$, then

$$\frac{\partial q_{01}}{\partial q_U} = \frac{\beta_1 \delta\omega\left(b + \lambda ra\, q_U^{\lambda-1}\right)}{r\omega\left(\beta_1 - 1\right)\left(\phi - c\right) + r\delta\nu\left(\chi + 1\right)\left(\beta_1 - 1 - \chi\right) q_{01}^{\chi}}. \qquad \text{(A13)}$$

Also, we define from (A11):

$$A_{01}\left(q_{01}, q_U\right) = \frac{\left(\nu\delta\left(\chi + 1\right) q_{01}^{\chi+1} + \omega\left(\phi - c\right) q_{01}\right)}{\beta_1 \delta\omega} q_{01}^{-\beta_1} - \frac{\Omega_{11} + \Lambda_{11} q_U^{\chi}}{r\left(\beta_1 - \beta_2\right)\delta\omega} q_U^{1-\beta_1}. \qquad \text{(A14)}$$

Then:

$$\frac{\partial A_{01}}{\partial q_{01}} = -\frac{q_{01}^{-\beta_1}}{\beta_1 \delta\omega}\left(\nu\delta\left(\chi + 1\right)\left(\beta_1 - 1 - \chi\right) q_{01}^{\chi+1} + \omega\left(\beta_1 - 1\right)\left(\phi - c\right)\right),$$

$$\frac{\partial A_{01}}{\partial q_U} = \frac{q_U^{-\beta_1}}{r\left(\beta_1 - \beta_2\right)\delta\omega}\left(\Omega_{11}\left(\beta_1 - 1\right) + \Lambda_{11}\left(\beta_1 - 1 - \chi\right) q_U^{\chi}\right).$$

Differentiating $A_{01}\left(q_{01}, q_U\right)$ with respect to q_U yields $\frac{\mathrm{d}A_{01}}{\mathrm{d}q_U} = \frac{\partial A_{01}}{\partial q_{01}}\frac{\partial q_{01}}{\partial q_U} + \frac{\partial A_{01}}{\partial q_U}$, The first order condition, $0 = \left.\frac{\mathrm{d}A_{01}}{\mathrm{d}q_U}\right|_{q_U = \hat{q}_U, q_{01} = \hat{q}_{01}}$, can be expressed as $0 = \left. -\frac{\partial A_{01}}{\partial q_{01}}\frac{\partial H_1}{\partial q_U} + \frac{\partial A_{01}}{\partial q_U}\frac{\partial H_1}{\partial q_{01}}\right|_{q_U = \hat{q}_U, q_{01} = \hat{q}_{01}}$ to yield after substituting from above and simplifying:

$$\left.\begin{aligned}
0 &= \frac{\hat{q}_{01}^{-\beta_1}\hat{q}_U^{-1-\beta_1}}{r\beta_1\left(\beta_1 - \beta_2\right)\delta^2\omega^2} \\
&\quad \times\left(\Omega_{11}\left(\beta_1 - 1\right)\hat{q}_{01}^{\beta_1}\hat{q}_U + \Lambda_{11}\left(\beta_1 - 1 - \chi\right)\hat{q}_{01}^{\beta_1}\hat{q}_U^{\chi+1} - \delta\omega\left(\beta_1 - \beta_2\right)\left(b\hat{q}_U + \lambda ra\, q_U^{\lambda}\right)\hat{q}_U^{\beta_1}\right) \\
&\quad \times\left(\delta\nu\left(1 + \chi\right)\left(\beta_1 - 1 - \chi\right)\hat{q}_{01}^{\chi} + \omega\left(\beta_1 - 1\right)\left(\phi - c\right)\right) \\
&= \left(\Omega_{11}\left(\beta_1 - 1\right)\hat{q}_{01}^{\beta_1}\hat{q}_U + \Lambda_{11}\left(\beta_1 - 1 - \chi\right)\hat{q}_{01}^{\beta_1}\hat{q}_U^{\chi+1} - \delta\omega\left(\beta_1 - \beta_2\right)\left(b\hat{q}_U + \lambda ra\, q_U^{\lambda}\right)\hat{q}_U^{\beta_1}\right) \\
&= \left.\frac{\partial NPV_1}{\partial q_U}\right|_{q_U = \hat{q}_U, q_{01} = \hat{q}_{01}}.
\end{aligned}\right\} \qquad \text{(A15)}$$

The solutions for $\hat{q}_{01}$ and $\hat{q}_U$ are obtained simultaneously from (A10) and (A15) using an appropriate numerical method. Also, (A15) establishes that identical optimal capacity solutions are obtained whether the optimal NPV or optimal option method is applied.

With-Breach Optimal NPV Method

The optimal capacity is found from maximising the net-present-value. Differentiating (16) with respect to the capacity yields:

$$\frac{\partial NPV_{11}}{\partial q_U} = -\lambda a\, q_U^{\lambda-1} + \frac{(\phi - c - b)}{r} + \frac{\nu(\chi+1)q_U^{\chi}}{r} - \frac{\Omega_{112}(1-\beta_2) + \Lambda_{112}(1-\beta_2+\chi)q_U^{\chi}}{r(\beta_1-\beta_2)\delta\omega} q^{\beta_2} q_U^{-\beta_2}. \tag{A16}$$

From (A16), the first order condition evaluated at the optimal capacity $\hat{q}_U$ and investment threshold $\hat{q}_{01}$ can be expressed as:

$$\hat{q}_{011}^{\beta_2} = \frac{(\beta_1-\beta_2)\delta\omega\left((\phi-c-b)\hat{q}_U + \nu(\chi+1)\hat{q}_U^{\chi+1} - \lambda ra\,\hat{q}_U^{\lambda}\right)}{\Omega_{112}(1-\beta_2) + \Lambda_{112}(1-\beta_2+\chi)\hat{q}_U^{\chi}} \hat{q}_U^{\beta_2-1}. \tag{A17}$$

With-Breach Optimal Threshold

For the optimal capacity level $\hat{q}_U$, the optimal threshold $\hat{q}_{011}$ is obtained from the value-matching relationship and smooth-pasting condition assuming the capacity is not breached:

$$0 = V_{11}(q) - V_0(q) - K\Big|_{q=\hat{q}_{011},\, q_U=\hat{q}_U},\; 0 = \frac{\partial V_{11}(q)}{\partial q} - \frac{\partial V_0(q)}{\partial q}\bigg|_{q=\hat{q}_{011},\, q_U=\hat{q}_U}, \tag{A18}$$

or respectively as:

$$0 = \frac{\nu\,\hat{q}_U^{\chi+1}}{r} + \frac{(\phi-c)\hat{q}_U}{r} - \frac{f}{r} + A_{112}\hat{q}_{011}^{\beta_2} - A_{011}\hat{q}_{011}^{\beta_1} - K, \tag{A19}$$

$$0 = \beta_2 A_{112}\hat{q}_{011}^{\beta_2-1} - \beta_1 A_{011}\hat{q}_{011}^{\beta_1-1}. \tag{A20}$$

Substituting

$$A_{011} = \frac{\beta_2}{\beta_1} A_{112}\hat{q}_{011}^{-\beta_1+\beta_2} = -\frac{\beta_2\left(\Omega_{112} + \Lambda_{112}\hat{q}_U^{\chi}\right)}{\beta_1(\beta_1-\beta_2)\delta\omega} \hat{q}_{011}^{-\beta_1+\beta_2} \hat{q}_U^{1-\beta_2} \tag{A21}$$

from (A21) into (A19) and simplifying yields:

$$\hat{q}_{011}^{\beta_2} = \frac{\beta_1 \delta\omega\left(\left(\phi - c - b\right)\hat{q}_U - ra\,\hat{q}_U^{\lambda} + \nu\,\hat{q}_U^{\chi+1}\right)}{\Omega_{112} + \Lambda_{112}\hat{q}_U^{\chi}}\,\hat{q}_U^{\beta_2 - 1}. \tag{A22}$$

The solutions for $\hat{q}_{01}$ and $\hat{q}_U$ are obtained simultaneously from (A17) and (A22) using an appropriate numerical method.

With-Breach Optimal Option Method

The optimal capacity is found from maximising the option coefficient A_{011}. Differentiating (A21) with respect to the capacity has to attend to the relationship between $\hat{q}_{01}$ and $\hat{q}_U$ as specified by (A22). We substitute (A21) in (A19) and define:

$$\begin{aligned} H_{11}\left(q_{011}, q_U\right) = {} & \frac{q_U^{-\beta_2}}{r\beta_1\delta\omega} \times \\ & \left(-\left(\Omega_{112} q_U + \Lambda_{112} q_U^{\chi+1}\right) q_{011}^{\beta_2} + \beta_1\delta\omega\left(\left(\phi - c - b\right) q_U - ra\,q_U^{\lambda} + \nu\,q_U^{\chi+1}\right) q_U^{\beta_2}\right). \end{aligned} \tag{A23}$$

From (A23):

$$\frac{\partial H_{11}}{\partial q_{011}} = \frac{q_U^{-\beta_2}}{r\beta_1\delta\omega} \times \left(-\beta_2\left(\Omega_{112} q_U + \Lambda_{112} q_U^{\chi+1}\right) q_{011}^{\beta_2 - 1}\right),$$

$$\begin{aligned} \frac{\partial H_{11}}{\partial q_U} = \frac{q_U^{-\beta_2 - 1}}{r\beta_1\delta\omega} \Big[& \Omega_{112}\left(\beta_2 - 1\right) q_U + \Lambda_{112}\left(\beta_2 - 1 - \chi\right) q_U^{\chi+1} \\ & + \beta_1\delta\omega q_U^{\beta_2}\left(\left(\phi - c - b\right) q_U - \lambda ra\,q_U^{\lambda} + \nu\left(\chi + 1\right) q_U^{\chi+1}\right)\Big], \end{aligned}$$

and since $\dfrac{\partial q_{011}}{\partial q_U} = -\dfrac{\partial H_{11}}{\partial q_U} \Big/ \dfrac{\partial H_{11}}{\partial q_{011}}$, then

$$\begin{aligned} \frac{\partial q_{011}}{\partial q_U} = {} & \frac{\beta_1\delta\omega q_U^{\beta_2} q_{011}^{1-\beta_2}\left[\left(\phi - c - b\right) q_U - \lambda ra\,q_U^{\lambda} + \nu\left(\chi + 1\right) q_U^{\chi+1}\right]}{\beta_2\left(\Omega_{112} q_U + \Lambda_{112} q_U^{\chi+1}\right) q_U^2} \\ & - \frac{q_{011}\left[\Omega_{112}\left(1 - \beta_2\right) q_U + \Lambda_{112}\left(1 - \beta_2 + \chi\right) q_U^{\chi+1}\right]}{\beta_2\left(\Omega_{112} q_U + \Lambda_{112} q_U^{\chi+1}\right) q_U^2}. \end{aligned} \tag{A24}$$

Also, we define from (A21):

$$A_{011}\left(q_{011}, q_U\right) = -\frac{\beta_2\left(\Omega_{112} + \Lambda_{112} q_U^{\chi}\right)}{r\beta_1\left(\beta_1 - \beta_2\right)\delta\omega} q_{011}^{-\beta_1 + \beta_2} q_U^{1-\beta_2}, \tag{A25}$$

so:

$$\frac{\partial A_{011}}{\partial q_{011}} = \frac{\beta_2\left(\Omega_{112} + \Lambda_{112} q_U^{\chi}\right)}{r\beta_1 \delta\omega} q_{011}^{-\beta_1+\beta_2-1} q_U^{1-\beta_2},$$

$$\frac{\partial A_{011}}{\partial q_U} = \frac{\beta_2\left(\Omega_{112}\left(\beta_2 - 1\right) + \Lambda_{112}\left(\beta_2 - 1 - \delta\right) q_U^{\chi}\right)}{r\beta_1\left(\beta_1 - \beta_2\right)\delta\omega} q_{011}^{-\beta_1+\beta_2} q_U^{-\beta_2}.$$

Then, as before:

$$\left.\begin{aligned}
0 &= -\frac{\partial A_{011}}{\partial q_{011}}\frac{\partial H_{11}}{\partial q_U} + \frac{\partial A_{011}}{\partial q_U}\frac{\partial H_{11}}{\partial q_{011}}\bigg|_{q_U=\hat{q}_U,\, q_{011}=\hat{q}_{011}} \\
&= \frac{\beta_2\left(\Omega_{112} + \Lambda_{112} q_U^{\chi}\right)}{r^2\beta_1\left(\beta_1 - \beta_2\right)\delta^2\omega^2} q_{011}^{-1-\beta_1+\beta_2} q_U^{-2-\beta_2} \\
&\quad \times\Big[\left(\Omega_{112}\left(\beta_2 - 1\right) q_U + \Lambda_{112}\left(\beta_2 - 1 - \delta\right) q_U^{\chi+1}\right) q_{011}^{\beta_2} \\
&\quad -\left(\beta_1 - \beta_2\right)\delta\omega\left(q_U\left(\phi - c - b\right) + \nu\left(\chi + 1\right) q_U^{\chi+1} - \lambda ra q_U^{\lambda}\right) q_U^{\beta_2}\Big] \\
&= \left(\Omega_{112}\left(\beta_2 - 1\right) q_U + \Lambda_{112}\left(\beta_2 - 1 - \delta\right) q_U^{\chi+1}\right) q_{011}^{\beta_2} \\
&\quad -\left(\beta_1 - \beta_2\right)\delta\omega\left(q_U\left(\phi - c - b\right) + \nu\left(\chi + 1\right) q_U^{\chi+1} - \lambda ra q_U^{\lambda}\right) q_U^{\beta_2} \\
&= \frac{\partial NPV_{11}}{\partial q_U}.
\end{aligned}\right\} \tag{A26}$$

The solutions for $\hat{q}_{011}$ and $\hat{q}_U$ are obtained simultaneously from (A22) and (A26) using an appropriate numerical method. Also, (A26) establishes that identical optimal capacity solutions are obtained whether the optimal NPV or optimal option method is applied.

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