

Investments with Declining Cost Following a Lévy Process

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Abstract

We consider an optimal investment problem in which the cost of the investment decreases over time. This decrease is modelled using the negative of a non-decreasing Lévy process. The decreasing cost is a way of modelling that innovations drive down the cost of the investment. Several explicit examples of how different Lévy processes influence the value of the investment are given.

Keywords: Optimal stopping, irreversible investments, innovations, Lévy processes.

JEL Classification: G11, G13

1 Introduction

An optimal irreversible investment problem with technical innovation is studied. We revisit and extend the model used in Murto [10], where an investment problem that generalizes the classical optimal investment problem in McDonald & Siegel [8] is considered. In Murto [10], the cost of making the investment is assumed to decrease by a given fraction each time a Poisson process (independent of the Brownian motion driving the cash flows generated by the investment) jumps. The idea is that as time passes innovations occur (represented by the jump in the Poisson process) that drives down the cost of the investment. This model has also recently been studied in Nunes et. al. [11]. We extend the class of models of innovation by also considering innovations where not only the timing of when innovations occur is uncertain, but also the size of the innovations. This is done by modelling the innovations according to a non-decreasing Lévy process.

The rest of this paper is organized as follows: In Section 2 the model is presented, in Section 3 the investment problem is solved, and in Section 4 several examples are given.

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2 The model

We let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a complete filtered probability space where the filtration (\mathcal{F}_t) is assumed to satisfy the usual conditions of \mathcal{F}_0 containing all P -null sets of \mathcal{F} , and the filtration being right-continuous. There exists a bank account with constant interest rate r , i.e. there exists a financial asset whose value B_t at $t \geq 0$ satisfies

$$dB_t = rB_t dt \text{ with } B_0 = 1.$$

We also assume the existence of a risk-neutral probability measure Q , locally equivalent to P , and with B as its numeraire.

An investment generates per unit cash flows of C_t at time $t \geq 0$. We assume that (C_t) evolves according to a geometric Brownian motion,

$$dC_t = \mu C_t dt + \sigma C_t dW_t^Q \text{ with } C_0 = c > 0,$$

where (W_t^Q) is a one-dimensional Q -Brownian motion, $\sigma \geq 0$ and

$$-\frac{\sigma^2}{2} \leq \mu < r.$$

The value of the stream of cash flows at time $t \geq 0$ is given by

$$X_t = E^Q \left[\int_t^\infty e^{-r(s-t)} C_s ds \middle| \mathcal{F}_t \right] = \frac{C_t}{r - \mu}$$

with $X_0 = x = c/(r - \mu)$. Since we assume that $\mu < r$, (X_t) is a well-defined and strictly positive stochastic process. If the value (X_t) is the value of a traded asset, then $\mu = r - \delta$, where $\delta > 0$ is the yield the investment generates. If (X_t) is not the value of a traded asset, then $\mu - r$ can be interpreted as an implied yield; see Armerin & Song [3] for a discussion.

The cost of making the investment at time $t \geq 0$ is I_t , where

$$I_t = ie^{-Z_t} \text{ with } i > 0 \tag{1}$$

and (Z_t) is a non-decreasing Lévy process satisfying $Z_0 = 0$ and that is independent of (C_t) (and thus also of (X_t)). For more on the facts of Lévy processes needed, see e.g. Kyprianou [6]. In this model innovation drives down the cost of the investment through the change in (Z_t) . In Murto [10] and Nunes et. al. [11] the cost of investment is written as $I_t = i\varphi^{N_t}$ with $\varphi \in (0, 1)$ and where (N_t) is a Poisson process with constant intensity $\lambda > 0$. Since $i\varphi^{N_t} = ie^{-\ln(1/\varphi)N_t}$, we still have their specific form as the special case $Z_t = \ln(1/\varphi)N_t$.

The goal of the investor is to maximize the net present value

$$E_{x,i}^Q [e^{-r\tau}(X_\tau - I_\tau)]$$

over the set of stopping times. We let

$$V(x, i) = \sup_{\tau} E_{x,i}^Q [e^{-r\tau}(X_\tau - I_\tau)], \tag{2}$$

where the supremum is taken over the set of stopping times, and this is the function we will determine given different Lévy processes (Z_t) . We are also interested in determining optimal stopping times, were we recall that an optimal stopping time is a stopping time τ^* such that

$$V(x, i) = E_{x,i}^Q [e^{-r\tau^*}(X_{\tau^*} - I_{\tau^*})].$$

3 The solution to the investment problem

To solve the optimal investment problem, i.e. to find the function given by Equation (2) and to find an optimal stopping time, we start by rewriting the problem as that of an American put option with constant strike price. First of all we observe that we can write

$$\begin{aligned} X_t - I_t &= X_t(1 - I_t/X_t) \\ &= xe^{\mu t} e^{\sigma W_t - \frac{\sigma^2}{2}t} \left(1 - \frac{i}{x} e^{U_t}\right) \\ &= e^{\mu t} L_t (x - i e^{U_t}), \end{aligned}$$

where

$$U_t = \left(\frac{\sigma^2}{2} - \mu\right)t - \sigma W_t - Z_t$$

and

$$L_t = e^{\sigma W_t - \frac{\sigma^2}{2}t}$$

is a Radon-Nikodym process. Using (L_t) to change measure from Q to a new measure, which we call \hat{Q} , we can write Equation (2) as

$$V(x, i) = \sup_{\tau} E^{\hat{Q}} \left[e^{-(r-\mu)\tau} (x - i e^{U_\tau}) \right]. \quad (3)$$

Furthermore, using the Girsanov theorem (see e.g. Jeanblanc et. al. [5]) we have

$$U_t = -\left(\mu + \frac{\sigma^2}{2}\right)t - \sigma \hat{W}_t - Z_t,$$

where \hat{W} is a \hat{Q} -Brownian motion (still independent of (Z_t) under \hat{Q}), and (Z_t) has the same distribution under \hat{Q} as under Q . Note that the condition $\mu \geq -\sigma^2/2$ imposed above, implies that the drift of (U_t) is non-positive.

For $\gamma > 0$ we let $T(\gamma)$ denote an exponentially distributed random variable independent of (U_t) with mean $1/\gamma$; all of this under \hat{Q} . We also introduce

$$\underline{U} = \inf_{0 \leq t < T(r-\mu)} U_t.$$

Mordecki [9] has shown that the function in Equation (3) is given by

$$V(x, i) = E^{\hat{Q}} \left[\left(x - i \cdot \frac{e^{\underline{U}}}{E^{\hat{Q}}[e^{\underline{U}}]} \right)^+ \right],$$

and that an optimal stopping time is given by

$$\tau^* = \inf \left\{ t \geq 0 \mid i e^{U_t} \leq x E^{\hat{Q}}[e^{\underline{U}}] \right\}$$

(see Theorem 2 in Mordecki [9]). Since the random variable \underline{U} has support on the negative real line, it follows that

$$x - i \cdot \frac{e^{\underline{U}}}{E^{\hat{Q}}[e^{\underline{U}}]} \geq 0 \quad \text{when} \quad x \geq \frac{i}{E^{\hat{Q}}[e^{\underline{U}}]} = iA_c,$$

where

$$A_c = \frac{1}{E^{\hat{Q}}[e^U]},$$

from which it follows that

$$V(x, i) = x - i \quad \text{when } x \geq iA_c.$$

When $x < iA_c$ we have

$$V(x, i) = \int_{-\infty}^{\ln(\frac{x}{iA_c})} \left(x - i \cdot \frac{e^y}{E^{\hat{Q}}[e^U]} \right) dF_{\underline{U}}(y),$$

where $F_{\underline{U}}(y) = \hat{Q}(\underline{U} \leq y)$ is the distribution function of \underline{U} under \hat{Q} . Since \underline{U} has support on $(-\infty, 0]$, we can write the optimal value function for every $(x, i) \in \mathbb{R}_{++}^2$ as

$$V(x, i) = \int_0^{\ln(\frac{x}{iA_c})} \left(x - i \cdot \frac{e^y}{E^{\hat{Q}}[e^U]} \right) dF_{\underline{U}}(y).$$

Now

$$\frac{I_t}{X_t} = \frac{i}{x} e^{U_t},$$

so the optimal stopping rule is to stop as soon as

$$X_t \geq \frac{1}{E^{\hat{Q}}[e^U]} I_t \quad \Leftrightarrow \quad X_t \geq A_c I_t.$$

In order to calculate V and the optimal stopping rule, we need to determine $E^{\hat{Q}}[e^U]$. To do this we start by defining the Laplace exponent ψ :

$$\psi(z) = \frac{1}{t} \ln E^{\hat{Q}}[e^{zU_t}],$$

which is finite at least for $z \geq 0$. To continue, we need to distinguish between two different cases.

- (a) When $\sigma > 0$, the process (U_t) is a spectrally negative Lévy process (i.e. a Lévy process with only negative jumps and that is not the negative of a subordinator). With our model we have

$$\psi(z) = \frac{\sigma^2 z^2}{2} - \left(\mu + \frac{\sigma^2}{2} \right) z + \psi^-(z),$$

where ψ^- is the Laplace exponent of $-Z_t$:

$$\psi^-(z) = \frac{1}{t} \ln E^{\hat{Q}}[e^{-zZ_t}].$$

Again using that (U_t) is a spectrally negative Lévy process when $\sigma > 0$, it follows from Equation (8.4) in Kyprianou [6] that for $z \geq 0$

$$E^{\hat{Q}}[e^{zU}] = \frac{r - \mu}{\Phi(r - \mu)} \cdot \frac{\Phi(r - \mu) - z}{r - \mu - \psi(z)},$$

where $\Phi(r - \mu)$ is the largest root of the equation

$$\psi(z) = r - \mu.$$

We have $\psi(1) = -\mu + \psi^-(1)$, so

$$E^{\hat{Q}} [e^{\underline{U}}] = \frac{r - \mu}{\Phi(r - \mu)} \cdot \frac{\Phi(r - \mu) - 1}{r - \psi^-(1)}.$$

Hence,

$$A_c = \frac{1}{E^{\hat{Q}} [e^{\underline{U}}]} = \frac{\Phi(r - \mu)}{\Phi(r - \mu) - 1} \cdot \frac{r - \psi^-(1)}{r - \mu}.$$

- (b) When $\sigma = 0$, then $U_t = -\mu t - Z_t$ with $\mu \geq 0$. This means that (U_t) in this case is the negative of a subordinator, which is a type of process not in the class of spectrally negative Lévy processes. The general result by Mordecki [9] is still valid, and in this case

$$\underline{U} = \inf_{0 \leq t < T(r - \mu)} U_t = -\mu T(r - \mu) - Z_{T(r - \mu)}.$$

We further have

$$\begin{aligned} E^{\hat{Q}} [e^{z\underline{U}}] &= E^{\hat{Q}} \left[e^{-z\mu T(r - \mu) - zZ_{T(r - \mu)}} \right] \\ &= E^{\hat{Q}} \left[\int_0^\infty e^{-z\mu y - zZ_y} (r - \mu) e^{-(r - \mu)y} dy \right] \\ &= (r - \mu) \int_0^\infty E^{\hat{Q}} [e^{-zZ_y}] e^{-(r - \mu + z\mu)y} dy \\ &= (r - \mu) \int_0^\infty e^{-(r - \mu + z\mu - \psi^-(z))y} dy \\ &= \frac{r - \mu}{r - \mu(1 - z) - \psi^-(z)}. \end{aligned}$$

Since in this case

$$\psi(z) = -\mu z + \psi^-(z),$$

it follows that

$$E^{\hat{Q}} [e^{z\underline{U}}] = \frac{r - \mu}{r - \mu - \psi(z)}$$

and

$$A_c = \frac{1}{E^{\hat{Q}} [e^{\underline{U}}]} = \frac{r - \psi^-(1)}{r - \mu}.$$

To summarize, we have the following result:

Proposition 3.1 *With notation and assumptions introduced above, the optimal value function V in Equation (2) is given by*

$$V(x, i) = \int_0^{\ln(\frac{x}{iA_c})} (x - iA_c e^y) dF_{\underline{U}}(y),$$

where

$$A_c = \frac{\Phi(r - \mu)}{\Phi(r - \mu) - 1} \cdot \frac{r - \psi^-(1)}{r - \mu}$$

when $\sigma > 0$ and

$$A_c = \frac{r - \psi^-(1)}{r - \mu}$$

when $\sigma = 0$. An optimal stopping time is in both cases given by

$$\tau = \inf \{t \geq 0 \mid X_t \geq A_c I_t\}.$$

To be able to numerically calculate the optimal value function V we can use Laplace transform techniques. We will now derive an expression where the value function is written using inverse Laplace transforms. In order to conform with standard Laplace transform methods we will work with the negative of \underline{U} (since this is a positive random variable). Letting $J = -\underline{U}$, we have

$$V(x, i) = E^{\hat{Q}} \left[\left(x - i \cdot \frac{e^{-J}}{E^{\hat{Q}}[e^{-J}]} \right)^+ \right] = E^{\hat{Q}} [(x - iA_c e^{-J})^+].$$

Now,

$$x - iA_c e^{-y} \geq 0 \Leftrightarrow y \geq \ln(iA_c/x).$$

Since J has support on $[0, \infty)$, we can use the same argument as in the case where we represented the value function using the random variable \underline{U} , to see that it can be written

$$V(x, i) = \int_{\ln(iA_c/x)}^{\infty} (x - A_c i e^{-y}) dF_J(y) = x \int_{\ln(iA_c/x)}^{\infty} dF_J(y) - iA_c \int_{\ln(iA_c/x)}^{\infty} e^{-y} dF_J(y). \quad (4)$$

We can also write the value function as

$$\begin{aligned} V(x, i) &= \int_{\ln(iA_c/x)}^{\infty} (x - A_c i e^{-y}) dF_J(y) \\ &= \int_0^{\infty} (x - A_c i e^{-y}) dF_J(y) - \int_0^{\ln(iA_c/x)} (x - A_c i e^{-y}) dF_J(y) \\ &= x - i - \int_0^{\ln(iA_c/x)} (x - A_c i e^{-y}) dF_J(y) \\ &= x - i - x \int_0^{\ln(iA_c/x)} dF_J(y) + A_c i \int_0^{\ln(iA_c/x)} e^{-y} dF_J(y). \end{aligned}$$

From this expression for $V(x, i)$ it is obvious that if we know

$$L(b; c) = \int_0^c e^{-by} dF_J(y)$$

for all $c \geq 0$ and $b = 0, 1$, then we can calculate the value function V . Since we know that

$$\int_0^{\infty} e^{-zy} dF_J(y) = E^{\hat{Q}} [e^{-zJ}] = \begin{cases} \frac{r-\mu}{\Phi(r-\mu)} \cdot \frac{\Phi(r-\mu)-z}{r-\mu-\psi(z)}, & \text{when } \sigma > 0 \\ \frac{r-\mu}{r-\mu-\psi(z)}, & \text{when } \sigma = 0, \end{cases}$$

we can use inverse Laplace transform techniques to calculate $L(0; \ln(iA_c/x))$ and $L(1; \ln(iA_c/x))$. We now use that if F is the distribution function of a positive random variable, then the Laplace transform of the function $y \mapsto \int_0^y e^{-bt} dF(t)$ is given by

$$\frac{\tilde{F}(z+b)}{z},$$

where $\tilde{F}(z) = \int_0^\infty e^{-zt} dF(t)$ is the Laplace-Stieltjes transform of F (this follows from changing the order of integration in the definition of the Laplace transform). Using this result, and with \mathcal{L} denoting the Laplace transform, we have

$$\mathcal{L} \left[\int_0^\cdot dF_J(y) \right] (z) = \frac{1}{z} \cdot \frac{r-\mu}{\Phi(r-\mu)} \cdot \frac{\Phi(r-\mu)-z}{r-\mu-\psi(z)},$$

and

$$\mathcal{L} \left[\int_0^\cdot e^{-y} dF_J(y) \right] (z) = \frac{1}{z} \cdot \frac{r-\mu}{\Phi(r-\mu)} \cdot \frac{\Phi(r-\mu)-(z+1)}{r-\mu-\psi(z+1)}$$

when $\sigma > 0$ (see below for the case when $\sigma = 0$). By inverting these two Laplace transforms and evaluate at the point $\ln(iA_c/x)$ will give us the value function. Writing $\mathcal{L}_z^{-1}[f(z)](t)$ for the inverse Laplace transform of f evaluated at t , we can write the value function as

$$\begin{aligned} V(x, i) &= x - i - x \mathcal{L}_z^{-1} \left[\frac{1}{z} \cdot \frac{r-\mu}{\Phi(r-\mu)} \cdot \frac{\Phi(r-\mu)-z}{r-\mu-\psi(z)} \right] (\ln(iA_c/x)) \\ &\quad + iA_c \mathcal{L}_z^{-1} \left[\frac{1}{z} \cdot \frac{r-\mu}{\Phi(r-\mu)} \cdot \frac{\Phi(r-\mu)-(z+1)}{r-\mu-\psi(z+1)} \right] (\ln(iA_c/x)) \\ &= x - i - x(r-\mu) \mathcal{L}_z^{-1} \left[\frac{1}{z} \cdot \frac{1}{r-\mu-\psi(z)} - \frac{1}{\Phi(r-\mu)} \cdot \frac{1}{r-\mu-\psi(z)} \right] (\ln(iA_c/x)) \\ &\quad + iA_c(r-\mu) \mathcal{L}_z^{-1} \left[\frac{1}{z} \cdot \frac{\Phi(r-\mu)-1}{\Phi(r-\mu)} \cdot \frac{1}{r-\mu-\psi(z)+1} - \frac{1}{\Phi(r-\mu)} \cdot \frac{1}{r-\mu-\psi(z+1)} \right] (\ln(iA_c/x)). \end{aligned}$$

Since the evaluation is at the point $\ln(iA_c/x)$, two of the terms in this expression cancels, and after some simplifications we get

$$\begin{aligned} V(x, i) &= x \left(1 + (r-\mu) \mathcal{L}_z^{-1} \left[\frac{1}{z} \cdot \frac{1}{\psi(z) - (r-\mu)} \right] (\ln(iA_c/x)) \right) \\ &\quad - i \left(1 + (r-\psi^-(1)) \mathcal{L}_z^{-1} \left[\frac{1}{z} \cdot \frac{1}{\psi(z+1) - (r-\mu)} \right] (\ln(iA_c/x)) \right). \end{aligned}$$

This is the expression of the solution (written here using inverse Laplace transforms, but usually written using scale functions) that occurs in many places in the literature (see Kyprianou [6], specifically Corollary 11.3, and references therein). When $\sigma = 0$, it is straightforward to see that we will arrive directly at the above expression for V , so it will hold for any $\sigma \geq 0$.

4 Examples

4.1 Introduction

In this section we consider the presented model under several assumption on the Lévy process (Z_t) driving down the cost of the investment. The general result in Proposition 3.1 results in the above expression involving Laplace inversions, and this in many cases the best way of numerically calculate the value function. In this section we focus on models where we can get analytically explicit formulas.

4.2 Models with $\sigma > 0$

A compound Poisson process with mixed-exponentially distributed jump sizes

When (Z_t) is a compound Poisson process where the compounding distribution is a convex combination of exponential distributions (i.e. the compounding distribution is mixed-exponential or hyperexponential), then the explicit solution to the optimal stopping problem can be found in Mordecki [9]. Here we illustrate this class of models by considering the investment problem when the jump size is exponentially distributed. Hence, we assume that

$$Z_t = \sum_{\ell=1}^{N_t} Y_\ell,$$

where (N_t) is a Poisson process with constant intensity $\lambda > 0$ and the random variables Y_1, Y_2, \dots are independent of each other, independent of (N_t) and have common density

$$f(y) = \beta e^{-\beta y}, \quad y \geq 0$$

for some $\beta > 0$. We also assume that $\sigma > 0$. In this case, the solution is given by Corollary 2 in Mordecki [9]. We need the two strictly negative roots $-\rho_1 > -\rho_2$ to the equation

$$\frac{\sigma^2 z^2}{2} - \left(\mu + \frac{\sigma^2}{2} \right) z - \lambda \frac{z}{z + \beta} = r - \mu.$$

Given these two, we calculate

$$B_1 = \frac{\frac{\rho_1}{\beta} - 1}{\frac{\rho_1}{\rho_2} - 1} \quad \text{and} \quad B_2 = \frac{\frac{\rho_2}{\beta} - 1}{\frac{\rho_2}{\rho_1} - 1}.$$

In this case

$$\psi^-(z) = -\frac{\lambda z}{z + \beta},$$

so

$$\frac{1}{A_c} = E^{\hat{Q}} [e^U] = \frac{r - \mu}{r + \lambda/(1 + \beta)} \cdot \frac{\Phi(r - \mu) - 1}{\Phi(r - \mu)}.$$

An alternative expression is proved in Mordecki [9]:

$$E^{\hat{Q}} [e^U] = \frac{B_1 \rho_1}{\rho_1 + 1} + \frac{B_2 \rho_2}{\rho_2 + 1}.$$

Given the four parameters ρ_1, ρ_2, B_1 and B_2 , the solution can be written (see Mordecki [9] for details)

$$V(x, i) = \begin{cases} x - i & \text{when } x \geq A_c i \\ \frac{x B_1}{\rho_1 + 1} \left(\frac{x}{A_c i} \right)^{\rho_1} + \frac{x B_2}{\rho_2 + 1} \left(\frac{x}{A_c i} \right)^{\rho_2} & \text{when } x < A_c i, \end{cases}$$

where

$$A_c = \frac{1}{\frac{B_1 \rho_1}{\rho_1 + 1} + \frac{B_2 \rho_2}{\rho_2 + 1}}.$$

A scaled Poisson process

The case where (Z_t) is a scaled Poisson process and $\sigma > 0$ is the main model considered in Murto [10] and in Nunes et. al. [11]. It is also considered in Aase [1] and Aase [2]. With $Z_t = kN_t$, where $k > 0$ is a constant and (N_t) is a Poisson process with constant intensity $\lambda > 0$, we have

$$\psi^-(z) = \lambda(e^{-kz} - 1),$$

and thus

$$\psi(z) = \frac{\sigma^2 z^2}{2} - \left(\mu + \frac{\sigma^2}{2} \right) z + \lambda(e^{-\lambda z} - 1).$$

It follows that

$$E^{\hat{Q}} [e^{zU}] = \frac{r - \mu}{\Phi(r - \mu)} \cdot \frac{\Phi(r - \mu) - z}{r - \mu - \frac{\sigma^2 z^2}{2} + \left(\mu + \frac{\sigma^2}{2} \right) z + \lambda(1 - e^{-zk})}$$

and we get

$$A_c = \frac{1}{E^{\hat{Q}} [e^U]} = \frac{\Phi(r - \mu)}{r - \mu} \cdot \frac{r + \lambda(1 - e^{-k})}{\Phi(r - \mu) - 1}.$$

We now use the fact that we in this case can write the value function as

$$\begin{aligned} V(x, i) &= x \left(1 + (r - \mu) \mathcal{L}_z^{-1} \left[\frac{1}{\frac{1}{z} \cdot \frac{\sigma^2 z^2}{2} - \left(\mu + \frac{\sigma^2}{2} \right) z + \lambda(e^{-\lambda z} - 1) - (r - \mu)} \right] (\ln(iA_c/x)) \right) \\ &\quad - i \left(1 + (r - \psi^-(1)) \mathcal{L}_z^{-1} \left[\frac{1}{\frac{1}{z} \cdot \frac{\sigma^2 (z+1)^2}{2} - \left(\mu + \frac{\sigma^2}{2} \right) (z+1) + \lambda(e^{-\lambda(z+1)} - 1) - (r - \mu)} \right] (\ln(iA_c/x)) \right) \\ &= x - i + x(r - \mu) \mathcal{L}_z^{-1} \left[\frac{1}{\frac{1}{z} \cdot \frac{\sigma^2 z^2}{2} - \left(\mu + \frac{\sigma^2}{2} \right) z + \lambda(e^{-\lambda z} - 1) - (r - \mu)} \right] (\ln(iA_c/x)) \\ &\quad - i(r - \psi^-(1)) \mathcal{L}_z^{-1} \left[\frac{1}{\frac{1}{z} \cdot \frac{\sigma^2 (z+1)^2}{2} - \left(\mu + \frac{\sigma^2}{2} \right) (z+1) + \lambda(e^{-\lambda(z+1)} - 1) - (r - \mu)} \right] (\ln(iA_c/x)). \end{aligned}$$

To get an analytical expression for the solution of the optimal stopping problem in this case, we follow the proof of the general compound Poisson process case studied in Landriault & Willmot [7]. Let

$$C(z) = \frac{\sigma^2}{2} \cdot \frac{1}{\psi(z) - (r - \mu)} = \frac{1}{z^2 - \left(\frac{2\mu}{\sigma^2} + 1 \right) z - \frac{2(r - \mu)}{\sigma^2} - \Lambda(1 - e^{-zk})},$$

where $\Lambda = 2\lambda/\sigma^2$. We can write

$$C(z) = \frac{1}{(z + z_1)(z - z_2) + \Lambda e^{-kz}},$$

where

$$z_1 = - \left(\frac{\mu}{\sigma^2} + \frac{1}{2} \right) + \sqrt{\left(\frac{\mu}{\sigma^2} + \frac{1}{2} \right)^2 + \frac{2(r - \mu + \lambda)}{\sigma^2}} > 0$$

and

$$z_2 = \frac{\mu}{\sigma^2} + \frac{1}{2} + \sqrt{\left(\frac{\mu}{\sigma^2} + \frac{1}{2} \right)^2 + \frac{2(r - \mu + \lambda)}{\sigma^2}} > 0.$$

Now

$$\begin{aligned} C(z) &= \frac{1}{(z+z_1)(z-z_2)} \cdot \frac{1}{1 + \frac{\Lambda e^{-kz}}{(z+z_1)(z-z_2)}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \Lambda^n e^{-knz}}{(z+z_1)^{n+1}(z-z_2)^{n+1}}. \end{aligned}$$

Let g_n be the inverse Laplace transform of $\frac{1}{(z+z_1)^{n+1}(z-z_2)^{n+1}}$. Then

$$\mathcal{L}_z^{-1} \left[\frac{(-1)^n \Lambda^n e^{-knz}}{(z+z_1)^{n+1}(z-z_2)^{n+1}} \right] (t) = (-1)^n \Lambda^n g_n(t-kn) \mathbf{1}(t \geq kn).$$

Hence,

$$c(t) := \mathcal{L}_z^{-1}[C(z)](t) = \sum_{n=0}^{\infty} (-1)^n \Lambda^n g_n(t-kn) \mathbf{1}(t \geq kn)$$

and, for $p \geq 0$,

$$\begin{aligned} \int_0^t e^{-pu} c(u) du &= \sum_{n=0}^{\infty} (-1)^n \Lambda^n \int_0^t e^{-pkn} e^{-p(u-kn)} g_n(u-kn) \mathbf{1}(t \geq kn) du \\ &= \sum_{n=0}^{\infty} (-1)^n \Lambda^n e^{-pkn} \int_{kn}^{\max(kn,t)} e^{-p(u-kn)} g_n(u-kn) du \\ &= \sum_{n=0}^{\infty} (-1)^n \Lambda^n e^{-pkn} \int_0^{(t-kn)^+} e^{-pv} g_n(v) dv \\ &= \sum_{n=0}^{\lfloor \frac{t}{k} \rfloor} (-1)^n \Lambda^n e^{-pkn} \int_0^{t-kn} e^{-pv} g_n(v) dv. \end{aligned}$$

The inverse Laplace transform g_n is given by

$$g_n(t) = e^{-z_1 t} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(z_2+z_1)^j t^{2n+j+1}}{(2n+j+1)!}$$

(this follows from Equation (29) in Landriault & Willmot [7]), and we get

$$c(t) = \sum_{n=0}^{\infty} (-1)^n \Lambda^n e^{-z_1(t-kn)} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(z_2+z_1)^j (t-kn)^{2n+j+1}}{(2n+j+1)!} \mathbf{1}(t \geq kn).$$

Since

$$\begin{aligned} e^{-pt} g_n(t) &= e^{-(z_1+p)t} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(z_2+z_1)^j t^{2n+j+1}}{(2n+j+1)!} \\ &= e^{-(z_1+p)t} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(z_2+z_1)^j}{(z_1+p)^{2n+j+2}} \cdot \frac{(z_1+p)^{2n+j+2} t^{2n+j+1}}{(2n+j+1)!}, \end{aligned}$$

we can write

$$\begin{aligned} \int_0^t e^{-pu} c(u) du &= \sum_{n=0}^{\lfloor \frac{t}{k} \rfloor} (-1)^n \Lambda^n e^{-pkn} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(z_2+z_1)^j}{(z_1+p)^{2n+j+2}} \\ &\quad \cdot \left(1 - e^{-(z_1+p)(t-kn)} \sum_{\ell=0}^{2n+j+1} \frac{(z_1+p)^\ell (t-kn)^\ell}{\ell!} \right), \end{aligned}$$

where we have used the fact that

$$f(t) = \frac{(z_1 + p)^{2n+j+2} t^{2n+j+1} e^{-(z_1+p)t}}{(2n+j+1)!}$$

is the density function of an Erlang distributed random variable with shape parameter $2n+j+1$ and rate parameter z_1+p . Now

$$\mathcal{L}_z^{-1} \left[\frac{1}{z} \cdot \frac{1}{\psi(z) - (r-\mu)} \right] (t) = \frac{2}{\sigma^2} \int_0^t c(u) du$$

and

$$\mathcal{L}_z^{-1} \left[\frac{1}{z} \cdot \frac{1}{\psi(z+1) - (r-\mu)} \right] (t) = \frac{2}{\sigma^2} \int_0^t e^{-u} c(u) du,$$

and from this

$$V(x, i) = x - i + x \frac{2(r-\mu)}{\sigma^2} \int_0^{\ln(\frac{iAc}{x})} c(u) du - i \frac{2(r+\lambda(1-e^{-k}))}{\sigma^2} \int_0^{\ln(\frac{iAc}{x})} e^{-u} c(u) du. \quad (5)$$

More explicitly we can write

$$\begin{aligned} V(x, i) = & x - i + x \frac{2(r-\mu)}{\sigma^2} \sum_{n=0}^{\lfloor \frac{1}{k} \ln(\frac{iAc}{x}) \rfloor} (-1)^n \left(\frac{2\lambda}{\sigma^2} \right)^n \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(z_2+z_1)^j}{z_1^{2n+j+2}} \\ & \cdot \left(1 - e^{-z_1 \cdot (t-kn)} \sum_{\ell=0}^{2n+j+1} \frac{z_1^\ell (t-kn)^\ell}{\ell!} \right) \\ & - i \frac{2(r+\lambda(1-e^{-k}))}{\sigma^2} \sum_{n=0}^{\lfloor \frac{1}{k} \ln(\frac{iAc}{x}) \rfloor} (-1)^n \left(\frac{2\lambda}{\sigma^2} \right)^n e^{-kn} \sum_{j=0}^{\infty} \binom{n+j}{j} \frac{(z_2+z_1)^j}{(z_1+1)^{2n+j+2}} \\ & \cdot \left(1 - e^{-(z_1+1) \cdot (t-kn)} \sum_{\ell=0}^{2n+j+1} \frac{(z_1+1)^\ell (t-kn)^\ell}{\ell!} \right). \end{aligned}$$

As a by-product of these calculations, and with notation as above, we have the following corollary (for the definition of the scale functions, see e.g. Kyprianou [6]), and using the corollary together with Equation (8.24) in Kyprianou [6] we can also get the distribution of J .

Corollary 4.1 *The scale functions $\mathcal{W}^{(r-\mu)}$ and $\mathcal{Z}^{(r-\mu)}$ of the stochastic process*

$$Z_t = - \left(\mu + \frac{\sigma^2}{2} \right) t - \sigma \hat{W}_t - k N_t$$

are for $x \geq 0$ given by

$$\mathcal{W}^{(r-\mu)}(x) = \frac{2}{\sigma^2} c(x)$$

and

$$\mathcal{Z}^{(r-\mu)}(x) = 1 + \frac{2(r-\mu)}{\sigma^2} \int_0^x c(u) du.$$

respectively.

Proof. The result follows from combining Equation (5) with the formula in Corollary 11.3 in Kyprianou [6]. \square

Deterministic innovations

When $Z_t = \gamma t$ for a constant $\gamma > 0$, the problem has no jump component, and is reduced to a pure diffusion setting. This case is solved already in McDonald & Siegel [8], and is also considered and solved in Murto [10].

4.3 Models with $\sigma = 0$

A scaled Poisson process

One of the special cases considered in Murto [10] is when $\sigma = 0$ and (Z_t) is a Poisson process times a positive constant: $Z_t = kN_t$, where $k > 0$ is a constant and (N_t) is a Poisson process with constant intensity $\lambda > 0$, then both the distribution of I , and the function $V(x, i)$ can be explicitly calculated. In this case

$$\psi^-(z) = \lambda(e^{-zk} - 1),$$

$$E^{\hat{Q}} [e^{zU}] = \frac{r - \mu}{r - \mu(1 - z) + \lambda(1 - e^{-zk})}$$

and

$$E^{\hat{Q}} [e^U] = \frac{r - \mu}{r + \lambda(1 - e^{-k})}.$$

With

$$J = -U = kN_{T(r-\mu)} + \mu T(r - \mu).$$

we recall that

$$V(x, i) = E^{\hat{Q}} \left[\left(x - i \cdot \frac{e^{-J}}{E^{\hat{Q}} [e^{-J}]} \right)^+ \right],$$

and the optimal stopping time is

$$\tau^* = \inf \left\{ t \geq 0 \mid ie^{Ut} \leq xE^{\hat{Q}} [e^{-J}] \right\}$$

(note that $\hat{Q} = Q$ in this case since $\sigma = 0$). Straightforward calculations show that for $x \geq 0$

$$f_J(x) = \frac{r - \mu}{\mu} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \cdot \left(\frac{\lambda(x - k\ell)}{\mu} \right)^\ell e^{-\frac{r-\mu+\lambda}{\mu}(x-k\ell)} \mathbf{1}(x \geq k\ell).$$

Let for $a, b \geq 0$

$$L_J(a, b) := \int_a^{\infty} e^{-bx} f_J(x) dx.$$

Using Fubini's theorem we get

$$L_J(a, b) = \frac{r - \mu}{\mu} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \underbrace{\int_a^{\infty} e^{-bx} \left(\frac{\lambda(x - k\ell)}{\mu} \right)^\ell e^{-\frac{r-\mu+\lambda}{\mu}(x-k\ell)} \mathbf{1}(x \geq k\ell) dx}_{=: I_\ell}.$$

For $\ell = 0, 1, \dots$ we have

$$\begin{aligned}
I_\ell &= \int_{\max(a, k\ell)}^{\infty} \left(\frac{\lambda(x - k\ell)}{\mu} \right)^\ell e^{-\frac{r-\mu+\lambda}{\mu}(x-k\ell)-bx} dx \\
&= \left(\frac{\lambda}{\mu} \right)^\ell e^{\frac{r-\mu+\lambda}{\mu}k\ell} \int_{\max(a, k\ell)}^{\infty} (x - k\ell)^\ell e^{-(\frac{r-\mu+\lambda}{\mu}+b)x} dx \\
&= \left(\frac{\lambda}{\mu} \right)^\ell e^{-bk\ell} \int_{(a-k\ell)^+}^{\infty} y^\ell e^{-(\frac{r-\mu+\lambda}{\mu}+b)y} dy.
\end{aligned}$$

We now use the fact that for $u, v \geq 0$ and n a non-negative integer it holds that

$$\int_u^{\infty} y^n e^{-vy} dy = n! \frac{e^{-uv}}{v^{n+1}} \sum_{m=0}^n \frac{(uv)^m}{m!};$$

see e.g. Jameson [4]. Using this result, we get

$$\int_{(a-k\ell)^+}^{\infty} y^\ell e^{-(\frac{r-\mu+\lambda}{\mu}+b)y} dy = \ell! \frac{e^{-(\frac{r-\mu+\lambda}{\mu}+b) \cdot (a-k\ell)^+}}{\left(\frac{r-\mu+\lambda}{\mu}+b\right)^{\ell+1}} \sum_{m=0}^{\ell} \frac{\left(\left(\frac{r-\mu+\lambda}{\mu}+b\right) \cdot (a-k\ell)^+\right)^m}{m!}.$$

It follows that

$$\begin{aligned}
L_J(a, b) &= \frac{r-\mu}{r-\mu+\lambda+b\mu} \sum_{\ell=0}^{\infty} \left(\frac{\lambda}{r-\mu+\lambda+b\mu} \right)^\ell e^{-bk\ell - (\frac{r-\mu+\lambda}{\mu}+b) \cdot (a-k\ell)^+} \cdot \\
&\quad \sum_{m=0}^{\ell} \frac{\left(\left(\frac{r-\mu+\lambda}{\mu}+b\right) \cdot (a-k\ell)^+\right)^m}{m!}
\end{aligned}$$

To get a formula for the value function V using Equation (4), we need to evaluate L_J for general $a \geq 0$ and $b = 0, 1$:

$$\begin{aligned}
L_J(a, 0) &= \frac{r-\mu}{r-\mu+\lambda} \sum_{\ell=0}^{\infty} \left(\frac{\lambda}{r-\mu+\lambda} \right)^\ell e^{-(\frac{r-\mu+\lambda}{\mu}) \cdot (a-k\ell)^+} \sum_{m=0}^{\ell} \frac{\left(\left(\frac{r-\mu+\lambda}{\mu}\right) \cdot (a-k\ell)^+\right)^m}{m!} \\
L_J(a, 1) &= \frac{r-\mu}{r+\lambda} \sum_{\ell=0}^{\infty} \left(\frac{\lambda e^{-k}}{r+\lambda} \right)^\ell e^{-(\frac{r+\lambda}{\mu}) \cdot (a-k\ell)^+} \sum_{m=0}^{\ell} \frac{\left(\left(\frac{r+\lambda}{\mu}\right) \cdot (a-k\ell)^+\right)^m}{m!}.
\end{aligned}$$

We recall that in this case

$$A_c = \frac{1}{E\hat{Q}[e^{-J}]} = \frac{r+\lambda(1-e^{-k})}{r-\mu}.$$

It now follows from Equation (4) that

$$V(x, i) = xL_J(\ln(iA_c/x), 0) - i \frac{r+\lambda(1-e^{-k})}{r-\mu} L_J(\ln(iA_c/x), 1).$$

By introducing

$$\alpha = \frac{r+\lambda}{\mu}, \quad \beta_0 = \frac{\lambda}{r-\mu+\lambda} \quad \text{and} \quad \beta_1 = \frac{\lambda e^{-k}}{r+\lambda}$$

we can write the value of the optimal stopping problem in this case as

$$\begin{aligned}
V(x, i) = & x(1 - \beta_0) \sum_{\ell=0}^{\infty} \beta_0^\ell e^{-(\alpha-1) \cdot (\ln(iA_c/x) - k\ell)^+} \sum_{m=0}^{\ell} \frac{((\alpha - 1) \cdot (\ln(iA_c/x) - k\ell)^+)^m}{m!} \\
& - i(1 - \beta_1) \sum_{\ell=0}^{\infty} \beta_1^\ell e^{-\alpha(\ln(iA_c/x) - k\ell)^+} \sum_{m=0}^{\ell} \frac{(\alpha(\ln(iA_c/x) - k\ell)^+)^m}{m!}.
\end{aligned}$$

References

- [1] Aase, K. K. (2005), 'The perpetual American put option for jump-diffusions with applications', *Working paper, Finance, Anderson Graduate School of Management, UC, Los Angeles*.
- [2] Aase, K. K. (2010), 'The Perpetual American Put Option for Jump-Diffusions', in E. Bjordal et. al. (eds.), 'Energy, Natural Resources and Environmental Economics', Energy Systems, *Springer-Verlag*.
- [3] Armerin, F. & Song, H.-S. (2018), 'Valuation of real options in incomplete models – an implied yield approach', *Fuzzy Economic Review*, Vol. 23, No. 1, pp. 19-32, DOI: 10.25102/fer.2018.01.02.
- [4] Jameson, G. J. O. (2016), 'The incomplete gamma functions', *The Mathematical Gazette*, 100(548), pp. 298-306, DOI: 10.1017/mag.2016.67.
- [5] Jeanblanc, M. Yor, M. & Chesny, M. (2009), 'Mathematical Methods for Financial Markets', *Springer-Verlag London*.
- [6] Kyprianou, A. E. (2014), 'Fluctuations of Lévy Processes with Applications. Introductory Lectures', Second Edition, *Springer-Verlag*.
- [7] Landriault, D. & Willmot G. E. (2019), 'On series expansions for scale functions and other ruin-related quantities', *Scandinavian Actuarial Journal*, DOI: 10.1080/03461238.2019.1663444.
- [8] McDonald, R. & Siegel, D. (1986), 'The Value of Waiting to Invest', *The Quarterly Journal of Economics*, 101(4), pp. 707-727.
- [9] Mordecki, E. (2002), 'Optimal stopping and perpetual options for Lévy processes', *Finance and Stochastics*, 6, 473-493.
- [10] Murto P. (2007), 'Timing of investment under technological and revenue-related uncertainties', *Journal of Economic Dynamics & Control*, 31, pp. 1473-1497.
- [11] Nunes, C., Oliveira, C. & Pimentel R. (2020), 'Quasi-analytical solution of an investment problem with decreasing investment cost due to technological innovations', *Working paper*.