# Investment problem with switching modes

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**Abstract**

In this paper we study the optimal control problem of a firm that may operate in two different modes, one being more risky than the other, in the sense that in case the demand decreases, the return of the risky mode is lower than with the more conservative mode. On the other side, in case the demand increases, the opposite holds. The switches between these two alternative modes have associated costs. In both modes, there is the option to exit the market.

We focus on two different parameter scenarios, that describe particular (and somehow extreme) economic situations. In the first scenario, we assume that the market is expected to increase in such a way that once the firm is producing in the more risky mode, it is never optimal to switch to the more conservative one. In the second scenario, there is a hysteresis region, where the firm is waiting in the more risky mode, in production, until some drop or increase in the demand leads to an exit or changing to the more conservative mode. This hysteresis region cannot be attained under continuous production.

We then address the problem of the optimal time to invest when the firm knows, *a priori*, that may invest in one of these two modes and then may switch. Depending on the relation between the switching costs (equal or different from one mode to another), it may happen that the firm invests in the hysteresis region.

*Keywords.* Real options, investment under uncertainty, optimal switching, optimal stopping

## Extended Abstract

In this paper we consider the classical real option problem, the investment decision under uncertainty (see e.g. Arrow and Fisher (1974), McDonald and Siegel (1986), Trigeorgis et al. (1996)). We will focus on the choice between two investments (or modes). This problem is usually studied under assumptions of partially irreversible investments with simple functions for the payoffs.(see, e.g. Dixit et al. (1994), Décamps et al. (2006)). In our set-up the firm can invest in one of the two projects (or modes) and after the investment has a possibility to switch as many times as necessary between them. We want to study how this possibility of switching will impact the investment

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decision. We will use the developed techniques for the valuation of mothballing option, which is characterized in the literature as sequential switching problem. (see e.g. Duckworth and Zervos (2001), Zervos (2003), Zervos et al. (2018), Guerra et al. (2018a)).
The uncertainty is modelled by a geometric Brownian motion for the price $P_t$ (the only economic indicator)

$$dP_t = \mu P_t dt + \sigma P_t dB_t,$$

on a filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where $\mu \in \mathbb{R}$ is the instantaneous drift that satisfies $r - \mu > 0$ for the risk free rate $r$ and $\sigma > 0$ is the instantaneous volatility. The two possible investments for the firm, (the only risk-neutral decision maker), denoted by $I_1$ (the risky one) and $I_2$ (the safe one). The payoff of the investment in $I_i$ during the period of time $[t_1, t_2]$ is $\int_{t_1}^{t_2} e^{-rs}\Pi_i(P_s)ds$, where $\Pi_i(p)$ are the instantaneous profit functions set as $\Pi_i(p) = \alpha_i p - \beta_i$. The coefficients $\beta_i$ can be interpreted as (instantaneous) costs of production for the project $i$, $\beta_i > 0$, also, since the project $I_1$ is the risky one the relationship that hold are $\alpha_1 > \alpha_2 \geq 0$ and $\beta_1 > \beta_2$.
After the investment, the firm can exit the market, the exit is permanent. The state space after investment is $\{I_1, I_2, ex\}$, where $I_i$ are transitory states and $ex$ is a stationary (absorbing) state. The firm can switch between investments incurring in costs: $K_{12}$ to transition between the states $I_1$ and $I_2$, and $K_{21}$ in the opposite direction. The cost of exiting either of the investments is the same and is denoted by $K_{ex}$.
The infinitesimal generator of the process is denoted by $\mathcal{L}$. The state of the firm process is denoted by $Z_t$. The transition between states times are denoted by $T_j^{a,b}$, the exit times by $\tau_i$ and $\mathcal{S}$ denotes the set of all admissible strategies. For each $s \in \mathcal{S}$ the payoff is given by

$$J_{\mathfrak{s}}(z,p) = \mathbb{E}_x \Bigg[ \int_0^\infty e^{-rt} \left( \Pi_1(P_t)\mathcal{I}_{\{Z_t = I_1\}} + \Pi_2(P_t)\mathcal{I}_{\{Z_t=I_2\}} \right) dt - K_{12} \sum_{j=1}^{\infty} e^{-rT_j^{12}} \mathcal{I}_{\{T_j^{12} < \infty\}}$$
$$K_{12} \sum_{j=1}^{\infty} e^{-rT_j^{12}} \mathcal{I}_{\{T_j^{12} < \infty\}} - K_{21} \sum_{j=1}^{\infty} e^{-rT_j^{21}} \mathcal{I}_{\{T_j^{21} < \infty\}} - K_{ex} e^{-r\tau_1} \mathcal{I}_{\{\tau_1 < \infty\}} - K_{ex} e^{-r\tau_2} \mathcal{I}_{\{\tau_2 < \infty\}} \Bigg]$$

where $\mathcal{I}_A$ is the indicator function.
To avoid the continuous switching we assume that $K_{12} + K_{21} > 0$.
**Problem 1 (Switching problem)** *Find the value function $V \in C^1(0, +\infty)$*

$$V(z,p) = \sup_{s \in \mathcal{S}} J_s(z,p) \tag{1}$$

For the convenience will divide the function (1) into: $v_1 := V(1,p)$ and $v_2 := V(2,p)$. After solving the problem 1 and equipped with its solution we will approach the investment problem. Starting with a simpler case, when entering the market the firm pays $K$ for either of the investments. Let us denote by

$$v^* = max(v_1, v_2) \tag{2}$$

and by $\mathcal{T}$ the set of all (Markov) stopping times. Then, we can formulate our first investment problem as follows.

**Problem 2 (Investment problem - same costs)** *Find the value function $W_s \in C^1(0, +\infty)$*

$$W_s(p) = \sup_{\tau \in \mathcal{T}} E_p \left[ e^{-r\tau}(v^*(P_\tau) - K) \right] \tag{3}$$

A more general case, when entering market has different costs, investing in project $I_1$ costs $K_{e1}$ and $K_{e2}$ in $I_2$, can be formulated as follows.

**Problem 3 (Investment problem - different costs)** *Find the value function* $W_d \in C^1(0, +\infty)$

$$W_d(p) = \sup_{\tau \in \mathcal{T}} E_p \left[ e^{-r\tau} \max\{v_1(P_\tau) - K_{e1}, v_2(P_\tau) - K_{e2}\} \right] \tag{4}$$

## Switching problem (the firm is already on the market)

We will solve the problem 1 by proposing the solution to it and then show that it verifies the general verification theorem (see e.g. Knudsen et al. (1998), (Zervos, 2003), Lamberton et al. (2013), (Guerra et al., 2018b) and the references therein).
The associated HJB equations (coupled quasi-variational inequalities) will take form:

$$\max\{\overset{[1]}{\mathcal{L}v_1 - rv_1 + \Pi_1}, \overset{[2]}{v_2 - v_1 - K_{12}}, \overset{[3]}{-v_1 - K_{ex}}\} = 0 \tag{5}$$

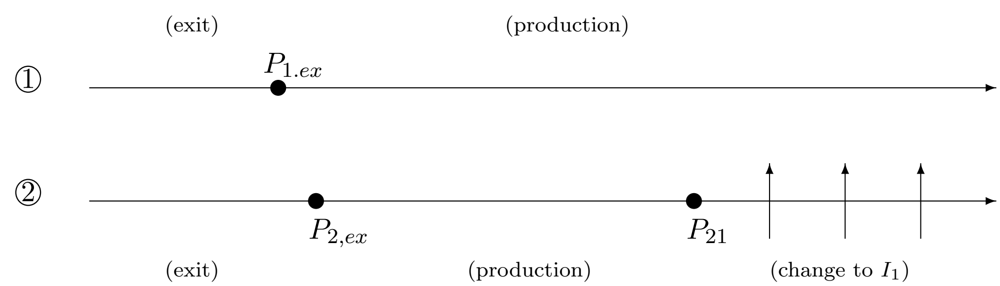
$$\max\{\mathcal{L}v_2 - rv_2 + \Pi_2, v_1 - v_2 - K_{21}, -v_2 - K_{ex}\} = 0 \tag{6}$$

The HJB naturally divide the space into several 'action' regions, depending on where each of the parcels of the above equations is equal to zero. These are: *production region, switching region, stopping region, hysteresis region.* We will not give a complete classification (division of the parameter set) as it was done in Zervos et al. (2018)[Section 4.4], as the proofs are similar and calculations are long, our objective is to illustrate the two interesting (in our opinion) cases.

### Case: No downgrading

This is the case illustrated in Figure 1, where it is never optimal to switch from $I_1$ to $I_2$. We introduce the following price points for the space division: $P_{i,ex}$ - for lower values of $p$ it is more profitable to exit $I_i$, $P_{21}$ - for higher values it is more profitable to switch from $I_2$ to $I_1$.

Figure 1: Case 2




**Lemma 4** *For the parameter conditions:*

*i)* $rK_{ex} - \beta_1 < 0, K_{12} > 0, K_{12} + K_{21} > 0$

*ii)* $rK_{ex} + \alpha_2 P_{2,ex} - \beta_2 < 0$

*iii)* $K_{21} \geq K_{21}^\dagger$, *where* $K_{21}^\dagger = AP_{2,ex}^{d_1} + \frac{\alpha_1}{r-\mu} P_{2,ex} - \frac{\beta_1}{r} + K_{ex}$, *iv)*

$$\frac{\alpha_1}{\alpha_2} > \frac{K_{ex} - (\frac{\beta_1}{r} + K_{21})}{K_{ex} - \frac{\beta_2}{r}}, \tag{7}$$

*there are unique constants $A, C, D > 0$, $P_{21} > P_{2,ex} > 0$ and $P_{21} > P_{1,ex} > 0$, such that the value function* (1) *is given by:*

$$v_1(p) = \begin{cases} -K_{ex}, & p < P_{1,ex} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r}, & p \geq P_{1,ex} \end{cases} \tag{8}$$

$$v_2(p) = \begin{cases} -K_{ex}, & p < P_{2,ex} \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r}, & P_{2,ex} \leq p < P_{21} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} - K_{21}, & p \geq P_{21} \end{cases} \tag{9}$$

The proof is similar to (Zervos et al., 2018), the sketch is as follows.

1. From the verification theorem we know that functions $v_i$ have to be of class $C^1$ in $p$, we use continuity of the functions and their derivatives to set up a system of equations, in this case two equations with two unknowns for $v_1$ and four equations with four unknowns for $v_2$.
2. We prove that the system has a unique solution for the conditions $P_{21} > P_{2,ex} > 0$ and $P_{21} > P_{1,ex} > 0$.
3. Then we show that constructed this way functions (8) and (9) satisfy the HJB (5) and (6).

It is worth to give a special attention to condition (7) as it is specific to our case and serves a purpose to eliminate 'bad behaviour' of the solutions. The relative gain $\frac{\alpha_1}{\alpha_2}$ is higher then the relative increase in costs $\frac{K_{ex}-(\frac{\beta_1}{r}+K_{21})}{K_{ex}-\frac{\beta_2}{r}}$. If (7) was not true, there would be no advantage from switching to a more risky project.

**Case: Hysteresis**

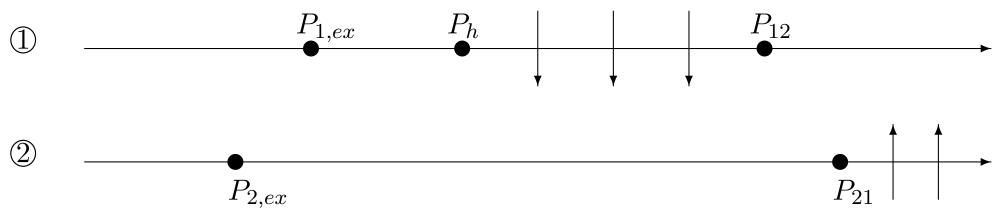
In this case there is an hysteresis region, where the firm is producing although waiting for the price either drop then exit, or go up then switch to $I_2$. This region cannot be attained under continuous production.

**Lemma 5** *For the set of parameter conditions. There are unique constants $A, C_1, C_2, D_1, D_2 > 0$, $P_{21} > P_{12} > P_h > P_{1,ex} > 0$ and $P_{21} > P_{2,ex} > 0$, such that the value function* (1) *is given by:*

$$v_1(p) = \begin{cases} -K_{ex} & p < P_{1,ex} \\ C_1p^{d_1} + D_1p^{d_2} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} & P_{1,ex} \leq p < P_h \\ C_2p^{d_1} + D_2p^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r} - K_{12} & P_h \leq p < P_{12} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} & P_{12} \leq p \end{cases}$$

$$v_2(p) = \begin{cases} -K_{ex} & p < P_{1,ex} \\ C_2p^{d_1} + D_2p^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r} & P_{1,ex} \leq p < P_{21} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} - K_{21} & P_{21} \leq p \end{cases}$$

Figure 2: Case - Hysteresis




## Investment problem

For problem 2 we will use the functions (8)- (9) to construct the function (2). This is a classical optimal stopping problem (see e.g. Peskir and Shiryaev (2006)).

### Case no downgrading - same costs

The function $W_s$ has to satisfy the quasi-variational inequality

$$\max\left\{\mathcal{L}W_s - rW_s, (v^* - K) - W_s\right\} = 0.$$

We have to look into two different scenarios

**1)** $P_{1,ex} < P_{2,ex}$

$$v^*(p) = v_1 \tag{10}$$

**Lemma 6** *Under the conditions of lemma 4, assuming that* $P_{1,ex} < P_{2,ex}$ *and* $-K_{ex} - K < 0$[1], *then there are constants* $B_1 > 0$ *and* $\gamma > 0$ *such that*

$$W_s(p) = \begin{cases} B_1 p^{d_2} & p < \gamma \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} - K & p \geq \gamma \end{cases} \tag{11}$$

[1] If this condition is not verified it is always optimal to invest, independently of $p$, i.e. $\tau \equiv 0$, making the problem trivial

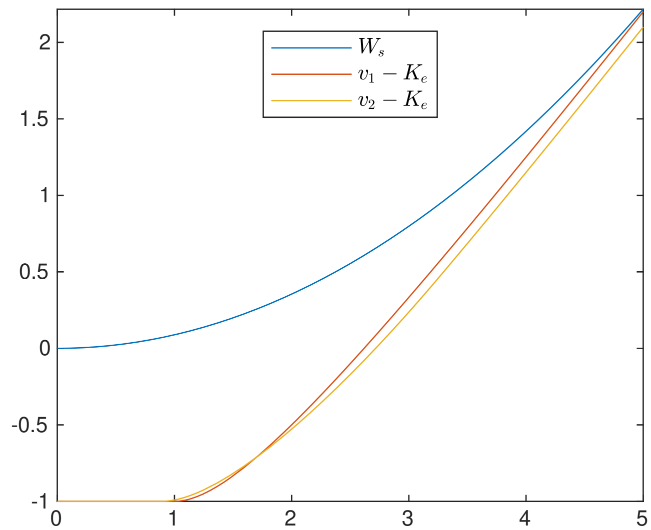



Figure 3: Case: no downgrading, presented in lemma 6, the values for cut points are $P_{1,ex} = 1$, $P_{2,ex} = 0.89$, $P_{21} \approx 3.26$, $\hat{p} \approx 1.7$,$\gamma \approx 5.45$.

**2)** $P_{2,ex} < P_{1,ex} < P_{21}$

$$v^*(p) = \begin{cases} -K_{ex} & p < P_{2,ex} \\ Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r}, & P_{2,ex} \le p < \hat{p} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} & p \ge \hat{p} \end{cases} \tag{12}$$

Where the $\hat{p}$ is the solution to the equation $Cp^{d_1} + Dp^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r} = Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r}$. Although theoretically it can be possible to have solutions of the form (13) described later, economically (without free lunches) every solution will be of the form (11) as illustrated on figure 3 There will be three different situations in this case:

i) $-K_{ex} - K > 0$ free lunch

ii) $-K_{ex} - K < 0$ and $K > K^*$ only invest in $I_1$

iii) $-K_{ex} - K < 0$ and $K < K^*$ it might be optimal to invest in $I_2$

**Case Hysteresis - same costs**

$$v_*(p) = \begin{cases} -K_{ex} & p < P_{2,ex} \\ C_2p^{d_1} + D_2p^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r} & P_{2,ex} \le p < \bar{p} \\ Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r} & p \ge \bar{p} \end{cases},$$

where $\bar{p}$ is a solution to $C_2p^{d_1} + D_2p^{d_2} + \frac{\alpha_2}{r-\mu}p - \frac{\beta_2}{r} = Ap^{d_1} + \frac{\alpha_1}{r-\mu}p - \frac{\beta_1}{r}$.

**Lemma 7** *Under the conditions of lemma 4, assuming that $P_{2,ex} < P_{1,ex}$ and $-K_{ex} - K > 0$, then there are constants $A_1, B_1, B_2 > 0$ and $\gamma_3 > \hat{p} > \gamma_2 > \gamma_1 > 0$ such that*

$$W_s(p) = \begin{cases} B_1 p^{d_2} & p \in [0, \gamma_1) \\ C_2 p^{d_1} + D_2 p^{d_2} + \frac{\alpha_2}{r-\mu} p - \frac{\beta_2}{r} - K & p \in [\gamma_1, \gamma_2] \\ A_1 p^{d_1} + B_2 p^{d_2} & p \in (\gamma_2, \gamma_3) \\ A p^{d_1} + \frac{\alpha_1}{r-\mu} p - \frac{\beta_1}{r} - K & p \in [\gamma_3, +\infty). \end{cases} \tag{13}$$

**Case Hysteresis - different costs**

This is problem 3. The preliminary results indicate that it might be possible to be optimal to invest during the hysteresis region.

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