# Optimization of an investment project portfolio: A real options approach using the Omega measure

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**Abstract:**

Investment decisions usually involve the assessment of more than one project. The most appropriate way to study the feasibility of a project is not to study the project on its own but as part of a portfolio, with correlations between the project inputs and outputs, so that the risks and gains are different from those that would be observed if the projects were studied in isolation. In light of this, the present study proposes a methodology for optimizing a portfolio of investment projects with real options based on the maximization of the Omega performance measure. Classical portfolio optimization methodologies, such as the Markowitz mean-variance formulation, normally use maximization of returns or minimization of risk as the objective function. The great advantage of using Omega as the objective function is that the best relationship between the weighted mean returns and weighted mean losses for the complete distribution of the net present values (NPVs) of the portfolio can be obtained, as the distribution is not restricted to its mean and variance as it is in the Markowitz formulation (1952). Furthermore, real options add value to the portfolio and can be included by extending the marketed asset disclaimer assumption (Copeland & Antikarov, 2003) for a project to all the projects in the portfolio. We give an example to illustrate the proposed methodology. We use Monte Carlo simulation as a tool because of its high level of flexibility in modeling uncertainties. The results show that the best risk-return relationship is obtained by optimizing Omega.

**Keywords:** Risk vs Return, Real Options, Monte Carlo Simulation, Portfolio Performance, Omega Measure.

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## 1. Introduction

It is well known in the financial literature that investors always seek to maximize the return on their investments while minimizing the associated risk as much as possible. Markowitz (1952) developed the basis of investment portfolio optimization theory and proposed the mean-variance model. According to his theory, investors can identify all the optimal portfolios by constructing an efficient frontier, which is the geometric locus with the best possible combination of portfolio assets, corresponding to the lowest level of risk (variance) for a given level of return. Investors concentrate on selecting a portfolio along this frontier.

Mean-variance theory assumes that an investor's risk preference is a quadratic utility function. Hence, the only properties that matter in the distribution of returns are the two first moments: the expected return and the variance. The investor does not need to know other moments of the distribution of returns as the distribution is assumed to be normal. Although the Markowitz theory (1952) is easy to apply and effective in determining portfolio composition, it does not take into account the real characteristics of a distribution. Historically it can be seen that the distributions of the returns of most financial assets are generally not normal distributions.

When a portfolio is made up of investment projects, evaluating it becomes more complex as, strictly speaking, there are no historical records of returns to allow the moments to be calculated. In addition, future management decisions about investments, such as the best time to begin investing, expand, reduce operations or stop investing, can also be taken into consideration. When there is the possibility of exercising these options, which are known as real options because they involve real assets, the modeling becomes more realistic and the attractiveness of projects can, therefore, be improved.

While the most popular index for evaluating portfolio performance is the Sharpe index, which is derived from the Markowitz (1952) theory and assumes that the distribution of returns is normal, there are other measures of performance (risk vs return) that are more consistent with the distributions of returns observed in practice, i.e., non-normal distributions. Among these, the Omega measure, introduced by Keating & Shadwick (2002), is an interesting approach as it takes into account all the moments of the distribution of returns to evaluate the risk and return expected from an asset without assuming a normal distribution.

We propose a methodology for evaluating a portfolio of investment projects that maximizes the Omega performance measure and allows real options to be included in the projects. The methodology, therefore, has two main advantages: the use of the Omega measure as an objective function—ensuring that the distribution of returns is not misinterpreted—and the possibility of including and evaluating real options in projects, with correlated input and output variables.

The article is organized as follows: In section 2 we present a review of the literature on portfolios of assets and real options and in section 3 we describe the main performance measures used to evaluate a portfolio and focus on the Omega measure. Section 4 introduces the proposed methodology for optimization of investment project portfolios with real options and section 5 illustrates the methodology with a numerical application. Finally, we conclude.

## 2. Assets Portfolios with Real Options

An asset portfolio is defined as a set of investments an investor holds in order to obtain the desired return within a given time for the use of his capital. When real assets, or investment projects, as they are also known, are involved, portfolio management and evaluation become more complex tasks than when the assets are financial, mainly because there are no historical records of yields for similar projects (each project has unique characteristics) to help predict the future behavior of the real asset.

The Project Management Institute (PMI) is the main international association that sets standards for the management of investment projects. According to the PMI (2017), a portfolio is a set of projects, programs and sub-portfolios managed as a group to achieve certain strategic objectives. Projects are carried out according to a schedule, and it is the manager’s duty to control the allocation of the required resources while satisfying any deadlines and cost and quality requirements. The PMI suggests various qualitative and quantitative methods for selecting the projects to make up a portfolio. These include the use of weighted ranking based on selection criteria and traditional economic evaluation techniques such as net present value (NPV), internal rate of return (IRR), payback and cost-benefit ratio. The PMI basically concentrates its efforts on establishing standards for the project implementation phase rather than carrying out in-depth studies on methods for selecting and prioritizing portfolio projects.

In academia, however, a number of projects proposing a variety of methodologies for selecting portfolio projects have been undertaken. Heidenberger and Stummer (1999), Carazo et al. (2010) and Mansini, Ogryczak, and Speranza (2014) all summarized the main methodologies currently available. These include methods that combine qualitative and quantitative criteria, such as comparative methods and methods based on scores or ranking, economic indicators, and group decision-making techniques. There are also more analytical methodologies, in which mathematical programming is used to select projects. The mean-variance methodology proposed by Markowitz (1952) stands out as a pioneering effort. Although it was originally intended to be used to optimize a portfolio of financial assets such as stocks and fixed income securities, its principles can also be extended to a portfolio of investment projects. Many studies, such as Hassanzadeh, Nemati & Sun (2014), have also explored non-linear and multi-objective programming, and others have focused on research and development projects, such as Hassanzadeh, MoHassanzadeh et al. (2014), Modarres and Hassanzadeh (2009), Bhattacharyya, Chatterjee, and Kar (2010) and Medaglia, Graves, and Ringuest (2007). The last two even introduce random variables in the optimization program.

As far as portfolios of real options are concerned, of particular note are: Brosch (2001), who describes the interactions that can exist between options and their correlations, especially in projects being carried out in stages; Anand, Oriani & Vassolo (2007), who carry out a theoretical review of the concept of real options inside a portfolio and recognize that there are significant effects when there is interdependence between the options and correlation between the expected returns on the assets; Smith & Thompson (2008), who analyze a portfolio of sequential options in an exploration project using a mathematical approach to assess how the options affect the value of the portfolio; Van Bekkum, Pennings & Smit (2009), who investigate what the effect on R&D projects is of financing that is conditional on results when the manager is responsible for deciding whether to focus on projects that yielded good results or diversify into others; Magazzini, Pammolli & Riccaboni (2015), who assess the case of a portfolio of R&D projects in pharmaceutical companies; and Maier, Pflug & Polak (2019) analyze a large portfolio of options (deferment, staging, mothballing, abandonment) under conditions of exogenous and endogenous uncertainties, developing an algorithm based on simulation and stochastic dynamic programming.

The methodology proposed here follows the spirit of the process of integrated risk analysis of a portfolio of projects and real options described in Mun (2010) using Monte Carlo simulation as the main tool to calculate the real options in the portfolio. Mun (2010) begins his analysis by selecting a potential set of projects that meet the strategic aims of the business. He then models the stochastic variables, quantifies the risks and uncertainties present and adds the real options. Finally, he performs stochastic optimization of the group of projects and strategies (options). Optimization programs generally seek to maximize a measure of return or minimize a measure of risk, such as the mean of the returns or their variance. The main advantage of using Omega as the objective function is that it takes into account the complete distribution of returns rather than reducing the distribution to its mean and variance as in classical theory.

The distribution of returns for the investment project is obtained using the marketed asset disclaimer (MAD) assumption described in Copeland & Antikarov (2003) (based on Samuelson, 1965), which considers the present value of the project cash flows without real options to be the best estimate of the market value of the project. Although the stochastic components that determine the cash flow, such as prices, costs, and market indexes, may follow various stochastic processes (mean reversion, jump processes, processes with two or more stochastic factors etc.), the resulting distribution of the market value of the project (PV) tends toward a lognormal distribution, allowing stochastic paths for the expected values of the distribution the PV of projects in the portfolio to be simulated in a correlated manner while including real options. This is the essence of the methodological approach described in this study.

The optimization program proposed here was inspired by the programs described in Modarres & Hassanzadeh (2009) and Sefair and Medaglia (2005). The former uses a robust optimization process to deal with the uncertainty in a portfolio of staged projects (sequential options), while Sefair & Medaglia (2005) consider the possibility of a project being started within a time interval and take into account an important characteristic when building an investment project portfolio, which is that projects are chosen in a binary manner, i.e., a given project is either included in its entirety in the portfolio or not included at all (rather than only part of the project being included). In addition, the optimization program uses the performance measure Omega as its objective function, which it maximizes, and, with the aid of Monte Carlo simulation, models the future value of the projects and real options in a portfolio with correlated input and output variables.

## 3. Analysis of Portfolio Performance (Risk vs Return)

Risk is traditionally defined as the standard deviation of a series of historical returns and the expected return as the mean value of the series. Figure 1 highlights the pitfalls of this approach particularly well and shows the importance of considering higher-order moments.

Both distributions have the same mean (E[$R$] = 10) and variance (Var[$R$] = 152) but differ in symmetry, kurtosis and all higher moments. However, some traditional performance indicators, such as the Sharpe index, defined as the ratio of the mean to the standard deviation, would indicate that both distributions are equivalent. Omega is more comprehensive as it takes into account all the moments of the distribution of returns.

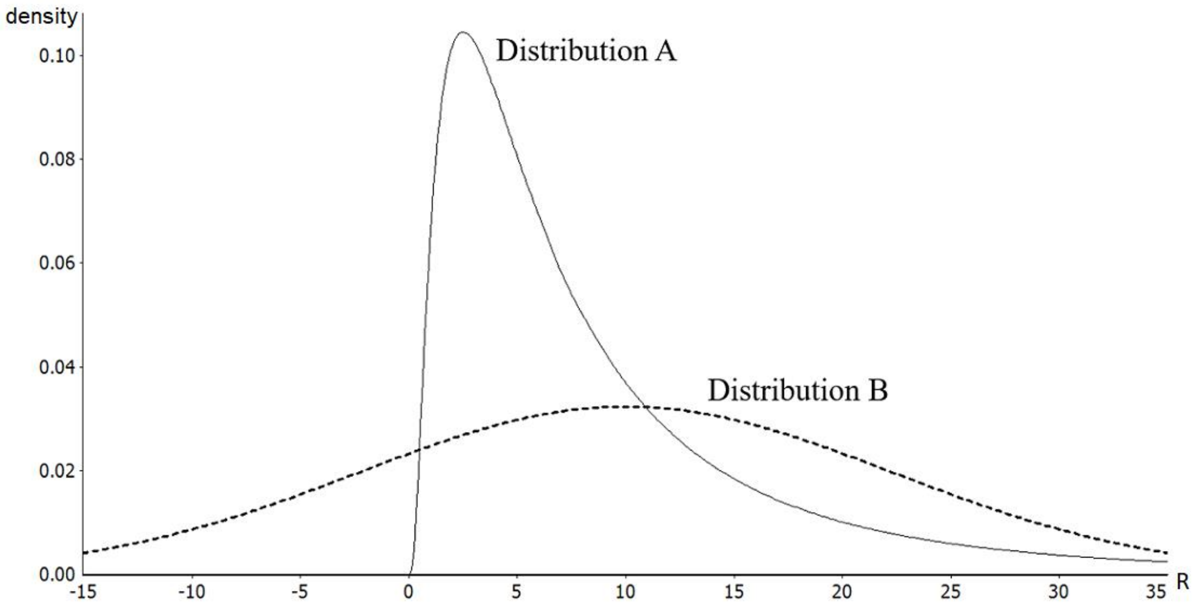


**Figure 1.** Distributions with equal means and variances (E[$R$]=10; Var[$R$]=152)

### 3.1 Sharpe Index

Formulated by Sharpe (1966), this index has gained widespread acceptance among academics and those working in the financial markets. It is based on Markowitz's modern portfolio theory (1952) and identifies points on the capital market line corresponding to optimal portfolios. The Sharpe Index (SI) is defined as

$$SI = \frac{E\left[R_p\right] - r_f}{\sigma_p} \tag{1}$$

where $E\left[R_P\right]$ and $\sigma_P$ , respectively, represent the expected return and standard deviation (volatility) of portfolio *P*, and $r_f$ is the risk-free interest rate.

Mean-variance theory identifies the portfolios with the maximum expected return for a given level of risk, which can then be plotted to form what is known as the efficient frontier. The portfolios with the highest SI are those along the efficient frontier if the distribution of returns is assumed to be normal.

### 3.2 Sortino Index

Sortino & Price (1994) observed that standard deviation only measures the risk of not achieving a mean. However, the most important thing is to capture the risk of not achieving a return above a target known as the minimum acceptable return ($R_{MA}$). The Sortino index (ISor) therefore differs from the Sharp index (SI) in that it uses the downside risk ($\sigma_{DR}$), defined in Eq. 2, to measure risk.

$$\sigma_{DR} = \sqrt{\sum_{i=1}^{n}\left[\min(0; R_{P,i} - R_{MA})^2\right] \Big/ n} \tag{2}$$

ISor is defined as

$$ISor = \frac{E[R_P] - R_{MA}}{\sigma_{DR}} \tag{3}$$

The $\sigma_{DR}$ in Eq. 2 is the standard deviation of the distribution of returns ($R_{P,i}$) of the portfolio *P* below $R_{MA}$, and n is the total number of observations (*i* = 1,...,n). The way in which risk is measured is the main difference between the Sortino and Sharpe indexes.

### 3.3 The Omega Performance Measure

The Markowitz mean-variance theory (1952) makes two important simplifications: (1) the investor's risk-return preferences are defined by a quadratic utility function and (2) the mean and variance are sufficient to describe a distribution of returns. These simplifications are valid if the distribution of returns is assumed to be normal. However, it is generally accepted as an empirical fact that investment returns do not have a normal distribution. Higher-order moments are therefore needed in addition to the mean and variance to describe the distribution better. The Omega measure proposed by Keating & Shadwick (2002) allows these higher moments to be taken into account and is given by

$$\Omega(L) = \frac{\int_L^b \left[1 - F(x)\right] dx}{\int_a^L F(x)} = \frac{\int_L^b (x - L) f(x) dx}{\int_a^L (L - x) f(x) dx} = \frac{E\left[\max(x - L;\ 0)\right]}{E\left[\max(L - x;\ 0)\right]} \tag{4}$$

where *F(x)* is the cumulative distribution function of the returns *x; a* and *b*, respectively, are the lower and upper limits of the distribution *f(x)* of returns; and *L* is the minimum return acceptable to the investor (defined exogenously). The numerator is thus the expected value of the excess return ($x$-$L$) for positive results, and the denominator the expected value of the shortfall ($L$-$x$) for negative results, as defined in Kazemi, Schneeweis, and Gupta (2003).

By taking into account the complete distribution, Omega has a major advantage over the Sharp index, which is derived from mean-variance theory and limits the distribution to essentially its first two moments. Furthermore, Omega is also intuitively attractive and easy to compute.

## 4. A Methodology for Optimizing Investment Portfolios with Real Options

This section describes a methodology for optimizing a portfolio of investment projects with real options. The methodology is divided into three stages: (1) information modeling; (2) optimization without real options; and (3) optimization with real options.

### 4.1 Stage I: Information Modeling

***<u>First Step:</u>* Identification of the Project Risk Variables**

Project variables whose behavior is uncertain are called risk variables. Uncertainty can be mainly of two kinds: economic uncertainty and technical uncertainty. The former is a result of general movements in the economy, over which there is almost no control (e.g., GDP, exchange rate and sale price of a commodity) and which are the source of the market risk associated with the project. Technical uncertainty depends on the steps taken by the company to reduce it and is the source of private risk associated with the project. Only economic uncertainty becomes apparent with the passing of time. As an example of technical uncertainty, the volume of oil reserves in an oil field will be directly proportional to the amount invested in exploration.

Hence, the first step is to identify the most important project variables that have non-deterministic behavior, are economically or technically uncertain and have a significant effect on cash flow.

***<u>Second Step:</u>* Modeling the Risk Variables**

Once the most important risk variables have been identified, their future behavior must be modeled. One simple approach is to assume that a variable follows some standard function for a random variable, such as a normal, lognormal or triangular function. Another possibility is econometric modeling, which is more sophisticated and uses mainly simple or multiple regression models. This type of modeling is recommended when there are seasonal cycles or when effects caused directly by past scenarios but with a time lag are identified. Another type of modeling is based on stochastic processes, the most widely used of these being geometric Brownian motion (GBM) and mean reversion (MR) (Dixit & Pindyck, 1994). This is the type of modeling considered in the present study.

***<u>Third Step:</u> Determining the Correlations between the Project Risk Variables***

To calculate the correlations between the variables, there are assumed to be J risk variables ($RV_1, RV_2, ..., RV_J$) with their respective history of realizations over time. First, the variance of each asset $j$ ($Var_j$) is calculated, as shown in Eq. 5.

$$Var_j = E\left[\left(RV_j - E[RV_j]\right)^2\right] \tag{5}$$

Then the covariance between two *RV*s, $j$ and $j'$, is calculated, as shown in Eq. 6.

$$Cov(RV_j, RV_{j'}) = E\left[\left(RV_j - E[RV_j]\right)\left(RV_{j'} - E[RV_{j'}]\right)\right] \tag{6}$$

The covariance quantifies the extent to which two *RV*s are related. The Pearson correlation coefficient ($\rho_{jj'}$) is a standardized covariance calculated as shown in Eq. 7.

$$\rho_{jj'} = Cov(RV_j, RV_{j'}) \Big/ \sqrt{Var_j \times Var_{j'}} \tag{7}$$

$\rho_{jj'}$ varies between -1 and 1. A value of -1 indicates a perfect negative correlation between the variables, 1 a perfect positive correlation and 0 that the variables do not depend linearly on each other. Once all the correlation coefficients between the pairs of variables have been calculated, the correlation matrix is built. This is a symmetrical matrix containing all the correlation coefficients.

## 4.2 Stage II: Optimization of the Portfolio without Real Options

First, the market value of each project (PV) is calculated based on the structure of the cash flow (CF) for that project. The cash-flow structure described in Brealey, Myers & Allen (2011) can be used as a reference. The risk variables are included in the CF and have realizations that depend on the model adopted and the correlations with the other variables.

Let the horizon of project $j$ have $\tau_j$ periods, $t$ = 0,1,..., $\tau_j$, with one CF for each $t$. The PV is obtained by adding the CFs from each simulation duly discounted by the estimated cost of capital for the project ($\mu_j$). Hence, the market value of project $j$ in a given simulation $i$ = 1,..., N expressed continuously is given by Eq. 8.

$$PV_{ij} = \int_0^{\tau_j} e^{-\mu_j t} CF_{ij}(t)dt \tag{8}$$

where $CF_{ij}(t)$ is the value of the cash flow of project $j$ in simulation $i$ in periods $t$ = 0,1,...,$\tau_j$. Once N simulations have been performed, a distribution of PVs can be obtained for each project $j$. The net present value (NPV) of project $j$ for simulation $i$ ($NPV_{ij}$) is calculated from the PV, as shown in Eq. 9.

$$NPV_{ij} = PV_{ij} - I_j \tag{9}$$

where $I_j$ is the initial investment in period $t$ = 0.

The proposed optimization model, which was introduced in Section 2, is based on the optimization models described by Modarres and Hassanzadeh (2009) and Sefair and Medaglia (2005). We change the objective function to the performance measure Omega, as in Favre-Bulle & Pache (2003), who use this measure to optimize a hedge fund portfolio.

Let $P$ be the portfolio of projects, and L the minimum acceptable NPV for the investors to invest in $P$. The objective function is given by

$$\max_P \ \Omega(L) = \frac{EC_P(L)}{EL_P(L)} \tag{10}$$

where

$EC_P(L) = E\left[\max(NPV_P - L;0)\right]$ is the expected chance for portfolio $P$, and

$EL_P(L) = E\left[\max(L - NPV_P;0)\right]$ is the expected loss for the portfolio.

The $NPV_P$ in a given simulation $i$ ($NPV_{P,i}$) is the sum of the NPVs of the projects in the portfolio (J projects), as defined in Eq. 11.

$$NPV_{P,i} = \sum_{j=1}^{J} \sum_{t'=t-}^{t+} w_{jt'} \times NPV_{ij0} \tag{11}$$

The variable $w_{jt'}$ is binary and is equal to 1 when project $j$ starts at a given time $t'$ within the interval [$t^-$, $t^+$], where $t^-$ is the earliest period in which the project can be started, and $t^+$ the last period the investment can be put off until. Both $t^-$ and $t^+$ should be specified beforehand for each project. When the project does not start in period $t'$, $w_{jt'} = 0$. The following constraint therefore applies:

$$\sum_{t'=t-}^{t+} w_{jt'} \leq 1 \tag{12}$$

In Eq. 11, $NPV_{ij0}$ is the NPV of project $j$ in simulation $i$ and period $t$=0 and is given by

$$NPV_{ij0} = e^{-r_f t'} NPV_{ij} \tag{13}$$

where $NPV_{ij}$ is given by Eq. 9 and $r_f$ is the risk-free rate. Note that the $NPV_{ij}$ is discounted $t'$ times at the risk-free rate ($r_f$) to bring it to $t$=0. Between $t$=0 and $t'$ the project has not yet started and does not have the same level of risk ($\mu_j$) as when it is underway. Another option would be to discount this waiting time by an opportunity cost the company would incur by not starting the project. Here we chose to use the risk-free rate, which is what the investor would earn by investing his money in a risk-free investment.

After $N$ simulations, the distribution of the NPV for each project $j$ and the distribution of $NPV_P$, the NPV of the portfolio, is obtained. The expected value of this distribution, $E[NPV_P]$, is the mean of the distribution of $NPV_P$. The resulting covariances and correlations between the NPVs of the projects can be calculated as in Eqs. 6 and 7 except that instead of using the risk variables, the distributions of the NPVs of the projects are used. Doing this is particularly useful for determining the variance of the portfolio, as shown in Eq. 14.

$$Var_P = \sum_{j=1}^{J} \sum_{j'=1}^{J} Cov(NPV_j, NPV_{j'}) w_{jt'} w_{j't'} \tag{14}$$

When $j \neq j'$ the equation gives the covariance between two assets, and when $j = j'$ it gives the variance of one asset. The Markowitz mean-variance methodology attempts to minimize $Var_P$.

In short, in this stage, the optimization program determines the values of the coefficients $w_{jt'}$, which indicate the period in which each project should be started.

## 4.3 Stage III: Optimization of the Portfolio with Real Options

***<u>First Step:</u> Determining the Market Value of each Project and its Volatility***

Based on the MAD assumption (Copeland & Antikarov, 2003), Brandão, Dyer, and Hahn (2005b) estimate the expected market value of a project ($\overline{PV}_{t'}$) according to Eq. 15.

$$\overline{PV}_{t'} = \sum_{t=t'}^{t'+\tau} \frac{E[CF_t]}{(1+\mu)^{t-t'}} \tag{15}$$

where $E\left[CF_t\right]$ is the expected value of the cash flow in period $t = t', t'+1, ..., t'+\tau$ discounted by the risk-adjusted rate ($\mu$), and $t'$ is the period when the project starts. Using Eq. 15 and the distribution of $PV_t$ (assumed to be lognormal), Smith (2005) and Brandão, Dyer, and Hahn (2005a) estimate the volatility ($\sigma$) of the market value of the project as the standard deviation of the return between the initial period and the subsequent period. We suggest that this procedure be adopted.

The mean initial value of the project, $\overline{PV}_{t'}$ and its volatility, $\sigma$, are the parameters needed to model the path of the market value of the project as GBM.

***<u>Second Step:</u> Determining the Correlation between Projects***

With the distributions of the market values of the projects (Eq. 8), the correlations between the outputs of the projects within the context of the portfolio can be calculated as shown in Eq. 7 for the risk variables. However, instead of using historical data for the RVs, the correlation is calculated with the simulated PVs of the projects.

***<u>Third Step:</u> Determining the Market Value of the Projects with Real Options***

Once $\overline{PV}_{t'}$ and $\sigma$ have been calculated for each project, these can be modeled as negotiable (risk-neutral) assets obeying GBM, as shown in Eq. 16.

$$PV_{j,t'+\Delta t} = \overline{PV}_{jt'} \exp^{\left[\left(\varphi_j - \sigma_j^2/2\right)\Delta t + \sigma_j \sqrt{\Delta t} N(0,1)\right]} \tag{16}$$

where $\overline{PV}_{j,t'+\Delta t}$ is the market value of project $j$ simulated in period $t'+\Delta t$, $\varphi_j = r_f - \delta_j$ is the drift or risk-neutral trend ($r_f$ is the risk-free rate and $\delta_j$ the dividend rate), $\sigma_j$ is the volatility of project $j$ and $N(0,1)$ is an i.i.d. normal distribution.

The simulations start at $t = t'$, the period when the project should be started, with the market value of the project in that period, $\overline{PV}_{jt'}$, and a path of values is generated until $t = t'+\tau_j$ ($\tau_j$ is the projected lifetime of the project). The real options are inserted along the paths simulated by Eq. 16 and evaluated according to the type of option.

Let $Op(PV_{jt'})$ be the function formed by the values in the set of real options each time that a path for the market value of project $j$ started at $t'$ ($\overline{PV}_{jt'}$) is simulated. Then its mean value is given by $\overline{RO}_{jt'}=E\left[Op(PV_{jt'})\right]$. Hence, $\overline{PV}^{+}_{jt'}$, the market value of project $j$ (started at $t'$) including the real options, can be calculated according to Eq. 17.

$$\overline{PV}^{+}_{jt'} = \overline{PV}_{jt'} + \overline{RO}_{jt'} \tag{17}$$

The minimum value of $\overline{PV}^{+}_{jt'}$ is $\overline{PV}_{jt'}$, when the real options have no value.

In this way, the market value of each project in the portfolio with options is calculated. The simulations are always done together using the matrix of correlations between projects.

***<u>Fourth step:</u> Determining the Net Present Value (NPV)***

Let $NPV^{+}_{ij0}$ be the NPV of project $j$ in simulation $i$ ($i$=1,..., N) of the path of the project's market value when the value of the real options is included. $NPV^{+}_{ij0}$ is given by

$$NPV^{+}_{ij0} = NPV_{ij0} + e^{-r_f t'} Op_i(PV_{jt'}) \tag{18}$$

where $NPV_{ij0}$ is defined in Eq. 13, and $Op_i(PV_{jt'})$ is the value of the real options in simulation $i$ of project $j$ started at $t'$ discounted at the risk-free rate to period $t$=0. After N simulations, a distribution of $NPV^{+}_{j0}$ is obtained for each project, and the expected value is given by

$$E\left[NPV^{+}_{j0}\right] = \overline{NPV}^{+}_{j0} = e^{r_f t'}(\overline{PV}^{+}_{jt'} - I_j) \tag{19}$$

where the initial investment ($I_j$) is subtracted from $\overline{PV}^{+}_{jt'}$, which is defined according to Eq. 17, and the result is adjusted to period $t$=0 using $r_f$. The expected value of the NPV of portfolio P with real options is the sum of the various $\overline{PV}^{+}_{jt'}$'s, as shown in Eq. 20.

$$E\left[NPV_P^{+}\right] = \sum_{j=1}^{J} \overline{NPV}^{+}_{j0} \qquad (20)$$

***<u>Fifth step:</u> The Portfolio Optimization Model with Real Options***

Let P be the portfolio of projects and L the minimum acceptable NPV for the investors to invest in the portfolio. Let the objective function be defined as

$$\max_P \ \Omega(L) = \frac{EC_P(L)}{EL_P(L)} \qquad (21)$$

where

$EC_P(L) = E\left[\max(NPV_P^{+} - L;0)\right]$ is the expected chance for the portfolio $P$ with real options,

$EL_P(L) = E\left[\max(L - NPV_P^{+};0)\right]$ is the expected loss for the portfolio $P$ with real options

and $NPV_P^{+}$ is the distribution function of the NPV of portfolio $P$ with real options. This function consists of N results from the simulation of the paths of the market values of the J projects (Eq. 16) with the real options included.

The $NPV_P^{+}$ in a given simulation $i$ ($NPV_{P,i}^{+}$) is the sum of the $NPV_{ij0}^{+}$'s (Eq. 18) of the J projects in portfolio P, as shown in Eq. 22.

$$NPV_{P,\,i}^{+} = \sum_{j=1}^{J} NPV_{ij0}^{+} \times v_j \qquad (22)$$

If $v_j$=1 project $j$ should be included; if $v_j$=0, it should not. The binary constraint therefore becomes

$$v_j = \{0 \ ou \ 1\}, \text{ for } i = 1,2,...,J \qquad (23)$$

The number of projects in the portfolio can be controlled by defining a minimum number ($N_{min}$) and a maximum number ($N_{max}$) that can be accepted. This constraint can be expressed as

$$N_{\min} \leq \sum_{j \in P} v_j \leq N_{\max} \ . \tag{24}$$

The portfolio may include projects that are mandatory, mutually associated or mutually exclusive. A mandatory project is one that has to be implemented for strategic reasons. For each mandatory project *j*, a constraint of the type shown in Eq. 25 is added.

$$v_j = 1, \text{ for a mandatory project } j \in P \tag{25}$$

When projects are associated with each other, all of them or none of them are carried out. Let *P*(a) be the set of mutually associated projects *j* and $N_a$ the number of such projects. Then the following constraint can be added to the model:

$$\sum_{j \in P(a)} v_j = \{0 \ ou \ N_a\} \tag{26}$$

When projects are mutually exclusive, only one of them can be considered. Let *P*€ be the set of mutually exclusive projects *j*. Then the following constraint can be added to the model:

$$\sum_{j \in P(e)} v_j = \{0 \ ou \ 1\} \tag{27}$$

Depending on the portfolio being evaluated, other constraints may exist. These must be modeled so as to comply with the characteristics of the portfolio.

## 5. Numerical Application

Consider an oil company with three oil fields, F1, F2 and F3, and three refineries. Basic information on the projects is provided in Tables 1 and 2.

**Table 1.** Basic information on the oil field projects

| Description | Unit | F1 | F2 | F3 |
|---|---|---|---|---|
| Oil reserves | MM bbls | 90 | 120 | 50 |
| Initial Production Rate | % of the reserves | 10% | 15.0% | 12% |
| Production Rate of Reduction (Year 2 to 10) | % per year | 15% | 13% | 17% |
| Variable Operating Cost (VOC) at t = 0 | US$ / bbl | 10 | 11 | 9 |
| Oil Sale Price (OP) at t = 0 | US$ / bbl | 25 | 24 | 26 |
| Fixed Costs | US$ MM /year | 5 | 7 | 5 |
| Profit Participation | % per year | 25% | 25% | 25% |
| Investment | US$MM | 250 | 500 | 135 |
| Maximum Time to Start the Project | years | 2 | 2 | 2 |
| Project Lifetime (τ) | year | 10 | 10 | 10 |

**Table 2.** Basic information on the refinery projects

| Description | Unit | R1 | R2 | R3 |
|---|---|---|---|---|
| Installed Capacity (IC) | MM bbls | 15 | 17 | 10 |
| Initial Operation Rate | % of the IC | 75% | 70% | 70% |
| Production Increase Rate (Year 2 to 5) | % per year | 6% | 5% | 5% |
| Production Rate of Reduction (Year 9 to 12) | % per year | 15% | 15% | 12% |
| Price of Brent Oil (PB) Price at t = 0 | US$ / bbl | 24 | 24 | 24 |
| Variable Operating Cost | % of PB | 115% | 116% | 113% |
| Mean Sale Price of the Petroleum Product (PP) at t = 0 | US$ / bbl | 28 | 29 | 28 |
| Fixed Costs | US$ MM/year | 3 | 4 | 2 |
| Profit Participation | % per year | 25% | 25% | 25% |
| Investment | US$MM | 150 | 160 | 80 |
| Maximum Time to Start the Project | years | 2 | 2 | 2 |
| Project lifetime ($\tau$) | years | 12 | 12 | 12 |

## Stage I: Information Modeling

*First Step: Identification of the Project Risk Variables*

There are two risk variables (*RV*) in the oil field projects: the variable operating cost (VOC) and the oil sale price (OP). The risk variables for the refinery projects are the price of Brent oil (PB) (the internationally negotiated price that is the basis for calculating the variable operating cost of the refineries) and the mean sale price of the petroleum product (PP).

*Second Step: Modeling the Risk Variables*

The risk variables (*RV*) follow GBM, the parameters of which are specified in Table 3.

**Table 3.** Parameters used to model the GBM of the risk variables (*RV*)

| RV Oil Fields | Parameters | F1 | F2 | F3 | RV Refineries | Parameters | R1 | R2 | R3 |
|---|---|---|---|---|---|---|---|---|---|
| Variable Operating Cost (VOC) | Drift ($\alpha_c$) | 1.98% | 1.98% | 1.98% | Price of Brent Oil (PB) | Drift ($\alpha_b$) | 2.76% | 2.76% | 2.76% |
| | Volatility ($\sigma_c$) | 10.00% | 10.00% | 10.00% | | Volatility ($\sigma_b$) | 14.00% | 14.00% | 14.00% |
| Oil Sale Price (OP) | Drift ($\alpha_{op}$) | 2.96% | 3.73% | 3.25% | Mean Sale Price of the Petroleum Product (PP) | Drift ($\alpha_{pp}$) | 3.92% | 4.02% | 4.11% |
| | Volatility ($\sigma_{op}$) | 15.00% | 12.00% | 13.00% | | Volatility ($\sigma_{pp}$) | 19.00% | 19.00% | 20.00% |

*Third Step: Determining the Correlations between the Project Risk Variables*

Using Eq. 7, we calculate the correlation coefficients shown in Table 4. OP-Fi (*i*=1,2,3) denotes the oil sale price at the oil field (F) “*i*”, and PP-Ri (*i*=1,2,3) denotes the mean sale price of the petroleum product at the refinery (R) “*i*”.

**Table 4.** Correlation matrix for the risk variables (*RV*)

| | VOC | OP-F1 | OP-F2 | OP-F3 | PB | PP-R1 | PP-R2 | PP-R3 |
|---|---|---|---|---|---|---|---|---|
| VOC | 1 | 0.5 | 0.5 | 0.5 | 0.45 | 0.1 | 0.1 | 0.1 |
| OP-F1 | 0.5 | 1 | 0.8 | 0.9 | 0.9 | 0.3 | 0.2 | 0.3 |
| OP-F2 | 0.5 | 0.8 | 1 | 0.7 | 0.85 | 0.2 | 0.15 | 0.25 |
| OP-F3 | 0.5 | 0.9 | 0.7 | 1 | 0.9 | 0.3 | 0.2 | 0.25 |
| PB | 0.45 | 0.9 | 0.85 | 0.9 | 1 | 0.3 | 0.3 | 0.3 |
| PP-R1 | 0.1 | 0.3 | 0.2 | 0.3 | 0.3 | 1 | 0.7 | 0.6 |
| PP-R2 | 0.1 | 0.2 | 0.15 | 0.2 | 0.3 | 0.7 | 1 | 0.8 |
| PP-R3 | 0.1 | 0.3 | 0.25 | 0.25 | 0.3 | 0.6 | 0.8 | 1 |

## Stage II: Optimization of the Portfolio without Real Options

For the oil field projects, the cost of capital, μ, is assumed to be 10% pa, and for the refinery projects, 9% pa. The risk-free rate ($r_f$) is 5% pa. Using the data from Stage I, the expected cash flow for the project is set up and the risk variables are simulated using GBM (with their correlations). This gives the E[PV] and E[NPV] for each project for a given start year (no later than year 2, as shown in Tables 1 and 2).

By way of illustration, Table 5 shows the cash flows for F1 starting at *t*=0.

**Table 5.** Expected Cash Flows for Project F1 (in million USD)

| Period (year) | t = 0 | t = 1 | t = 2 | t = 3 | t = 4 | t = 5 | t = 6 | t = 7 | t = 8 | t = 9 | t = 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (a) Remaining Reserves (MM bbls) | | 90.00 | 81.00 | 73.35 | 66.85 | 61.32 | 56.62 | 52.63 | 49.23 | 46.35 | 43.90 |
| (b) Production Level (MM bbls) * | | 9.00 | 7.65 | 6.50 | 5.53 | 4.70 | 3.99 | 3.39 | 2.89 | 2.45 | 2.08 |
| (c) Variable Operating Cost (US $ / bbl) | 10.00 | 10.20 | 10.40 | 10.61 | 10.82 | 11.04 | 11.26 | 11.49 | 11.72 | 11.95 | 12.19 |
| (d) Oil Price (US $ / bbl) | 25.00 | 25.75 | 26.52 | 27.32 | 28.14 | 28.98 | 29.85 | 30.75 | 31.67 | 32.62 | 33.60 |
| (e) Revenue: (b) x (d) | | 231.75 | 202.90 | 177.64 | 155.52 | 136.16 | 119.21 | 104.37 | 91.37 | 80.00 | 70.04 |
| (f) Production Cost: (b)x(c) + 5 (Fixed Cost) | | 96.80 | 84.59 | 74.01 | 64.83 | 56.87 | 49.97 | 43.99 | 38.80 | 34.31 | 30.41 |
| (g) Operating Cash Flow: (e) - (f) | | 134.95 | 118.31 | 103.63 | 90.69 | 79.29 | 69.24 | 60.38 | 52.57 | 45.69 | 39.63 |
| (h) Profit Participation (25%) | | 33.74 | 29.58 | 25.91 | 22.67 | 19.82 | 17.31 | 15.09 | 13.14 | 11.42 | 9.91 |
| (i) Net Cash Flow (E[CF]): (g) - (h) | | **101.21** | **88.73** | **77.72** | **68.02** | **59.47** | **51.93** | **45.28** | **39.43** | **34.27** | **29.72** |
| (j) Present Value (E[PV]) $CF_t$ to $CF_{t=10}$ | **404.05** | **444.45** | **377.57** | **317.72** | **264.00** | **215.57** | **171.72** | **131.77** | **95.14** | **61.28** | **29.72** |
| (k) Rate E[CF] / E[PV]: (*i*) / (j) | | 0.23 | 0.24 | 0.24 | 0.26 | 0.28 | 0.30 | 0.34 | 0.41 | 0.56 | 1.00 |
| (l) Investments | 250.00 | | | | | | | | | | |
| **E[NPV] = E[PV] - I** | **154.05** | | | | | | | | | | |

* Production $\text{Level}_{t=1}$ = Initial Production Rate (10%) × $\text{Reserves}_{t=1}$. Between *t*=-2 and *t*=10, Production $\text{Level}_t$ = $\text{Reserves}_{t-1}$ × [1-Rate of Reduction (15%)].

Table 6 shows the E[PV] and E[NPV] of the projects. The nomenclature F$k$($t'$) and R$k$($t'$) is used to show that project F$k$ or R$k$ ($k$ = 1,2,3) starts in period $t'$ ($t'$ = 0,1,2). So that the different E[PV$_{t'}$]'s and E[NPV$_{t'}$]'s can be compared, these must all be in the same baseline period. These values are therefore discounted to period zero using the risk-free rate $r_f$. So, E[PV$_0$] = $(1+r_f)^{-t'}$E[PV$_{t'}$] and E[NPV$_0$] = $(1+r_f)^{-t'}$E[NPV$_{t'}$].

**Table 6.** Market Values (PV) and Net Present Values (NPV) of the Projects (in million USD)

| Project | I | E[PV$_{t'}$] | E[NPV$_{t'}$] | E[PV$_0$] | E[NPV$_0$] | Project | I | E[PV$_{t'}$] | E[NPV$_{t'}$] | E[PV$_0$] | E[NPV$_0$] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **F1(0)** | 250.00 | 404.05 | 154.05 | **404.05** | **154.05** | R1(0) | 150.00 | 161.88 | 11.88 | 161.88 | 11.88 |
| F1(1) | 262.50 | 419.54 | 157.04 | 399.56 | 149.56 | R1(1) | 157.50 | 194.95 | 37.45 | 185.66 | 35.66 |
| F1(2) | 275.63 | 435.56 | 159.93 | 395.06 | 145.06 | **R1(2)** | 165.38 | 230.06 | 64.68 | **208.67** | **58.67** |
| F2(0) | 500.00 | 808.65 | 308.65 | 808.65 | 308.65 | R2(0) | 160.00 | 224.69 | 64.69 | 224.69 | 64.69 |
| F2(1) | 525.00 | 851.94 | 326.94 | 811.37 | 311.37 | R2(1) | 168.00 | 262.71 | 94.71 | 250.20 | 90.20 |
| **F2(2)** | 551.25 | 897.09 | 345.84 | **813.69** | **313.69** | **R2(2)** | 176.40 | 303.08 | 126.68 | **274.90** | **114.90** |
| **F3(0)** | 135.00 | 282.76 | 147.76 | **282.76** | **147.76** | R3(0) | 80.00 | 139.32 | 59.32 | 139.32 | 59.32 |
| F3(1) | 141.75 | 294.82 | 153.07 | 280.78 | 145.78 | R3(1) | 84.00 | 164.01 | 80.01 | 156.20 | 76.20 |
| F3(2) | 148.84 | 307.31 | 158.47 | 278.74 | 143.74 | **R3(2)** | 88.20 | 190.25 | 102.05 | **172.56** | **92.56** |

*The largest E[PV$_0$] and E[NPV$_0$] for a given *t* for each project are in bold.

If the choice of the start time for a project were based exclusively on the largest E[NPV$_0$] for each $t'$= 0,1 and 2, there would be no need to optimize. However, analysis using the Omega measure is not based on the mean but on the complete distribution of the NPV$_0$'s of all the projects in portfolio *P*. The objective function in Eq. 10 is then optimized subject to the constraint in Eq. 12 and the stipulation that L=0, i.e., the investor does not want to make a loss by investing in this portfolio.

For comparison purposes, the portfolio was also optimized using the Markowitz mean-variance theory. In this case, the optimization program minimizes the variance of the portfolio (Eq. 14), and it is assumed that the NPV$_0$ of each project has a normal distribution. The results for both optimization models are summarized in Table 7.

**Table 7.** Results of optimization using the mean-variance and Omega methodologies with L=0

| Project | Start period | Mean-Variance | Omega (L=0) | Project | Start period | Mean-Variance | Omega (L=0) |
|---|---|---|---|---|---|---|---|
| F1 | $w_{10}$ | 0 | **1** | R1 | $w_{40}$ | **1** | **1** |
| | $w_{11}$ | 0 | 0 | | $w_{41}$ | 0 | 0 |
| | $w_{12}$ | **1** | 0 | | $w_{42}$ | 0 | 0 |
| F2 | $w_{20}$ | 0 | 0 | R2 | $w_{50}$ | **1** | **1** |
| | $w_{21}$ | 0 | 0 | | $w_{51}$ | 0 | 0 |
| | $w_{22}$ | **1** | **1** | | $w_{52}$ | 0 | 0 |
| F3 | $w_{30}$ | 0 | **1** | R3 | $w_{60}$ | **1** | **1** |
| | $w_{31}$ | 0 | 0 | | $w_{61}$ | 0 | 0 |
| | $w_{32}$ | **1** | 0 | | $w_{62}$ | 0 | 0 |

Mean-variance optimization:
$E[NPV_P] = E[NPV_{F1(2),0}] + E[NPV_{F2(2),0}] + E[NPV_{F3(2),0}] + E[NPV_{R1(0),0}] + E[NPV_{R2(0),0}] + E[NPV_{R3(0),0}] =$ **US$MM 738.4**
$\sqrt{Variance_P}$ = US$MM 1,926.29
Omega index (L=0) = EC/EL = 2.96

Omega optimization (L=0):
$E[NPV_P] = E[NPV_{F1(0),0}] + E[NPV_{F2(2),0}] + E[NPV_{F3(0),0}] + E[NPV_{R1(0),0}] + E[NPV_{R2(0),0}] + E[NPV_{R3(0),0}] =$ **US$MM 751,4**
$\sqrt{Variance_P}$ = US$ 1,934.21
Omega index (L=0) = EC/EL = 3.00

When $w_{jt'} = 1$, project $j$ should start in period $t'$. Using the mean-variance methodology, F1 would be started in period $t'$=2 ($w_{12}$=1), while with the Omega measure it would start in period $t$=0.

The distribution of the net present value of the portfolio, $NPV_P$, (Eq. 11) is made up of the sum of the distributions of the $NPV_0$'s of the projects. Hence, the mean of the distribution of the $NPV_P$ of the portfolio ($E[NPV_P]$) is the sum of the means of the $NPV_0$'s of the projects, as shown in the calculations at the end of Table 7. The mean-variance methodology finds the portfolio with the smallest variance, but its Omega index (EC/EL) is lower than that obtained when optimizing using Omega (L=0) (2.96 vs 3.00). The ratio of weighted returns (EC) to weighted shortfalls (EL) is always greater when optimization is performed with Omega. Only when the distributions are normal do the methodologies coincide.

### *Stage III: Optimization of the Portfolio with Real Options*

*First Step: Determining the Market Values of the Projects and their Volatility*

The market values of the projects ($E[PV_{t'}]$) were already calculated in the first stage of the methodology and are shown in Table 6.

The volatilities of the projects for their respective start times, i.e., F1(0), F2(2), F3(0), R1(0), R2(0) and R3(0), are shown in Table 8. These were obtained by simulating the cash flows of all the projects together to capture the effect of the correlation between the risk variables and applying the method described by Brandão et al. (2005a) (BDH method).

**Table 8.** Volatility of the projects, BDH method

| **Project** | F1(0) | F2(2) | F3(0) | R1(0) | R2(0) | R3(0) |
|---|---|---|---|---|---|---|
| **Volatilities** | 23.76% | 18.98% | 19.17% | 149.52% | 130.17% | 131.98% |

*Second Step: Determining the Correlation between Projects*

Applying Eq. 12 gives the market value $PV_{ij}$ for each project *j* in a given simulation *i*. A large number of simulations must be carried out to obtain a distribution of $PV_j$. The values obtained in the simulations are used to calculate the correlations between the projects (Table 9).

**Table 9.** Coefficients of correlation between the $PV_j$'s of the projects

| | $PV_{F1(0)}$ | $PV_{F2(2)}$ | $PV_{F3(0)}$ | $PV_{R1(0)}$ | $PV_{R2(0)}$ | $PV_{R3(0)}$ |
|---|---|---|---|---|---|---|
| $PV_{F1(0)}$ | 1.0000 | 0.7241 | 0.8644 | -0.2252 | -0.3250 | -0.2006 |
| $PV_{F2(2)}$ | 0.7241 | 1.0000 | 0.5942 | -0.3006 | -0.3421 | -0.2257 |
| $PV_{F3(0)}$ | 0.8644 | 0.5942 | 1.0000 | -0.1942 | -0.2956 | -0.2344 |
| $PV_{R1(0)}$ | -0.2252 | -0.3006 | -0.1942 | 1.0000 | 0.6778 | 0.5590 |
| $PV_{R2(0)}$ | -0.3250 | -0.3421 | -0.2956 | 0.6778 | 1.0000 | 0.7678 |
| $PV_{R3(0)}$ | -0.2006 | -0.2257 | -0.2344 | 0.5590 | 0.7678 | 1.0000 |

Note: the number between the parentheses indicates the period when the project was started.

*Third Step: Determining the Market Value of the Projects with Options*

Table 10 summarizes the market values of the projects and the initial investment, which, together with the volatilities in Table 8 and the correlations in Table 9, allow $\overline{PV}_{t'}$ to be modeled using GBM (Eq. 16). The risk-free rate ($r_f$) is 5% pa.

**Table 10.** Mean market value of the projects and initial investment (in million USD)

| Project (star period) | F1(*t'*=0) | F2(*t'*=2) | F3(*t'*=0) | R1(*t'*=0) | R2(*t'*=0) | R3(*t'*=0) |
|---|---|---|---|---|---|---|
| $\overline{PV}_{t'}$ | 404.05 | 897.09 | 282.76 | 161.88 | 224.69 | 139.32 |
| $I_{t'}$ | 250.00 | 551.25 | 135.00 | 150.00 | 160.00 | 80.00 |

We assume that in year 5 the company considers the possibility of exercising various options that could increase the value of the projects. These are shown in Table 11.

**Table 11.** Real options to be included in the projects in year 5

| | | Projects | | | | | |
|---|---|---|---|---|---|---|---|
| **Real Options** | **Parameters** | **F1(*t'*=0)** | **F2(*t'*=2)** | **F3(*t'*=0)** | **R1(*t'*=0)** | **R2(*t'*=0)** | **R3(*t'*=0)** |
| Option to expand | Expansion factor | 1.33 | 1.33 | 1.5 | 1.7 | 1.33 | 1.2 |
| | Cost to expand (US$MM) | 40 | 110 | 40 | 50 | 30 | 30 |
| Option to contract | Contraction factor | 0.75 | 0.75 | 0.75 | 0.8 | 0.75 | 0.5 |
| | Recovered value (US$MM) | 50 | 140 | 45 | 100 | 70 | 70 |
| Option to abandon | Salvage value (US$MM) | 100 | 350 | 110 | 120 | 90 | 80 |

These options are mutually exclusive, i.e., in year 5 the option that maximizes the value of the project that year will be exercised (or not). Risk-neutral simulations of the mean market value ($\overline{PV}_{t'}$) are performed for each project to allow the options to be modeled. Table 12 shows how the options in the projects were evaluated using project F1 as an example, in a given simulation (in total there were 10,000 simulations using the software @Risk®).

**Table 12.** Simulation of market value paths with options built-in for project F1 (in million USD)

| Project | F1(*t'*=0) |
|---|---|
| $\overline{PV}_0$ | 404.05 |
| $r_f$ | 5.00% |
| $\sigma_{F1}$ | 23.09% |
| I | 250.00 |
| $\overline{NPV}_0$ = | 154.05 |

| Year | *t* = 0 | *t* = 1 | *t* = 2 | *t* = 3 | *t* = 4 | *t* = 5 | *t* = 6 | *t* = 7 | *t* = 8 | *t* = 9 | *t* = 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Dividend ($\delta_i$) | 0 % | 22.8% | 23.5% | 24.5% | 25.8% | 27.6% | 30.2% | 34.4% | 41.4% | 55.9% | 100% |
| $PV_t$ | 404.05 | 413.59 | 326.94 | 256.02 | 197.95 | **173.19** | 128.37 | 91.67 | 61.59 | 36.92 | 16.66 |
| $E[PV_t]$ | 404.05 | 424.76 | 344.85 | 277.34 | 220.23 | 171.87 | 130.84 | 95.96 | 66.21 | 40.76 | 18.89 |
| $E[CF_t]$ | 0.00 | 96.73 | 81.04 | 67.85 | 56.74 | 47.41 | 39.57 | 32.97 | 27.44 | 22.79 | 18.89 |

| | | |
|---|---|---|
| | 146.73 | $PV_5$ of Expand |
| | **173.19** | $PV_5$ of Contract (in this simulation this option got better, and so, it was the PV in this year) |
| | 141.49 | $PV_5$ of Abandon |
| | 150.42 | $PV_5$ of No option ($E[PV_5]$= 171.87) |
| * $\overline{PV}_5^+$ | 201.22 | |

| | |
|---|---|
| Option Value $\overline{RO}$ | 22.86 |
| $\overline{PV}_0^+ = \overline{PV}_0 + \overline{RO}$ | 426.91 |
| $\overline{NPV}_0^+ = \overline{PV}_0^+ - I$ | 176.91 |

$$\overline{RO} = \exp(-5r_f)\times\left(\overline{PV}_5^+ - E\left[PV_5\right]\right)$$

* this is the mean value of the project in year 5 ($\overline{PV}_5^+$) considering the possibility of choosing between three real options or not, after 10,000 simulations (risk-neutral simulation). Without options, this mean value is $E[PV_5]$ = 171.87

*Fourth step: Determining the Net Present Value (NPV)*

Table 13 summarizes the results for the six projects. As expected, the NPV of the portfolio without any options in year zero (751.39) is less than the NPV of the portfolio with options (1,114.24).

**Table 13.** $\overline{\text{PV}}$ and $\overline{\text{NPV}}$ of the projects with and without real options (in million USD)

| | Projects (start period) | F1($t'$=0) | F2($t'$=2) | F3($t'$=0) | R1($t'$=0) | R2($t'$=0) | R3($t'$=0) | Portfolio |
|---|---|---|---|---|---|---|---|---|
| WITH OPTIONS | $\overline{PV}_{t'}$ | 404.05 | 897.09 | 282.76 | 161.88 | 224.69 | 139.32 | |
| | $I_{t'}$ | 250.00 | 551.25 | 135.00 | 150.00 | 160.00 | 80.00 | |
| | $\overline{NPV}_{t'}$ | 154.05 | 345.84 | 147.76 | 11.88 | 64.69 | 59.32 | |
| | $\overline{NPV}_0$ | 154.05 | 313.69 | 147.76 | 11.88 | 64.69 | 59.32 | 751.39 |
| WITHOUT OPTIONS | Options value: $\overline{RO}_{t'}$ | 23.10 | 66.83 | 32.90 | 121.49 | 73.52 | 51.33 | |
| | $\overline{PV}^+_{t'} = \overline{PV}_{t'} + \overline{RO}_{t'}$ | 427.15 | 963.92 | 315.67 | 283.38 | 298.21 | 190.65 | |
| | $\overline{NPV}^+_{t'}$ | 177.15 | 412.67 | 180.67 | 133.38 | 138.21 | 110.65 | |
| | $\overline{NPV}^+_0$ | 177.15 | 374.31 | 180.67 | 133.38 | 138.21 | 110.65 | 1,114.36 |

The presence of real options always adds value to the projects and even changes initial expectations about them. For example, the smallest $\overline{NPV}_0$ of the oil field projects without the real options corresponds to project F3 (147.76), but when the real options are taken into account the figure for F3 increases to 180.67, which is larger than the $\overline{NPV}^+_0$ for F1. With the refinery projects the situation is similar: project R1 has the lowest $\overline{NPV}_0$ (11.88), but when the real options are included the lowest $\overline{NPV}^+_0$ corresponds to R3 (110.65).

*Fifth step: Optimization of the Portfolio with Real Options*

The distribution of the NPV of the portfolio *P* with real options ($NPV^+_P$) is the sum of the distributions of the $\overline{NPV}^+_0$'s of the projects. The distribution of the $NPV^+_P$ is then used to calculate the measures EL and EC so that the portfolio can be optimized with the Omega measure. In the particular example considered here, all the projects are included in the portfolio and optimization is not required (objective function, Eq. 21) as there are no constraints that require a project to be excluded from the project. The binary variable $v_j$ is therefore equal to 1 for every project *j* (the constraint shown in Eq. 23).

Figure 2 shows the distribution of the NPV of the portfolio with and without real options and the distribution of the real options.

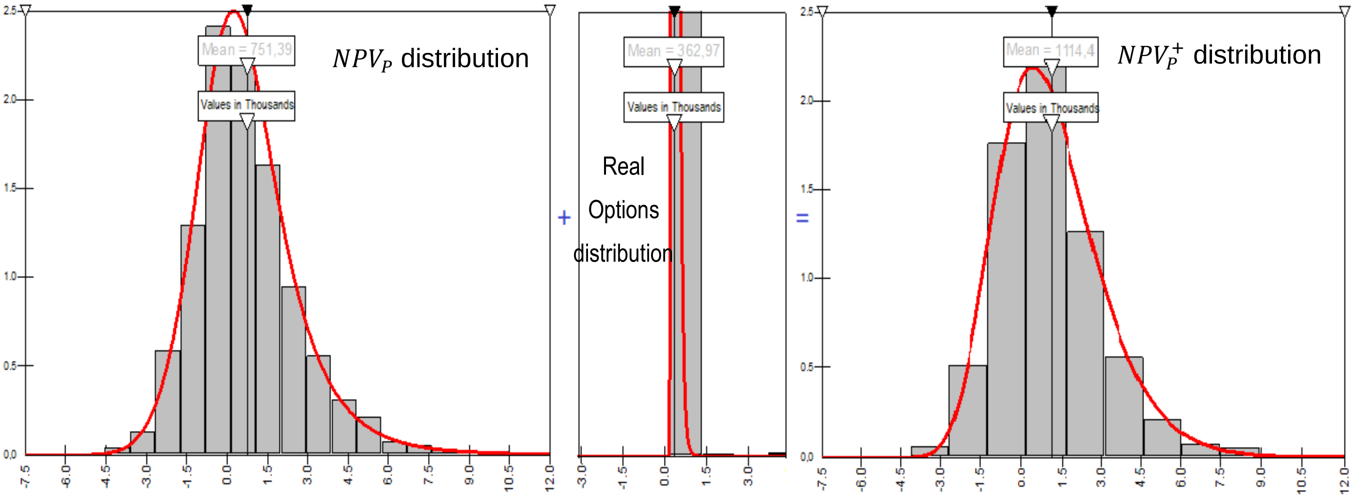


**Figure 2.** Distributions of the NPV of the portfolio without and with real options (in millions of USD)

Note that in Figure 2 the distribution of real options consists exclusively of positive values, increasing the mean of the distribution of the NPV from 751.39 to 1,114.36.

Figure 3 shows the expected loss (EL) and the log of the Omega measure against $L$, the minimum acceptable NPV, which divides the distribution into two areas corresponding to shortfalls and returns. The log of the Omega measure is used to make the vertical axis more manageable. As the value of $L$ increases, so the risk measure EL increases, since the area on the distribution of NPV corresponding to shortfalls increases and Omega therefore decreases.

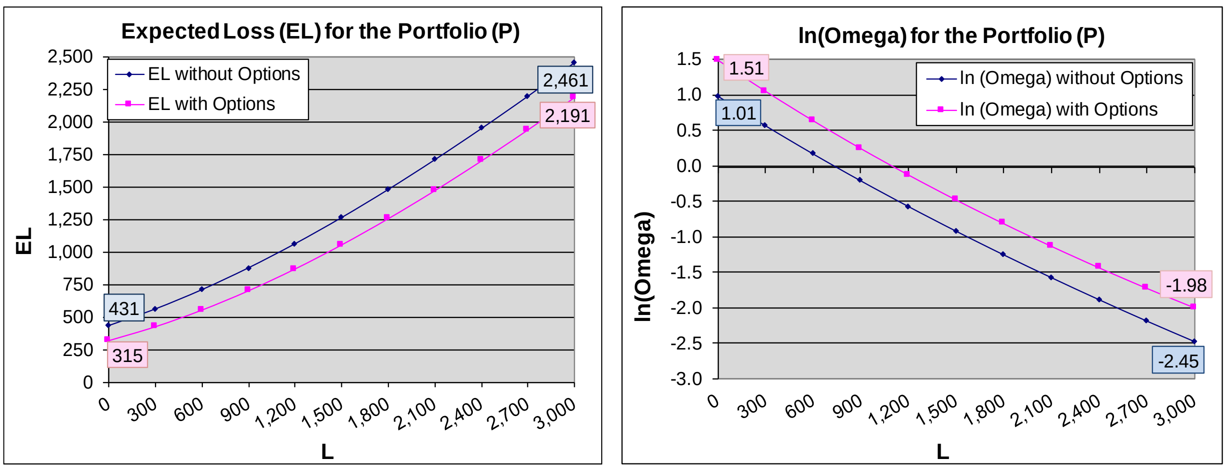


**Figure 3.** EL and ln(Omega) for the portfolio with and without real options.

According to Figure 3, for a given value of *L*, the portfolio with real options always has a lower risk (EL) and, therefore, a higher Omega. This clearly makes sense, as the distribution of the values of the real options for the portfolio contains only positive numbers, which, when added to the distribution of the $NPV_P$, reduce shortfalls in the negative scenarios and increase returns in the positive scenarios.

## 6. Conclusions

Correct analysis of the risk, returns, and performance of a portfolio of investment projects, or a portfolio of real assets, is of crucial importance in decision making. The more flexible the evaluation techniques and models used, the greater the company's ability to react to favorable or unfavorable circumstances.

The main aim of this study was to propose a methodology for optimizing a portfolio of investment projects using real options and the Omega risk measure. Notable among the main contributions of the proposed methodology are: (1) optimization by maximizing the Omega performance measure, which takes into account all the moments of the distribution of the NPV of the projects rather than just the mean and variance and (2) extension of the MAD model (Copeland & Antikarov, 2003) to a portfolio of various correlated projects.

The methodology was illustrated with a numerical application to the case of an oil company and European-type real options were included to increase or decrease the value of the project. The results show that the best ratio of expected returns to expected shortfalls was achieved with the optimization methodology proposed here. Other types of real options could also be analyzed, such as sequential options, simultaneous options and a switch in supplies. The more complex the real options for the projects in the portfolio, the greater the computational effort required.

The proposed methodology is flexible because it allows non-deterministic risk variables to be modeled while at the same time incorporating optimal exercising of available real options and is more robust because the Omega performance measure evaluates all the moments of the distribution of returns of the portfolio.

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