Volatility and Option Prices: Good, Bad, and Ugly[1]
(Preliminary Draft for RO2019 Review; Please Do NOT Distribute!)

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Abstract

We provide general results regarding the impact of volatility on option prices (i.e., *Vega*). Contrary to what is widely believed and taught, we show that the price of a large class of options is non-monotonic with respect to volatility: the value first increases but eventually *decreases* with the volatility. This seemingly counter-intuitive proposition is driven by a particular feature of Martingale processes bounded from below (including the Geometric Brownian Motion (GBM) and the CIR processes). We show that in such processes a higher variance parameter may *reduce* the probably mass of realizations above the expected value. When the volatility approaches infinity, the probability of hitting a barrier above the mean goes to zero. As a real-world case, we apply the results to managerial compensation in the presence of risk shifting. We show that risk-shifting can be mitigated by concavifying option-like components of managers' compensation scheme.

Keywords: Option Price, Vega, Option Compensation, Risk Shifitng

1. Introduction

Let us start the paper by a simple thought experiment: consider a fixed barrier point $K > 0$ and a bounded-from-below Markovian stochastic process (e.g., a Geometric Brownian Motion) starting at an initial point S_0, $S_0 < K$ at time $t = 0$. The setup, depicted in Figure 1, resembles the pay-off structure a digital (or binary) option, which can be observed in many financial and real options problems.[4]

Now, consider the following comparative statics question: how does the probability of hitting the upper barrier,B, at a given time $T > t -$ in other words, the probability of triggering the option

[1]Some parts of the current draft have been previously published in Ghoddusi and Fahim (2016). The draft is extending Ghoddusi and Fahim (2016) to a general setup.

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[4]The digital option pay-off structure is relevant whenever a fixed amount of positive/negative payment will be triggered once the underlying process passes a threshold. Examples of such pay-offs can be observed in problems such as penalty payment for exceeding a legal pollution level, debt restructuring cost, tax deductibility, and fixed prize winning. Conditional on passing the threshold, the size of the payment is independent of the level of the underlying process.

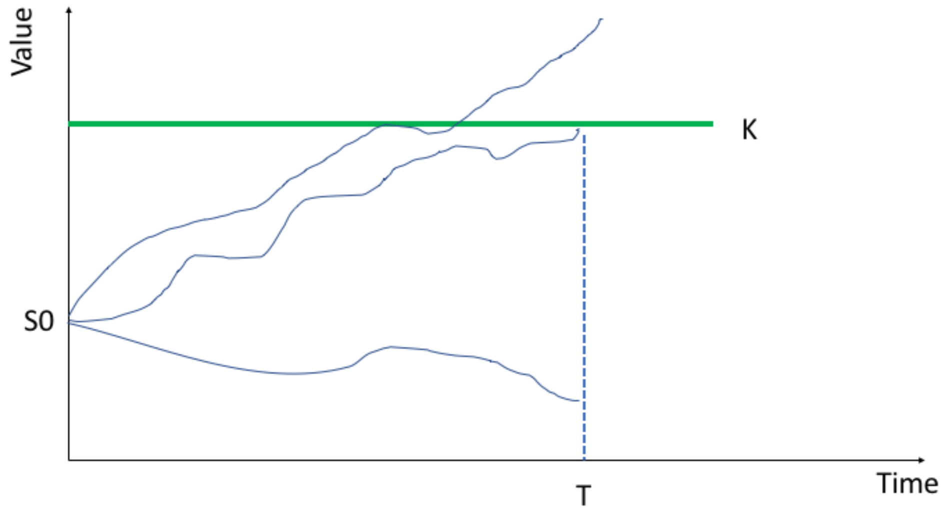

Figure 1: Hitting the Barrier K at time T

payment – change as we keep the mean of the GBM process constant and increase its volatility?

We have run this thought experiments in many occasions with rooms full of skilled financial economists, applied mathematicians, and statisticians. The almost-unanimous answer has always been: "of course, it goes up!". When we ask "why?", The intuitive answer is obvious: "because the process becomes somehow "wilder" and more likely to hit a barrier." Some financial economists also refer to the well-known result that the *Vega*of an option is a positive number.[5] We show that this universal intuition is *wrong!*Indeed, under a reasonably general setup, the probability of hitting an upper barrier *decreases* as the volatility of underlying process increases. In the limit, when the volatility is sufficiently large, the probability converges to zero!

One can see that this non-intuitive result has immediate implications in many domains of finance, including contingent claim pricing and dynamic investment decisions. For example, it is generally believed that an increase in the volatility of the underlying process will have a positive impact on option prices. The positive *Vega* is one of key summaries students remember from their mathematical finance courses. The Internet is also full of articles explaining "why option prices increase with volatility." Surprisingly, to the best of our knowledge, very little has been written on

[5]Vega is the derivative of the option price with respect to the underlying volatility.

the non-monotone relationship between volatility and option prices. The only paper that we are aware of is Ghoddusi and Fahim (2016) that suggest such a result for a simple binary option. We generalize their results to a much larger class of options by defining a metrics for the convexity of the pay-off. We demonstrate that for a large class of options – that are not sufficiently convex – the option price can be *decreasing* with volatility.

Another key contribution of our paper is to provide an algorithm to analyze complex option structure – which are neither a simple put or a simple call. In practice, many option-like payments can be approximated by a pay-off structure resembling a binary option. For example, as the pay-off of the mortgage interest deduction (MID), compensation packages of managers, mezzanine securities, venture capital, etc.

Real-World Application. To better demonstrate the practical relevance of our results, we apply the insights of the model to the optimal design of managerial compensation contracts. In the context of linear contracts with risk-neutral agents, volatility plays no role in shaping the manager's incentives. However, once one leaves the space of linear contracts, volatility will have an effect on the expected pay-off even if the manager is risk-neutral. For example, with a convex pay structure (e.g., a baseline payment plus options and bonuses), the manager will have incentives to increase the volatility of the cash-flow to maximize her expected payment.[6]

In summary, we offer the following contribution to the finance literature. First, we show that options *Vega* can be non-monotonic for a large class of options. We then provide asymptotic results for complex option structures. Finally, we apply the insights to the case of managerial compensation and show implications for the optimal contract design.

2. Basic Model

To provide the intuition behind the result, we first start discussing the result for a European digital option. A more general form of the results will be presented in the next section.

[6]Increasing cash-flow volatility can be achieved by reducing hedging activities, choosing riskier projects, increasing the leverage, and entering markets with higher volatility. It is typically assumed that volatility cannot be contracted upon.

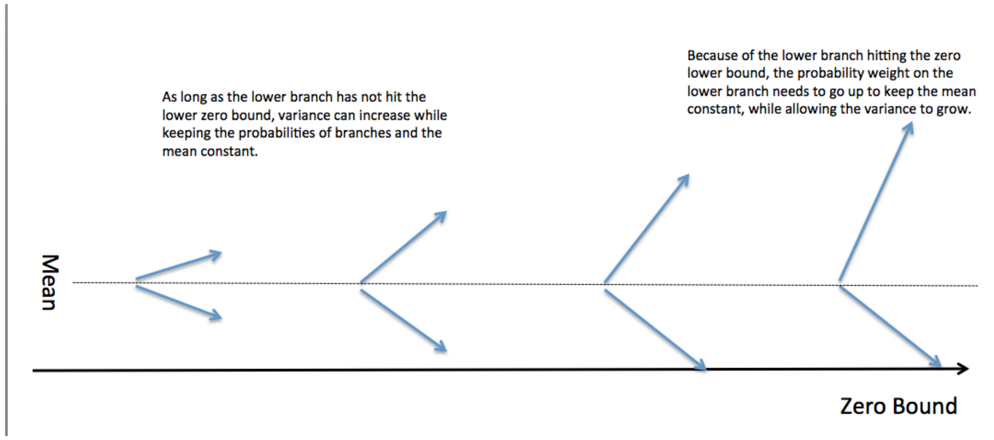

Figure 2: A schematic justification of non-monoticity on volatility for mean preserving variance enhancing shock applied to a binomial process.

2.1. Intuitive Description of the Problem

Consider a binomial random variable, which takes values and probabilities $(\overline{X}, P)$ and $(\underline{X}, 1-P)$. The process is bounded from the below; i.e. $\overline{X} > \underline{b}$. We fix the mean of the process and increase the variance. For small values of σ, increased variance widens the process symmetrically. However, once the lower branch hits the lower bound, the *probability* of the lower branch needs to increase.

Figure 2 shows the behavior of the binomial process.

To provide additional insights for continuous processes, Figure 3 shows samples of simulated GBM processes for four different levels of volatility. We note that when the volatility increases, more path are realized in the lower side.

2.2. Formal Proofs

We present the following result for the general distributions.

Theorem 2.1. *Assume X_μ is a stochastic process indexed with parameter $\mu \in [0, \infty)$ and bounded from below (i.e. mapping to a range $(\underline{b}, \infty)$), with a expected value bounded by $\overline{X}$ which does not depend on μ. Assume that*

1. *for any barrier $b(\mu)$ with $\lim_{\mu\to\infty} b(\mu) = \infty$, we have $\lim_{\mu\to\infty} E[X_\mu^2; X_\mu < b(\mu)] = \infty$ ($E[X_\mu^2; X_\mu < b(\mu)]$ is the second momentum of X_μ of the range $X_\mu < b(\mu)$),*
2. *$X_0 = \overline{X}$.*

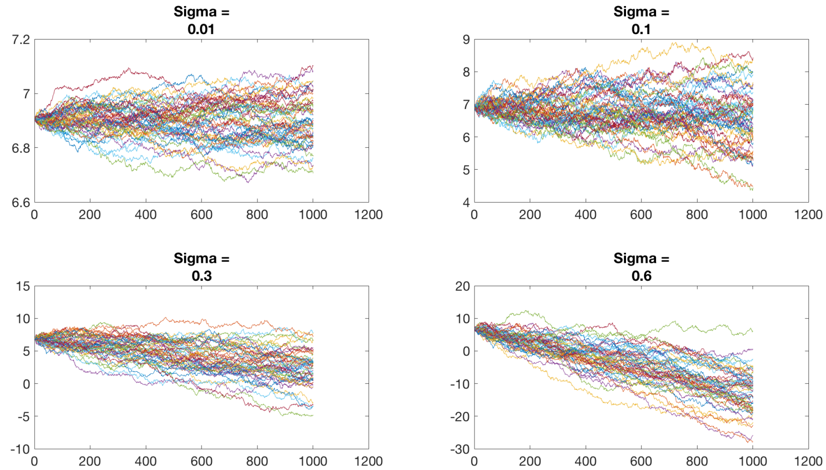

Figure 3: Realizations of GBM Process for Different Values of Volatility. When the volatility increases, more path are realized in the lower side.

Then, the probability of hitting a barrier $\bar{b} > \overline{X}$, *(i.e.* $P(X_t > \bar{b})$*) first increases and then decreases on* μ*.*

Heuristic proof. Without loss of generality we set $\underline{b} = 0$. We only present a proof in the case where X_μ is a non-negative discrete random variable which takes values $0 \leq y_M^\mu < ... < y_2^\mu < y_1^\mu < \overline{X} < x_1^\mu < x_2^\mu < ... < x_N^\mu$ with probabilities $q_M^\mu, ..., q_2^\mu, q_1^\mu, p_1^\mu, p_2^\mu, ..., p_N^\mu$, respectively for $\mu > 0$. In addition, condition (ii) implies that when $\mu = 0$, we have $y_M^0 = ... = y_2^0 = y_1^0 = x_1^0 = x_2^0 = ... = x_N^0 \leq \overline{X}$, therefore the probability of $X_\mu \geq \bar{b}$ is zero.

Notice that since $E[X_\mu^2] = \sum_i (x_i^\mu)^2 p_i^\mu$ diverges to infinity with μ but

$$E[X_\mu] = \overline{X} = \sum_i x_i^\mu p_i^\mu =: \overline{X} < \infty,$$

we must have $\lim_{\mu\to\infty} x_N^\mu = \infty$. Similarly, condition (ii) with $b(\mu) = x_N^\mu$ implies that $\lim_{\mu\to\infty} x_{N-1}^\mu = \infty$. Recursively, we conclude that $\lim_{\mu\to\infty} x_i^\mu = \infty$ for $i = 1, ..., N$.

On the other hand, since the mean of X_μ is $\overline{X} := \sum_i x_i^\mu p_i^\mu$ remains bounded, we have $\alpha_N := \lim_{\mu\to 1} x_N^\mu p_N^\mu < \infty$ and therefore $p_N^\mu \leq \frac{\alpha_N}{x_N^\mu}$. Therefore, the probability of $X_\mu \geq \bar{b}$ becomes positive as soon an $x_N^\mu > \bar{b}$ and as $\mu \to \infty$, it converges to 0. **End of proof**

Proposition 2.1. *Assume* X_t *is a Martingale process bounded from below (i.e. mapping to a range*

$(\underline{b}, \infty)$), *with a expected value* $\overline{X}$, *and*

(i) *for any barrier* $b(t)$ *with* $\lim_{t\to\infty} b(t) = \infty$, *we have* $\lim_{t\to\infty} E[X_t^2; X_t < b(t)] = \infty$,

(ii) $X_0 = \overline{X}$.

Then, for $\bar{b} > \overline{X}$, *the* $P(X_t > \bar{b})$ *first increases and then decreases over time. In particular, when* $t \to \infty$ *then* $P(X_t > \bar{b}) \to 0$.

2.3. Distribution of GBM Process Paths

A GBM process is the limit of a binomial process. Consider a binomial random variable, whose lower branch is bounded by zero. The underlying process is assumed to follow a Geometric Brownian Motion (GBM) process.

$$\frac{dX}{X} = \mu dt + \sigma dW, \quad X(0) = \overline{X}, \tag{2.1}$$

where μ is the drift and σ is the volatility parameters and dW is the standard Brownian shock. Working under the risk-neutral framework, μ is equal to risk-free interest rate r if the underlying asset does not pay a dividend. If the underlying asset pays dividends at rate q, then $\mu = r - q$. In all these cases, the GBM satisfies the properties listed in Proposition 2.1. One can also calculate the probability of hitting a barrier in closed form and study the monotonicity directly; which is done in the next Section.

2.4. European Digital Option

Given the fact the the future values of a GBM processes are distributed following a log-normal distribution we can derive a closed-form solution for the European option. The probability of hitting a barrier at the value of K, at time T by a GBM process starting at the initial value $X(0)$, drift μ, and volatility σ is given by $\mathcal{N}(D_+)$, where $\mathcal{N}$ the standard normal CDF and

$$D_+ = \frac{\log(\frac{X(0)}{K}) + \mu T}{v} + \frac{1}{2}v. \tag{2.2}$$

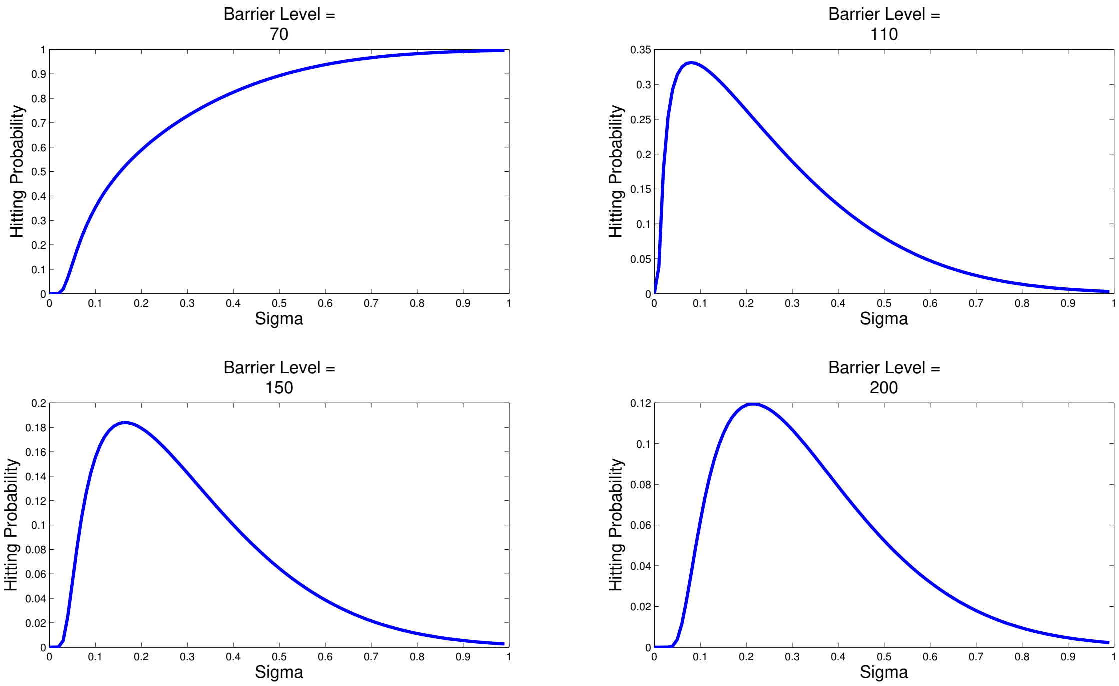

Figure 4: The probability of being above a barrier for a GBM Process for $X(0) = 100$ at a fixed time

The new variable $v := \sigma\sqrt{T}$ is the main variable in our study and represents the time-scaled volatility.

To analyze the effect of higher volatility and exercise time on the option value we take the first derivates of D with respect to σ. After a few algebraic steps:

$$\frac{\partial D_+}{\partial \sigma} = -\frac{\log(\frac{X(0)}{K}) + \mu T}{v^2} + \frac{1}{2}. \tag{2.3}$$

For $X(0) \leq Ke^{-\mu T}$, the sign of above derivative is always non-negative, and therefore the probability of hit is always increasing in v. Otherwise, for $X(0) > Ke^{-\mu T}$, the probability of hit is increasing on v in $[0, v^*]$ and decreasing in $[v^*, \infty)$ with $v^* := \sqrt{2(\log(\frac{X(0)}{K}) + \mu T)}$.

Figure 4 shows the sensitivity of probability of hitting four barriers at different values of σ. As we notice, if the barrier is below the starting value of the GBM process, the probability of hitting is monotonically increasing with σ. However, for all barrier above the starting value of the GBM, the hitting probability first increases and then decreases.

3. Extended Model

Let S be the underlying price following a Geometric Brownian Motion with drift μ, volatility σ and current value of S_0. Consider an European call option with maturity T and the final pay-off as $f(S_T)$, where we assume $f(.) \geq 0$, $f(x) = 0$ for $x \leq 0$ for simplicity. As in the Black-Scholes model, assume a constant interest rate r. Denote the price of this general call by $C_f(\sigma)$ which is a function of σ.

It is well known that the conventional European call with pay-off $f(S_T) = \max\{S_T - K, 0\}$ has a positive Vega which means the price is increasing with σ. The same is true for a put option. The price of a call starts from either 0 or $S - Ke^{-rT}$ (if $S - Ke^{-rT} > 0$) and increases to S when one changes σ from 0 to ∞. For a put option, price starts from either 0 or $Ke^{-rT} - S$ (if $Ke^{-rT} - S > 0$) and increases to Ke^{-rT}. Denote the price of such a European option by $C(\sigma, K)$ where we assumed r, T, μ, S_0 are constant. Note that the stock itself is a call option with strike 0 which means $C(\sigma, 0) = S_0$ is independent of σ. Another option structure that we need is a binary option which is given by pay-off function $f(x) = 1_{S_T > K}$. Let's denote the price of this option by $B(\sigma, K)$. Again, for $K = 0$ we have a fixed pay-off which resembles a bond and so $B(\sigma, 0) = e^{-rT}$ is independent of σ.

We first state a general result that for convex pay-off structures the positive Vega result holds.

Theorem 3.1. *If f is convex, then the price is always increasing with volatility.*

Proof. The price of any European claim can be written as $\int f(x)p(x)dx = \int f''(x)call(x)dx$ where $p(x)$ is the price of the Arrow security corresponding to event of $S_T = x$. Since call prices are increasing with volatility and f'' is positive when f is convex, the result follows. □

Another proof is based on piece-wise approximation:

Proof. A continuous payoff function is a limit of a piecewise linear continuous payoff which can be written as a summation of call and put options with the same maturity and different strikes. If the payoff is convex, this linear combination can be made with positive coefficients. Since the price of the call and put is increasing with volatility, the price of any linear combination of calls and puts with positive coefficients is so. In the limit, we obtain the result. □

Now, we state a result regarding a general pay-off form. First, note that any general pay-off function (we assume usual regularity assumptions) can be approximated by sum of calls. We can write $f(x) = \sum \alpha_i \max\{x - K_i, 0\}$ where we only focus on finite sums for simplicity and as a result the price for pay-off f is $C_f(\sigma) = \sum \alpha_i C(\sigma, K_i)$. Since we assumed $f(.) \geq 0$ we should have $\sum_i \alpha_i \geq 0$. When we change σ from 0 to ∞, $C_f(\sigma)$ starts from $\sum \alpha_i C(0, K_i) \geq 0$ and will tend to $\lim C_f = \sum \alpha_i S_0$. We are interested to investigate whether the maximum price by changing σ is attained for $\sigma < \infty$.

Note that

$$\frac{\partial C_f(\sigma)}{\partial \sigma} = S_0 \sqrt{T} \sum_i \alpha_i N'(d_1(\sigma, K_i)),$$

and for large values of σ the terms N's become close to zero while their proportion become close to proportion of $\sqrt{K_i}$. Therefore, always Vega goes to 0 for large values of σ and the price converges to $\lim C_f = \sum \alpha_i S_0$. Thus, we have the following theorem:

Theorem 3.2. *Suppose f is a finite combination of calls with coefficients α_i, strike prices K_i as explained above. Then, the price is eventually increasing with σ if $M = \sum \alpha_i \sqrt{K_i} > 0$ and the price is eventually decreasing with σ if $M = \sum \alpha_i \sqrt{K_i} < 0$ for which the price goes to 0.*

Now we investigate simple yet important special cases to understand the optimum amount of volatility for the price.

Special Case 1:. Assume a concavified call option where $\alpha_1 = K_1 = 1$, $0 < \alpha_2 < 1$ (here we flip the sign of α_2) and $S_0 < e^{-rT}$. The price starts from 0 and is increasing with σ for $\alpha^* = 1/\sqrt{K_2} < \alpha_2$. For higher values of α_2 the price has a peak at $\sigma^*(\alpha_2, K_2)$ and tends to $(1 - \alpha_2)S_0$.

This result means that by concavifying the option, the problem of risk shifting can be mitigated. Now, can one force the optimal σ be any value by changing α_2 and K_2? Another relevant question is when one can be assured that the maximum price is achieved for a finite value of σ?

Theorem: If the price at infinity is bigger than the price at 0 and if the price is eventually decreasing the maximum should have happened for a finite value of σ.

Theorem 3.3. *For a binary option $\sigma^*(1, K) = \sqrt{2\ln(Ke^{-rT}/S_0)}$ and by changing K one can* *change σ^* form 0 to ∞ and since binary option is a special case of the concavified call, the same is true for the latter.*

So there are many solutions to the equation $\sigma^*(\alpha_2, K_2) = \bar{\sigma}$, where $\bar{\sigma}$ is a given desired volatility level for the firm. By choosing a proper concavified call and issuing a proper number of options for the manager, the firms can achieve a compensation value that will lead to the choice of the desired volatility by the manager.

4. Implications for Managerial Compensation [To Be Completed]

The mainstream literature on managerial compensation has mainly focused on the incentive for effort. However, managers can also shift higher moments of the firm's cash-flow through hedging, investment strategies, and operational decision. Manager's incentive to hedge their exposure to corporate cash-flow risk is known and has been studied by the previous literature (Acharya and Bisin (2009), Armstrong and Vashishtha (2012), Gao (2010), Cvitanić et al. (2014)) . However, the focus has mainly been on managers' risk-aversion and their tendency to reduce risk (e.g., Akron and Benninga (2013)). Much less has been written on the incentive for *shifting* risk to alter the option-like component of a manager's compensation package.

Smith and Stulz (1985) argue that by providing risk-taking incentives, options-based contracts counterbalance the incentives of risk-averse managers to excessively reduce the volatility. It is widely believed that when managers' wealth is more sensitive to stock volatility, they choose riskier actions. (Coles et al. (2006)). Laux (2014) shows that convex executive pay plans (e.g. option-based contracts) increases managers' incentive for information manipulation.

Using a risk-aversion based mechanism, Ross (2004) provides insights on why offering stocks to managers may not necessarily induce them to take more risk. The mechanism we propose is different because our focus is on the value of the option rather than the risk-aversion of the agent.

Dittmann et al. (2017) provide empirical evidence for the assumption that "shareholders take into account both effort and risk-taking incentives when designing the compensation contract"

Now consider a basic managerial compensation framework: the manager should exert effort so he should be given an option. Also, there is the cost of volatility for the firm so the volatility should

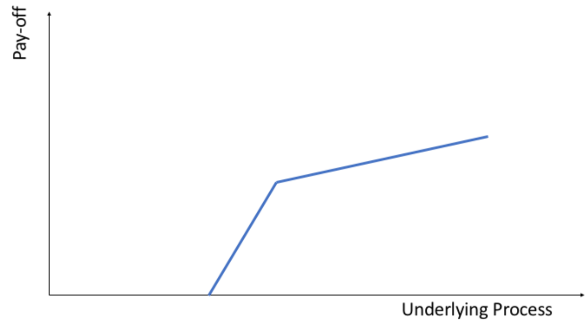

Figure 5: Three-Part Payment

not be blown up. These two imply that the optimum compensation is a concavified call.

4.1. A Three-Part Contract

As argued by Dittmann et al. (2017), the optimal managerial contract may consist of a flat payment for low performance, a convex part for medium performance, and a concave part for very high performance (see Figure 5).

Our results imply that that for such complex structure, higher volatility eventually reduces the expected pay-off of the manager, resulting in a risk-averse behavior.

4.2. Value-Maximizing Volatility

Will the manager increase the volatility in an unbounded fashion to maximize her compensation value? The answer to this question depends on the monotonicity of option value with respect to volatility.

To provide conjecture on this matter, Figure 6 shows plots of option values (with different strike levels) versus volatility. We observe that the value of sigma that maximizes the expected pay-off have a finite value.

Theorem 4.1. *The value of sigma that maximizes the expected pay-off of the manager is always a finite number.*

Proof. TBD □

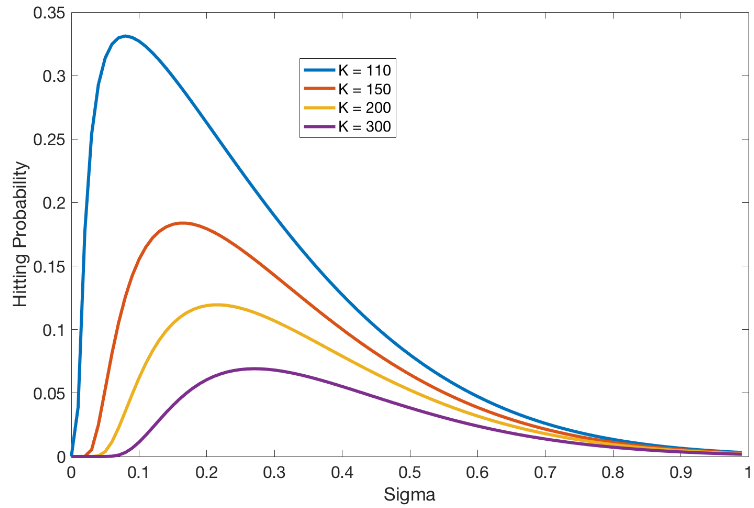

Figure 6: Value Maximizing Sigma

5. Conclusions

We show that the value of a large class of financial and real options has a non-monotonic relationship with the underlying volatility, and can eventually decrease with volatility.

There are numerous examples, in which the intuitions of our result apply. For example, Ghoddusi and Afkhami (2018) identifies option-like structures in the mortgage interest tax deductibility (MID) offered to households. When standard deductions are present, households will file for MID only if the sum of itemized expenses is larger than the standard deduction option. Ghoddusi and Afkhami (2018) show that higher volatility can *reduce* the value of MID for high-income households. As a new application, we show how the results can be applied to the optimal design of managerial compensation schemes.

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