Expanding the theory of the exercise boundary real options method into applications with real-world complexity

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1. Introduction

This extended abstract presents the work in progress devoted to expanding a recently introduced real option (RO) valuation method based on simulation and exercise boundary fitting (Bashiri, Davison, & Lawryshyn, 2018). The idea of the method is to fit the RO exercise boundary with the objective of maximizing the expected value. The method potentially allows to overcome limitations of the existing analytical and numerical RO approaches in the number of uncertainties and their modeling approach, a number of real option types, the number of time steps and decision points in the modeled time horizon. The utilization of the method has been proven to converge with theoretical analytic results and has been illustrated on several simplistic cases that involved a single source of uncertainty, two RO types, namely, to invest and to abandon, two-year time horizon and three decision points.

The aim of this research is to expand this model into a more realistic case in order to capture real-world complexity and to demonstrate how the method can overcome the limitations of other RO approaches. A copper mining investment is chosen as a case study. The following aspects of such an investment are captured:

- Realistic but limited time horizon of 20 years reflecting the licensing term;
- Stochastic development of the copper price that can be modeled as GBM, MRM or by any other customized process;
- Limited ore-body size;
- Three decision points over the time horizon;
- Investment timing RO, i.e. when to build;
- RO to choose the capacity size or how much to build;
- RO to stage investment (building one module after another up to maximum capacity);
- A second stochastic process that represents learning about the uncertain size of the ore-body.

The exercise boundary RO method optimizes the threshold price for every possible decision at every possible time, maximizing the expected value over the course of uncertainty development. This method enables the analysis of a case with multiple uncertainties and a complex decision tree. Current challenges and future research directions are discussed in the conclusions.

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2. The model

The general structure of the model is presented in Figure 1.

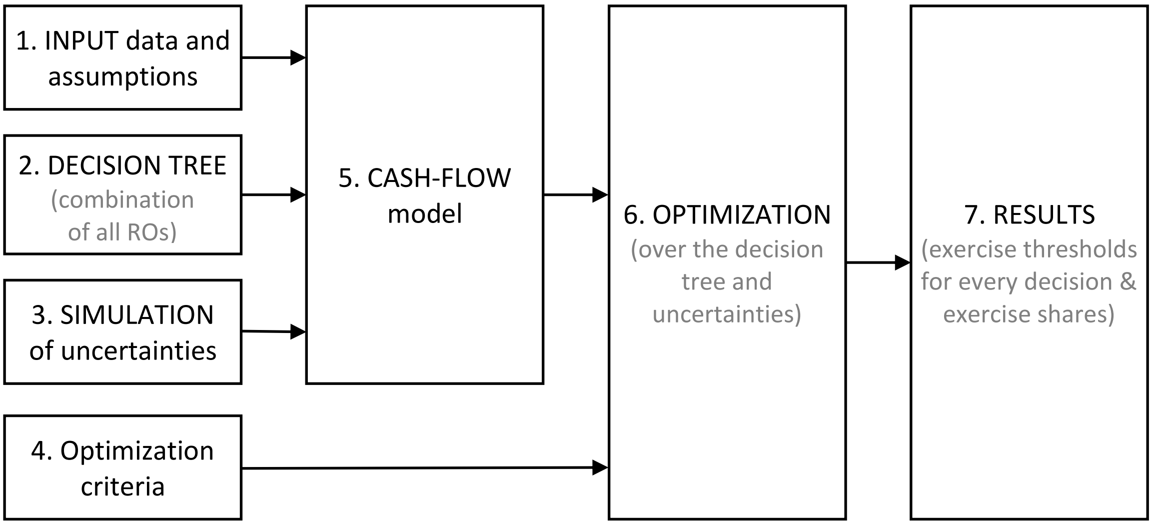

Figure 1. The schematic representation of the model

Essentially, the model requires numerical assumptions on some inputs (block 1), such as investment cost, operation and maintenance costs, the orebody etc. The second block refers to the structural input, here, practically, all possible real options are combined in a decision tree over a span of decision points. For example, if only an investment timing option is available, then at the first decision point there is a choice of either to invest or not, and at the second decision the same choice appears only from the 'not invested' choice of the first decision point. The larger the amount of ROs available, choices within them (e.g. capacity choice) and the number of decision points, the more complicated the tree becomes. For a given set of ROs, the construction of the decision tree can be automated.

In the third block, the Monte Carlo simulation of stochastic variables is implemented. If the problem were modeled using a lattice method, for example, as the number of possible future states increases, the number of lattices required to solve the problem must increase too. A similar issue occurs with LSTM and finite difference models. However, with the current Monte Carlo simulation approach, the computational complexity grows linearly with increasing complexity.

All three blocks are inputs into the cash flow model (block 5), which delivers a single expected value of the investment, which in essence is an average of NPV of one or another decision branch realized in each state of the future in accordance with simulated uncertainties. Which decision branch is realized at which uncertainty path depends on the expected value maximization (block 6). As a result, the model delivers (block 7) exercise thresholds for every decision making node of the decision tree and the corresponding exercise share, or how many paths out of the overall number of simulation runs, resulted in exercising a particular combination of ROs in a particular time. The exercise share helps to understand whether particular branches of the decision tree are relevant or not. In the case of two sources of uncertainty, the exercise threshold is represented by a line that defines a number of threshold combinations of two uncertainties for each decision node. In the case of three uncertainties, it is a surface etc.

3. Results

The model was developed in Matlab for the mining investment case study. The input values are estimated based on a recent copper project in Brazil (Barbosa, Mendonça, & Tomi, 2017) and presented in Table 1.

Table 1. Parameter estimates

Parameter	**Estimate**
License lifetime	20 years
Time to build	1 year
Capital expenses	300M USD/module
Operating expenses	1.5k USD/t
Extraction rate	22 t/year with one module
Orebody size	500kt
Initial copper price	138 cent/lb (3.0k USD/t)

Within this research two models have been developed. The first model starts with a single source of uncertainty, namely the copper price. The real options embedded include an investment timing RO, a capacity choice (1 or 2 modules) and a staging RO. Decisions are allowed to be at years 3, 6, and 9. Therefore the decision tree encapsulates 10 possible branches, Figure 2.

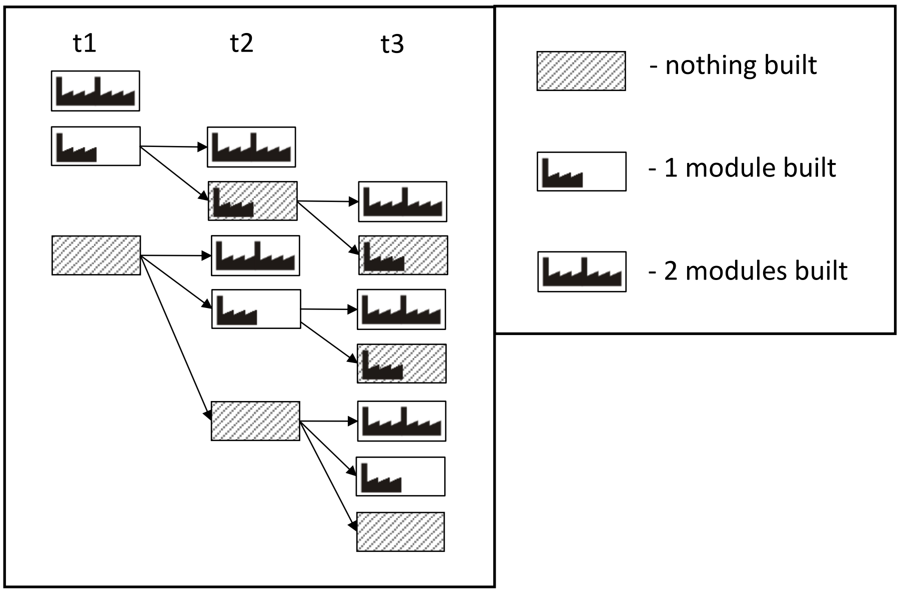

Figure 2. The decision tree used in the models

The model is robust due to providing consistent results through different simulations of the uncertain price with the same parameters. Conducted sensitivity analyses of the results to the input values and both parameters and the type of the price stochastic process provide interesting insights that are presented and discussed in the following subsection.

The second model aims to incorporate a second uncertainty, namely a learning-by-doing process with respect to the size of the ore-body. With the extraction in progress, more information is available regarding ore amount and quality, and therefore managers are able to optimize capacity accordingly. An important aspect of such an uncertainty source is that it is modeled not over time, but over the extracted amount, that depends on when and how many modules have been built. In this setting, at the first decision point (t1) the exercise boundary is represented only by a threshold copper price, whereas at the second decision point (t2) for the branch where 1 module has been built at t1, the exercise boundary is represented by a line in the 2D space of the price and the learning state. For transparency, the model is reduced to two decision points. The model is work in progress and requires further development. Some further details and preliminary results are presented in subsection 3.2.

3.1. Model 1

Models 1 represents a mining investment with inputs from Table 1 and the decision tree in Figure 2. While threshold prices depend heavily on the numerical assumptions, the exercise share displays which branches of the decision tree get exercised in a particular setting. Therefore, instead of deriving a crisp number saying e.g. that this is an investment threshold price for the decision whether to build one or two modules at the first decision point, we focus on the investment pattern that is based on the exercise share, and how it reacts to changes in the model setup. For this purpose, a sensitivity analysis to a number of parameters is conducted. In particular, we present how the investment pattern will change with different stochastic processes of the copper price (section 3.1.1), how changing parameters of a mean reverting process will affect the investment pattern (section 3.1.2), and finally, what is the effect of extraction rate (section 3.1.3).

3.1.1. Sensitivity to the stochastic process

GBM (Figure 3, left, on the top) and MRP (Figure 3, left, on the bottom) with the same initial price and volatility generate very different distributions after a 20-year development. On the right four decision trees are presented, on the top resulted from GBM, on the bottom – from MRP, left trees have the base-case investment cost (Table 1), on the right the investment cost is increased. The trees are of the same form as in Figure 2, but the nodes are color-coded in accordance with the exercise share obtained from the simulation with given inputs. To the nodes where nothing is built (hatched) no colors are applied to highlight only positive decisions. For example, the lower left tree is obtained from the simulation where copper prices follow MRP and investment cost is 600M USD for 2 modules. With these inputs the optimal investment pattern is to invest to two modules at once at the earliest. And this pattern was exercised on close to all 100% paths, the color of the corresponding node is dark blue. The overall value, that is the average NPV of all paths in this case was 585M USD.

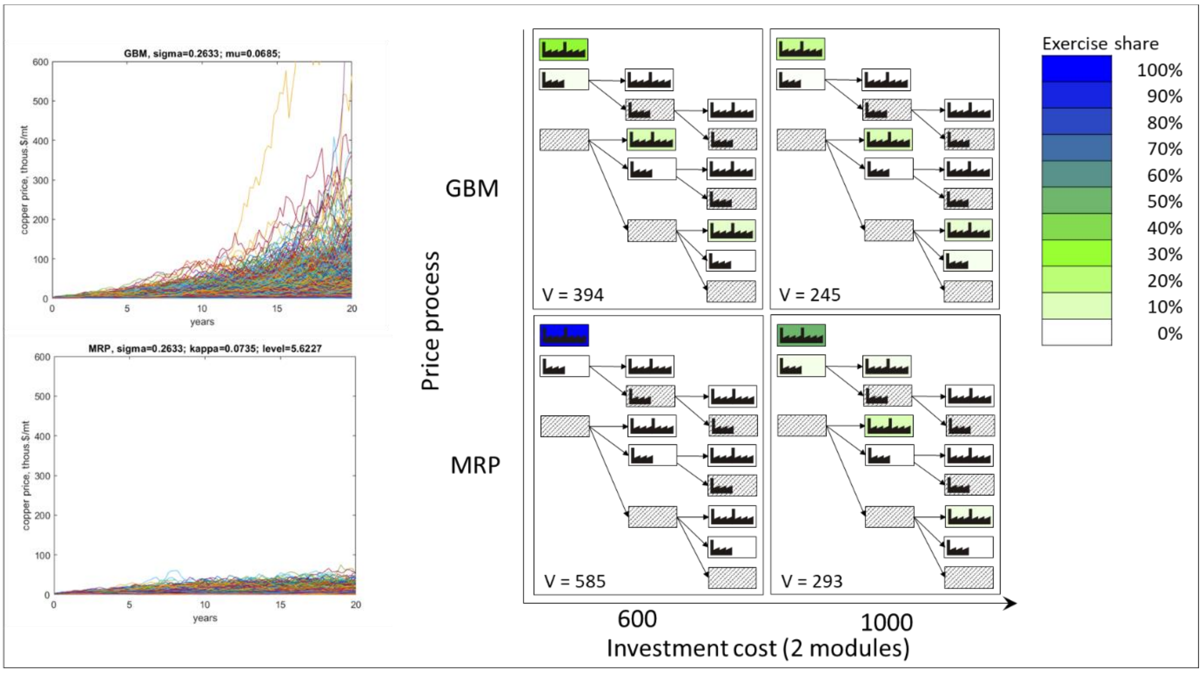

Figure 3. The sensitivity of the investment pattern to the price process

GBM produces an incentive for delaying investments, while MRP encourages to build two modules at once at the first decision point (Figure 3, two left decision trees), due to substantially higher expected price under GBM. Such a different investment pattern indicates the importance of choosing the stochastic process in real option studies. Apparently the most often choice of price process in studies of mining valuation is GBM, and the most popular real option studied is investment timing (Savolainen, 2016). Similar picture occurs in other industrial applications, e.g. in renewable energy studies (Kozlova, 2017).

Back to Figure 3, in case of increased investment cost, the investment becomes unprofitable more often and fewer paths get exercised, for the MRP also the delaying effect appears (Figure 3, two right decision trees). For all cases, whether the investment is delayed or not, the decision to build two modules at once prevails. No staging occurs.

3.1.2. Sensitivity to parameters of the mean reverting process

The investment pattern is not surprisingly sensitive to the parameters of a stochastic process. Here we adjust the mean reverting level, below or above the initial price and the volatility (Figure 4). Counterintuitively, the volatility does not affect the pattern itself, but the mean reverting level defines whether the investment tends to be delayed (MR level below the initial price) or not (MR level above the initial price). On the Figure 4, the four stochastic processes are presented, the upper charts represent the case when MR level is above the initial price, and the lower charts – below. The charts to the right are obtained from lower volatility, the charts to the right – higher. the decision trees displayed on the right of Figure 4 have the same dislocation logic.

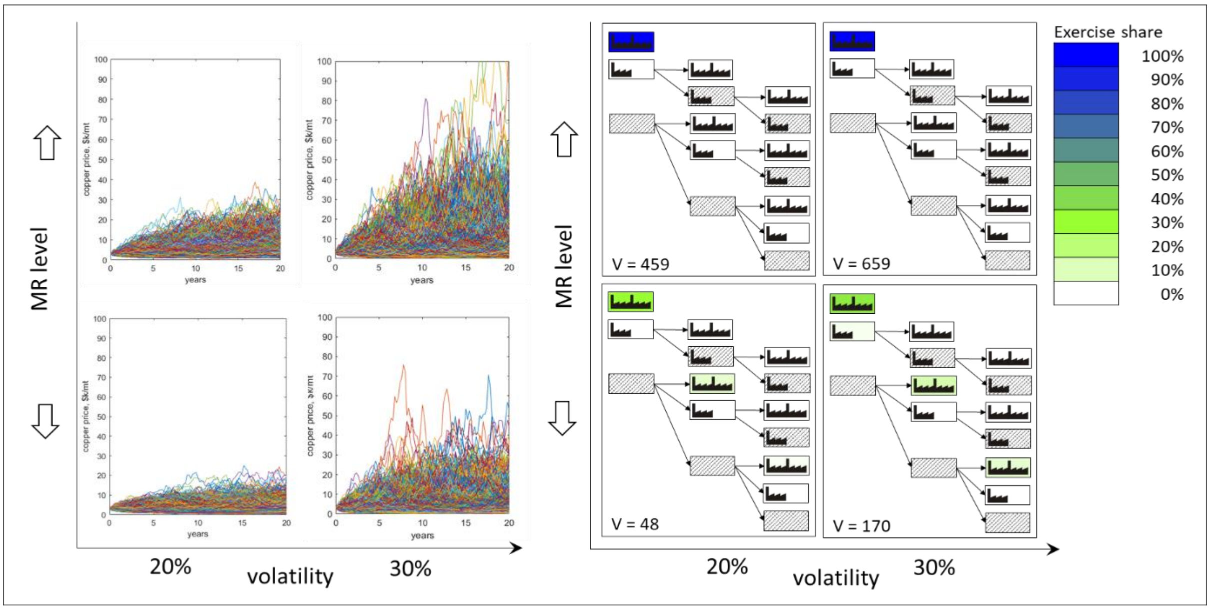

Figure 4. The sensitivity of the investment pattern to the MRP parameters

As before, independently of timing, the choice to build two modules at once prevails. No staging occurs.

3.1.3. Sensitivity to the extraction rate

The lower the extraction rate, the longer it takes to get the ore, the earlier and the bigger the investment tends to be (Figure 5). In Figure 5, the underlying price process is shown on the left and the tree decision trees correspond to different extraction rate, which is expressed in the total time that takes to extract all of the ore available with two modules.

When the extraction rate is such that it takes 20 years to extract the ore, the decision to build two modules at once at the earliest prevails (Figure 5, the decision tree on the right). With increased extraction rate the pattern to postpone investment is observed, but also the choice of building one module at the second decision point appears (Figure 5, the decision tree in the middle). When the extraction rate is further improved so it becomes possible to extract the whole ore-body in 5 years with 2 modules, or correspondingly in 10 years with 1 module, the winning strategy is to build one module at the latest in order to capture expectedly higher prices (Figure 5, the decision tree on the left). In other words, if prices are expected to grow, and the technology allows to extract the ore fast, the investment of the minimum sufficient size at the latest sufficient time will be chosen, because it will generate higher profits.

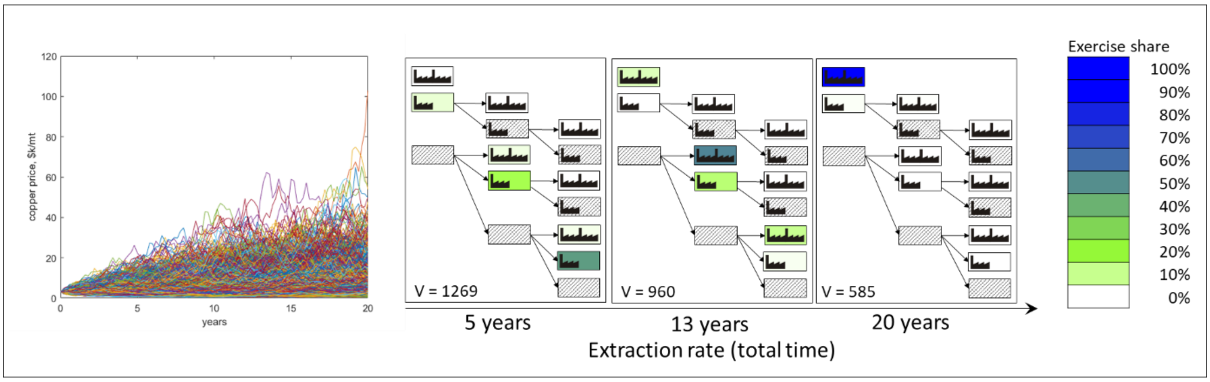

Figure 5. The sensitivity of the investment pattern to the extraction rate

Although the capacity choice now varies depending on the extraction rate, the RO to stage is not relevant in the current setting. Indeed, if there are only the uncertain price and the extraction schedule that matter, there should be an optimal investment size at an optimal timing. But there is no reason to stage the investment (or to choose another suboptimal timing) unless it brings an extra value. And there is an extra value of staging the investment in practice if the ore body size is uncertain and after some extraction, this uncertainty can be reduced. This is addressed in the Model 2.

3.2. Model 2

The learning process is modeled as Brownian motion of abstract knowledge state, which distribution is fitted to the expert estimates. We adopt here an approach of stopping time problem presented in (Davison & Lawryshyn, 2016), except instead of the time we use the amount of ore extracted.

For this case, we assume the following expert estimates on the ore body size, Table 2.

Table 2. Expert estimates on the ore-body size

Size, kt	300	500	700	1000
Cumulative probability	10%	60%	90%	99,9%

the resulting learning process is presented in Figure 6, where the chart on the right illustrates possible knowledge states over the span of extraction and the straight line represents the end of the extraction with the corresponding ‘realized’ ore-body size. The distribution of the possible ore-body sizes along the line coincides with the expert estimates, see the chart on the left on Figure 6.

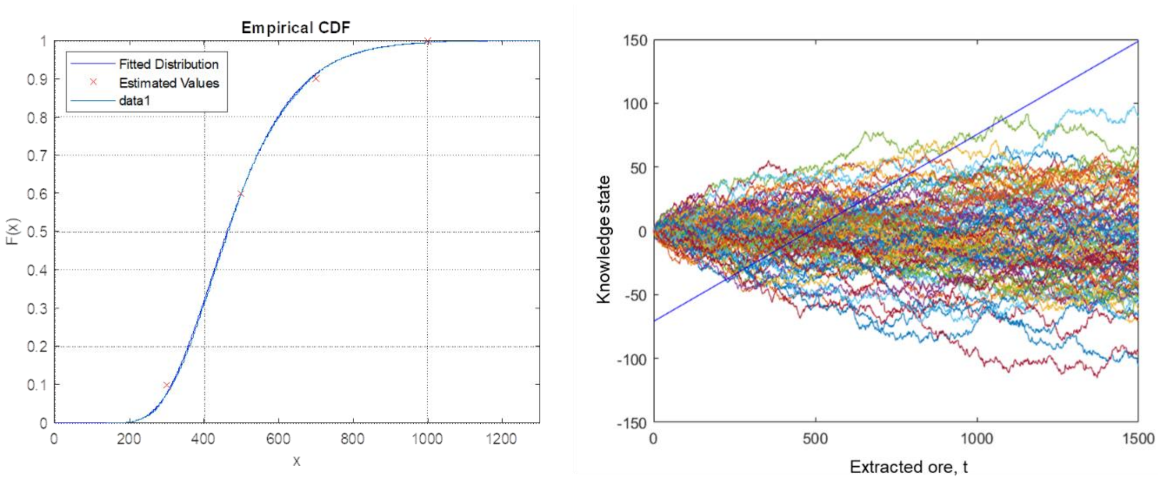

Figure 6. The learning process

When the first decision to build is made, no additional information is available and the knowledge state is 0. If two modules are built at once, and the extraction starts, the acquired information is not relevant, because the maximum size is realized already. Only is one module is built, by the next decision time the amount of extracted ore allowed to better estimate the ore-body size, and the knowledge state could be below 0, representing lower than average expectations and vice versa. Based on that information and the price development the decision of building another module is made. At some point, independently of the construction schedule, the extraction process can 'hit the bottom', when there is no more ore left and so the production stops.

The results of this model are presented in Figure 7. Figure 7A shows selected paths of price development (GBM, S_0 = 3.0k USD/t, σ = 26.3%, r= 2.8%, 10000 of paths). The diamonds show the price thresholds for the decisions based only on price. The green color represents those parts of paths where nothing has been built, the blue parts – one module built, and brown – two modules. For convenience, the same colors are used on the decision tree (Figure 7B). The threshold price is shown in a diamond, and the exercise share is displayed next to it as a percentage of the total amount of paths.

At the first decision point at year 3, 18% of paths build two modules and 32% build one. From the non-built half of the paths, later at the second decision point at year 10, 14% build two modules, none build one module (two threshold prices collapse into one), so the rest (36%) remain not exercised through the whole licensing term. The 32% of paths that has built one module, at the second decision point the choice whether to build another module depends not only on the price but also on the updated expectation of the ore-body size.

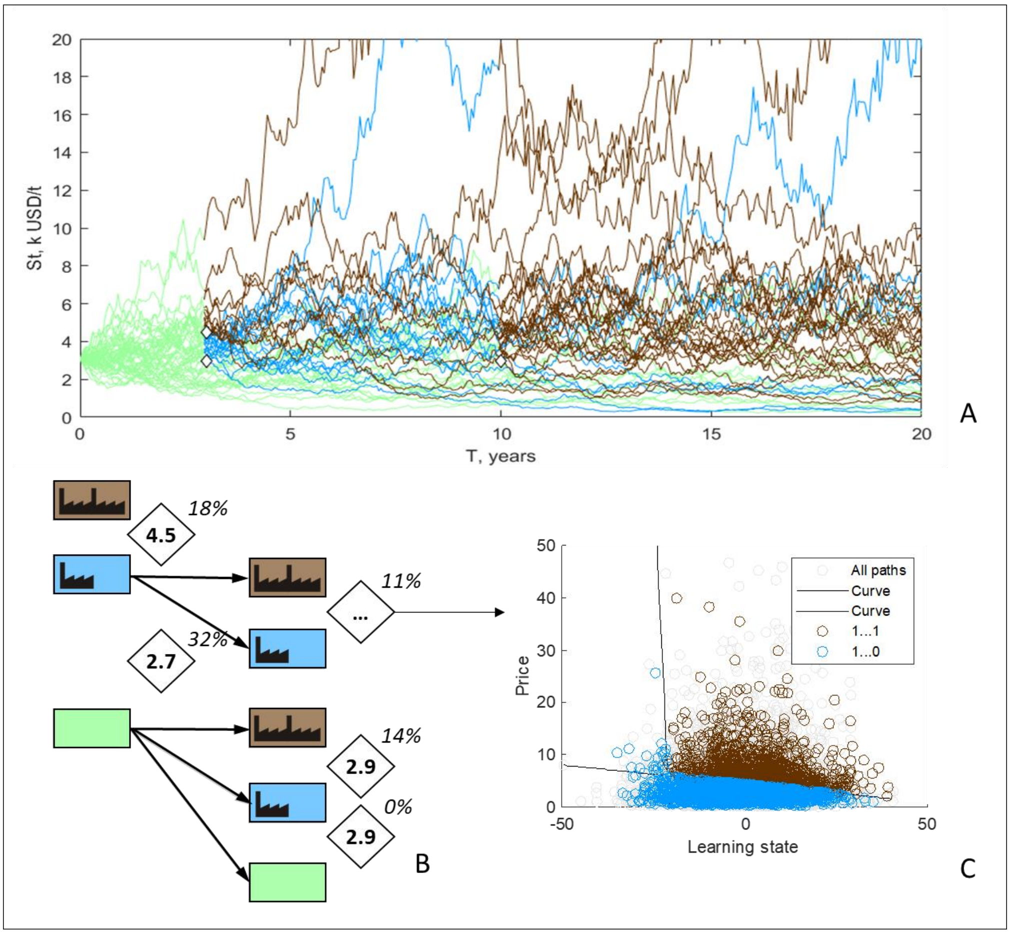

Figure 7. The investment pattern with the learning process

Therefore, the exercise boundary here should be represented by a line(s) on the 2D space of learning state and the price (Figure7C). Out of a single straight line, a parabola and two intersecting straight lines, the latter shown to provide the best so far results. Figure 7C represents a cut through Figure 7A at the second decision point.

Colored dots represent the 32% of paths where one module has been built at the first decision point and where the choice of whether to build another module is now in question. Other paths are shown in greyish. The parameters of the two lines are left to the optimization and everything that is to the upper right side of the lines are the paths where building the second module is relevant, those are 11% of the total paths. So, for some combinations of the states of knowledge and price, the RO to stage is relevant.

The further work includes conducting systematic sensitivity analyses on this model.

4. Conclusions

The new RO approach based on simulation and exercise boundary fitting, that was recently presented on a few simplified cases (Bashiri, Davison, & Lawryshyn, 2018), in this research is extended to more realistic and complex setups of mining investment. The two models are developed that provide insights into optimal investment patterns of multiple real option combinations. In the presence of single price uncertainty, the investment patterns are shown to depend heavily on the choice of a stochastic process and its parameters. In particular, GBM incentivizes RO to delay investment, whereas, under MRP, ceteris paribus, the optimal decision is to invest at the earliest. Downward mean reverting process also increases the value of postponing the investment. Investing not full size is only reasonable when full extraction is possible within the license term. The RO to stage the investment becomes relevant only when the learning process about the total ore amount is introduced. The added value of learning during extraction surpasses the otherwise suboptimal investment timing.

The models presented are the work in progress and require further development. Among future directions are selecting the best function for the exercise boundary for two stochastic processes, possibly involving neural networks; expanding the model into more decision points; repeating sensitivity analyses of obtained investment patterns to all factors presented here and the decision timing; capturing the complexity of the real learning process in mining, that adds an extra decision to each node e.g. to pay for seismic works and get more knowledge about the ore-body size, or not; finally, adding other relevant sources of uncertainty and flexibility.

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