An abatement investment strategy with ambiguous abatement technology*

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1 Introduction

In this paper, we investigate a pollutant abatement investment strategy when the prediction of the abatement technology is ambiguous. We consider a production economy, which consists of a representative consumer and a firm. The representative consumer has constant relative risk-averse preferences and tries to maximize her utility. The representative firm produces output using production capital and maximizes its profit. The production process, however, generates pollutant emissions proportional to the output, and these damage the consumer. Therefore, the firm must invest in pollutant abatement activities to reduce pollutant emissions. We formulate both agents' problems as a central planner's problem that maximizes social welfare.

We first examine the case in which the social planner has a confidence of predicting the abatement technology as a base case model. This base case model is based on Tsujimura (2017). The abatement technology is assumed to be governed by a geometric Brownian motion as in Steger (2005) and Wälde (2011). That is, the central planner's problem is a social welfare maximization problem under risk in terms of Knight (1921). We solve the Hamilton-Jacobi-Bellman (HJB) equation associated with the central planner's problem and obtain a nonlinear partial differential equation (PDE) that derives the optimal abatement investment strategy.

Next, we extend the base case model by incorporating abatement technology ambiguity. The social planner does not have perfect confidence in the distribution of the abatement technology. Then, the social planner's problem goes to be under ambiguity/Knightian uncertainty in terms of Knight (1921). To solve the central planner's problem, we employ the Hansen-Sargent (HS) type robust control approach (Hansen and Sargent, 2001; Hansen et al., 2002, 2006). We solve the Hamilton-Jacobi-Bellman-Issac (HJBI) equation associated with the central planner's problem and obtained the nonlinear PDE, which derives the optimal abatement investment strategy.

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Because of the nonlinearity, the both PDEs have to be solved numerically. We leave the numerical calculation for future work.

2 Base Case Model

In this section, we consider a social maximizing problem, excluding technology ambiguity, as a base case model by following the line of Tsujimura (2017). The economy consists of a representative consumer and a firm. The representative firm produces output Y_t using production capital K_t at time t. The firm's production function $F(K_t)$ is given by the following AK-form:

$$Y_t = F(K_t) = AK_t,$$

where A is the level of production technology. For analytical simplicity, A is assumed to be constant. The dynamics of the capital stock are given by:

$$\mathrm{d}K_t = (I_t - \delta K_t)\mathrm{d}t, \qquad K_0 = k, \tag{2.1}$$

where I_t is the capital investment, $\delta \in (0, 1)$ is the depreciation rate.

The output production process generates pollutant emissions proportional to the output level, $\eta F(K_t)$, where $\eta > 0$ is the emission conversion coefficient. As the pollutant damages the consumer, the firm invests in pollutant abatement activity $H(I_t^A)$:

$$H(I_t^A) = X_t(I_t^A)^2, \tag{2.2}$$

where I_t^A is the abatement investment and X_t is the level of abatement technology. As in Steger (2005) and Wälde (2011), we assume that the abatement technology is governed by the following geometric Brownian motion:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \qquad X_0 = x, \tag{2.3}$$

where $\mu > 0$ and $\sigma > 0$ are constants. W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, where $\mathcal{F}_t$ is generated by W_t. The abatement activity reduces the pollutant emissions, and the resulting net pollutant emissions E_t are expressed as:

$$E_t = \eta F(K_t) - H(I_t^A). \tag{2.4}$$

The pollutant emissions accumulate as:

$$\mathrm{d}P_t = (E_t - \delta_P P_t)\mathrm{d}t, \qquad P_0 = p, \tag{2.5}$$

where $\delta_P \in (0, 1)$ is the depreciation rate of the pollutant stock P_t.

The representative consumer, who has constant relative risk-averse preferences, receives utility from consumption C_t at time t. She suffers from the pollutant stock at the same time. As in Smulders and Gradus (1996), incorporating disutility from the pollutant, her utility function becomes:

$$U(C_t, P_t) = \frac{1}{1-\gamma}\left(C_t P_t^{-\phi}\right)^{1-\gamma}, \tag{2.6}$$

where $\gamma > 0$ is the degree of relative risk aversion and $\phi > 0$ is the disutility coefficient. The consumer faces the budget constraint:

$$\begin{aligned} Y_t &= I_t + I_t^A + C_t \\ &= I_t + \theta_t Y_t + C_t, \end{aligned}$$

where $\theta_t > 0$ is the abatement investment share, which is given by:

$$\theta_t = \frac{I_t^A}{Y_t}.$$

The net pollutant emissions flow is calculated as:

$$E_t = \eta A K_t - X_t \theta_t^2 (A K_t)^2. \tag{2.7}$$

Substituting (2.7) into (2.5), the dynamics of the pollutant stock can be rewritten as:

$$\mathrm{d}P_t = [\eta A K_t - X_t \theta_t^2 (A K_t)^2 - \delta_P P_t]\mathrm{d}t, \qquad P_0 = p. \tag{2.8}$$

Rewriting the budget constraint of the consumer, the capital investment is:

$$I_t = (1-\theta) Y_t - C_t. \tag{2.9}$$

It follows from (2.1) and (2.9) that the dynamics of the capital stock are rewritten as:

$$\mathrm{d}K_t = ((1-\theta)Y_t - \delta K_t - C_t)\mathrm{d}t, \qquad K_0 = k. \tag{2.10}$$

The representative firm maximizes its profits, while the representative consumer maximizes her utility, subject to the budget constraint. However, the firm's production activity generates a pollutant as a by-product, and the consumer suffers from the pollutant, which reduces her utility. Therefore, the central planner's problem is to choose a consumption level and an investment share for the abatement activity in order to maximize social welfare:

$$\hat{V}(k,p,x) = \max_{\{C_t,\theta_t\}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} U(C_t, P_t)\mathrm{d}t\right], \tag{2.11}$$

where $\hat{V}$ is the value function the central planner's problem.

The HJB equation of the central planner's problem (2.11) is:

$$\begin{aligned} r\hat{V} = \max_{c,\theta} \Bigg\{ &\frac{1}{1-\gamma}(cp^{-\phi})^{1-\gamma} + [(1-\theta)Ak - \delta k - c]\hat{V}_K + \\ &[(\eta Ak - x\theta^2(Ak)^2 - \delta_P p]\hat{V}_P + \mu x \hat{V}_X + \frac{1}{2}\sigma^2 x^2 \hat{V}_{XX} \Bigg\}. \end{aligned} \tag{2.12}$$

From the first-order condition for the optimality, if $\hat{V}_K > 0$ and $\hat{V}_P < 0$, we obtain the optimal consumption c^* and optimal abatement investment share θ^*:

$$c^* = p^{-\frac{\phi(1-\gamma)}{\gamma}} \hat{V}_K^{-\frac{1}{\gamma}}, \tag{2.13}$$

$$\theta^* = -\frac{1}{2}x^{-1}(Ak)^{-1}\frac{\hat{V}_K}{\hat{V}_P}. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), we obtain the following nonlinear PDE:

$$\begin{aligned} r\hat{V} =& \frac{\gamma}{1-\gamma}p^{-\frac{\phi(1-\gamma)}{\gamma}}\hat{V}_K^{-\frac{1-\gamma}{\gamma}} + \left[(A-\delta)k + \frac{1}{2}x^{-1}(Ak)^{-1}\frac{\hat{V}_K}{\hat{V}_P} - \frac{1}{4}x^{-1}\hat{V}_K\right]\hat{V}_K \\ &+ [\eta Ak\hat{V}_P - \delta_P p]\hat{V}_P + \mu x\hat{V}_X + \frac{1}{2}\sigma^2 x^2 \hat{V}_{XX}. \end{aligned} \tag{2.15}$$

The optimal consumption level and abatement investment share are derived from the PDE (2.15). Because of the nonlinearity, we have to solve the equation (2.15) numerically. The numerical and/or asymptotic results will be presented in the conference.

3 The Model under Ambiguity

In this section, we assume that the central planner does not have perfect confidence in the distribution of the abatement technology. In the presence of ambiguity, the central planner chooses a probability distribution based on her best possible estimate, which is referred to as the reference probability. The central planner is, however, concerned about the robustness of her decisions to misspecification of the reference probability. To resolve the concern, she considers a set of equivalent probability measures, $\mathcal{P}$, on $(\Omega, \mathcal{F})$ in order to incorporate the possible misspecification. Then, the reference probability measure $\mathbb{P}$ could be replaced by another equivalent probability measure $\mathbb{Q} \in \mathcal{P}$. The penalty for a difference between the reference probability $\mathbb{P}$ and another equivalent probability $\mathbb{Q}$ is imposed to avoid choosing a probability measure too far from the reference probability measure. As in Hansen et al. (2002), Skiadas (2003), Hansen et al. (2006), and Imai and Tsujimura (2018), we introduce a discounted relative entropy $R(\mathbb{Q})$ to measure the difference between $\mathbb{P}$ and $\mathbb{Q}$:

$$\begin{aligned} R(\mathbb{Q}) &= r\int_0^\infty \mathrm{e}^{-rt}\left(\int \log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\mathrm{d}\mathbb{Q}\right)\mathrm{d}t \\ &= \mathbb{E}_\mathbb{Q}\left[\int_0^\infty \mathrm{e}^{-rt}\frac{h_t^2}{2}\mathrm{d}t\right], \end{aligned} \tag{3.1}$$

where h_t is the measurable distortion between $\mathbb{P}$ and $\mathbb{Q}$. We assume that

$$R(\mathbb{Q}) < \infty. \tag{3.2}$$

Let $W_t^\mathbb{Q}$ denote a Brownian motion under the measure $\mathbb{Q}$. By Girsanov's theorem, this is given by

$$W_t^\mathbb{Q} = W_t - \int_0^t h_s \mathrm{d}s. \tag{3.3}$$

The dynamics of the abatement technology under the probability measure $\mathbb{Q}$ are written as:

$$\mathrm{d}X_t = (\mu + \sigma h_t)X_t\mathrm{d}t + \sigma X_t \mathrm{d}W_t^\mathbb{Q}, \quad X_0 = x > 0. \tag{3.4}$$

The central planner chooses a consumption level and an investment share for the abatement activity in order to maximize social welfare under the abatement technology ambiguity. To solve this problem, we adopt the HS type robust control approach (Hansen and Sargent, 2001; Hansen et al., 2002, 2006). In the HS type robust control approach, another decision-maker is introduced to deal with her misspecification as a hypothetical evil decision-maker. The hypothetical decision-maker chooses the worst possible abatement technology path for the central planner, so that it chooses a probability measure $\mathbb{Q}$ to minimize the expected social welfare. Then, the social planner's model can be expressed as a two-player zero-sum game between the social planner and the hypothetical decision-maker:

$$V(k,p,x) = \max_{\{C_t,\theta_t\}} \min_{\{h_t\}} \mathbb{E}_{\mathbb{Q}} \left[\int_0^\infty \mathrm{e}^{-rt} U(C_t, P_t) \mathrm{d}t + \psi R(\mathbb{Q}) \right], \tag{3.5}$$

where V is the value function of the central planner's problem and $\psi \geq 0$ is the multiplier on the penalty given as the relative entropy.

It follows from the central planner's problem (3.5) that we have the following Hamilton–Jacobi–Bellman–Isaac (HJBI) equation:

$$\begin{aligned} rV = \max_{c,\theta} \min_{h} \Bigg\{ & \frac{1}{1-\gamma}(cp^{-\phi})^{1-\gamma} + [(1-\theta)Ak - \delta k - c]V_K + \\ & [(\eta Ak - x\theta^2 (Ak)^2 - \delta_P p]V_P + (\mu + h\sigma)xV_X + \frac{1}{2}\sigma^2 x^2 V_{XX} + \psi \frac{h^2}{2} \Bigg\}. \end{aligned} \tag{3.6}$$

From the first-order conditions for the optimality with respect to c, θ, and h, if $\hat{V}_K > 0$ and $\hat{V}_P < 0$, we obtain the optimal consumption c^{**}, abatement investment share, θ^{**}, and distortion h^{**}:

$$c^{**} = p^{-\frac{\phi(1-\gamma)}{\gamma}} V_K^{-\frac{1}{\gamma}}, \tag{3.7}$$

$$\theta^{**} = -\frac{1}{2} x^{-1} (Ak)^{-1} \frac{V_K}{V_P}, \tag{3.8}$$

$$h^{**} = -\frac{\sigma x}{\psi} V_X. \tag{3.9}$$

It follows from (3.9) that the probability distortion h goes to zero as ψ goes to infinity. That is, the two probability measures coincide as ψ goes to infinity. This means that, when ψ goes to infinity, the central planner acts as if she has the perfect confidence the distribution of abetment technology. Then, the central planner's problem (3.5) degenerates to the benchmark model. On the other hand, the probability distortion h increases as ψ decreases. This means that the central planner concerns with the model misspecification more and chooses $\mathbb{Q}$ far away from $\mathbb{P}$. However, the cost of taking $\mathbb{Q}$ increases as ψ increases due to the existence of the relative entropy penalty. These considerations imply that the parameter ψ represents the robustness of the model.

Substituting (3.7), (3.8), and (3.9) into equation (3.6), we obtain the nonlinear degenerate PDE:

$$\begin{aligned} rV = & \frac{\gamma}{1-\gamma} p^{-\frac{\phi(1-\gamma)}{\gamma}} V_K^{-\frac{1-\gamma}{\gamma}} + \left[(A-\delta)k + \frac{1}{2} x^{-1} (Ak)^{-1} \frac{V_K}{V_P} - \frac{1}{4} x^{-1} V_K \right] V_K \\ & + [\eta Ak V_P - \delta_P p] V_P + \mu x V_X - \frac{1}{2} \frac{\sigma^2 x^2}{\psi} V_X^2 + \frac{1}{2} \sigma^2 x^2 V_{XX}. \end{aligned} \tag{3.10}$$

The optimal consumption level, abatement investment share, and distortion are derived from the nonlinear PDE (3.10). The difference between both nonlinear PDE (2.15) and (3.10) is the quadratic term $-\frac{1}{2}\frac{\sigma^2 x^2}{\psi}V_X^2$.

Because of the nonlinearity, we have to solve the equation (3.10) numerically as in Section 2. The numerical and/or asymptotic results will be presented in the conference.

4 Conclusion

In this paper, we analyzed a pollutant abatement investment strategy when the central planner does not have confidence with the dynamics of abatement technology. The central planner maximizes social welfare under the abatement technology ambiguity by choosing a consumption level and an investment share for the abatement activity. Due to the existence of abatement technology ambiguity, we adopt the HS type robust control approach to solve the central planner's problem. We obtained the nonlinear PDE, which derives the optimal consumption, abatement investment share, and distortion. Because the PDE is nonlinear, it has to be solved numerically. We leave the numerical calculation for future work. Appropriate boundary conditions should be discussed considering the characteristics of the coefficients along the boundaries (Oleinik, 2012)

There are several ways to extend this paper in future. First, we would show the HJBI equation admits a unique continuous viscosity solution as in Yoshioka and Tsujimura (2019). A viscosity solution approach for a certain economic growth model has recently been used in Yuan et al. (2018). Next, we would consider abatement technological progress that follows a jump diffusion process. Next, we would incorporate both the production and the abatement technology into the model. Finally, in practice, it is not always possible to track the economic dynamics to be optimized, motivating us to incorporate the discrete costly observation as in Dyrssen and Ekstrom (2018).

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