

# Choosing a Price and Cost Combination – the Role of Correlation

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## **Abstract**

Often, firms can choose among different combinations of price and cost processes. For example, they can choose between different production locations or technologies, between different products to produce, or between different locations to sell them. To study the choice of the optimal combination, we return to the Dixit and Pindyck (1994) model where both output price and production cost are stochastic processes, and add a novel focus on how the correlation between these processes affects the firm's decision. We find that, ceteris paribus, the firm prefers the combination with the lowest correlation between the processes, as it seeks a greater profitability variance which maximizes its value.

Key words: Investment, Real-options, correlation

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## 1. Introduction

*"Our goal is to produce cars where they're sold." (Paul Nolasco, a spokesman for Toyota in Tokyo, New York Times, September 3, 2010).*

Often, firms can choose among different combinations of price and cost processes. This variety of combinations spring, for example, from an ability to choose between different production locations or technologies, between different products to produce or between different locations to sell them. In this study we model such a case, assume that both output price and production cost are stochastic processes and focus on the role of the correlation between them in choosing the optimal price-cost combination.

The typical models of the literature about investment under uncertainty examine the optimal policy of a firm that has an option to invest and can choose the investment timing optimally. Usually these models also assume that the investment enables production of a certain good, and that the demand for this good stochastically rises and falls across time. The two main results of this literature regard how uncertainty (captured by the variance of the stochastic process governing the demand shifts) affects investment policy and firm value:

- Uncertainty delays investment: the higher the uncertainty – the higher the profitability threshold that hitting it triggers investment.
- Uncertainty is good for the firm – the higher the uncertainty the larger the value of the expected stream of profits from the investment.

The first result describes how the firm is more cautious under greater uncertainty. The second

result is based on the convexity of the value function – the greater uncertainty can be thought of as a mean-preserving spread that, due to endogenous investment policy, its improvement of the good outcomes benefits the firm's value by more than how it harms the bad outcomes.

To study the role of the correlation between the price process and the cost process in the firms' choices, we take a step further and assume that, alongside the demand, the production cost is a stochastic process too. Such cases have already been analyzed by a few studies such as Dixit and Pindyck (1994, pages 207-211), Tareen, Wetzstein and Duffield (2000) and Wiemers and Behan (2004). Yet, those studies have focused on finding the profitability investment threshold, and have not looked at how it is affected by the price-cost correlation. In the current study we fill this void by analyzing how the degree of correlation affects the optimal investment policy and the resulting firm value. Then we use these results to study how the price-cost correlation affects the choice of price and cost combination.

More specifically, with demand and cost being stochastic – so is profitability. We characterize the resulting profitability process and find that its variance is a decreasing function of the correlation between the two processes. This result is rather intuitive as the greater the correlation – the smaller the changes in the gap between price and cost.

Thus, the higher the demand-costs correlation – the lower the uncertainty in the profitability process and therefore the lower the profitability threshold which triggers investment, and the lower the value of the firm with the option to invest. This is the main result of this study.

One of the main applications of this result regards the case of a firm that can choose where to sell and where to produce. Such a firm will prefer, *ceteris paribus*, the combination with the

lesser price-cost correlation. Thus, for example, Toyota's dilemma, at the quotation above, regarding where to produce the cars it intends to sell in the USA. Under the reasonable assumption that the USA price is more positively correlated with production costs in the USA than with the production costs in Japan – the lesser positive correlation should drive Toyota to produce in Japan the cars it intends to sell in the USA.

This logic applies not merely in a spatial context, but could apply also in other case where a firm can choose between production processes, for example when a firm can choose between different production technologies, and should prefer, *ceteris paribus*, the one that generates a process with a lesser positive correlation with the price process.

Finally the results applies just the same to cases where the firm has a given production process and contemplates what product to produce with it, or at which market to sell it. As in the examples before, it will choose the combination with the lesser price-cost correlation.

Section 2 presents the model and its main results. Section 3 Shows a numerical illustration of the result. Section 4 offers some concluding results.

## **2. The Model and Its Solution**

Consider a risk neutral firm that can enter the market for a certain good at any point in time it wishes. Time in the model is continuous and once the firm enters it produces indefinitely at a rate of one unit of output per unit of time at the variable cost  $W$  per unit, and sells this unit at the price  $P$ . Both  $P$  and  $W$  change stochastically over time as geometric Brownian motions, under the following rules of motion:

$$(1) \quad dP_t = \alpha_P \cdot P_t \cdot dt + \sigma_P \cdot P_t \cdot dZ_t$$

$$(2) \quad dW_t = \alpha_W \cdot W_t \cdot dt + \sigma_W \cdot W_t \cdot dH_t$$

where  $\mu_P$  and  $\mu_W$  are drift parameters,  $\sigma_P$  and  $\sigma_W$  are the standard deviation of the incremental changes in  $P$  and in  $W$  respectively, and  $dZ_t$  and  $dH_t$  are increments of a standard Wiener processes, satisfying at each point in time:

$$(3) \quad E(dZ_t) = E(dH_t) = 0,$$

$$(4) \quad E(dZ_t^2) = E(dH_t^2) = dt,$$

$$(5) \quad E(dZ_t \cdot dH_t) = \rho \cdot dt,$$

where the parameter  $\rho$ , satisfying  $-1 \leq \rho \leq 1$ , denotes the coefficient of contemporaneous correlation between  $dZ_t$  and  $dH_t$ .

The interest rate is constant and denoted by  $r$ . Convergence of the firm's revenues and expenses requires  $r > \alpha_P$  and  $r > \alpha_W$ , as shall be seen later. The firm is risk-neutral and therefore maximizes its expected value.

To enter the project at time  $t$  the firm must incur a sunk cost,  $k \cdot W_t$ , where  $k$  is a positive constant. Thus, the expected value of its total costs upon entry at time  $t$  are:

$$(6) \quad I_t \equiv k \cdot W_t + E \left[ \int_t^{\infty} W_{\tau} \cdot e^{-r \cdot \tau} \cdot d\tau \right] = k \cdot W_t + \frac{W_t}{r - \alpha_W} = \left( k + \frac{1}{r - \alpha_W} \right) \cdot W_t,$$

where the calculation of the expectancy follows from the standard properties of the Geometrical Brownian Motion. From Itô's lemma, since the process  $I_t$  is proportionate to the process  $W_t$  then  $I_t$  is a Geometrical Brownian Motion too with drift and variance parameters identical to those of the process  $W_t$ , i.e.  $\alpha_W$  and  $\sigma_W$ .

Thus constructed the model is the same one studied in pages 207-211 of Dixit and Pindyck (1994).<sup>1</sup> The analysis from here to the end of this section merely presents their results. First, as their analysis show, the optimal policy in this case is based on a trigger function  $P_H(W)$  such that for every value of  $W$ :

- if  $P < P_H(W)$  then the firm remains inactive keeping the option to enter later
- if  $P > P_H(W)$  the firm exercises this option at a cost  $k \cdot W$  and becomes active

Let  $V(P, W)$  denote the value function of an active firm, given the current levels of  $P$  and  $W$ .<sup>2</sup>

Based on the assumptions above,  $V(P, W)$  satisfies:

$$(6) \quad V(P, W) = E \left[ \int_0^{\infty} (P - W) \cdot e^{-rt} \cdot dt \right] = \frac{P}{r - \alpha_P} - \frac{W}{r - \alpha_W}$$

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<sup>1</sup> More specifically, our model is a particular case in which the parameter  $\rho_{pm} = 0$  of the case that Dixit and Pindyck (1994) study. In their model this parameter captures the correlation between  $P$  and the return on the whole market portfolio. For simplicity we assume no such correlation.

<sup>2</sup> Time indexes will be omitted from now on.

Let  $F(P, W)$  denote the value function of an idle firm, given the current levels of  $P$  and  $W$ . Inactive firm has no operating profit, but it owns an investment opportunity that at each time instance  $dt$  yields an expected capital gain  $dF(P, W)$  in response to the fluctuations of  $P$  and  $W$  during that instance. The standard no-arbitrage condition requires that the expectancy of this capital gain should equal the instantaneous normal return, i.e.:

$$(7) \quad E[dF(P, W)] = r \cdot dt \cdot F(P, W)$$

Expanding  $dF(P, W)$  via Ito's lemma, taking the expectancy, applying (3), (4) and (5) and simplifying, turns (7) into the following differential equation:<sup>3</sup>

$$(8) \quad \begin{aligned} & F_P(P, W) \cdot \alpha_P \cdot P + F_W(P, W) \cdot \alpha_W \cdot W + \frac{1}{2} \cdot F_{PP}(P, W) \cdot \sigma_P^2 \cdot P^2 \\ & + \frac{1}{2} \cdot F_{WW}(P, W) \cdot \sigma_W^2 \cdot W^2 + F_{PW}(P, W) \cdot \sigma_P \cdot \sigma_W \cdot P \cdot W \cdot \rho - r \cdot F(P, W) = 0 \end{aligned}$$

In general, this type of a multivariate differential equation cannot be solved. Yet, as Dixit and Pindyck (1994) point out, the equation can be transformed into a single variable differential equation using the homogeneity of  $V(P, W)$  and  $F(P, W)$  which springs from the fact that multiplying  $P$  and  $W$  by the same amount also multiplies by the same amount both the expected value of discounted revenues and the expected value of total costs. To use this homogeneity, we define the markup  $M$  and the functions  $v(M)$  and  $f(M)$  as follows:

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<sup>3</sup> For a clear presentation of Ito's lemma, in general and also for the case of two correlated processes, see pages 79-81 in Dixit and Pindyck (1994).

$$(9) \quad M \equiv \frac{P}{W}$$

$$(10) \quad V(P, W) = W \cdot V(M, 1) = W \cdot v(M)$$

$$(11) \quad F(P, W) = W \cdot F(M, 1) = W \cdot f(M)$$

From (10) and (6) it follows that:

$$(12) \quad v(M) \equiv W \cdot \left( \frac{M}{r - \alpha_P} - \frac{1}{r - \alpha_W} \right),$$

From (11) it follows that:

$$(13a) \quad F_P(P, W) = f'(M),$$

$$(13b) \quad F_{PP}(P, W) = \frac{1}{W} \cdot f''(M),$$

$$(13c) \quad F_W(P, W) = f(M) - M \cdot f'(M),$$

$$(13d) \quad F_{WW}(P, W) = \frac{M^2}{W} \cdot f''(M),$$

$$(13e) \quad F_{PW}(P, W) = -\frac{M}{W} \cdot f''(M).$$



In the appendix we show that as the ratio of two such processes,  $M$  is a Geometrical Brownian Motion too, and with the variance parameter  $\sigma_M^2$  which satisfies:

$$(14) \quad \sigma_M^2 = \sigma_P^2 + \sigma_W^2 - 2 \cdot \sigma_P \cdot \sigma_W \cdot \rho$$

Applying (11) and (13a)-(13e) in (8), dividing both sides by  $W$ , applying (9) and (14) and simplifying, yields:

$$(15) \quad \frac{\sigma_M^2}{2} \cdot f''(M) \cdot M^2 + f'(M) \cdot M \cdot (\mu_P - \mu_W) - (r - \mu_W) \cdot f(M) = 0$$

To solve to this single-variable second-order homogenous differential equation we first try the general solution  $f(M) = M^X$  which turns (13) into:

$$(16) \quad \frac{\sigma_M^2}{2} \cdot X^2 + \left( \alpha_P - \alpha_W - \frac{\sigma_M^2}{2} \right) \cdot X - (r - \alpha_W) = 0$$

The LHS of this equation is a quadratic function of  $X$ . This is a quadratic with a minimum point, because  $\sigma_M^2 > 0$ , as shown in the appendix. In addition, from  $r > \alpha_P$  and  $r > \alpha_W$  it follows that the LHS is negative both at  $X = 0$  and at  $X = 1$ . This leads to the conclusion that (16) has one negative root and one root which is larger than 1. We denote the negative root by  $\gamma$  and the positive one by  $\beta$ . Thus, the general solution of the differential equation (15) is:

$$(17) \quad f(M) = A \cdot M^\gamma + B \cdot M^\beta$$

where  $A$  and  $B$  are parameters to be determined via boundary conditions. The first one of these boundary conditions is:

$$(18) \quad \lim_{(P/W) \rightarrow 0} F(P, W) = 0$$

This condition implies that if  $P$  approaches 0 than, by the properties of the Geometric Brownian Motion, the probability of it ever rising above a much larger  $W$  so that the stream of profits will become positive is zero and therefore the value of the option to become active is worthless. A similar interpretation of the condition rises from looking at the case where  $W$  goes to infinity (and in particular that it is infinitesimally larger than  $P$ ).

From (11) and (18) and from  $\gamma < 0$  and  $\beta > 1$  it follows that  $A = 0$ , which leads to:

$$(19) \quad f(M) = B \cdot M^\beta$$

The next boundary condition is the following *Value Matching Condition* which refers to the investment done when  $P$  and  $W$  are such that  $P$  hits the entry threshold:

$$(20) \quad F[P_H(W), W] = V[P_H(W), W] - k \cdot W$$

Applying (10) and (11) in (20) and simplifying yields:

$$(21) \quad f(M_H) = v(M_H) - k$$

where:

$$(22) \quad M_H \equiv \frac{P_H(W)}{W}.$$

(21) also leads to the high-order contact condition known as the *Smooth Pasting Condition*:

$$(23) \quad f'(M_H) = v'(M_H)$$

Using the functional forms (12) and (19) in (21) and (23) gives the following solution for the trigger markup  $M_H$  :

$$(24) \quad M_H = \frac{\beta}{\beta-1} \cdot \frac{r-\alpha_P}{r-\alpha_W} \cdot [1 + (r-\alpha_W) \cdot k].$$

Note that  $M_H$  is independent of  $W$ . Thus, the firm's optimal policy can be presented solely in terms of the markup  $M$ : to delay investment as long as  $M$  , which is a Geometric Brownian Motion, is below the constant value  $M_H$  and to invest when  $M$  finally hits  $M_H$ .

### 3. Firm value and correlation

In this section we go beyond the analysis of Dixit and Pindyck (1994) and find the firm's value, and how the correlation between price and cost affects this value.

We start by noticing that applying (12) and (19) in (21) and (23) also yields, alongside the threshold  $M_H$ , the following expression for the parameter  $B$ :

$$(25) \quad B = \frac{1 + (r - \mu_W) \cdot k}{(\beta - 1) \cdot (r - \mu_W) \cdot M_H^\beta}$$

Applying it in (19) yields:

$$(26) \quad f(M) = \frac{1 + (r - \mu_W) \cdot k}{r - \mu_W} \cdot \frac{1}{\beta - 1} \cdot \left( \frac{M}{M_H} \right)^\beta$$

The following *proposition* 1 states that the more positive the correlation coefficient the lower the entry threshold,  $M_H$ , and also the lower the firms value.

$$\text{Proposition 1:} \quad (i) \quad \frac{dM_H}{d\rho} < 0 \quad (ii) \quad \frac{dF(P, W)}{d\rho} < 0.$$

*Proof:* From (24) it follows that:

$$(27) \quad \frac{dM_H}{d\beta} = -\frac{M_H}{\beta \cdot (\beta - 1)} < 0.$$

In addition, an implicit derivation of (16), evaluated at  $X = \beta$ , yields:

$$(28) \quad \frac{d\beta}{d\sigma_M^2} = -\frac{\frac{1}{2} \cdot \beta^2}{\sigma_M^2 \cdot \beta + \alpha_P - \alpha_W - \frac{\sigma_M^2}{2}} = -\frac{\frac{1}{2} \cdot \beta^2}{\frac{r - \alpha_W}{\beta} + \frac{\sigma_M^2}{2} \cdot \beta} < 0.$$

The first equality follows from implicit derivation of (16). The second equality follows from rearranging terms in (16). The inequality springs from  $\sigma_M^2 > 0$ ,  $\beta > 1$ , and  $r > \alpha_w$ .

From (14) it immediately follows that  $\frac{d\sigma_M^2}{d\rho} < 0$ . Thus:

$$(29) \quad \frac{dM_H}{d\rho} = \frac{dM_H}{d\beta} \cdot \frac{d\beta}{d\sigma_M^2} \cdot \frac{d\sigma_M^2}{d\rho} < 0,$$

which proves (i). To prove (ii) note that:

$$(30) \quad \frac{d \ln \left[ \frac{1}{\beta-1} \cdot \left( \frac{M}{M_H} \right)^\beta \right]}{d\beta} = \frac{d \ln \left( \frac{1}{\beta-1} \right)}{d\beta} + \frac{d \left[ \beta \cdot \ln \left( \frac{M}{M_H} \right) \right]}{d\beta}$$

$$= -\frac{1}{\beta-1} + \ln \left( \frac{M}{M_H} \right) + \beta \cdot \frac{M_H}{M} \cdot \frac{-M}{M_H^2} \cdot \frac{dM_H}{d\beta} = \ln \left( \frac{M}{M_H} \right) < 0,$$

where the third equality springs from (27) and the inequality springs from  $M < M_H$  which holds throughout the definition range of  $F(P, W)$ .

From (30), together with (26), and  $r > \alpha_w$ , it follows that  $\frac{df(M)}{d\beta} < 0$ .

$\frac{df(M)}{d\beta} < 0$ ,  $\frac{d\sigma_M^2}{d\rho} < 0$ , (11) and (28) completes the proof of (ii) via:

$$(31) \quad \frac{dF(P, W)}{d\rho} = W \cdot \frac{df(M)}{d\beta} \cdot \frac{d\beta}{d\sigma_M^2} \cdot \frac{d\sigma_M^2}{d\rho} < 0. \quad \square$$

With this result at hand we can now objective stated at the beginning of this article –verifying the role of price-costs correlation in optimally choosing production possibilities. As we can see from *Proposition 1*, ceteris paribus, the firm shall prefer the production possibility with the lowest positive correlation (hopefully negative, and as close to -1 as can be) with the price process.

To be more explicit consider a non-USA firm that sells in the USA but contemplates whether to produce at home or in the USA where it sells. Many factors influence this decision: difference in production costs in the USA and at home, shipping costs, different managerial efficiency in controlling production done at the USA or at home, etc. Yet, among these relevant variables there is also the correlation between the swings in the prices at the USA market and the swings in production costs. Taking the reasonable assumption that the USA price has a greater positive correlation with the USA production cost, compared to its correlation with the cost of production at the firm's home country, leads to the conclusion that with everything else equal – it will prefer to produce at home. Producing in the USA will be undertaken in this case only if production costs are not equal between the two locations, but cheaper in the USA.

#### **4. A numerical illustration**

To illustrate the point made in the previous section more clearly we use the following numerical example of the model. In this example there is a firm that plans to sell at a certain market and also has two possible production locations named A and B. Production in location

$i$ , where  $i \in \{A, B\}$  entails the cost process  $W^i$  which follows the model assumption regarding  $W$ . While idle and waiting for the optimal time to become active, the firm cannot monitor two locations at the same time and therefore cannot efficiently preserve both its option to become active by producing at A and the option to become active by producing at B. Thus, it needs to decide already at time 0 about the location in which it will produce. Interested in maximizing its value, it will choose to commit at time 0 to producing at A if, and only if  $F(P_0, W_0^A) \geq F(P_0, W_0^B)$ . Otherwise it will choose to commit to production at B. After the time 0 choice of production location  $i$ , the firm waits until  $W^i$  and  $P$  are such that the entry threshold (24) is reached.

Indifference in time 0 between choosing production in A or in B happens if:

$$(32) \quad F(P_0, W_0^A) = F(P_0, W_0^B)$$

Note that there is no location index on  $P$ , implying that the selling location needs not be A or B but could be a different one.

For simplicity assume  $r$ ,  $\alpha_W$ ,  $k$  and  $\sigma_W$  are the same in both locations. This enhances the focus on the role of the difference between  $\rho^A$  and  $\rho^B$  in choosing between A and B.

Applying (11), (19), (24) and (25) in (32) and rearranging terms, the indifference condition becomes:

$$(33) \quad W_0^A = \left[ \frac{\beta^A - 1}{\beta^B - 1} \cdot \frac{(M_H^A)^{\beta^A}}{(M_H^B)^{\beta^B}} \cdot P_0^{\beta^B - \beta^A} \cdot (W_0^B)^{1 - \beta^B} \right]^{\frac{1}{1 - \beta^A}}.$$

Note that  $\rho^A$  and  $\rho^B$  appear in this equation within  $\beta^A$  and  $\beta^B$ , as follows from (14) and (16).

Equation (33) shows the "cost of correlation" by showing how higher levels of  $\rho^A$  require lower production cost in A at time 0 in order to prevent the firm from preferring B and to preserve indifference between choosing A or B.

We use (33) via these parameter values:  $r^A = r^B = 0.025$ ,  $\sigma_P^2 = (\sigma_W^A)^2 = (\sigma_W^B)^2 = 0.01$ ,

$\alpha_P = \alpha_W^A = \alpha_W^B = 0$ ,  $k^A = k^B = 4$ ,  $P_0 = 1$ ,  $W_0^B = 1$  and  $\rho^B = 0$ .

From (33), if  $\rho^A = 0$  (just like  $\rho^B$ ) then the value of  $W_0^A$  required to preserve indifference is

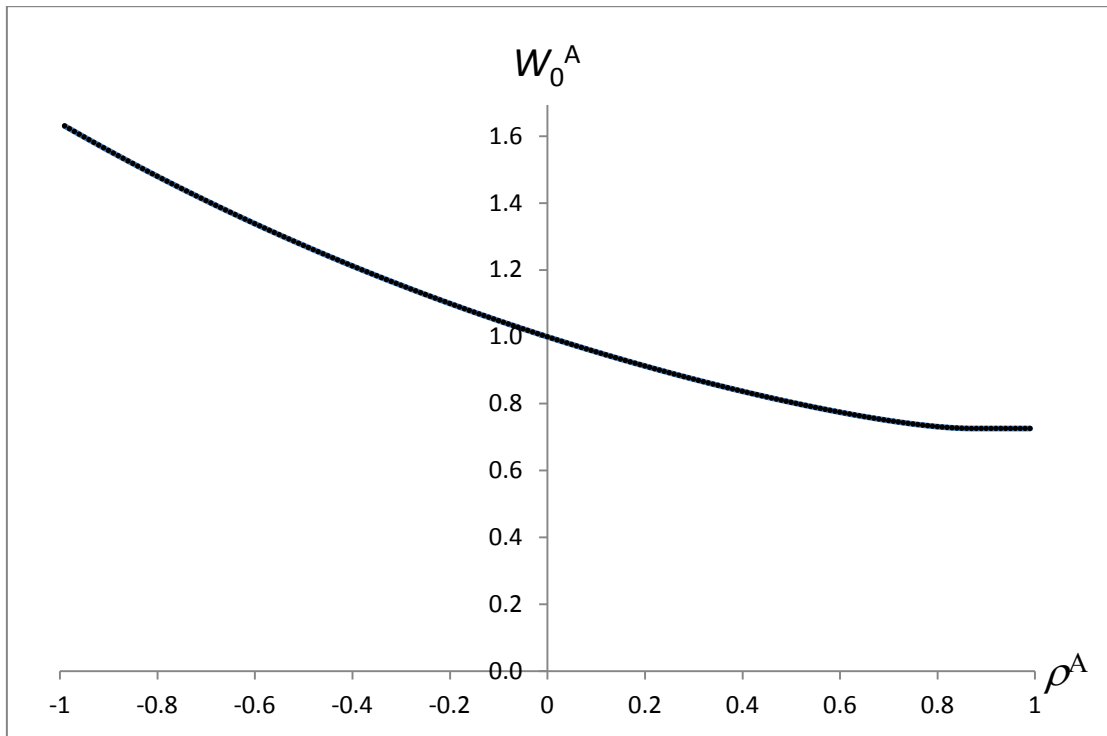
$W_0^A = 1$  (also as the time 0 cost in B).

Yet, with a medium-size positive correlation of  $\rho^A = 0.4$ , location A is as good as B only if its time 0 cost is  $W_0^A = 0.84$ , which represents a 16% "correlation cost".

As  $\rho^A$  approaches 1, the  $W_0^A$  required to preserve indifference between A and B converges down to  $W_{0,A} = 0.73$ , implying that A can be as good as B despite the large correlation its cost has with the price process only if its cost is 27% smaller than the cost in B.



The following figure shows this trade-off between correlation and cost, based on (33). Note that if  $\rho^A < 0$  then the firm is indifferent between A and B even though production cost at A exceeds the production cost at B.



**Figure 1: The correlation cost.** The larger the positive correlation between  $P$  and  $W^A$ , the lower  $W_0^A$  should be for the firm not to prefer B and for indifference between A and B to remain.

### Conclusion

In this article we have shown that when a firm can choose different price-cost combinations – it will tend to choose the combination with the lowest price-cost correlation, preferably a negative correlation, and the closer to -1 the better.

The reasons for that was explained already at the introduction for this article – the more positively correlated are the price and cost processes, the lower the volatility of profitability and the therefore the lower the firm value. This negative effect of profit volatility on firm

value is a classic result of the literature about investment under uncertainty, and it reflects the asymmetric effect that a mean preserving spread of the distribution of profitability values has on the firm's value. The reason for this asymmetry is that the firm can optimally choose whether to invest or not, and also to choose when to invest. Thus, it can enjoy the increase in the probability of very high profits while the parallel increase in the probability of very low profits does not harm it that much because it is less likely to invest if the profitability process is in the range of these low profits.

In the analysis we simplified and assumed that prior to entering the market, while the firm is idle it waits for the optimal market condition for entry - it cannot monitor two different markets and therefore it has to choose the combination of price and cost already at time zero. Analyzing a more complicated case where the firm preserves both its options until making one of the possible investments may be an interesting topic for future research.

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## Appendix

In order to find the stochastic process for the markup  $M$  we apply Ito's lemma to  $M$ , as defined by (9), and simplify. This yields:

$$(A.1) \quad dM(P, W) = \alpha_M \cdot M \cdot dt + \sigma_M \cdot M \cdot dW,$$

where:

$$(A.2) \quad \alpha_M \equiv \alpha_P - \alpha_W + \sigma_W^2 - \sigma_P \cdot \sigma_W \cdot \rho$$

$$(A.3) \quad \sigma_M^2 \equiv \sigma_W^2 + \sigma_P^2 - \sigma_P \cdot \sigma_W \cdot \rho$$

$$(A.4) \quad dW \equiv \frac{\sigma_P \cdot dZ - \sigma_W \cdot dH}{\sigma_M}.$$

Since  $dZ$  and  $dH$  are normally distributed – so is  $dW$ . From (3) and (A.4) it follows that  $E(dW) = 0$ , and from (4), (5) and (A.4) it follows that  $E[(dW)^2] = dt$ . Thus,  $dW$  is a standard Wiener process and, by (A.1),  $M$  is therefore a Geometric Brownian Motion with the drift parameter  $\alpha_M$  and the variance parameter  $\sigma_M$ .