Dynamic Capital Structure Choice and Investment Timing*

Dockner, Engelbert J.[1], Hartl, Richard F.[2], Kort, Peter.M.[3]
[1]Deceased
[2]Institute of Business Administration, University of Vienna, Vienna, Austria
[3]Department of Econometrics & Operations Research and CentER, Tilburg University, Tilburg, the Netherlands, and Department of Economics, University of Antwerp, Antwerp, Belgium

February 9, 2018

Abstract

The paper considers the problem of an investor that has the option to acquire a firm. Initially this firm is run as to maximize shareholder value, where the shareholders are risk averse. To do so it has to decide each time on investment and dividend levels. The firm's capital stock can be financed by equity and debt, where less solvable firms pay a higher interest rate on debt. Revenue is stochastic.

We find that the firm is run such that capital stock and dividends develop in a fixed proportion to the equity. In particular, it turns out that more dividends are paid if the economic environment is more uncertain. We also derive an explicit expression for the threshold value of the equity above which it is optimal for the investor to acquire the

*This paper was started several years ago at the moment that Engelbert Dockner got the idea to combine Hartl et al. (2002) with the real options approach. Unfortunately, before we were able to finish the paper, he passed away. His coauthors feel honored to finish the paper in his memory.

firm. This threshold increases in the level of uncertainty reflecting the value of waiting that uncertainty generates.

Key words: real options, optimal control, capital accumulation

1 Introduction

A huge literature deals with the topic of mergers and acquisitions (see, e.g., Loukianova et al. (2017)). Our paper offers a theoretical framework of analyzing when to acquire a firm. The firm can be bought after incurring a fixed investment cost. Upon investing the investor obtains the firm, the value of which is determined based on a dynamic firm model. Before being acquired by the investor, the firm is run as to maximize shareholder value. The shareholders are risk averse and the objective of the firm is to maximize a discounted utility stream of dividends. At each moment in time the firm has to decide how much to invest and pay out dividends, which at the same time determines how much the firm will borrow. Borrowing is more expensive the higher the debt-equity ratio is.

We start out formulating the dynamic model of the firm, which is based on Steigum (1983) (see also Hartl et al. (2002)). Where the Steigum model is deterministic we extend it to a stochastic framework by considering a stochastic revenue process. We show that after an initial impulse investment, the firm invests such that capital stock and equity develop over time in a proportional way. Part of the capital stock is financed by debt. Also the debt-equity ratio is fixed over time, and this ratio is high in case of large marginal revenue, a low depreciation rate, and a low interest rate. Moreover, also the dividend is paid out in fixed proportion to the equity. This proportion is higher in case the economic environment is more uncertain and the shareholder time preference rate is large. The dependence of dividend payout on the degree of risk aversion is less clear.

The investor acquires the firm once the firm's equity has reached a certain threshold value, for which we obtain an explicit expression. This threshold value reveals the real options logic that uncertainty generates a value of waiting, since the threshold value goes up if the economic environment gets more uncertain. Again the dependence of the threshold value on the risk aversion parameter is less clear. In this sense we could not confirm the result in Hugonnier and Morellec (2007), who find that risk aversion provides an incentive for the investor to delay investment.

Technically, the analysis of the model is new in the sense that it combines continuous and lumpy investments in one framework. Continuous investments are for the first time considered in the capital accumulation model in Jorgensen (1963). Lucas (1967) and Gould (1968) are among the first to introduce a convex adjustment cost function of investment in such a framework that spreads out the growth of the capital stock over time. Later on, authors introduce non-concavities in the revenue function, which results in history-dependent long run equilibria separated by Skiba points (see, e.g., Davidson and Harris (1981) and Dechert (1983)). A dynamic model of the firm with a completely convex revenue function, caused by for instance increasing returns to scale, is studied in Barucci (1998) (see also Hartl and Kort (2000)).Next to an adjustment cost function being convex, also adjustment costs with a (partly) concave shape are considered (see, e.g., Jorgensen and Kort (1993)). The just mentioned contributions all assume a perfect capital market, indicating that firms can lend and borrow as much as they want against a fixed interest rate. Jorgensen and Kort (1997) depart from this assumption by letting the interest rate increase with debt. Alternatively, Steigum (1983) and also the present paper considers a framework in which the interest rate increases with the debt-equity ratio, reflecting that it is more risky to lend to firms that are less solvable, so a higher interest rate is demanded then.

The literature stated in the previous paragraph has in common that all work in a deterministic framework, and, as stated, investments are continuous or incremental. We extend these contributions, first, by also considering the lumpy investment of the investor needed to acquire the firm. Second, we introduce uncertainty by letting revenue be stochastic. As such we combine capital accumulation models with real options analysis. The seminal work in this area is Dixit and Pindyck (1994). In a real options model firms determine the optimal time to undertake a lumpy investment, where it is assumed that investment is irreversible and subject to ongoing uncertainty. The key result is that firms delay investing in a more uncertain world since uncertainty generates a value of waiting with investment. Later on this theory was extended to include competition, in which several firms have the option to invest in the same market (see Grenadier (2000) for a survey). The standard real options literature considers an investment problem as a timing problem, where the firm should find the optimal time to undertake an investment of given size. More recently, contributions arise that, besides the timing, also optimize the size. This literature is surveyed in Huberts et al. (2015).

The paper is organized as follows. The model is formulated in Section 2.

Section 3 contains the analysis of the firm model while the investor's problem is solved in Section 4.

2 The Model

We consider the problem of an investor that has the option to acquire a company. To obtain the company, it has to incur an investment cost equal to I. The value of the company fluctuates over time. In particular, it is given by $V(X)$ where X is the amount of equity of the targeted firm.

The firm under consideration operates such that it maximizes its shareholder value. The shareholder value equals a discounted utility steam of dividends. Shareholders are risk averse, so that the utility function is concavely increasing with dividends, D. We impose that this utility function is given by $U(D) = \frac{1}{\gamma} D^{\gamma}$. The shareholder time preference rate is constant and equal to i. Hence the value of the firm is given by

$$V(X_0) = \max E\left[\frac{1}{\gamma}\int_0^{\infty} e^{-it} D^{\gamma} dt\right]. \tag{1}$$

where X_0 is the initial equity at time zero.

Equity, X, increases over time due to revenue obtained from selling products on the market, and decreases due to depreciation, δK, where K is the capital stock, interest payment $r(B/X)B$, where B is the amount of debt and $r(B/X)$ is the interest rate, and paying dividends to the shareholders. Due to demand uncertainty, revenue, $R(K)$, is stochastic:

$$R(K)\,dt = aK dt + \tilde{\sigma} K dz.$$

Hence the revenue consists of a deterministic part being linearly dependent on the capital stock, K, and a stochastic part governed by the increment of a Wiener process dz, where the standard deviation is equal to $\tilde{\sigma}K$. The linearity of the deterministic part of revenue, aK, is valid e.g. if the firm is a price taker on the output market and there is constant returns to scale in the production process. Revenue is stochastic because of demand shocks.

Following e.g. Steigum (1983), we impose that the interest rate $r(B/X)$ is increasing in the debt to equity ratio, B/X. Later on, we use the specification

$$r(B/X) = \beta \frac{B}{X}. \tag{2}$$

We conclude that the evolution of equity satisfies the following stochastic differential equation:

$$dX = [aK - \delta K - r(B/X)B - D]\,dt + \tilde{\sigma} K dz. \tag{3}$$

Capital stock follows the standard evolution such that it increases with investment, J, and decreases with depreciation, δK, i.e.

$$\dot{K} = J - \delta K.$$

The balance equation of the firm is such that capital stock can be financed by equity and debt:

$$K = X + B.$$

From the last three equations it can be derived that

$$dB = [J - R(K) + r(B/X)B + D]\,dt - \tilde{\sigma} K dz.$$

Hence, given that the firm holds no cash, this equation represents the cash balance: at each moment in time money is spent on investments, paying interest and dividends, while the cash inflow consists of revenue and extra borrowing (if $dB > 0$). On the other hand, dB can also be negative, in which case the firm pays off debt.

Of course we need some upper bound on B which we introduce indirectly by imposing that equity needs to be non-negative:

$$X \geq 0. \tag{4}$$

We conclude that the dynamic model of the firm can be presented as follows:

$$\begin{aligned}
V(X_0) &= \max_{J,D} E\left[\frac{1}{\gamma}\int_0^\infty e^{-it} D^\gamma dt\right], & (5)\\
\dot{K} &= J - \delta K, & (6)\\
dX &= \left[aK - \delta K - \beta\frac{B^2}{X} - D\right] dt + \tilde{\sigma} K dz, & (7)\\
B &= K - X, & (8)\\
D &\geq 0, X \geq 0. & (9)
\end{aligned}$$

The above model is based on Steigum (1983), where our interest rate function is a simplified version. On the other hand the depreciation rate δ is not explicitly included in the Steigum formulation, and our revenue function is stochastic, while in Steigum (1983) it is deterministic[1].

The problem of the investor is when, if ever, to acquire the firm. To do so, it has to solve

$$\max_T E\left[e^{-iT}\left(V\left(X\left(T\right)\right)-I\right)\right] \tag{10}$$

in which T is the acquisition time and i is the time preference rate of the investor, which is assumed to be the same as the time preference rate of the current shareholders.

The next section analyzes the optimization problem of the firm, $(5-9)$, while the investor's problem (10) will be solved in the subsequent section.

3 Analysis of the Firm Model

Like Steigum (1983), we apply a two-step approach. In Step 1, a static optimization problem is solved to determine, for every value of equity, X, the optimal level of the capital stock, K. In this way, the function $K\left(X\right)$ is obtained. In Step 2 the remaining stochastic optimal control problem is solved using the function $K\left(X\right)$ as input.

Mathematically, the two steps can be described as follows.

3.1 Step 1: Capital Accumulation

For a given equity level, X, we maximize the deterministic instantaneous profit rate with respect to K. This leads to the following problem:

$$\begin{aligned} \pi\left(X\right) &= \max_K\left[aK-\delta K-\beta\frac{B^2}{X}\right], \\ \text{s.t. } B &= K-X. \end{aligned} \tag{11}$$

The optimal level of K is therefore given by

$$K\left(X\right)=\left(1+\frac{a-\delta}{2\beta}\right)X. \tag{12}$$

[1] A deterministic variant of the Steigum [16] model is studied in Hartl et al. [8].

which implies that debt $B = K - X$ is always positive:

$$B(X) = \frac{a-\delta}{2\beta}X > 0. \tag{13}$$

If expected marginal revenue, a, is large relative to depreciation, δ, and/or if the cost of debt, β, is small, the firm has a high debt-equity ratio. In such a situation it is good for the firm to borrow a substantial amount to enhance further growth.

Results (12) and (13) mean that the firm should always keep a constant capital to equity ratio, and a constant debt to equity ratio. Since the model (5) to (9) has no adjustment costs for capital stock, K, impulse investments or disinvestments are performed to reach these levels.

3.2 Step 2: Optimal Dividend Policy

Next, we proceed to solve the following stochastic optimal control problem in order to obtain the dividend rate:

$$\max E\left[\frac{1}{\gamma}\int_0^\infty e^{-it}D^\gamma dt\right],$$

subject to

$$\begin{aligned} dX &= \left[aK(X) - \delta K(X) - \beta\frac{B(X)^2}{X} - D\right]dt + \tilde{\sigma}K(X)\,dz, \qquad (14) \\ D &\geq 0, \end{aligned}$$

$$B = K(X) - X.$$

Substitution of (12) and (13) into expression (14) gives

$$dX = \left[[a-\delta]\left[1 + \frac{a-\delta}{4\beta}\right]X - D\right]dt + \sigma X dz, \tag{15}$$

with

$$\sigma = \tilde{\sigma}\left(1 + \frac{a-\delta}{2\beta}\right). \tag{16}$$

To derive the optimal dividend rate, we employ the Bellman equation

$$iV(X) = \max_D\left[\frac{1}{\gamma}D^\gamma + V'(X)\left[[a-\delta]\left[1+\frac{a-\delta}{4\beta}\right]X - D\right] + \frac{1}{2}\sigma^2X^2V''(X)\right], \tag{17}$$

where we recall that $V(X)$ is the value of the firm. The dividend rate is, therefore, given by

$$D = V'(X)^{\frac{1}{\gamma-1}}.$$

To solve the Bellman equation, we postulate that

$$V(X) = \frac{c}{\gamma} X^{\gamma}, \tag{18}$$

where c is some constant to be determined. This gives a linear dividend payment rule

$$D = c^{\frac{1}{\gamma-1}} X. \tag{19}$$

After substitution of (19) in (17), we obtain that

$$c = \left(\frac{\left(\frac{i}{\gamma} + \frac{\sigma^2}{2}(1-\gamma)\right) - [a-\delta]\left[1 + \frac{a-\delta}{4\beta}\right]}{\frac{1}{\gamma} - 1} \right)^{\gamma-1}. \tag{20}$$

This implies that the dividend payment rule is given by

$$D = \frac{\left(\frac{i}{\gamma} + \frac{\sigma^2}{2}(1-\gamma)\right) - [a-\delta]\left[1 + \frac{a-\delta}{4\beta}\right]}{\frac{1}{\gamma} - 1} X. \tag{21}$$

From (20) and (21) we see that we have to assume that

$$i + \frac{\sigma^2}{2}\gamma(1-\gamma) > \gamma[a-\delta]\left[1 + \frac{a-\delta}{4\beta}\right], \tag{22}$$

must hold in order to ensure that dividend is positive. Note that this condition is always satisfied for a sufficiently large time preference rate. This implies that the shareholders are sufficiently impatient that dividend payments are required from the beginning.

Summing up the results so far, we have obtained a complete characterization of the firm's investment and dividend policy as follows:

Proposition 1. *If the time preference rate is sufficiently large* (22), *the firm always pays dividends proportional to equity, given by the explicit rule* (21). *Consequently, also debt is proportional to equity, as given by* (13). *This results in a concave value function, explicitly given by* (18).

In order to perform sensitivity analysis, first we see that the dividend-equity ratio increases in the shareholder time preference rate, i. This makes sense since a large value of i means that the shareholder has high opportunity costs. A risk averse shareholder does not like to be subject to uncertainty, σ. Therefore, the shareholder prefers a higher dividend payout when the firm operates in a more uncertain economic environment. This effect (D/X increases with σ) is more pronounced if the shareholder is more risk-averse, which is reflected by a smaller γ.

From the previous section we recall that the firm wants further growth especially when expected marginal revenue, a, is large relative to depreciation, δ, and/or if the cost of debt, β, is small. It makes sense that in such a situation the firm pays fewer dividends per unit of equity.

The dependence of the dividend-equity ratio in the risk aversion parameter, γ, is more involved. We obtain that

$$\frac{\partial D/X}{\partial \gamma} = \frac{1}{4\beta\left(\gamma-1\right)^{2}}\left(-\left(a-\delta\right)^{2}-4\beta\left(a-\delta-i\right)+2\sigma^{2}\beta\left(\gamma-1\right)^{2}\right).$$

From this equality we get that for γ close to one, the dividend equity ratio decreases with gamma. This is because a less risk averse investor (γ larger) is not bothered by the uncertain revenues and is willing to keep more money in the firm.

4 The Investment Timing Problem

Here we consider the problem of the investor that has the opportunity to acquire this firm at investment cost I. The question is whether he should do so and if yes, when. In fact, he has to solve the optimization problem (10).

As we learned from the previous section, the value of the firm is

$$V\left(X\right) = \frac{c}{\gamma}X^{\gamma}, \tag{23}$$

with c from (20). After taking into account that dividend satisfies (21), we obtain from (15) that equity, X, develops according to

$$dX = AXdt + \sigma X dz, \tag{24}$$

with

$$A = \frac{1}{1-\gamma}\left[\left(a-\delta\right)\left(1+\frac{a-\delta}{4\beta}\right)-\left(i+\frac{\sigma^{2}}{2}\gamma\left(1-\gamma\right)\right)\right]. \tag{25}$$

From the theory of real options (see, e.g., Dixit and Pindyck, 1994) it is known that investment is delayed until infinity, thus will never be undertaken, when

$$A > i.$$

In case $A < i$, a threshold value for X, say X^*, can be identified, above which it is optimal to invest. So, for $X > X^*$ the firm invests immediately and the payoff of this transaction equals

$$V(X) - I. \tag{26}$$

For $X < X^*$ the investor refrains from investment, so that he keeps the investment option alive. Let us denote the value of the investment option by $F(X)$. Following e.g. Dixit and Pindyck [4], application of Ito´s lemma and the Bellman equation implies that

$$iF(X) = AXF'(X) + \frac{1}{2}\sigma^2 X^2 F''(X).$$

The solution to this differential equation is given by

$$F(X) = GX^{\varepsilon}, \tag{27}$$

where G is an unknown constant and ε is the positive root of the fundamental quadratic

$$\frac{1}{2}\sigma^2\varepsilon\left[\varepsilon - 1\right] + A\varepsilon - i = 0,$$

i.e.

$$\varepsilon = \frac{-2A + \sigma^2 + \sqrt{4A^2 + \sigma^4 - 4A\sigma^2 + 8i\sigma^2}}{2\sigma^2}, \tag{28}$$

Note that the negative root cancels due to the boundary condition

$$F(0) = 0,$$

and that the positive root always exceeds one.

The threshold value X^* can be obtained by the value matching and the smooth pasting conditions. From the expressions (26) and (27) it follows that the value matching condition is given by

$$GX^{*\varepsilon} = V(X^*) - I,$$

and the smooth pasting condition equals

$$\varepsilon G X^{*\varepsilon-1} = V'\left(X^*\right)$$

respectively. From these two equalities it is obtained that

$$V\left(X^*\right) - I = \frac{X^*}{\varepsilon} V'\left(X^*\right).$$

Plugging in (23) leads to

$$X^* = \left[\frac{\gamma\varepsilon}{c\left(\varepsilon-\gamma\right)} I\right]^{\frac{1}{\gamma}}. \tag{29}$$

To interpret this result, let us first consider the risk neutral case, $\gamma = 1$. This would mean that

$$X^*\big|_{\gamma=1} = \frac{\varepsilon}{c\left(\varepsilon-1\right)} I.$$

The fraction $\frac{\varepsilon}{\varepsilon-1}$ stands for the hurdle rate in real options theory, reflecting the value of waiting with investment due to irreversibility and uncertainty. Our formulation, in fact, modifies this threshold value for the fact that the investor is risk averse. We state our finding in the following proposition:

Proposition 2. *It is optimal for the investor to acquire the firm, once the firm's equity exceeds the threshold* X^* given by (29).

We illustrate this result in a short example. We consider the following parameters

$$\gamma = 0.7; \quad \tilde{\sigma} = 0.2; \quad i = 0.1; \quad \delta = 0.2; \quad a = 0.25; \quad \beta = 0.1; \quad I = 1.$$

From this we derive the outcome:

$$\begin{aligned} c &= 1.566; \quad \sigma = 0.25; \quad A = -0.167; \quad \varepsilon = 6.834; \\ D^* &= 0.117; \quad X^* = 0.369; \quad D^*/X^* = 0.319; \quad K\left(X^*\right) = 0.461; \\ V\left(X^*\right) &= 1.114. \end{aligned}$$

The investor is willing to pay $I = 1$, as soon as the equity of the firm has reached $X^* = 0.369$ and the corresponding capital stock has reached $K\left(X^*\right) = 0.461$. This is profitable, since the investor gets a value of the firm of $V(X^*) = 1.114$ for his investment. Hence, there is an option value of waiting $V(X^*) - I = 0.114 > 0$. The trend, A, is negative, i.e. under these parameter values the firm pays high dividends, $D^*/X^* = 0.319$, so that expected growth of equity is negative. Nevertheless, due to the stochastic nature of the problem, equity can also go up and eventually exceed X^*, so that the firm is bought by the investor.

5 Conclusion

The paper is innovative in the sense that it combines continuous investment for capital accumulation with the lumpy investment needed to acquire the firm, while the considered framework is stochastic. As such it combines optimal control with the theory of real options. This paper is the first in a series of analyses that combines both approaches. As the model is now, in optimizing its behavior the managers maximize shareholder value but with the restriction that they ignore the possibility that the firm will be sold to an outside investor. Future work will address a similar framework in which the managers of the firm will explicitly anticipate the existence of this outside investor that has the aim to acquire the firm once its underlying value is big enough.

References

[1] Barucci, E., 1998, Optimal investments with increasing returns to scale, International Economic Review, 39, 789-808.

[2] Davidson, R., Harris, R., 1981, Non-convexities in continuous-time investment theory, Review of Economic Studies, 48, 235-253.

[3] Dechert, D.W., 1983, Increasing returns to scale and the reverse flexible accelerator, Economic Letters, 13, 69-75.

[4] Dixit, A.K., Pindyck, R.S., Investment under Uncertainty, Princeton University Press, Princeton, 1994.

[5] Gould, J.P., 1968, Adjustment costs in the theory of investment of the firm, Review of Economic Studies, 35, 47-56.

[6] Grenadier, S.R., Game choices: the intersection of real options and game theory, Risk Books, London, 2000.

[7] Hartl, R.F., Kort, P.M., 2000, Optimal investments with increasing returns to scale: a further analysis, in E.J. Dockner, R.F. Hartl, M. Luptacik, and G. Sorger (eds.): Optimization, Dynamics and Economic Analysis: Essays in Honor of Gustav Feichtinger, Springer/Physica, 228-240.

[8] Hartl, R.F., Kort, P.M., Novak, A., 2002, Capital Accumulation Model with Debt Financing: The Steigum Model Revisited. in: G. Zaccour (ed.), Optimal Control and Differential Games - Essays in Honour of Steffen Jørgensen, Kluwer, Boston, pp. 23-38.

[9] Hugonnier, Julien and Morellec, Erwan, Real Options and Risk Aversion (September 1, 2007). Swiss Finance Institute Research Paper. Available at SSRN: https://ssrn.com/abstract=422600 or http://dx.doi.org/10.2139/ssrn.422600

[10] Huberts, N.F.D., Huisman, K.J.M., Kort, P.M., Lavrutich, M.N., 2015, Capacity choice in (strategic) real options models: as survey, Dynamic Games and Applications, 5, 424-439.

[11] Jorgensen, S., Kort, P.M., 1993, Optimal dynamic investment policies under concave-convex adjustment costs, Journal of Economic Dynamics and Control, 17, 153-180.

[12] Jorgensen, S., Kort, P.M., 1997, Optimal investment and financing in renewable resource harvesting, Journal of Economic Dynamics and Control, 21, 603-630.

[13] Jorgenson, D.W., 1963, Capital theory and investment behavior, American Economic Review, 52, 247-259.

[14] Anna Loukianova, Egor Nikulin, Andrey Vedernikov, Valuing synergies in strategic mergers and acquisitions using the real options approach, Investment Management and Financial Innovations 14(1-1):236-247, 2017.

[15] Lucas, R.E. jr., 1967, Optimal investment policy and the flexible accelerator, International Economic Review, 8, 78-85.

[16] Steigum, E. jr., 1983, A financial theory of investment behavior, Econometrica, 51, 637-645.