Decision Horizon and Idiosyncratic Risk*

Kyoung Jin Choi[†], Minsuk Kwak[‡], Gyoocheol Shim[§], Wei Wei[¶]

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Abstract

We consider a decision maker's problem in a real option framework with several projects that can only be sequentially undertaken within a "decision horizon" and characterize the optimal sequence of exercises. We show that the length of the decision horizon, time until projects' expiration, directly affects the order of execution of the projects and the risk exposures of the firm. Limited decision horizon is a time constraint that leads to early executions of projects with high idiosyncratic volatility. Consequently, the decision maker's current value is lower and more volatile than when he faces an ample decision horizon. We empirically document that firms with a short (long) decision horizon are associated with high (low) idiosyncratic volatility. We also verify that this relationship depends on the decision makers' exposure to idiosyncratic risk and is stronger for firms more heavily dependent on real option projects. Our paper opens a new perspective on firm investment theory and advocates for more research on the determinants and effects of the decision horizon.

JEL classification: G11; G31; G32; E2

Keyword: Real options; Idiosyncratic volatility; Firm investments; Project selection; Incomplete markets

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†Haskayne School of Business, University of Calgary, 2500 University Drive N.W., Calgary, Alberta, T2N 1N4, Canada, email: kjchoi@ucalgary.ca

‡Department of Mathematics, Hankuk University of Foreign Studies, Yongin, 449-791, Republic of Korea, email: mkwak@hufs.ac.kr, corresponding author.

§Department of Financial Engineering, Ajou University, Suwon, 443-749, Republic of Korea, email: gshim@ajou.ac.kr

¶Haskayne School of Business, University of Calgary, 2500 University Drive N.W., Calgary, Alberta, T2N 1N4, Canada, email:wei.wei@haskyane.ucalgary.ca, corresponding author.

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1 Introduction

Decision makers (e.g., CEOs, managers, and entrepreneurs) are often presented several new projects (business plans) with positive net present values (NPVs). Due to limited resource or attention, they can not carry out all projects at once. Rather than executing some projects arbitrarily in front, it is more optimal for them to select the best projects and observe how the projects' values evolve then execute them along the way. Like random walks, some projects may become badly perceived while others become more promising as time goes by. Gradually, the decision makers sequentially execute projects whose values grow sufficiently high. The question is what type of projects will be executed first. Given the various constraints faced by the decision makers, how should they order the executions of the projects?

Our model suggests that the optimal order of execution depends on the idiosyncratic volatilities of the projects and the "decision horizon", which is the time allowed for execution before the projects expire. When a risk-averse decision maker is allowed a short (long) time to implement all projects, he will try to execute the projects with high (low) volatility first. The key to understand this result is to recognize that the degree of convexity of the project payoffs depends on the time until expiration. Consequently, the risk attitude of the decision maker depends on the decision horizon. More specifically, in an incomplete market, a risk-averse decision maker bears a considerable amount of idiosyncratic risk from the projects he executes (see for example Miao and Wang (2007) and Chen, Miao, and Wang (2010)). On one hand, the concavity from the utility function makes the decision maker favor projects with low idiosyncratic volatility. On the other hand, the convexity from the payoffs of the real option projects makes him favor projects with high idiosyncratic volatility. Since the convexity of the project payoffs increases with the length of the decision horizon, convexity (concavity) dominates concavity (convexity) when decision horizon is long (short) (see Figure 8 for a simple illustration). Therefore, the decision maker acts as if he were more (less) risk-averse when decision horizon is shortened (extended).

In practice, time constraints (decision horizons), which alter our risk attitudes, can arise from many aspects. First, the natural and legal environments oftentimes impose time constraints on a business decision. A typical example is a petroleum and gas (PNG) drilling license. An upstream company[1] in the exploration and production (E&P) sector enters the waiting stage for drilling a

[1]The American Petroleum Institute divides the petroleum industry into five sectors: upstream, downstream, pipeline, marine, and service and supply. The upstream sector includes searching for potential underground or underwater crude oil and natural gas fields, drilling exploratory wells, and subsequently drilling and operating the wells.

well after acquiring a license for a specific tract from a local government. Then, the company must explore the tract and start to drill the well within a predetermined period of time,[2] otherwise the drilling license will expire, which means the decision horizon ends.

In other cases, time constraints are imposed by external financing conditions. For example, a business venture has a finite decision horizon because venture capital financing is staged and the capitalist will exit after a certain period of time.[3] Hence, the entrepreneur has to execute the best projects within the financing period.

Furthermore, firms in industries that depend on differentiated goods often have to undertake new projects before their competitors have a chance to come up with similar products. Or an entrepreneur in a R&D-intensive firm often has to implement a new project within a short period of time since other firms can quickly identify similar or superior ideas, which effectively forces her investment opportunity to expire. Similarly, if the entrepreneur holds several patents, which are the main source of generating current revenue, she needs to invest to create new patents before the existing ones expire.

Finally, even on the human capital side, the time constraint is an important factor. For example, employee stock options (ESO) have the same feature as real options with uninsurable risk and a finite horizon.[4] CEO tenure can also be interpreted as a time constraint (decision horizon) which can affect the firm's strategy. And in the world of academics, the tenure clock is a time constraint that might influence the types of papers academics work on.

In any of the above cases, the finite nature of the decision time imposes a constraint on our optimization problems. This time constraint is just as important as the resource (budget) and financial constraints which have been extensively researched in the literature. This paper is a first attempt to provide a tractable model to tackle the impacts of time constraints. For simplicity we consider an entrepreneur's problem. Our model departs from the standard real option model by incorporating the following three key features: multiple real options, a finite decision horizon, and idiosyncratic risk. Since the full model is fairly challenging to characterize all at once, we first consider a simple model whereby the entrepreneur has a single option, and we extend it into

[2] For example, the initial term of the licenses in Alberta, Canada is (a) two years if the location is in the Plains Region, (b) four years if the location is in the Northern Region, or (c) five years if the location is in the Foothills Region. The initial lease term of offshore central Gulf of Mexico is (a) five years in water depths less than 800 meters, (b) seven years in water depths between 800 and 1600 meters, and (c) 10 years in water depths more than 1600 meters. This kind of license expiration rule is common in many countries.

[3] Most venture capital funds have a 10-year life.

[4] There is uninsurable risk of holding ESOs since the CEO is not allowed to freely trade the underlying stock. ESOs usually have a maximum maturity of 10 years from the date of issue.

the general model in which the entrepreneur has several projects (options) with different levels of idiosyncratic risk that can be sequentially executed within a finite decision horizon. We explain how the length of the decision horizon changes the entrepreneur's optimal order of execution and hence affects the idiosyncratic volatility of the entrepreneur's overall value.

Our paper builds upon the classical real investment model as in McDonald and Siegel (1986) and is closely related to the current literature on the effects of uninsurable idiosyncratic risk in private business's investment decisions (e.g., Henderson (2007), Miao and Wang (2007), and Chen, Miao, and Wang (2010)). McDonald and Siegel (1986) prove that in a complete market, an increase in the payoff volatility always increases the real option value hence delays the execution of the project. Incorporating finite maturity does not change any theoretical predictions from their infinite horizon setup. The reason is that in a complete market, the decision maker only benefits from the convexity of the payoff hence is always risk-loving. An increase in the decision horizon only makes the decision maker favor riskier project even more. Miao and Wang (2007) show that under incomplete markets, the decision maker faces a trade-off between the concavity of the utility function and the convexity of the project payoffs. Hence, an increase in the idiosyncratic volatility of the project may result in early execution of the project. Our paper points out that the decision horizon is crucial in determining the dominance between concavity and convexity and the overall risk attitude of the decision maker. We also show that in a model with sequential executions of projects this finite time constraint (decision horizon) determines the order of execution of the real investment opportunities and generates novel investment dynamics.

Our model suggests that the decision maker tends to execute higher (lower) idiosyncratic volatility projects first when facing a short (long) decision horizon. In addition, we show that the decision maker's current value is more (less) sensitive to the projects' idiosyncratic volatilities when facing a short (long) decision horizon. Since the idiosyncratic volatility of the decision maker's value function is the sum of the squared products of each project's volatility and decision makers' sensitivity to the idiosyncratic component of the project payoff, our model predicts that firms with a short (long) decision horizon have high (low) idiosyncratic volatility.[5] Moreover, if the project execution decisions are what determine the firms' idiosyncratic volatility, the inverse relationship between the decision horizon and volatility should be stronger for firms that are more dependent on real option type of projects. Finally, according to our model, the decision makers' exposure

[5]We believe this prediction can be extended to the idiosyncratic volatility of firm stock returns. Cao, Simin, and Zhao (2008) and Grullon, Lyandres, and Zhdanov (2012) also suggest that a firm's real option holdings and decisions are naturally reflected in its stock return dynamics.

to a firm's idiosyncratic risk should play a major role in this inverse relationship. That is, the relationship should be strong in firms with decision makers who are highly exposed to idiosyncratic risk. The relationship should be hard to detect if decision makers in the firms are barely exposed to idiosyncratic risk.

We empirically show that public firms behave consistently according to the above model predictions during the sample period from 1997 to 2015 with data from CRSP, Compustats, Thomson Reuters, and Hoberg et al. (2014). Although, in the model, the decision maker (entrepreneur) is assumed to be risk-averse and exposed to idiosyncratic risk, our theory not only fits private firms[6] but also describes project execution decisions in public firms since many public firms are controlled and operated by members of a founding family with concentrated ownership, as pointed out by Chen, Miao, and Wang (2010).[7] In non-family-owned public firms, the decision makers are also exposed to the projects' risk when they have a considerable stake in the firms. In theory, the decision makers' exposure should be proportional to the ratio of the wealth and income they derive from the firms to their total wealth. Due to data limitations, we use the ratio of company stocks owned by them to total shares outstanding (insider ownership) as a proxy for their exposure to the idiosyncratic risk of their firms. We are able to verify that the decision makers' exposure to idiosyncratic risk is crucial in the relationship between the decision horizon and idiosyncratic volatility.

Our paper contributes to the literature in three ways. First, our sequential option model opens up new ways for considering the optimal order of executions when the decision makers face various real investment opportunities. To the best of our knowledge we are the first to model the time constraint (decision horizon) and consider its effect on project order selection. Our model points out that due to the time constraint, an entrepreneurial firm not only receives less value from the projects, but also is forced to bear a higher idiosyncratic risk. The short decision horizon constrains the entrepreneur from enjoying the benefit of a limited downside risk since the firm with a short decision horizon is likely to prematurely exercise projects with a potentially high implied option value. Moreover, the firm's current value is more sensitive to changes in the option value of the projects when the decision horizon is short. Along with the fact that the firm is forced to execute high volatility projects early, the firm's current return is more volatile. Hence, our paper can speak

[6] The decision makers (owners) of private firms are risk-averse individuals whose income and wealth heavily depends on the performance of the firms; hence they bear a large part of the firm's idiosyncratic risk, which can not be fully diversified away.

[7] For example, Anderson and Reeb (2003) observe, by using Standard & Poor's 500 firms from 1992 through 1999, that founding families are a prevalent and important class of investors. It shows that family firms constitute over 35% of the S & P 500 Industrials and, on average, families own nearly 18% of their firms' outstanding equity (see also Bennedsen et al. (2007) and Miller et al. (2007)).

specifically about how exactly and to what degree firms can be worse off due to a shortening of the decision horizon. Given the prevalence of the time constraint in all individual and firm decision problems, perhaps more convoluted individual behaviors and firm policies can be understood by considering the effects of time constraints.

Second, our model brings a new perspective to the real option investment literature by showing that the time horizon matters in a real option framework. It explains how the finite time horizon can affect the implied real option values of projects differently depending on idiosyncratic volatilities. Furthermore, the model explains that the decision maker can change her risk-taking behavior depending on the length of time that remains until the end of the decision horizon. Consequently, the dynamics of the firm's value (e.g., cash flow or stock return dynamics) are affected by economic factors that influence the decision horizon, such as the natural resource development license period, the venture capital financing timing, the remaining time on a patent, the product life cycle, the competition pressure on a new product, the tenure of the CEO or other key officers, and so on. This latter point can have many policy implications at the government and institutional levels.

Finally, we confirm in the data that the decision horizon is negatively related to idiosyncratic volatility. Moreover, the inverse relationship crucially depends on the decision makers' exposure to idiosyncratic risk and is stronger for firms more heavily dependent on real option projects. These findings are related to the literature on aggregate idiosyncratic volatility (see e.g., Morck, Yeung, and Yu (2000),Campbell, Lettau, Malkiel, and Xu (2001), and Brandt, Brav, Graham, and Kumar (2010)). Our paper offers a potential alternative explanation from the angle of a firm's real option investment decisions[8] to the debate on the pattern of increasing idiosyncratic volatility. The pattern in idiosyncratic volatility may be related to the change of firm decision horizons as technologies advance and market competition evolves.[9] Our results may also add to the literature on the link between idiosyncratic volatility and returns[10] by pointing out the decision horizon as an important determinant for firm-level idiosyncratic volatility.

[8]Cao, Simin, and Zhao (2008) have a similar aspect with our paper in the option pricing perspective. Cao, Simin, and Zhao (2008) empirically show that the trends in idiosyncratic risk are related to growth options, using the option pricing framework of Galai and Masulis (1976) viewing equity as the call option on the firm's value (Merton (1974)). However, our approach hinges on the decision horizon rather than on the option pricing.

[9] Similar views are expressed by Gaspar and Massa (2006) and Irvine and Pontiff (2009), who argue that the recent increase in idiosyncratic return volatility is attributed to the increase in the product market competition. We use fluidity as a proxy for the decision horizon. Fluidity is partly related to product market competition. Hence, in a way, our empirical analysis connects the industry-level result of Gaspar and Massa (2006) and Irvine and Pontiff (2009) to the individual firm-level case.

[10]Since the seminal work done by Ang et al. (2006), many others have offered explanations for the relationship between past idiosyncratic volatility and future returns. For example, see Ang et al. (2009), Fu (2009), Chen and Petkova (2012), Stambaugh et al. (2015) and Hou and Loh (2016)).

We hope our paper will encourage more studies on time constraints (decision horizon). The effects of the decision horizon does not stop at idiosyncratic volatility. Plenty of new effects on firm characteristics and policies are waiting to be discovered. In the current literature, we know little about the determinants of decision horizons and their effects. In this paper, we focus on the fact that the competitors' time and ability to undertake the project will inversely affect a firm's decision horizon. For that, *Fluidity*, suggested by Hoberg et al. (2014), is a good measure.[9] Nevertheless, more direct measures, perhaps from survey data, are helpful to advance our understanding of the decision horizon and to further investigate the effects of the decision horizon on firm's policies.

The rest of the paper unfolds as follows. Section 2 presents the baseline model and the solution. It explains the properties of the solution and provides baseline ideas that will be used and extended for the general case presented in Section 3. Then, Section 3 explains the main sequential option model and solution analysis including its implications. In Section 4, we discuss our empirical designs and explain our choice of main variables. We present the testing results for each of our model predictions. Section 5 concludes the paper. All the proofs are provided in Appendix A. The robustness of our results, including other extensions, is discussed in Appendix B.

2 The Baseline Model: Single Option Case

2.1 Model

Tractability is a big challenge in incomplete market models like ours because the utility-based approach is required.[11] In order to have an explicit solution, we construct a discrete time model incorporating a random walk binomial approach similar to that of Detemple and Sundaresan (1999). In particular, the implied option value, which is key part of the value function, can be recursively obtained by using the binomial setup. A simple continuous time version of the model is presented in Appendix B.3.

The baseline model is the case in which the decision maker has a single option to exercise. The model, solution analysis, and intuition developed in this model will be used for generalization to the main sequential model in the next section. Specifically, we consider a consumption and investment

[11] Explicit solutions are hardly obtained in standard continuous time models of individual's consumption and the investment problem (see Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998), Svensson and Werner (1993), Henderson (2005) and references therein). We also have tried several versions of continuous time models including random horizon models and find that we eventually need to discretize the corresponding Hamilton-Jacobi-Bellman (HJB) equation. In fact, by doing so we reached a similar discrete time setup as in the current paper, while there is no guarantee for the convergence to a solution to the HJB equation. The current discrete time model provides more concrete closed form solutions.

problem of an infinitely-lived economic agent having a real option with finite maturity T. There are N periods with step size h. Letting $t_k = kh$, for $k = 0, 1, \cdots, N$, we have $T = t_N = Nh$. Let $(b^1, b^2) \triangleq (b^1_{t_k}, b^2_{t_k})_{k=0,1,\cdots,\infty}$ be a two-dimensional binomial random walk on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t_k})_{k=0}^N, \mathbb{P})$. The random walk is symmetric under $\mathbb{P}$, that is,

$$\mathbb{P}(\Delta b^i_{t_k} = 1) = \mathbb{P}(\Delta b^i_{t_k} = -1) = 1/2, \quad i = 1, 2. \tag{1}$$

There are one risk-free asset and one risky asset in the financial market. The risk-free rate over one period, h, is constant $r > 0$. Let $P_k \triangleq P_{t_k}$, $k = 0, 1, \cdots$, be the price of the risky asset at time t_k and satisfy

$$\frac{P_{k+1}}{P_k} = \exp\left(\alpha h + \Sigma\sqrt{h}\Delta b^1_{t_k}\right), \ P_0 = p_0 > 0,$$

where $\alpha > r$ and $\Sigma > 0$ are assumed to be constant. Let $X_k \triangleq X_{t_k}$, $k = 0, 1, \cdots$, be the payoff process of the real option that follows

$$\Delta X_k = X_{k+1} - X_k = \alpha_x h + \rho\sigma_x\sqrt{h}\Delta b^1_{t_k} + \sqrt{1-\rho^2}\sigma_x\sqrt{h}\Delta b^2_{t_k}, \quad X_0 = x, \tag{2}$$

for $k = 0, 1, \cdots, N-1$ and constant $\alpha_x \in \mathbb{R}$, $\sigma_x > 0$, and $\rho \in (-1, 1)$. Since $\rho \in (-1, 1)$, the idiosyncratic risk of the real option cannot be fully hedged by the risky asset in the financial market. Notice that the underlying payoff process follows an arithmetic Browinan motion. Our result is robust to the extension to the geometric Brownian case (see Appendix B.2).

For simplicity of exposition, let us define

$$\begin{aligned}
&\alpha_u \triangleq \exp\left(\alpha h + \Sigma\sqrt{h}\right), \quad \alpha_d \triangleq \exp\left(\alpha h - \Sigma\sqrt{h}\right), \\
&u_1 \triangleq \alpha_x h + \rho\sigma_x\sqrt{h} + \sqrt{1-\rho^2}\sigma_x\sqrt{h}, \quad u_2 \triangleq \alpha_x h + \rho\sigma_x\sqrt{h} - \sqrt{1-\rho^2}\sigma_x\sqrt{h}, \\
&d_1 \triangleq \alpha_x h - \rho\sigma_x\sqrt{h} + \sqrt{1-\rho^2}\sigma_x\sqrt{h}, \quad d_2 \triangleq \alpha_x h - \rho\sigma_x\sqrt{h} - \sqrt{1-\rho^2}\sigma_x\sqrt{h}.
\end{aligned}$$

Then, by (1), we have

$$\begin{aligned}
&\mathbb{P}\left(\frac{P_{k+1}}{P_k} = \alpha_u, \ \Delta X_k = u_1\right) = \mathbb{P}\left(\frac{P_{k+1}}{P_k} = \alpha_u, \ \Delta X_k = u_2\right) \\
&= \mathbb{P}\left(\frac{P_{k+1}}{P_k} = \alpha_d, \ \Delta X_k = d_1\right) = \mathbb{P}\left(\frac{P_{k+1}}{P_k} = \alpha_d, \ \Delta X_k = d_2\right) = 1/4.
\end{aligned}$$

Assumption 1. *The following condition is necessary to exclude arbitrage:*

$$\alpha_d < 1 + r < \alpha_u, \quad \frac{\alpha_u + \alpha_d}{2} > 1 + r. \tag{3}$$

Let $W_k \triangleq W_{t_k}$, $k = 0, 1, \cdots$, be the agent's wealth process and τ be the exercise time of the real option, which is a stopping time. Then, at exercise time τ, the agent receives lump-sum payment X_τ by paying the investment cost I, which is assumed to be constant. Thus, the agent's wealth increases by $X_\tau - I$ right after the exercise of the real option. For $k = 0, 1, \cdots$, let $\pi_k \triangleq \pi_{t_k}$ be the amount of money the agent invests in the risky asset at time t_k, and $c_k \triangleq c_{t_k}$ be the consumption at time t_k. Then, the agent's wealth process W_k, $k = 0, 1, \cdots$ with the initial wealth $W_0 = w$ evolves according to

$$W_{k+1} = (1 + r)(W_k - c_k - \pi_k) + \pi_k \exp\left(\alpha h + \Sigma\sqrt{h}\Delta b^1_{t_k}\right) + (X_\tau - I)\mathbf{1}_{\{t_k = \tau\}}. \tag{4}$$

Now, at time t_k, the agent's problem is to maximize her expected utility:

$$V(w, x, k) \triangleq \max_{\{c_j\}_{j \geqslant k}, \{\pi_j\}_{j \geqslant k}, \tau} \mathbb{E}\Big[\sum_{j=k}^{\infty} \beta^{j-k} U(c_j) | W_k = w,\ X_k = x\Big],$$

subject to (2) and (4). $V(w, x, k)$ is the value function at time t_k if the wealth at time t_k is w and the value of the underlying process of the real option at time t_k is x. $\beta \in (0, 1)$ is the subjective discount factor of the agent's utility and is assumed to be constant. We consider the constant absolute risk aversion (CARA) utility function with a coefficient of absolute risk aversion $\gamma > 0$, that is, $U(c) = -\exp(-\gamma c)/\gamma$. For simplicity, we assume $\beta(1 + r) = 1$.[12]

The approach to find the value function $V(w, x, k)$ is to compute the value at maturity first and to go backward by comparing the continuation value and the exercise value. The details are as follows. We define $V_e(w, k)$ which is the maximized expected utility at time t_k with wealth w if the real option is already exercised or expired. Then, $V_e(w, k)$ is the value function of a standard

[12]We can obtain all the results without this assumption.

Merton problem (without real options) and thus is written as

$$V_e(w,k) = \max_{\{c_j\}_{j\geqslant k},\{\pi_j\}_{j\geqslant k}} \mathbb{E}\Big[\sum_{j=k}^{\infty}\beta^{j-k}U(c_j)|W_k = w\Big], \tag{5}$$

subject to

$$W_{j+1} = (1+r)(W_j - c_j - \pi_j) + \pi_j \exp\left(\alpha h + \Sigma\sqrt{h}\Delta b^1_{t_j}\right), \quad j \geqslant k.$$

For $k = 0,1,\cdots,N-1$, let $V_n(w,x,k)$ be the maximized expected utility if the option is not exercised until time t_k and the wealth at time t_k is w and the value of the payoff process at time t_k is x. Then, $V_n(w,x,k)$ is given by

$$V_n(w,x,k) = \max_{c_k,\pi_k}\mathbb{E}\Big[U(c_k) + \beta V(W_{k+1},x_{k+1},k+1)|W_k = w,\ X_k = x\Big],$$

subject to

$$W_{k+1} = (1+r)(W_k - c_k - \pi_k) + \pi_k \exp\left(\alpha h + \Sigma\sqrt{h}\Delta b^1_{t_k}\right),$$

$$X_{k+1} = X_k + \alpha_x h + \rho\sigma_x\sqrt{h}\Delta b^1_{t_k} + \sqrt{1-\rho^2}\sigma_x\sqrt{h}\Delta b^2_{t_k}.$$

Note that if $W_{t_k} = w$ and $X_{t_k} = w$ at time t_k, the exercise value is $V_e(w+(x-I)^+,k)$[13], whereas the continuation value is $V_n(w,x,k)$. Thus, the value function $V(w,x,k)$ is determined as

$$V(w,x,N) = V_e(w+(x-I)^+,N) \quad \text{and}$$
$$V(w,x,k) = \max\left(V_e(w+(x-I)^+,k), V_n(w,x,k)\right), \quad k = 0,1,\cdots,N-1.$$

2.2 Solution: Value Function and Exercise Threshold

Now we summarize the explicit solution for the value function as follows.

Theorem 1. *For $k = 0,1,\cdots,N$, the value function is given by*

$$V(w,x,k) = \frac{1+r}{r}U\left(\frac{r}{1+r}(w + K + Y_k(x))\right), \tag{6}$$

where

$$K = -\frac{1+r}{\gamma r^2}\ln\left[\frac{1}{2}\left\{\frac{\alpha_u-\alpha_d}{(1+r)-\alpha_d}\left(\frac{(1+r)-\alpha_d}{\alpha_u-(1+r)}\right)^{\frac{\alpha_u-(1+r)}{\alpha_u-\alpha_d}}\right\}\right]$$

[13] $X^+ = \max(X,0)$.

and $Y_k(x)$ is the implied option value defined recursively as follows:

$$
\begin{aligned}
Y_N(x) &= (x-I)^+ = \max(x-I,0),\\
Y_k(x) &= \max\left\{ -\frac{1}{\gamma r}\ln\left[\left(\frac{e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+u_1)}+e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+u_2)}}{2}\right)^{\frac{(1+r)-\alpha_d}{\alpha_u-\alpha_d}}\times\right.\right.\\
&\qquad \left.\left.\left(\frac{e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+d_1)}+e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+d_2)}}{2}\right)^{\frac{\alpha_u-(1+r)}{\alpha_u-\alpha_d}}\right], (x-I)^+\right\},
\end{aligned}
\tag{7}
$$

for $k = N-1, N-2, \cdots, 0$.

Proof. See the Appendix. □

Note that the implied option value $Y_k(x)$ is the certainty equivalent in the CARA setup. It is intuitively straightforward to see that the implied option value is decreasing in time (k). We characterize how the implied option value is sensitive to the size of idiosyncratic volatility σ_x in Section 2.4. This property is important to understand the result of the main (sequential option) model.

The optimal consumption, risky investment, and exercise threshold level, $(\bar{c}_k(w,x), \bar{\pi}_k(w,x), \bar{x}_k)$, are explicitly presented in the following theorem.

Theorem 2. *Define $\bar{x}_k$ recursively as follows: $\bar{x}_N = I$. For $k = N-1, N-2, \cdots, 0$, $\bar{x}_k$ is the unique solution of the following equation:*

$$
\begin{aligned}
(\bar{x}_k - I)^+ = -\frac{1}{\gamma r}\ln\Big[\Big(&\frac{e^{-\frac{\gamma r}{1+r}Y_{k+1}(\bar{x}_k+u_1)}+e^{-\frac{\gamma r}{1+r}Y_{k+1}(\bar{x}_k+u_2)}}{2}\Big)^{\frac{(1+r)-\alpha_d}{\alpha_u-\alpha_d}}\times\\
&\Big(\frac{e^{-\frac{\gamma r}{1+r}Y_{k+1}(\bar{x}_k+d_1)}+e^{-\frac{\gamma r}{1+r}Y_{k+1}(\bar{x}_k+d_2)}}{2}\Big)^{\frac{\alpha_u-(1+r)}{\alpha_u-\alpha_d}}\Big],
\end{aligned}
\tag{8}
$$

where $Y_{k+1}(\cdot)$ is defined in Theorem 1. Then, at time t_k, it is optimal to exercise the real option if

$$X_k \geqslant \bar{x}_k, \quad k = 0, 1, \cdots, N.$$

Moreover, if the real option is not exercised before time t_k, the optimal consumption $\bar{c}_k(w,x)$ and investment $\bar{\pi}_k(x)$ at time t_k with wealth w and the value of payoff x are given as follows:

$$\bar{c}_k(w,x) = \frac{r}{1+r}(w + K + Y_k(x)), \tag{9}$$

$$\bar{\pi}_k(x) = \begin{cases} \frac{1+r}{\gamma r(\alpha_u-\alpha_d)} \ln\left(\frac{\alpha_u-(1+r)}{(1+r)-\alpha_d}\right) + \frac{1+r}{\gamma r(\alpha_u-\alpha_d)} H(x,k,\rho), & x < \bar{x}_k, \\ \\ \frac{1+r}{\gamma r(\alpha_u-\alpha_d)} \ln\left(\frac{\alpha_u-(1+r)}{(1+r)-\alpha_d}\right), & x \geqslant \bar{x}_k. \end{cases} \tag{10}$$

where the intertemporal hedging term $H(x,k,\rho)$ *is defined by*

$$H(x,k,\rho) \triangleq \ln\Big(\frac{e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+u_1)} + e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+u_2)}}{e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+d_1)} + e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+d_2)}}\Big). \tag{11}$$

Proof. See the Appendix. □

See Lemma 1 in the Appendix for the comparison between the optimal policies before and after the option exercise. First, note that if there were no real option, the optimal consumption would not have the implied option component. In other words, the agent can increase the consumption by $\frac{r}{1+r}Y_k(x)$, by smoothing out the future income, generated by exercising the option over time. The optimal risky investment, when there is no real option, is constant. With the real option, there is an additional component, i.e. the intertemporal hedging demand against the future income risk, as seen in (10). We discuss this hedging component in Appendix B.1. More specifically, we investigate how the hedging need affects the exercise threshold.

The following remark is the immediate consequence of the implied option value $Y_k(x)$, which is decreasing in both time and risk aversion in (7) by using the property of the log and exponential functions.

Remark 1. *The optimal exercise threshold* $\bar{x}_k$ *defined in* (8) *has the following properties:*

(a) $\bar{x}_k$ *is decreasing in time (k).*

(b) An increase in risk aversion (γ) *decreases* $\bar{x}_k$.

2.3 Idiosyncratic Volatility Over the Decision Horizon

This section investigates how the firm's idiosyncratic volatility changes over the decision horizon. In our discrete time framework, the idiosyncratic volatility is the percent change of the agent's value relative to Δb^2, the idiosyncratic component of the random walk in (1). For a simple example, consider the case in which $\rho = 0$. Then, the size of idiosyncratic volatility is defined by

$$Var\left(\frac{V(w,x+\xi,k) - V(w,x,k)}{V(w,x,k)}|(w,x)\right),$$

where $Var(\cdot|(w,x))$ means the conditional variance given (w,x) and ξ is the random variable that takes either u_1 or u_2.[14] To compute the exact form of the variance is quite complicated. However, if the length of interval h is small enough, then we can use the continuous time approach as an approximation. For small h, the payoff process in equation (2) can be rewritten as the following arithmetic Brownian motion:

$$dX = \alpha_x dt + \rho\sigma_x dB_t + \sigma_x\sqrt{1-\rho^2}dB_t^x,$$

where B is the systematic component and B^x is the idiosyncratic component of the risk. Since the value function should be written as $V = V(w, x, T-t)$, by using Ito's lemma the volatility part of the return $\frac{dV}{V}$ with respect to B^x is $\frac{\partial V/\partial x}{V}\sigma_x$. Then the magnitude of the idiosyncratic volatility is $\left(\frac{\partial V/\partial x}{V}\sigma_x\right)^2$, which is the product of the idiosyncratic part of the project volatility and the agent's sensitivity to the project payoff (in particular, the idiosyncratic part of the payoff). Using (6) in Theorem 1, a bit of algebra gives us

$$\left(\frac{\partial V/\partial x}{V}\sigma_x\right)^2 = \left(\frac{\gamma r}{1+r}\right)^2 Y_k'(x)^2\sigma_x^2 \tag{12}$$

since $U'(\cdot) = -\gamma U(\cdot)$. In other words, to understand the size of the idiosyncratic volatility in time, it is sufficient to characterize $Y_k'(x)$ in time.

Proposition 1. $0 < Y_k'(x) < 1$ *for* $x < \bar{x}_k$. *Furthermore,* $\lim_{k\to N} Y_k'(x) = 1$ *for* $I < x$.

Proof. See the Appendix. □

See Theorem 2 and Remark 1 about the exercise threshold $\bar{x}_k$. We believe that $Y_k'(x)$ monotonically increases in k because the intuition is quite clear, as follows. The implied option value $Y_k(x)$ is the sum of the time value and the intrinsic value. As the remaining time becomes shorter, the time value of the option becomes smaller and $Y_k(x)$ becomes closer to the intrinsic value $(x-I)^+$. Thus, as k increases, $Y_k'(x)$ becomes closer to 1 if the option is in the money. We have confirmed this monotone increasing property of $Y_k'(x)$ by numerical examples with a wide range of parameter sets. Figure 1 is a typical example. Consequently, the firm's idiosyncratic volatility tends to increase, as k increases or as the remaining decision time becomes shorter (see (12)).

There are two more comments. First, Proposition 1 implies that the implied option value, specially when in-the-money or near at-the-money, changes more relative to a dollar increase in

[14] $u_1 = d_1 = \alpha_x h + \sigma_x\sqrt{h}$ and $u_2 = d_2 = \alpha_x h - \sigma_x\sqrt{h}$ if $\rho = 0$.

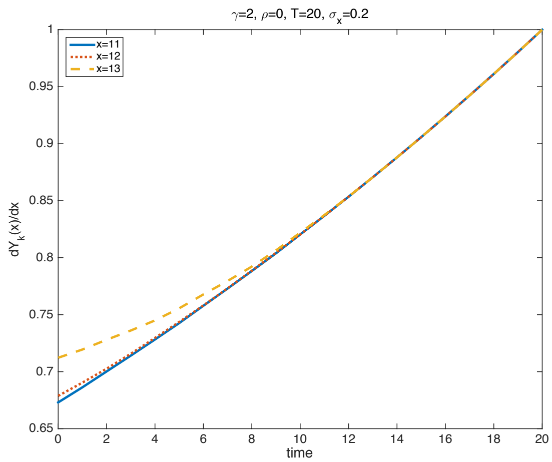

Figure 1: $Y_k'(x)$ in time for different values of x. The maturity of the project is T=20.

the underlying asset value when maturity is short than when maturity is long. In other words, the option delta with shorter maturity tends to be higher for in the money or near at-the-money options. Notice that the sensitivity is not about the systematic risk, but about the idiosyncratic risk in our model and the concavity of the utility function plays an important role as well as the convexity of the payoff. We will show that this intuition can be extended and applied to the more general case of the sequential model in Section 3.

Second, Proposition 1 does not hold when x is smaller than I (out-of-the-money case). One might be concerned that our empirical result is affected if there are firms having this type of option. On the contrary, note $I < x$ in the latter part of Proposition 1, which means that the result is applied for any options in-the-money or near at-the-money. Firms normally do not consider the real options with negative net present values. Furthermore, we do not explicitly model the cost of creating options in this paper while in reality there exists an initial option-creating cost. For example, it is not free to obtain the PNG drilling license. A manager or an entrepreneur should be fairly reluctant to *create* a currently out-of-the-money option or such a project if the firm needs to spend a nontrivial amount for the option creation. Therefore, we believe that the concern about out of the money options does not matter much for our empirical results.

2.4 Implied Real Option Value

This section investigates how sensitive the real option value is to the size of idiosyncratic risk depending on the remaining time. More precisely, we explain how and why the decision maker's revealed risk attitude toward the idiosyncratic risk changes in time. The fundamental result is that the implied option value is higher for a option with a higher (lower) idiosyncratic risk when the remaining time is long (short). In particular, this result is used as a key idea in understanding how to determine the sequence of the project execution in the main sequential option model (see Section 3.3).

On the one hand, as in a common financial (call) option, the payoff $(X_\tau - I)^+$ of the real option is a convex function of the underlying asset value X_τ at the exercise time τ, which means it is risk-preferred. On the other hand, the implied option value is obtained by the maximization of the risk-averse agent's lifetime utility, that is, the utility function is concave in consumption so that the value function $V_e(w)$ after exercising the option is a concave function. Thus, the two effects of the risk-preferred convex payoff and the risk-averse agent's concave utility compete with each other. The agent's risk attitude revealed by the implied option value toward the idiosyncratic risk will depend on which of the two effects dominates the other. If the current value of the underlying asset is far below the cost I, i.e., the real option is out-of-the-money (OTM), then there is no downside risk and only the opportunity of the upside benefit increases with high idiosyncratic risk. Thus, in this case, high risk will be preferred, that is, the convexity effect always dominates the concavity effect, as shown in Figure 2: the higher the idiosyncratic risk (σ_x), the higher the implied option value. That is, in this case, the agent's risk attitude revealed by the implied option value is risk-loving.

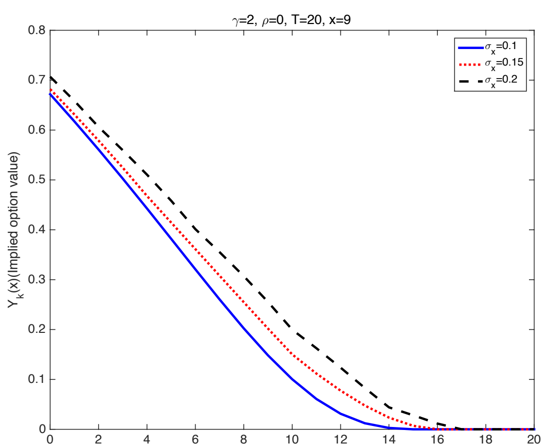

Figure 2: The implied option value as a function of time for different levels of idiosyncratic risk: OTM case with $x = 9$ and $I = 10$. The maturity of the projects is T=20.

If the current value of the underlying asset is near or above the cost I, i.e., the real option is at

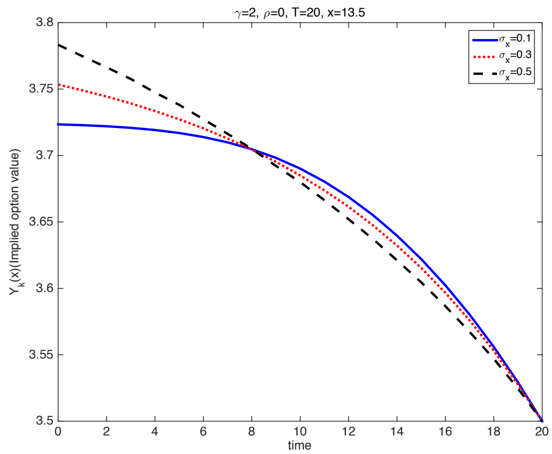

Figure 3: The implied option value as a function of time for different levels of idiosyncratic risk

or in-the-money, then both the downside risk and the opportunity of an upside benefit exist with the idiosyncratic risk, which is a more realistic and interesting case as we pointed out at the end of Section 2.3.[15]

If the time to maturity is long, then the agent has enough time to enjoy the opportunity of a lot of upside benefit with the loss limited at zero, and the exercise threshold will be relatively high, as stated in Remark 1 so that the agent prefers the high risk to reach the high threshold, that is, the convexity effect will dominate the concavity effect. If the time to maturity is short, then the advantage of limited downside risk is less than in the case of long horizon and the threshold is relatively low and the current value of the underlying asset is close enough to the threshold so that the increase of the underlying asset value by the positive drift with less risk will be preferred. Figure 3 illustrates a typical case of this. As shown in Figure 3, the implied real option value decreases much more sharply as the time approaches maturity when the idiosyncratic risk is high than when the risk is low. Thus, two curves cross each other at some point in time before maturity and the implied option value curve with higher σ_x becomes lower than the one with lower σ_x. In other words, the implied real option value increases (decreases) in the idiosyncratic risk if the time to maturity is long (short). This means that the agent's risk attitude, represented by the implied option value, is risk loving (averse) when the time to maturity is long (short).

Notice that a high degree of the agent's risk aversion makes the precautionary savings motive stronger (Miao and Wang (2007) and Henderson (2007)). This risk aversion effect can be observed

[15] Note that we use $I = 10$ in most of the numerical examples and figures in the paper.

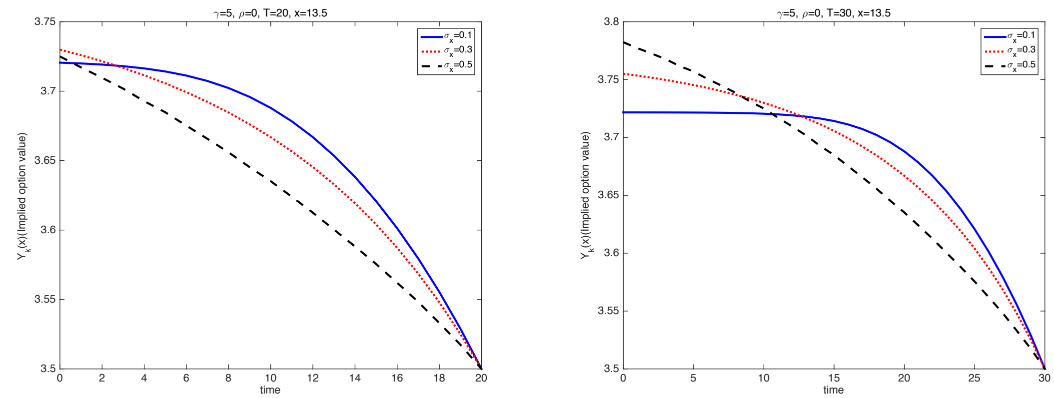

Figure 4: The implied option value as a function of time for different levels of idiosyncratic risk: The case of intermediate risk aversion with short and long horizons

by comparing Figure 3 when $\gamma = 2$, the left panel of Figure 4 when $\gamma = 5$, and the left panel of Figure 5 when $\gamma = 10$. The time horizon effect seems to disappear as the coefficient of risk aversion increases. However, the right panels of Figures 4 and 5 show the similar pattern as in Figure 3. More explicitly, there is a cross-over of the curves if we set a larger T. This result confirms that the precautionary savings motive becomes stronger as time to maturity becomes shorter.

Note that so far the results are explained in terms of the implied option value. One might be curious about the threshold dynamics or the exercise boundary. In fact, the properties of the exercise boundary are very similar to those of the implied option value. This also helps to understand the comparison between the complete market real option literature and the incomplete market real option literature. We present the results regarding the threshold dynamics in Appendix B.1.

Before we conclude this section, we emphasize that the decision horizon effect, explained above, does not exist in the complete market. As an example, we provide Figure 6 showing American call option values in time (with dividends). The figure shows that higher volatility is always preferred regardless of the time to maturity or the moneyness of the option. In our real option case, if the option is out-of-the-money, the option with a higher idiosyncratic risk is preferred. However, if the option is in the money, the option with a lower idiosyncratic risk can be preferred depending on the remaining time. This comparison between the complete market case and our case confirms that the limited decision horizon is much more important for the case the decision maker faces idiosyncratic risk.

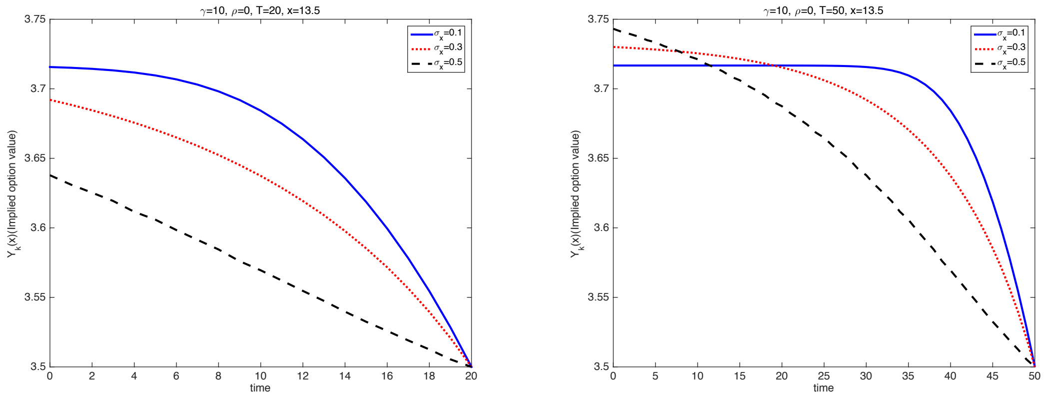

Figure 5: The implied option value as a function of time for different levels of idiosyncratic risk: The case of high risk aversion with short and long horizons

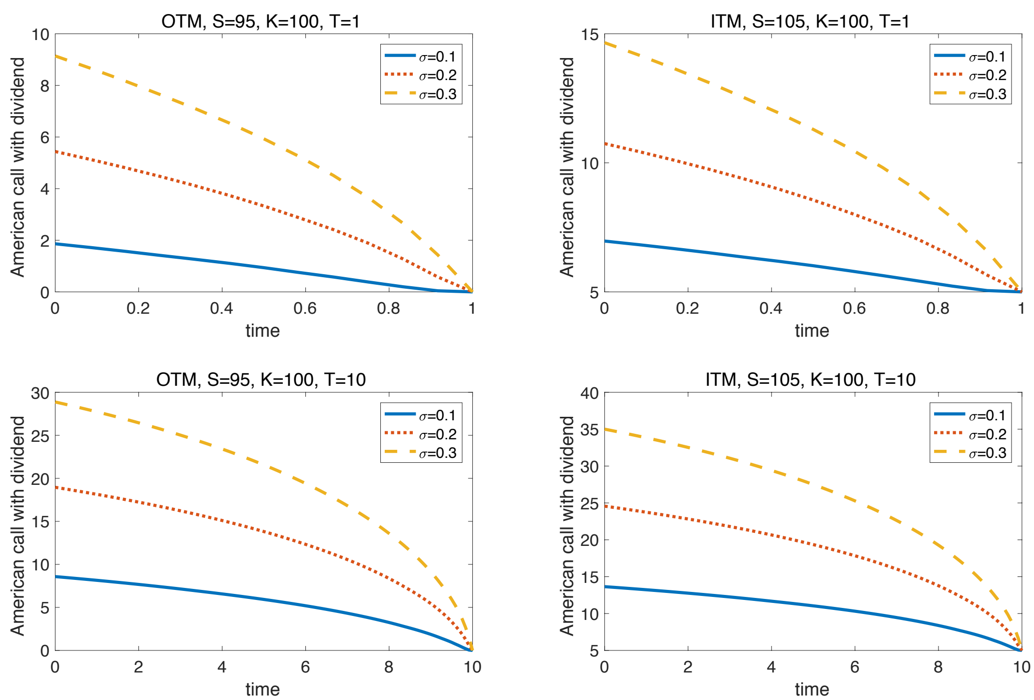

Figure 6: A complete market example (American call option (with dividend) values in time): The left panels are in-the-money cases, while the right panels are out-of-the-money cases. The upper panels are for the short horizon, while the lower panels are for the long horizon.

3 Main Model: Sequential Option Model

3.1 Model

Now we extend the baseline model to the more general and realistic case in which the firm has multiple real options that can be exercised sequentially within the finite decision horizon. More precisely, the order of the option exercises should be determined initially, and then the agent optimally chooses the timing of each option by following the predetermined order. To avoid the time inconsistency problem, the sequence of the option exercise is assumed to be unchanged over time, once determined initially.

Before we introduce the detailed mathematical model, to facilitate an understanding of the framework, let us provide an example of a medical venture having several experimentation ideas. All these ideas can be tested (or executed) as long as they are exercised within a finite time horizon or a predetermined deadline. However, due to limited laboratory space and a limited number of employees, the venture cannot exercise more than one kind of experimentation (option) at the same time.

Note that if the firm has several projects and can execute them simultaneously, the total implied option value in our framework turns out to be the additive sum of those single option values. The reason is that the implied option value is the same as the certainty equivalent, and there is no wealth effect in the CARA utility framework. However, the total implied option value is not the sum of single option values anymore when options can only be exercised in a specific order. Therefore, the sequential option problem is more relevant to those firms that have limited resources to carry out real investment. The sequential option model can also be relevant to a large firm when the cost of exercising real options is large enough that the firm cannot handle all the projects in hand at the same time.

Now the mathematical description of the model is as follows. Suppose that an agent has two real investment opportunities that can be exercised sequentially within the time horizon $[0, T]$. We refer to those two real investment opportunities as Option I and Option II. We assume that Option I must be exercised first, and Option II can only be exercised after the exercise of Option I. Let τ_1 and τ_2 be the exercise (stopping) time of Options I and II, respectively. The payoff of Option I upon being exercised is given by

$$(A_{\tau_1} - I_1)^+ \tag{13}$$

at $\tau_1 \in \{0, t_1, \cdots, t_N\}$, where $A_k \triangleq A_{t_k}$ follows

$$\Delta A_k = A_{k+1} - A_k = \alpha_a h + \tilde{\rho}\sigma_a \sqrt{h} \Delta b^1_{t_k} + \sqrt{1-\tilde{\rho}^2}\sigma_a \sqrt{h} \Delta b^2_{t_k}. \tag{14}$$

$b^1_{t_k}$ and $b^2_{t_k}$ are independent random walks and h is the size of the time step introduced in Section 2. $\alpha_a \in \mathbb{R}$, $\sigma_a > 0$, and $\tilde{\rho} \in (-1, 1)$ are constant.

Remark 2. *It is important to notice that by assuming the payoff of Option I to be* (13)*, we allow the agent to be able to nullify or abandon Option I before maturity if the option is not valuable. For example, if Option I is out-of-the-money while Option II is deep in-the-money, the agent can nullify Option I and proceed to Option II.*

Option II pays $(X_{\tau_2} - I_2)^+$ at $\tau_2 \in \{\tau_1, \cdots, t_N\}$ upon being exercised, where $X_k \triangleq X_{t_k}$ evolves as follows:

$$\Delta X_k = X_{k+1} - X_k = \alpha_x h + \sigma_x \sqrt{h} \Delta b^1_{t_k}. \tag{15}$$

Let us define

$$\begin{aligned}
&u_x \triangleq \alpha_x h + \sigma_x \sqrt{h}, \quad d_x \triangleq \alpha_x h - \sigma_x \sqrt{h}, \\
&u_1 \triangleq \alpha_a h + \tilde{\rho}\sigma_a \sqrt{h} + \sqrt{1-\tilde{\rho}^2}\sigma_a \sqrt{h}, \quad u_2 \triangleq \alpha_a h + \tilde{\rho}\sigma_a \sqrt{h} - \sqrt{1-\tilde{\rho}^2}\sigma_a \sqrt{h}, \\
&d_1 \triangleq \alpha_a h - \tilde{\rho}\sigma_a \sqrt{h} + \sqrt{1-\tilde{\rho}^2}\sigma_a \sqrt{h}, \quad d_2 \triangleq \alpha_a h - \tilde{\rho}\sigma_a \sqrt{h} - \sqrt{1-\tilde{\rho}^2}\sigma_a \sqrt{h}.
\end{aligned}$$

Here $\tilde{\rho}$ is the correlation between the two payoff processes. For simplicity, we assume that neither A nor X is correlated to the market index. Similar to (1) in the baseline model of Section 2, we assume

$$\begin{aligned}
&\mathbb{P}\left(\Delta A_k = u_1,\ \Delta X_k = u_x\right) = \mathbb{P}\left(\Delta A_k = u_2,\ \Delta X_k = u_x\right) \\
&= \mathbb{P}\left(\Delta A_k = d_1,\ \Delta X_k = d_x\right) = \mathbb{P}\left(\Delta A_k = d_2,\ \Delta X_k = d_x\right) = 1/4.
\end{aligned}$$

Then, the agent's wealth process satisfies the following recursive equation:

$$W_{k+1} = (1+r)(W_k - c_k) + (A_{\tau_1} - I_1)^+ \mathbf{1}_{\{t_k = \tau_1\}} + (X_{\tau_2} - I_2)^+ \mathbf{1}_{\{t_k = \tau_2\}},$$

where $W_k \triangleq W_{t_k}$ and $c_k \triangleq c_{t_k}$ are the wealth level and the consumption at time t_k.

3.2 Solution

Value function after τ_1: To solve the problem in a recursive manner, we first need to find the value function after Option I is exercised. The value function of the agent at $t_k \geqslant \tau_1$ is defined by

$$V_2(w,x,k) \triangleq \max_{(\{c_j\}_{j\geqslant k}, \tau_2 \in \{t_k,\cdots,t_N\})} \mathbb{E}\Big[\sum_{j=k}^{\infty} \beta^{j-k} U(c_j) | W_k = w,\ X_k = x\Big],$$

subject to

$$W_{j+1} = (1+r)(W_j - c_j) + (X_{\tau_2} - I_2)^+ \mathbf{1}_{\{t_j = \tau_2\}}.$$

By using Theorem 1 in Section 2, we can easily obtain $V_2(w,x,k)$ as follows.

Corollary 1. *For $k = 0,1,\cdots,N$, $V_2(w,x,k)$, is given by*

$$V_2(w,x,k) = \frac{1+r}{r} U\left(\frac{r}{1+r}(w + Y_k(x))\right), \tag{16}$$

where $Y_k(x)$ is the implied option value defined recursively by

$$\begin{aligned} Y_N(x) &= (x - I_2)^+ = \max(x - I_2, 0), \\ Y_k(x) &= \max\left\{Y_k^c(x), (x - I_2)^+\right\}, \quad \textit{for } k = N-1, N-2, \cdots, 0, \end{aligned} \tag{17}$$

where the continuation value $Y_k^c(x)$ is given by

$$Y_k^c(x) = -\frac{1}{\gamma r} \ln \left[\frac{e^{-\frac{\gamma r}{1+r} Y_{k+1}(x+u_x)} + e^{-\frac{\gamma r}{1+r} Y_{k+1}(x+d_x)}}{2}\right]. \tag{18}$$

Value function before τ_1: Next, we find the value before τ_1. If $t_k = t_N = T$ and the first option is not exercised yet, then both options should be exercised if they are in-the-money. If not, out-of-the-money options will be nullified. Then the value function of the agent at time t_N becomes:

$$V_1(w,a,x,N) = \frac{1+r}{r} U\left(\frac{r}{1+r}(w + (a - I_1)^+ + (x - I_2)^+)\right). \tag{19}$$

If $t_k \leqslant t_{N-1}$ and the first option is not exercised until t_{k-1}, that is, $t_{k-1} < \tau_1$, the agent's value function at the time t_k is given by

$$V_1(w,a,x,k) \triangleq \max_{(\{c_j\}_{j\geqslant k}, \tau_1 \leqslant \tau_2)} \mathbb{E}\Big[\sum_{j=k}^{\infty} \beta^{j-k} U(c_j) | W_k = w,\ A_k = a,\ X_k = x\Big],$$

subject to

$$W_{j+1} = (1+r)(W_j - c_j) + (A_{\tau_1} - I_1)^+ \mathbf{1}_{\{t_j=\tau_1, t_k \leqslant \tau_1 \leqslant t_N\}} + (X_{\tau_2} - I_2)^+ \mathbf{1}_{\{t_j=\tau_2, \tau_1 \leqslant \tau_2 \leqslant t_N\}}.$$

The following theorem explicitly characterizes the implied option value $B_k(a,x)$ over time and the value function $V_1(w,a,x,k)$.

Theorem 3. *For $k = 0, 1, \cdots, N$, $V_1(w,a,x,k)$ is*

$$V_1(w,a,x,k) = \frac{1+r}{r} U\left(\frac{r}{1+r}(w + B_k(a,x))\right), \tag{20}$$

where the implied option value $B_k(a,x)$ is defined recursively as

$$B_N(a,x) = (a - I_1)^+ + (x - I_2)^+, \tag{21}$$

and for $k = N-1, N-2, \cdots, 0$,

$$B_k(a,x) = \max\left\{B_k^c(a,x), (a - I_1)^+ + Y_k(x)\right\}, \tag{22}$$

where the continuation value $B_k^c(a,x)$ is given by

$$\begin{aligned} B_k^c(a,x) = -\frac{1}{\gamma r} \ln \Big[& \frac{e^{-\frac{\gamma r}{1+r} B_{k+1}(a+u_1, x+u_x)}}{4} + \frac{e^{-\frac{\gamma r}{1+r} B_{k+1}(a+u_2, x+u_x)}}{4} \\ & + \frac{e^{-\frac{\gamma r}{1+r} B_{k+1}(a+d_1, x+d_x)}}{4} + \frac{e^{-\frac{\gamma r}{1+r} B_{k+1}(a+d_2, x+d_x)}}{4} \Big] \end{aligned} \tag{23}$$

and $Y_k(x)$ is defined by Eq (17).

Proof. See the Appendix. □

3.3 How to Determine the Option Exercising Sequence

Note that in the sequential option model the order of the exercises of the two options is determined first at the beginning, and then the exercise timing of each option is chosen optimally. Our focus is how the decision horizon or the remaining time affects the order of option exercises. Specifically, we assume that two options have different levels of idiosyncratic volatility, other things being equal. Let us denote the option with higher idiosyncratic volatility and the option with lower idiosyncratic

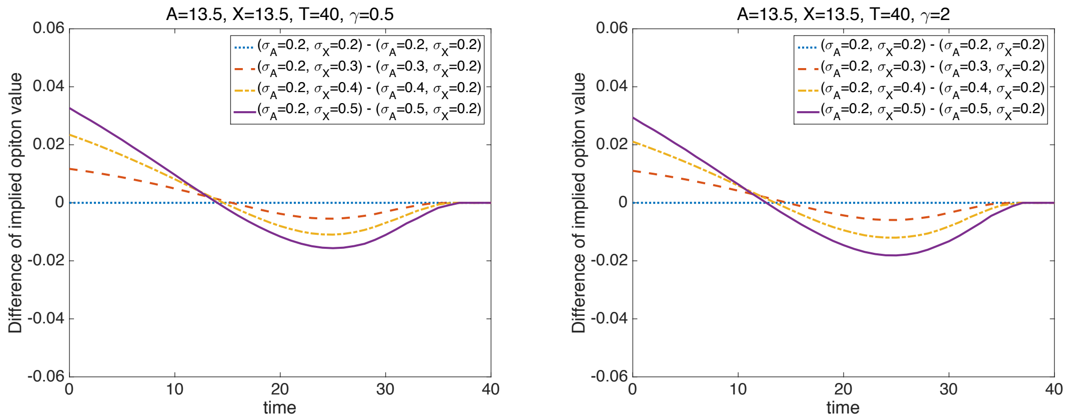

Figure 7: The figures plot $V^{LH} - V^{HL}$, which is the difference between the agent's values for two cases over time. The left panel is when $\gamma = 0.5$ and the right panel is when $\gamma = 2$.

volatility by Option H and Option L, respectively. Then, we compute the agent's values for the following two cases:

- case (LH): exercise Option L first, and Option H later.
- case (HL): exercise Option H first, and Option L later.

Let us define the value functions of case (LH) and case (HL) by V^{LH} and V^{HL}, respectively. Then, Figure 7 shows the difference between the agent's values of the two cases, $V^{LH} - V^{HL}$. When the remaining time to maturity is long enough in Figure 7, the difference between the agent's values is positive, i.e., the agent's value for the case (LH) is higher than for the case (HL). Therefore, the agent decides to exercise Option L first, and Option H later, if there is ample time remaining. However, if the remaining time is short enough, the difference becomes negative, which means that it is better for the agent to exercise Option H first, and Option L later.

The above result in the sequential option case is consistent with the result from the baseline model with a single option. Recall the result from the baseline model with a single option (see Section 2.4): for a long horizon, the option with a higher idiosyncratic risk provides a higher implied option value. Consider the total (idiosyncratic) risk when the agent has two options that should be exercised sequentially. Computing or quantifying the total risk in each case precisely is not an easy task in this instance since the distribution of the exercise time, which is endogenously determined, should be characterized first. However, notice that deferring the exercise of a specific option implies holding that option for a longer time. Thus, it is intuitive to see that the total level of idiosyncratic risk in the case (LH) is higher than that of case (HL) because case (LH) carries Option H for a longer time than for case (HL). Our result with sequential options means that if the time to maturity is long enough, the case with higher total idiosyncratic risk, case (LH), provide a higher value (therefore, is preferred). On the other hand, the case with lower total risk, case (HL), provides a higher value if the remaining time is short. Thus, in terms of the total idiosyncratic risk, the result shown in Figure 7 is very consistent with the result obtained from the single option model. Exercising the option with a higher idiosyncratic risk later creates a higher implied option value when there is ample remaining time. Readers can also refer to Figure 8 for a simple illustration.

3.4 Idiosyncratic Volatility

In Section 2.3, we have shown that the shorter the decision horizon, the higher the firm's idiosyncratic volatility when there is only one project. Here we generalize that result for sequential project selection. We will highlight how the optimal order of project execution reinforces and ensures the higher idiosyncratic volatility of firms with a shorter decision horizon in most cases.

Similar to the single option case, let's consider a continuous time version of the model. Without loss of generality, assume $\tilde{\rho} = 0$. Then, the payoff processes (14) for A and (15) for X are rewritten, respectively, as follows:

$$dA = \alpha_a dt + \sigma_a dB_a \quad \text{and} \quad dX = \alpha_x dt + \sigma_x dB_x,$$

where B_a and B_x are independent Brownian motions with no correlation with the stock market. In this case, let us rewrite the value function $V_1(w, a, x, T-t)$ in (20) in Theorem 3 as V. The idiosyncratic volatility part of the return $\frac{dV}{V}$ has two components:

$$\frac{\partial V/\partial a}{V}\sigma_a dB_a + \frac{\partial V/\partial x}{V}\sigma_x dB_x.$$

Therefore, the size of the total idiosyncratic volatility is

$$\left(\frac{\partial V/\partial a}{V}\right)^2 \sigma_a^2 + \left(\frac{\partial V/\partial x}{V}\right)^2 \sigma_x^2.$$

Based on the above argument, by using (22) in Theorem 3, we have

$$\left(\frac{\partial V_1/\partial a}{V_1}\right)^2 \sigma_a^2 + \left(\frac{\partial V_1/\partial x}{V_1}\right)^2 \sigma_x^2 = C\left[\left(\frac{\partial B_k(a,x)}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial B_k(a,x)}{\partial x}\right)^2 \sigma_x^2\right],$$

for some constant C. This means that the instantaneous idiosyncratic volatility is the sum of the squared products of the sensitivity of each project payoff and its volatility.

First, we extend Proposition 1 to the two project case by ignoring the optimal project selection. That is, without loss of generality, we assume the decision maker will execute project a first and x second and prove the following.

Proposition 2. *For each a and x at which the options are in-the-money or near at-the-money before being exercised,* $0 < \frac{\partial B_k(a,x)}{\partial a} < 1$ *and* $0 < \frac{\partial B_k(a,x)}{\partial x} < 1$. *Furthermore, for such a and x,* $\lim_{k \to N} \frac{\partial B_k(a,x)}{\partial a} = 1$ *and* $\frac{\partial B_k(a,x)}{\partial x} = 1$.

Proof. See the Appendix. □

The proposition suggests that if two decision makers with different decision horizons have the same optimal order of execution, the one with shorter decision horizon will have more idiosyncratic volatility than the one with longer decision horizon.

Next, we incorporate the project order selection. Suppose $\sigma_x > \sigma_a$ and $\alpha_a = \alpha_x$. Let us consider two firms (Firm 1 and Firm 2) that are identical except that they have different decision horizons. The decision maker in Firm 1 has horizon T_s and the decision maker in Firm 2 has horizon T_l with $T_s < T_l$. The two decision makers are also identical in terms of preference. From the optimal exercising sequence discussed in Section 3.3, we know that Firm 1 with a shorter horizon (T_s) will choose to exercise X first while Firm 2 with a longer horizon (T_l) will choose to exercise A first because $\sigma_x > \sigma_a$. So, let us define the implied option value for the decision maker in Firm 1 by $B_k(x, a; T_s)$ and the implied option value for the decision maker in Firm 2 by $B_k(a, x; T_l)$ (meaning that the first argument is exercised first).

We will show that the overall volatility attributed from both projects is higher for Firm 1 with the shorter horizon (T_s) which is formally:

$$\left(\frac{\partial B_k(x, a; T_s)}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial B_k(x, a; T_s)}{\partial x}\right)^2 \sigma_x^2 > \left(\frac{\partial B_k(a, x; T_l)}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial B_k(a, x; T_l)}{\partial x}\right)^2 \sigma_x^2. \quad (24)$$

Since Firm 1 chooses to exercise X first and has a shorter decision horizon T_s, Firm 1 on average exercises X way before Firm 2 exercises X. Combining this with Proposition 2, we have

$$\frac{\partial B_k(x, a; T_s)}{\partial x} > \frac{\partial B_k(a, x; T_s)}{\partial x} > \frac{\partial B_k(a, x; T_l)}{\partial x} \quad (25)$$

In other words, Firm 1 has higher sensitivity to the project with higher idiosyncratic volatility because he is on average guaranteed to execute it before Firm 2 does. Hence, the crucial impact of the optimal order of execution is that it ensures the higher idiosyncratic volatility project will attribute more to Firm 1's overall volatility than to that of Firm 2.

The remaining question is whether Firm 1 or Firm 2 has a higher sensitivity to project A. For this, there are two opposite forces. On one hand, the decision horizon is shorter for Firm 1. On the other hand, Firm 2 will choose to execute project A before project X. Therefore, let's consider the following three cases.

First, when T_l is sufficiently longer than T_s, we have $\frac{\partial B_k(x,a;T_s)}{\partial a} > \frac{\partial B_k(a,x;T_l)}{\partial a}$, despite the fact that Firm 2 chooses to execute A first and Firm 1 chooses to execute A second. That is, when the decision horizon is sufficiently longer for Firm 2 than for Firm 1, Firm 1 will on average execute both projects before Firm 2 does regardless of the order of execution. Firm 1's return is more sensitive to both projects than that of Firm 2. Hence, (24) easily holds.

Second, if T_l is not much longer than T_s, then $\frac{\partial B_k(x,a;T_s)}{\partial a} < \frac{\partial B_k(a,x;T_l)}{\partial a}$ might occur, that is, Firm 2's return is more sensitive to project A than Firm 1's return is. However, this reverse equality means that the expected waiting time for Project A to be exercised in Firm 1 is shorter than that for Firm 2. Therefore, the expected time to exercise Project X in Firm 1 is much shorter than that in Firm 2, which makes $\frac{\partial B_k(x,a;T_s)}{\partial x}$ much bigger than $\frac{\partial B_k(a,x;T_l)}{\partial x}$. Thus, (24) is still likely to hold because project X's volatility, σ_x, is higher than project A's volatility, σ_a.

Finally, let's consider the case when T_l is very close to T_s. If T_l is very close to T_s, the order of real option execution should be the same for both firms. Then, simply by Proposition 2, Firm 1 has a higher return volatility than Firm 2.

In summary, we have shown that for most cases, the shorter the decision horizon, the higher the idiosyncratic volatility of the firm's return. The optimal order of project execution plays an important role in these results.

3.5 Loss from Suboptimal Order of Execution: Numerical Examples

Here we suggest a way to quantify the impact of the horizon constraint. More precisely, we compute loss from an incorrect order choice of the project execution by using counterfactual analysis. Suppose there are two projects: projects A and X. Assume that it is optimal to exercise (A, X) (meaning project A first and project X later) since project A has a lower volatility than X does, other things being equal. Suppose, in this case, the decision maker, before deciding the order of execution, mistakenly learns that the remaining horizon is much shorter than he thought, so he picks the order (X, A), which is suboptimal. Immediately after the choice, the firm realizes the actual decision horizon is longer, but for whatever reason, it is impossible to change the order. As a result, loss occurs. In this case, we define the percent value of loss by $\frac{V^{AX}-V^{XA}}{V^{AX}}$. The examples are as follows.

Example 1. *Consider the first panel of Figure 7 with $\sigma_a = 0.2$ and $\sigma_x = 0.5$ with $T = 40$. Note $X = A = 13.5$ at $t = 0$. It is optimal to exercise (A, X). The firm mistakenly picks the order (X, A). In this case, the percent value loss, $\frac{V^{AX}-V^{XA}}{V^{AX}}$, is* 0.39%*.*

Example 2. *Suppose $X = A = 14$ at $t = 0$. Everything else is the same as in Example 1. Again, the optimal order is (A, X). If the entrepreneur mistakenly learns that the horizon is much shorter, prompting her to select (X, A)-order, then the percentage loss is* 0.52%*.*

Example 3. *Suppose $X = A = 14.5$ at $t = 0$. Everything else is the same as in Example 1. If the same thing happens, the percentage loss is* 0.54%*.*

Examples 1, 2, and 3 indicate that the loss becomes greater as the options are further in-the-money. Notice that the underlying process follows the arithmetic Brownian motion and the step size is $h = 1$. This means that $(\sigma_a, \sigma_x) = (0.2, 0.5)$ is fairly small relative to the current value of (A, X) in each example. $\sigma_x = 0.5$ contributes only about $\frac{0.5}{14} \approx 3.6\%$ changes in return when $A = X = 14$. Therefore, the loss computed in each example looks small. In the following examples we consider the case with 22.2%, 21.4%, and 20.7% change, respectively, in the payoff return by setting $\sigma_x = 3$.

Example 4. *Suppose $X = A = 13.5$ at $t = 0$ and $\sigma_a = 0.5$ and $\sigma_x = 3$ with $T = 40$. Again, it is optimal to exercise (A, X). Suppose the decision maker, before he decides the order of execution, mistakenly learns that the remaining horizon is very short so he mistakenly picks the order (X, A). The percentage loss in this case is 1.08%.*

Example 5. *Suppose $X = A = 14$ at $t = 0$. Everything else is the same as in Example 4. If the same thing happens, the percentage loss is* 1.72%*.*

Example 6. *Suppose $X = A = 14$ at $t = 0$. Everything else is the same as in Example 4. If the same thing happens, the percentage loss is* 2.30%*.*

Examples 4, 5, and 6 show that the suboptimal choice of the project execution sequence can lead to more than a 2% loss in the firm's value. These numbers are arguably significant enough while we admit that the current model is rather simple and thus a more elaborate structural model is required to quantify the impact. We leave this issue for exploration in future research.

4 Empirical Analysis

4.1 Hypothesis development

Our model makes several predictions regarding the relationship between the decision horizon and idiosyncratic volatility. In this section, we empirically examine those predictions. We use data on public firms because the data are easily available and our theory also applies to public firms. First, public firms owned and controlled by members of founding families are likely to exhibit a pattern of project selection similar to that described in our model: the founding family members are risk-averse and exposed to idiosyncratic risk since a large portion of their total wealth is vested in the firm. Hence, they are likely to behave as the entrepreneur did in our model. Second, non-family-controlled public firms may also exhibit similar patterns of project selection. Although the firms are owned by diverse shareholders and each shareholder can diversify his wealth portfolio through the market, the decision makers in each firm are a small handful of executives and managers. If those decision makers have a large stake in the firm, they are exposed to idiosyncratic risk associated with the projects. To what extent our model can depict the behavior of public firms with low ownership concentration is a more involved empirical question. In Section 4.5, we use insiders' ownership as a proxy for decision makers' risk exposure, and investigate whether firms with a high level of insider ownership follow our model more closely than firms with a low level of insider ownership. For now, we state our basic null hypothesis:

> $\mathbf{H}_{1,0}$: There is no relationship between firms' idiosyncratic volatility and their decision horizon.

In order to make sure the empirical analysis examines our model predictions accurately, it is important to understand how we think about the decision horizon in practice. The decision horizon in our model refers to the time period the decision maker has to execute all his projects. For example, Exxon Mobil may have six months to decide to open several new plants. Different types and locations of the plants come with different idiosyncratic risks. By the end of the six months, Exxon Mobil needs to start building them. Note in this case that the decision horizon is not related to the life of the assets or the time to build (time until the capital deployment and construction are completed). Hence, the traditional measures for project time horizons based on life of assets or time to build from the investment literature do not apply to our analysis. Instead, we choose to use "Product Market Fluidity" by Hoberg et al. (2014) as the measure for the decision horizon. Many factors can affect a firm's decision horizon. We believe the most important aspects are the product market conditions and threats from its competitors. For instance, let's imagine that 5G network technology will be ready next year. What is the decision horizon for Verizon to decide on how it would like to build its 5G network? The most reasonable guess is probably until the time AT& T, T-Mobile, and Sprint start building their 5G network. According to Hoberg et al. (2014), "Fluidity" measures "how intensively the product market around a firm is changing in each year." We believe when firms face high product market fluidity, they are likely to be forced to make quicker decisions. Therefore, "Fluidity" can be viewed as the inverse of the decision horizon.

In practice, it is impossible to observe the idiosyncratic volatility of each potential real option project faced by a firm.[16] The structure of idiosyncratic volatility from the underlying real option projects should be reflected in the firms' stock return dynamics, as stated by Cao, Simin, and Zhao (2008) and Grullon, Lyandres, and Zhdanov (2012). Hence, we choose to focus on stock return idiosyncratic volatility. Finally, since our model speaks to the selection and execution of the real option type of projects, firms with more real options should follow our model more closely. If there is evidence of a negative relationship between idiosyncratic volatility and fluidity from the full sample, the results should be stronger for firms that possess more real options or have a substantial component of their value constituted by real options. In Section 4.4, we examine whether the relationship between idiosyncratic volatility and fluidity depends on the extent to which firms rely on real options.

4.2 Data description

The data on product market fluidity are from the Hoberg-Phillips Data Library. The data include Fluidity calculated following Hoberg et al. (2014) for each firm (gvkey) each year.[17] We obtain daily stock returns from CRSP and the Fama French daily three factors from French's data library. Idiosyncratic volatility (IVOL) is calculated for each firm each year as the standard deviation of the residuals from regressing daily excess returns on the Fama French three factors:

$$\mathrm{r}_t^i = \alpha^i + \beta_{MKT}^i \mathrm{MKT}_t + \beta_{SMB}^i \mathrm{SMB}_t + \beta_{HML}^i \mathrm{HML}_t + \varepsilon_t^i$$

[16] For executed projects, one may be able to derive the idiosyncratic volatility from ex-post realized cash flows of the project. However, in the model, the executions of the projects depend on the evolution of the implied option values. There is no guarantee that the entrepreneur will execute any of the potential projects. Nevertheless, the volatility of the value of the projects is reflected in the current value function regardless of whether the entrepreneur will execute the project in the future. Therefore, if we want to consider idiosyncratic volatility of the individual projects, we need to pin down the volatility of all potential projects not just the ones that get executed eventually. Such a task is impossible.

[17] Please refer to Hoberg et al. (2014) and Hoberg-Phillips data library (http://hobergphillips.usc.edu/) for the details on the construction of Fluidity.

We obtain accounting data such as total assets, total long-term debt, debt in current liabilities, and sales from Compustats Annual. Finally, we obtain shares held by company insiders from Thomson Reuters Insider Fillings. Hence, our sample is the intersection of Hoberg's fluidity data, CRSP, Compustats, and Thomson Reuters Insider data. The sample period is from 1997 to 2015 since 1997 is the earliest year for which we have data on fluidity. All variable definitions are summarized in Table 1. We winsorize the variables at 1% and 99% levels to reduce the impacts of outliers.[18] We also require the firms to be present for at least 10 out of the 19 years in our sample. The summary statistics are given in Table 2. There are around 56,333 firm year observations. The idiosyncratic volatility ranges from 0.754% to 6.716%, with a mean of 2.541% and a standard deviation of 1.239%. Our main independent variable in interest, Fluidity ranges from 0.062 to 27.728 with a mean of 7.008 and a standard deviation of 3.584. Since the unit of Fluidity is not meaningful, we will use its standard deviation when assessing the effects of Fluidity in our subsequent analysis.

4.3 Effects of Decision Horizon

As a basic test of our model prediction, we regress idiosyncratic volatility on fluidity with year fixed effects and standard error clustered on firms, as shown in the following equation:

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t} \tag{26}$$

As we explained in section 4.1, Fluidity inversely measures the decision horizon. Thus, Fluidity should be positively related to IVOL. However, since it is uncertain when the projects are selected and how fast the market incorporates such information, we not only test relations between the current IVOL and Fluidity, as represented by $\tau = 0$ in (26), but also between the current IVOL and Fluidity lagged up to five years, as represented by $\tau = 1$ to 5 in (26). From Panel A in Table 3, we see that all the β_1's are positive and significant at the 1% level. This suggests on average, firms with a shorter decision horizon are associated with higher idiosyncratic volatility. To assess the magnitude of the effects, we scale the estimates of β_1 by 3.5842, the standard deviation of Fluidity. Thus, one standard deviation increase in lagged and current Fluidity is associated with increases in idiosyncratic volatility, from 0.078% to 0.104%, respectively. We think those effects are economically significant given that the mean and standard deviation of idiosyncratic volatility are 2.541% and 1.239%.

[18] Exc. Fluidity is not winsorized. Tobin's Q and returns are winsorized at 2% and 98% levels.

4.4 Decision Horizon and Real Options

To show that the relationship between Fluidity and IVOL are driven by real options, we further test whether firms that more heavily depend on real options exhibit stronger relations. Following Grullon, Lyandres, and Zhdanov (2012), we identify firms that have a substantial component of their value constituted by real options or possess more real options by their growth opportunities, size, and industry. First, firms with more growth opportunities are associated with more real options. Hence, we run Regression (26) for each of the quartiles of firms by Tobin's Q. From Tables 4 and 5, we see that as we move from the first (lowest Tobin's Q) to the fourth (highest Tobin's Q) quartile, the estimates of β_1 increase in magnitude for the current and all lagged Fluidity. The results from the fourth quartile (Panel D in Table 5) indicate that one standard deviation increase in lagged and current Fluidity is associated with increases in idiosyncratic volatility from 0.326% to 0.358% respectively. Those effects are economically significant compared to the mean, 2.541%, and the standard deviation, 1.239%, of idiosyncratic volatility. The results from the first quartile (Panel A in Table 4) are quite unusual as the estimates are negative. This does not seem to follow our model. However, since those firms all have a Tobin's Q less than 0.721, they are likely experiencing distress and may even disinvest rather than taking on new projects. Therefore, they may be radically different from the ordinary firms.

Second, small firms are associated with more real options. Hence, we run Regression (26) for each of the quartiles of firms by total assets. From Tables 6 and 7, we see that as we move from the first (smallest) to the fourth (largest) quartile, the estimates of β_1 decrease in magnitude for the current and all lagged Fluidity. This means the relation between the decision horizon and idiosyncratic volatility is stronger for smaller firms, which are likely associated with more real options. Once again, the results from the first quartile (Panel A in Table 6) are slightly inconsistent with our story as the estimate for the coefficient of current Fluidity is smaller than that from larger firms (Panel B in Table 6). However, similar to previous cases with Tobin's Q, the smallest firms may also behave unconventionally since they are likely to be extremely constrained.

Finally, firms from the pharmaceutical products, chemicals, petroleum/natural gas, computers, and electronic equipment industries highly depend on their real option projects as suggested by Grullon, Lyandres, and Zhdanov (2012). Therefore, we check if stronger results are obtained for firms within those industries than for firms from other industries. First, we run Regression (26) separately for those high real option intensive firms with codes 13, 14, 30, 35, and 36 from the Fama

French 48 industries and for firms from the rest of the Fama French 48 industries and compare results from those two subsamples. Then we create a dummy variable Real.Opt (RO) that equals one for firms in industries 13, 14, 30, 35, 36 and zero otherwise. We add both Real.Opt and its interaction with fluidity, RO $\times$ *Fluidity*, to the base regression and run the following:

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \beta_2 \text{Real.Opt}_{i,t-\tau} + \beta_3 \text{RO} \times \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t} \qquad (27)$$

As shown in Panel A of Table 8, the β_1 estimates range from 0.058 to 0.060 and all are significant at the 1% level for firms from industries associated with high real options. This means one standard deviation increase in lagged and current Fluidity is associated with increases in idiosyncratic volatility from 0.208% to 0.215%, respectively. Those effects are economically significant compared to the mean, 2.541%, and the standard deviation, 1.239%, of idiosyncratic volatility. On the other hand, the β_1 estimates range from 0.003 to 0.011 and are only significant at the 5% level for current and last year Fluidity, as presented in Panel B of Table 8. Hence, the inverse relation between decision horizon and idiosyncratic volatility is indeed stronger for firms that are more related to real option projects and is barely significant for firms that are not real option intensive.

To test whether the differences in estimates between the panels in the previous table are significant, we examine the estimates for β_3 from Regression (27), which captures the difference in slopes between the high and the low real option intensive firms. From Table 9, we see that the coefficients for Fluidity, β_1, are similar to those from Panel B in Table 8 since they all represent the effects of Fluidity on IVOL for low real option intensive firms.[19] The coefficients for Real.Opt, β_2, are all insignificant which suggest there are no differences in idiosyncratic volatility on average between the high and the low real option intensive firms. More importantly, the coefficients on the interaction RO $\times$ Fluidity, β_3, are all positive and significant at the 1% level. This means the effects of Fluidity on IVOL are stronger for firms more related to real option projects. Hence, the relation between decision horizon and idiosyncratic volatility observed in the data is likely driven by firms' project execution decisions as described in our model.

[19] In Table 9, the estimates of the constants and the slopes on Fluidity are not exactly the same as in Panel B of Table 8 because we include the year fixed effects in all regressions. When the year fixed effects are excluded, the results and patterns are qualitatively similar to those presented here.

4.5 Decision Horizon and Risk Aversion

An important element of our model is the risk averse of the decision maker. It is the interplay between the concavity of the utility function and the convexity of the real option payoff over time that drives our results. In the model, an increase in the decision maker's risk aversion has similar effects as a shortening of decision horizon. Therefore, our model suggests that more "risk-averse" firms should have higher idiosyncratic volatility. What we mean by risk aversion in the context of firms is how much the decision makers (e.g., executives and managers) care about the idiosyncratic volatility. When the decision makers are unable to diversify in the market, for example when they have a large fraction of their wealth vested in the company, they will be risk-averse towards idiosyncratic volatility. Hence, they will select projects and execute them according to our model. Measuring the degree of risk aversion of each decision maker in the data is never an easy task. However, ceteris paribus, we believe firms with a higher insider ownership will resemble entrepreneurs with nontrival risk aversion because higher ownership in the firm means that the decision makers will have more risk exposure and will be less able to diversify their wealth in the market. Ideally, we would like to compare the decision makers' wealth inside the company to their total wealth; however, due to data limitations, we use insider ownership as a proxy for decision makers' risk exposure.

Insider Ownership (IO) is the fraction of shares owned by insiders out of the total shares outstanding. For each firm in each year, we take the sum of the shares held by all its insiders listed in the Thomson Reuters Insider Filing. Then we divide the total shares held by insiders by the total shares outstanding for the year. We take the time series average of Insider Ownership (IO) for each firm to form "Average IO." As a first test, we regress idiosyncratic volatility, IVOL, on current and lagged Fluidity and Insider Ownership with year fixed effects and standard errors clustered on firms. From Table 10 we see that the estimates for all current and lagged Fluidity remain positive and significant. Moreover, the estimates for all current and lagged Insider Ownership are also positive and significant. From columns 1 to 6, when lagged and current Insider Ownership increase by 1%, idiosyncratic volatility increases by 3.929% to 5.151%. This suggests that as risk exposure increases (i.e. as insiders own a larger fraction of the firm) idiosyncratic volatility goes up.

Although the above results are fairly consistent with our model's prediction about the effect of risk aversion on idiosyncratic volatility, we carefully note that the level of insider ownership might not reflect the degree of risk aversion. What we want to stress is that since the precautionary savings motive is crucial in our model, a certain level of risk aversion is required. Thus, when risk aversion is too low, there is no relationship between the decision horizon and idiosyncratic volatility. Hence, it should be more difficult to detect the relationship in firms with low insider ownership. To test that prediction, we run Regression (26) separately for firms with an Average IO of less than 0.75% and for firms with an Average IO of greater than 3%. Those thresholds are arbitrarily picked to ensure a similar number of observations in each sample. From Panels A and B in Table 11, we see that the estimates of the coefficients on lagged and current Fluidity are much smaller for low insider ownership firms (Panel A) than those for high insider ownership firms (Panel B).

To illustrate the point further, we add the product of Fluidity and Average IO to the equation and run the following regression:

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \beta_2 \text{Insider Ownership}_{i,t-\tau} + \beta_3 (\text{Fluidity} \times \text{Average IO})_{i,t-\tau} + \nu_t + \varepsilon_{i,t}. \tag{28}$$

As shown in Table 12, the coefficients on Fluidity become smaller overall. For some of the lagged Fluidity, the coefficients even become insignificant. When the interaction Fluidity $\times$ Average IO is added, those coefficients on current and lagged Fluidity can be interpreted as the effects of Fluidity on idiosyncratic volatility when Average IO approaches zero. In other words, those coefficients capture the effects of Fluidity in firms with no insider ownership. The small and insignificant estimates for the coefficients of Fluidity documented here suggest that some minimal level of risk aversion is required for a decision horizon to affect idiosyncratic volatility.

4.6 Robustness Checks: Herfindahl-Hirschman Index and Industry Fixed Effect

Finally, we address some potential alternative explanations and show that our results are robust. First, we recognize that the Product Market Fluidity we used to measure the decision horizon may also be a proxy for market competition. Hence, to control for the effects of competition on idiosyncratic volatility, we add the Herfindahl-Hirschman index (HHI) to the regressions. [20] As shown in Table 13, the coefficients on HHI are all negative. The sign seems to be consistent with the story that higher product market competition leads to more volatile returns. However, only the estimates in columns 2 and 3 are significant at the 5% level. Moreover, the estimates for the coefficients of current and lagged Fluidity all become slightly larger in magnitude compared to those from Table 3. Therefore, after controlling for the Herfindahl-Hirschman index, our results on the

[20] The Herfindahl-Hirschman index (HHI) is calculated as the sum of squared market shares for each of the 3-digit SIC industries.

relationship between the decision horizon and idiosyncratic volatility remain strong.

Another alternative explanation would be that due to the nature of business, firms may be facing projects with very different idiosyncratic volatility distributions. For example, construction companies may always have projects with much lower volatility than those of high-tech companies. No matter how construction companies select their projects, they will never have higher volatility than high-tech companies. To address this concern, we add industry fixed effects to the regressions. In this way, any differences in the volatility associated with the nature of business will be captured by the industry dummies. As shown in Panel B of Table 3, the coefficients of Fluidity remain positive and significant with both the industry (3-digit SIC) and year fixed effects. Therefore, the effect of the decision horizon on idiosyncratic volatility is not driven by the fact that different businesses always have projects with different levels of idiosyncratic risk.

5 Conclusions

The effect of the time constraint has rarely been studied in the literature. This paper first tries to investigate the effect of the decision horizon (time until project maturity) by using a real option investment framework under incomplete markets. The base model highlights that the decision horizon affects the risk attitude toward project payoffs. More specifically, since the convexity (from limited downside risk) of project payoffs increases with time to maturity, the entrepreneur in our model will act more risk-averse as the remaining time to execution becomes shorter. This means risk-averse decision makers with long (short) decision horizon will favor projects with high (low) idiosyncratic volatility.

We extend this intuitive into a case where the entrepreneur has several options that can be sequentially exercised. In the sequential model, the decision maker exercises project with high (low) volatility first, when his decision horizon is short (long). As a result, his current value becomes more volatility as the decision horizon shortens. We provide empirical evidence that firms with a shorter decision horizon are associated with higher idiosyncratic volatility. We also show that this inverse relationship between the decision horizon and idiosyncratic volatility is stronger in firms that rely more on real option projects and in firms with higher insider ownership. Our results may shed light on a new aspect of firm real option investment decisions driven by the time constraint and its impact on the firm's return characteristics.

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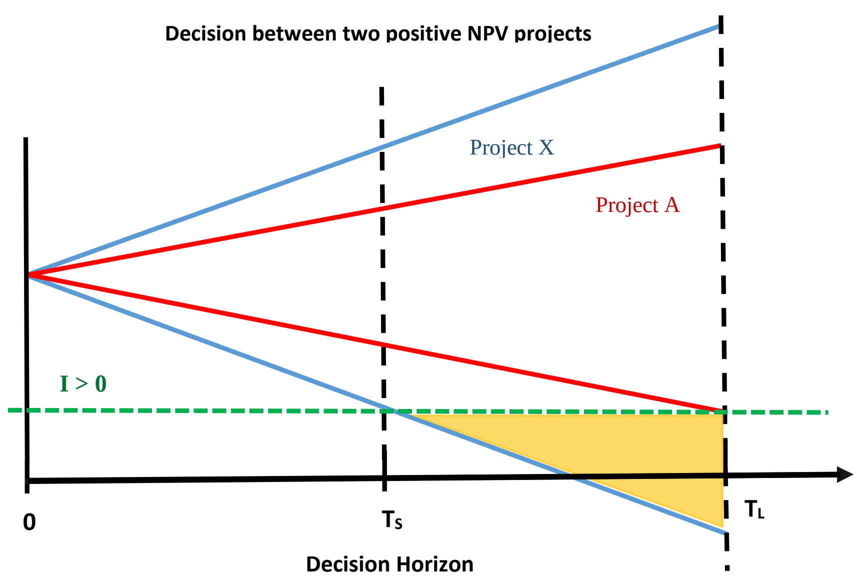
Figure 8: **Project selection with short and long decision horizons**

Here we discuss briefly about how the decision horizon affects the payoff convexity and the optimal order of execution. Let's imagine that the decision maker faces Project A with low idiosyncratic volatility, Project X with high idiosyncratic volatility, and execution cost "I". Suppose the decision maker has a long time to decide on the projects, that is, he can choose to execute the projects at any time from now (time 0) to time T_L. Project X has a lot to gain in the long run and little to lose since the lowest payoff is limited at zero as for the yellow shaded region. In this case, the convexity of the option payoff dominates the concavity of the risk-averse decision maker's utility function. It is more beneficial to wait for Project X than for Project A. Hence, Project A will be executed before Project X. Now suppose the decision maker has a short time to decide on the projects, that is he can choose to execute the projects at any time from now (time 0) to time T_S. Since the decision horizon is so short, Project X no longer has the advantage of limited downside risk over Project A. In this case, the concavity of the utility function dominates the convexity of the option payoffs. The decision maker chooses to execute Project X before Project A. Certainly, in practice, the value supports, execution cost, and deadlines would not line up exactly in the way shown in the figure. So, the contrast between the short and long deadlines are not always as perfect as shown in the figure. Nevertheless, the same intuition applies as long as the decision maker faces projects with different idiosyncratic volatilities and decision horizons.



The values of both projects are assumed to follow discrete random walks. The supports for the value of Project A (X) are indicated by the red (blue) lines. The green dotted line denotes the execution cost "I". Time $t = T_S$ indicates the deadline when the decision maker has a short decision horizon. Time $t = T_L$ indicates the deadline when the decision maker has a long decision horizon. The projects may be executed at any time between 0 and the deadline depending on the evolution of their values. The projects that are not executed expire (disappear) after the deadline.

Table 1: **Definitions of variables**

Variables	Definition
Idio.Vol	The idiosyncratic volatility of return is the standard deviation of the residuals from the following Fama French regression: $\mathrm{r}_t^i = \alpha^i + \beta_{MKT}^i \mathrm{MKT}_t + \beta_{SMB}^i \mathrm{SMB}_t + \beta_{HML}^i \mathrm{HML}_t + \varepsilon_t^i$ For each firm in each year, we regress its excess daily returns on the daily Fama French three factors. We take the standard deviation of ε_t^i as the idiosyncratic volatility of the firm for the year.
Fluidity	The data for Product Market Fluidity is kindly provided by Gerard Hoberg (See Hoberg et al. (2014) for more details). Fluidity measures the intensity of product market change around a firm in each year. From a theoretic prospective, firms facing high intensities of product market change are likely to have shorter decision horizon. Therefore, we use Fluidity to measure the decision horizon of a firm around a given year. (See Sec. 4.1 for more details). The higher the fluidity, the shorter the decision horizon for the firm to choose a project in a given year.
Insider Ownership (IO)	The fraction of shares owned by insiders divided by total shares outstanding. For each firm in each year, we take the sum of the shares held by all its insiders listed in Thomson Reuters Insider Filing-Table 1. Then we divide the total shares held by insiders by the total shares outstanding of the year.
Average IO	Average insider ownership (IO) is the time-series mean of Insider Ownership (IO) for each firm. It indicates how risk-averse the decision makers are within a firm on average over the sample period.
HHI	The Herfindahl-Hirschman index (HHI) is calculated as the sum of squared market shares for each industry based on the three digit SIC code each year. We use it to control for the effects of market competition on idiosyncratic volatility.
Log(size)	The natural logarithm of Total Assets (in millions) from Compustats Annual data. We use Total Assets to measure firm size.
Tobin's Q (TQ)	The ratio of market value of asset to book value of asset. The market value of asset is calculated as the sum of market value of equity, total long-term debt, and debt in current liability. The book value of asset is the Total Assets from Compustats.

This table presents the measures used. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for the decision horizon. The higher the Fluidity the shorter the decision horizon. The sample period is 1997–2015.

Table 2: **Summary statistics of variables**

	IVOL	Fluidity	Insider Ownership	Tobin's Q	Log(size)	Average IO
mean	2.540698	7.008304	.0269321	1.634607	6.283864	.0301868
sd	1.239183	3.584208	.0563566	1.723112	2.169071	.041169
min	.7541607	.0618753	3.60e-07	.1305778	-2.453408	3.82e-06
p25	1.577275	4.329622	.0016201	.7281535	4.709692	.0051641
p50	2.273255	6.339113	.0062734	1.111972	6.299798	.0157353
p75	3.301502	9.056975	.0249185	1.868851	7.772567	.0402106
max	6.716109	27.72773	.5341814	12.36874	11.83993	.4438793
N	56333	56333	53387	46705	54201	55751

This table reports the mean, standard deviation, minimum, 25 percentile, median, 75 percentile, and the maximum value of the variables used in our analysis. The Tobin's Q and Log(size) are used to form subsamples. The sample period is 1997–2015.

Table 3: **Decision horizon effect on idiosyncratic volatility (full)**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
Panel A: Full sample without industry fixed effects						
L5.Fluidity	0.022***					
	(0.004)					
L4.Fluidity		0.024***				
		(0.004)				
L3.Fluidity			0.026***			
			(0.004)			
L2.Fluidity				0.026***		
				(0.004)		
L.Fluidity					0.028***	
					(0.004)	
Fluidity						0.029***
						(0.004)
Constant	2.754***	2.956***	3.339***	3.027***	2.861***	2.574***
	(0.036)	(0.037)	(0.035)	(0.036)	(0.035)	(0.036)
Observations	38340	41805	45272	48805	52425	56333
Adjusted R^2	0.171	0.186	0.215	0.221	0.223	0.218
Panel B: Full sample with industry fixed effects						
L5.Fluidity	0.015***					
	(0.004)					
L4.Fluidity		0.021***				
		(0.004)				
L3.Fluidity			0.026***			
			(0.004)			
L2.Fluidity				0.029***		
				(0.004)		
L.Fluidity					0.034***	
					(0.004)	
Fluidity						0.039***
						(0.004)
Constant	2.833***	3.007***	3.378***	3.044***	2.839***	2.523***
	(0.040)	(0.039)	(0.038)	(0.039)	(0.038)	(0.038)
Observations	37646	40855	44049	47299	50639	54201
Adjusted R^2	0.307	0.328	0.363	0.378	0.386	0.389

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for the decision horizon. The higher the Fluidity the shorter the decision horizon. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity 5, 4, 3, 2, and 1 year(s) ago, respectively. Panel A presents the base results without industry (3-digit SIC) fixed effects. Panel B results are with industry (3-digit SIC) fixed effects in addition to year fixed effects and standard errors clustered on firms. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 4: **Effect of decision horizon by Tobin's Q**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
Panel A: Lowest quartile of Tobin's Q						
L5.Fluidity	-0.046***					
	(0.006)					
L4.Fluidity		-0.045***				
		(0.006)				
L3.Fluidity			-0.044***			
			(0.006)			
L2.Fluidity				-0.046***		
				(0.005)		
L.Fluidity					-0.046***	
					(0.005)	
Fluidity						-0.044***
						(0.005)
Constant	3.513***	3.648***	3.730***	3.606***	3.599***	3.544***
	(0.076)	(0.067)	(0.067)	(0.080)	(0.078)	(0.077)
Observations	9150	9983	10707	11321	11516	11676
Adjusted R^2	0.287	0.289	0.291	0.286	0.285	0.282
Panel B: Second quartile of Tobin's Q						
L5.Fluidity	0.016**					
	(0.007)					
L4.Fluidity		0.021***				
		(0.007)				
L3.Fluidity			0.020***			
			(0.007)			
L2.Fluidity				0.020***		
				(0.007)		
L.Fluidity					0.018***	
					(0.007)	
Fluidity						0.017***
						(0.007)
Constant	2.700***	2.852***	3.125***	3.066***	3.069***	3.083***
	(0.064)	(0.069)	(0.064)	(0.065)	(0.064)	(0.063)
Observations	8992	9742	10538	11365	11542	11677
Adjusted R^2	0.150	0.161	0.178	0.184	0.185	0.185

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for the decision horizon. The higher the Fluidity the shorter the decision horizon. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity 5, 4, 3, 2, and 1 year(s) ago, respectively. Panel A is for firm-year observations with Tobin's Q in the lowest quartile. Panel B is for firm-year observations with Tobin's Q in the second quartile. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 5: **Effect of decision horizon by Tobin's Q (Cont.)**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
Panel C: Third quartile of Tobin's Q						
L5.Fluidity	0.056***					
	(0.005)					
L4.Fluidity		0.062***				
		(0.006)				
L3.Fluidity			0.064***			
			(0.006)			
L2.Fluidity				0.066***		
				(0.006)		
L.Fluidity					0.067***	
					(0.006)	
Fluidity						0.065***
						(0.006)
Constant	2.488***	2.620***	3.119***	2.715***	2.717***	2.734***
	(0.058)	(0.063)	(0.067)	(0.058)	(0.057)	(0.057)
Observations	9318	9956	10583	11340	11546	11675
Adjusted R^2	0.179	0.205	0.246	0.259	0.263	0.260
Panel D: Highest quartile of Tobin's Q						
L5.Fluidity	0.091***					
	(0.006)					
L4.Fluidity		0.094***				
		(0.006)				
L3.Fluidity			0.095***			
			(0.006)			
L2.Fluidity				0.098***		
				(0.006)		
L.Fluidity					0.100***	
					(0.006)	
Fluidity						0.097***
						(0.006)
Constant	2.444***	2.661***	3.343***	2.760***	2.755***	2.787***
	(0.063)	(0.063)	(0.059)	(0.059)	(0.059)	(0.060)
Observations	8263	9124	10080	11057	11504	11677
Adjusted R^2	0.241	0.277	0.364	0.384	0.401	0.395

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table is a continuation of Table 4. It reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for the decision horizon. The higher the Fluidity the shorter the decision horizon. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity 5, 4, 3, 2, and 1 year(s) ago, respectively. Panel C is for firm-year observations with Tobin's Q in the third quartile. Panel D is for firm-year observations with Tobin's Q in the highest quartile. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 6: **Effect of decision horizon by total assets**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
Panel A: Lowest quartile of total assets						
L5.Fluidity	0.044***					
	(0.006)					
L4.Fluidity		0.048***				
		(0.006)				
L3.Fluidity			0.053***			
			(0.006)			
L2.Fluidity				0.054***		
				(0.006)		
L.Fluidity					0.056***	
					(0.006)	
Fluidity						0.052***
						(0.005)
Constant	3.719***	3.898***	4.211***	3.927***	3.676***	3.442***
	(0.068)	(0.066)	(0.063)	(0.062)	(0.060)	(0.056)
Observations	7739	8655	9652	10761	12018	13550
Adjusted R^2	0.197	0.213	0.253	0.262	0.258	0.247
Panel B: Second quartile of total assets						
L5.Fluidity	0.038***					
	(0.004)					
L4.Fluidity		0.043***				
		(0.004)				
L3.Fluidity			0.047***			
			(0.004)			
L2.Fluidity				0.051***		
				(0.004)		
L.Fluidity					0.054***	
					(0.004)	
Fluidity						0.055***
						(0.004)
Constant	2.800***	3.035***	3.302***	2.818***	2.577***	2.149***
	(0.051)	(0.051)	(0.052)	(0.046)	(0.045)	(0.043)
Observations	9008	9900	10810	11729	12664	13550
Adjusted R^2	0.251	0.272	0.302	0.305	0.308	0.302

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for decision horizon. The higher the Fluidity the shorter the decision horizon. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity 5, 4, 3, 2, and 1 year(s) ago, respectively. Panel A is for firm-year observations with total assets in the lowest quartile. Panel B is for firm-year observations with total assets in the second quartile. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 7: **Effect of decision horizon by total assets (Cont.)**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
Panel C: Third quartile of total assets						
L5.Fluidity	0.038***					
	(0.004)					
L4.Fluidity		0.042***				
		(0.004)				
L3.Fluidity			0.044***			
			(0.004)			
L2.Fluidity				0.044***		
				(0.004)		
L.Fluidity					0.046***	
					(0.004)	
Fluidity						0.047***
						(0.004)
Constant	2.202***	2.386***	2.664***	2.340***	2.086***	1.620***
	(0.049)	(0.050)	(0.047)	(0.046)	(0.044)	(0.041)
Observations	10143	10899	11599	12273	12940	13550
Adjusted R^2	0.282	0.294	0.313	0.312	0.310	0.311
Panel D: Fourth quartile of total assets						
L5.Fluidity	0.013***					
	(0.004)					
L4.Fluidity		0.014***				
		(0.004)				
L3.Fluidity			0.016***			
			(0.004)			
L2.Fluidity				0.016***		
				(0.004)		
L.Fluidity					0.017***	
					(0.004)	
Fluidity						0.019***
						(0.004)
Constant	1.950***	2.004***	2.465***	2.006***	1.825***	1.409***
	(0.047)	(0.045)	(0.044)	(0.042)	(0.041)	(0.037)
Observations	10756	11401	11988	12536	13017	13551
Adjusted R^2	0.306	0.313	0.340	0.339	0.338	0.337

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table is a continuation of Table 6. It reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for decision horizon. The higher the Fluidity the shorter the decision horizon. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity 5, 4, 3, 2, and 1 year(s) ago, respectively. Panel C is for firm-year observations with total assets in the third quartile. Panel D is for firm-year observations with total assets in the highest quartile. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 8: **Decision horizon effect on idiosyncratic volatility (by Real Options)**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
Panel A: High Real option projects focused industries						
L5.Fluidity	0.058***					
	(0.008)					
L4.Fluidity		0.058***				
		(0.008)				
L3.Fluidity			0.060***			
			(0.008)			
L2.Fluidity				0.060***		
				(0.008)		
L.Fluidity					0.060***	
					(0.008)	
Fluidity						0.060***
						(0.007)
Constant	3.003***	3.221***	3.708***	3.322***	3.110***	2.761***
	(0.085)	(0.084)	(0.083)	(0.083)	(0.081)	(0.082)
Observations	7538	8205	8868	9529	10209	10934
Adjusted R^2	0.180	0.211	0.269	0.284	0.289	0.287
Panel B: Low Real option projects focused industries						
L5.Fluidity	0.003					
	(0.004)					
L4.Fluidity		0.005				
		(0.004)				
L3.Fluidity			0.007*			
			(0.004)			
L2.Fluidity				0.008*		
				(0.004)		
L.Fluidity					0.010**	
					(0.004)	
Fluidity						0.011***
						(0.004)
Constant	2.742***	2.943***	3.307***	3.017***	2.864***	2.592***
	(0.040)	(0.040)	(0.038)	(0.039)	(0.038)	(0.039)
Observations	30802	33600	36404	39276	42216	45399
Adjusted R^2	0.180	0.191	0.215	0.219	0.219	0.213

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for decision horizon. The higher the Fluidity the shorter the decision horizon. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity 5, 4, 3, 2, and 1 year(s) ago, respectively. Panel A results are for industries 13, 14, 30, 35, and 36 from the Fama French 48 industries. These are: pharmaceutical products, chemicals, petroleum/natural gas, computers, and electronic equipment industries which are heavily focused on real option types of projects. Panel B results are for industries other than 13, 14, 30, 35, and 36 from the Fama French 48 industries. These are non-real option focused industries. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 9: **Decision horizon and real option effects on idiosyncratic volatility**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
L5.Fluidity	0.005					
	(0.004)					
L4.Fluidity		0.007*				
		(0.004)				
L3.Fluidity			0.008**			
			(0.004)			
L2.Fluidity				0.009**		
				(0.004)		
L.Fluidity					0.011***	
					(0.004)	
Fluidity						0.012***
						(0.004)
L5.Real.Opt	-0.027					
	(0.079)					
L4.Real.Opt		0.018				
		(0.077)				
L3.Real.Opt			0.041			
			(0.076)			
L2.Real.Opt				0.082		
				(0.075)		
L.Real.Opt					0.118	
					(0.074)	
Real.Opt						0.117
						(0.073)
L5.RO × Fluidity	0.051***					
	(0.009)					
L4.RO × Fluidity		0.048***				
		(0.008)				
L3.RO × Fluidity			0.049***			
			(0.008)			
L2.RO × Fluidity				0.047***		
				(0.008)		
L.RO × Fluidity					0.045***	
					(0.008)	
RO × Fluidity						0.047***
						(0.008)
Constant	2.800***	2.995***	3.377***	3.060***	2.888***	2.602***
	(0.038)	(0.038)	(0.036)	(0.037)	(0.036)	(0.037)
Year Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
Observations	38340	41805	45272	48805	52425	56333
Adjusted R^2	0.192	0.206	0.238	0.245	0.248	0.244

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for decision horizon. The higher the Fluidity the shorter the decision horizon. "Real.Opt" is an indicator variable that equals one for firms in industries 13, 14, 30, 35, and 36 and zero for other industries. "RO × Fluidity" is the interaction between "Real.Opt" and "Fluidity". Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity, Real.Opt, and RO × Fluidity 5, 4, 3, 2, and 1 year(s) ago, respectively. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \beta_2 \text{Real.Opt}_{i,t-\tau} + \beta_3 \text{RO} \times \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 10: **Effects of decision horizon and insider ownership**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
L5.Fluidity	0.026***					
	(0.004)					
L4.Fluidity		0.028***				
		(0.003)				
L3.Fluidity			0.030***			
			(0.003)			
L2.Fluidity				0.031***		
				(0.003)		
L.Fluidity					0.034***	
					(0.003)	
Fluidity						0.035***
						(0.003)
L5.Insider Ownership	3.929***					
	(0.200)					
L4.Insider Ownership		4.110***				
		(0.208)				
L3.Insider Ownership			4.406***			
			(0.209)			
L2.Insider Ownership				4.592***		
				(0.211)		
L.Insider Ownership					4.834***	
					(0.215)	
Insider Ownership						5.151***
						(0.215)
Constant	2.558***	2.750***	3.120***	2.796***	2.615***	2.318***
	(0.036)	(0.036)	(0.034)	(0.034)	(0.034)	(0.034)
Year Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
Observations	36232	39539	42860	46247	49700	53387
Adjusted R^2	0.216	0.232	0.267	0.276	0.282	0.283

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for decision horizon. The higher the Fluidity the shorter the decision horizon. "Insider Ownership" is the fraction of shares owned by insiders. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity and Insider Ownership 5, 4, 3, 2, and 1 year(s) ago, respectively. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \beta_2 \text{Insider Ownership}_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 11: **Effect of decision horizon with different insider ownership**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
	Panel A: Low Inside Ownership (IO) firms					
L5.Fluidity	0.014***					
	(0.005)					
L4.Fluidity		0.015***				
		(0.005)				
L3.Fluidity			0.017***			
			(0.005)			
L2.Fluidity				0.018***		
				(0.005)		
L.Fluidity					0.020***	
					(0.005)	
Fluidity						0.022***
						(0.005)
Constant	2.161***	2.314***	2.737***	2.372***	2.204***	1.849***
	(0.045)	(0.045)	(0.043)	(0.040)	(0.039)	(0.038)
Observations	12391	13436	14468	15528	16607	17787
Adjusted R^2	0.246	0.260	0.303	0.304	0.301	0.291
	Panel B: High Inside Ownership (IO) firms					
L5.Fluidity	0.026***					
	(0.007)					
L4.Fluidity		0.030***				
		(0.007)				
L3.Fluidity			0.034***			
			(0.006)			
L2.Fluidity				0.034***		
				(0.006)		
L.Fluidity					0.037***	
					(0.006)	
Fluidity						0.036***
						(0.006)
Constant	3.427***	3.632***	3.963***	3.696***	3.497***	3.281***
	(0.065)	(0.066)	(0.063)	(0.065)	(0.065)	(0.065)
Observations	12494	13704	14925	16174	17470	18856
Adjusted R^2	0.178	0.198	0.233	0.246	0.250	0.246

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for decision horizon. The higher the Fluidity the shorter the decision horizon. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity 5, 4, 3, 2, and 1 year(s) ago, respectively. The sample in Panel A consists of firms with average insider ownership lower than 0.75%. The sample in Panel B consists of firms with average insider ownership higher than 3%. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 12: **Effects of decision horizon, insider ownership, and average interaction**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
L5.Fluidity	-0.002					
	(0.004)					
L4.Fluidity		-0.001				
		(0.004)				
L3.Fluidity			0.003			
			(0.004)			
L2.Fluidity				0.004		
				(0.004)		
L.Fluidity					0.007*	
					(0.004)	
Fluidity						0.010***
						(0.004)
L5.Insider Ownership	0.545***					
	(0.176)					
L4.Insider Ownership		0.684***				
		(0.166)				
L3.Insider Ownership			1.032***			
			(0.171)			
L2.Insider Ownership				1.298***		
				(0.170)		
L.Insider Ownership					1.589***	
					(0.170)	
Insider Ownership						2.081***
						(0.172)
L5.Average IO $\times$ Fluidity	1.269***					
	(0.105)					
L4.Average IO $\times$ Fluidity		1.271***				
		(0.108)				
L3.Average IO $\times$ Fluidity			1.239***			
			(0.108)			
L2.Average IO $\times$ Fluidity				1.216***		
				(0.108)		
L.Average IO $\times$ Fluidity					1.212***	
					(0.107)	
Average IO $\times$ Fluidity						1.151***
						(0.107)
Constant	2.638***	2.830***	3.197***	2.869***	2.686***	2.387***
	(0.034)	(0.034)	(0.033)	(0.033)	(0.032)	(0.033)
Year Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
Observations	36232	39539	42860	46247	49700	53387
Adjusted R^2	0.275	0.288	0.317	0.323	0.328	0.325

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for decision horizon. The higher the Fluidity the shorter the decision horizon. "Insider Ownership" is the fraction of shares owned by insiders. "Average IO" is the time-series mean of Insider Ownership for each firm. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity, Insider Ownership, and Average IO $\times$ Fluidity 5, 4, 3, 2, and 1 year(s) ago, respectively. The sample consists of firms year observations with total assets in the highest quartile. The sample period is 1997–2015.

$$\text{IVOL}_{i,t} = \beta_0 + \beta_1 \text{Fluidity}_{i,t-\tau} + \beta_2 \text{Insider Ownership}_{i,t-\tau} + \beta_3 (\text{Average IO} \times \text{Fluidity})_{i,t-\tau} + \nu_t + \varepsilon_{i,t}$$

Table 13: **Decision horizon effect (full) controlling for HHI**

	(1) IVOL b/se	(2) IVOL b/se	(3) IVOL b/se	(4) IVOL b/se	(5) IVOL b/se	(6) IVOL b/se
L5.Fluidity	0.025***					
	(0.004)					
L4.Fluidity		0.027***				
		(0.004)				
L3.Fluidity			0.030***			
			(0.004)			
L2.Fluidity				0.031***		
				(0.004)		
L.Fluidity					0.033***	
					(0.004)	
Fluidity						0.034***
						(0.004)
L5.HHI	-0.175*					
	(0.089)					
L4.HHI		-0.175**				
		(0.089)				
L3.HHI			-0.174**			
			(0.088)			
L2.HHI				-0.154*		
				(0.086)		
L.HHI					-0.144*	
					(0.085)	
HHI						-0.135
						(0.085)
Constant	2.823***	3.017***	3.398***	3.078***	2.894***	2.602***
	(0.042)	(0.042)	(0.041)	(0.041)	(0.040)	(0.041)
Year Fixed Effects	Yes	Yes	Yes	Yes	Yes	Yes
Observations	36328	39764	43205	46725	50326	54201
Adjusted R^2	0.163	0.188	0.224	0.233	0.237	0.234

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses.

This table reports the results of OLS estimations with year fixed effects and standard errors clustered on firms. The outcome variable is the idiosyncratic return volatility, "IVOL." "Fluidity" is a measure for decision horizon. The higher the Fluidity the shorter the decision horizon. "HHI" is the Herfindahl-Hirschman index constructed for each 3-digit SIC industry each year. Columns 1, 2, 3, 4, and 5 present the estimates for the coefficients of Fluidity and HHI 5, 4, 3, 2, and 1 year(s) ago, respectively. The sample period is 1997–2015.

Appendix

A Proofs

Before the proof of Theorem 1, let us consider $V_e(w,k)$, defined in (5), which is the value function at t_k with wealth w if the real option is already exercised or expired. The explicit formula of $V_e(w,k)$, the optimal consumption and the optimal investment in the risky asset, which gives $V_e(w,k)$, are presented by the following lemma. The proof is omitted.

Lemma 1. *After exercise or maturity of the real option, the value function $V_e(w,k)$ at time t_k with wealth w is*

$$V_e(w,k) = \frac{1+r}{r} U\left(\frac{r}{1+r}(w+K)\right), \tag{A1}$$

and the optimal consumption $c_k^*(w)$ and investment in the risky asset π_k^* are given by

$$(c_k^*(w), \pi_k^*) = \left(\frac{r}{1+r}(w+K), \frac{1+r}{\gamma r(\alpha_u - \alpha_d)} \ln\left(\frac{\alpha_u - (1+r)}{(1+r) - \alpha_d}\right)\right), \tag{A2}$$

where $K = -\frac{1+r}{\gamma r^2} \ln\left[\frac{1}{2}\left\{\frac{\alpha_u - \alpha_d}{(1+r)-\alpha_d}\left(\frac{(1+r)-\alpha_d}{\alpha_u-(1+r)}\right)^{\frac{\alpha_u-(1+r)}{\alpha_u-\alpha_d}}\right\}\right]$.

Proof of Theorem 1

Proof. The proof is done by backward induction. Since $t_N = T$ is maturity of the real option, the option should be exercised if $X_N \geqslant I$ and the option expires without any payoff if $X_N < I$. Hence we have

$$V(w,x,N) = V_e(w + Y_N(x), N) = \frac{1+r}{r} U\left(\frac{r}{1+r}(w + K + Y_N(x))\right)$$

by using $Y_N(x) = (x-I)^+$ and (A1). Now, suppose $V(w,x,k+1) = \frac{1+r}{r}U\left(\frac{r}{1+r}(w+K+Y_{k+1}(x))\right)$. Then it is enough to show that $V(w,x,k) = \frac{1+r}{r}U\left(\frac{r}{1+r}(w+K+Y_k(x))\right)$. In this case, $V_n(w,x,k)$ can be written as

$$V_n(w,x,k) = \max_{c_k,\pi_k} F_k(c_k,\pi_k),$$

where

$$\begin{aligned} F_k(c_k,\pi_k) =& U(c_k) + \frac{1}{4r}U\Big(r(w-c_k) + \pi_k\Big(\frac{r\alpha_u}{1+r} - r\Big) + \frac{r}{1+r}K + \frac{r}{1+r}Y_{k+1}(x+u_1)\Big) \\ &+ \frac{1}{4r}U\Big(r(w-c_k) + \pi_k\Big(\frac{r\alpha_u}{1+r} - r\Big) + \frac{r}{1+r}K + \frac{r}{1+r}Y_{k+1}(x+u_2)\Big) \\ &+ \frac{1}{4r}U\Big(r(w-c_k) + \pi_k\Big(\frac{r\alpha_d}{1+r} - r\Big) + \frac{r}{1+r}K + \frac{r}{1+r}Y_{k+1}(x+d_1)\Big) \\ &+ \frac{1}{4r}U\Big(r(w-c_k) + \pi_k\Big(\frac{r\alpha_d}{1+r} - r\Big) + \frac{r}{1+r}K + \frac{r}{1+r}Y_{k+1}(x+d_2)\Big). \end{aligned} \tag{A3}$$

Let us define $\{\tilde{c}_k, \tilde{\pi}_k\} = \operatorname*{argmax}_{c_k,\pi_k} F_k(c_k,\pi_k)$. Since $U(\infty) = 0$ and $U(-\infty) = -\infty$, we have

$$F_k(\infty,\pi_k) = F_k(-\infty,\pi_k) = -\infty, \quad F_k(c_k,\infty) = F_k(c_k,-\infty) = -\infty.$$

Therefore, the maximum of the function F_k is a critical point. The first-order condition with respect to c_k with $U'(\cdot) = -\gamma U(\cdot)$ implies that $V_n(w,x,k) = \frac{1+r}{r}U(\tilde{c}_k)$ and

$$\tilde{c}_k = \frac{r}{1+r}w + \frac{r}{(1+r)^2}K - \frac{1}{\gamma(1+r)} \ln\left(\frac{1}{4}M_k(x)\right), \tag{A4}$$

where

$$\begin{aligned} M_k(x) \triangleq\ & e^{-\gamma\tilde{\pi}_k\left(\frac{r\alpha_u}{1+r}-r\right)}\left(e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+u_1)} + e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+u_2)}\right) \\ & + e^{-\gamma\tilde{\pi}_k\left(\frac{r\alpha_d}{1+r}-r\right)}\left(e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+d_1)} + e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+d_2)}\right). \end{aligned}$$

The first-order condition with respect to π_k provides

$$\tilde{\pi}_k = \tilde{\pi}_k(x) = \frac{1+r}{\gamma r(\alpha_u-\alpha_d)} \ln\left(\frac{\alpha_u-(1+r)}{(1+r)-\alpha_d}\right) + \frac{1+r}{\gamma r(\alpha_u-\alpha_d)}H(x,k,\rho), \tag{A5}$$

where $H(x,k,\rho)$ is defined by (11). By substituting (A5) into $M_k(x)$ in (A4), we have

$$\begin{aligned} \tilde{c}_k = \tilde{c}_k(w,x) = & \frac{r}{1+r}(w+K) \\ & - \frac{1}{\gamma(1+r)} \ln\Bigg[\left(\frac{e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+u_1)} + e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+u_2)}}{2}\right)^{\frac{(1+r)-\alpha_d}{\alpha_u-\alpha_d}} \left(\frac{e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+d_1)} + e^{-\frac{\gamma r}{1+r}Y_{k+1}(x+d_2)}}{2}\right)^{\frac{\alpha_u-(1+r)}{\alpha_u-\alpha_d}}\Bigg]. \end{aligned} \tag{A6}$$

Note that

$$V(w,x,k) = \max(V_e(w+(x-I)^+,k), V_n(w,x,k))$$

where $V_e(w+(x-I)^+,k) = \frac{1+r}{r}U\left(\frac{r}{1+r}(w+K+(x-I)^+)\right)$ and $V_n(w,x,k) = \frac{1+r}{r}U(\tilde{c}_k(w,x))$. Therefore, define $Y_k(x)$ as in (7) and we have $V(w,x,k) = \frac{1+r}{r}U\left(\frac{r}{1+r}(w+K+Y_k(x))\right)$, which completes the proof. □

Proof of Theorem 2

Proof. At maturity T of the real option, it is optimal to exercise the real option if $X_N \geqslant I$. Hence, the optimal exercise threshold at maturity is $\bar{x}_N = I$. For $k = 0,1,\ldots,N-1$, we can verify that there is a unique $\bar{x}_k$ satisfying equation (8). Moreover, it can be shown that $x < \bar{x}_k$ is equivalent to $Y_k(x) > (x-I)^+$ or $V(w,x,k) = V_n(w,x,k) > V_e(w,x,k)$. On the other hand, $x \geqslant \bar{x}_k$ is equivalent to $Y_k(x) = (x-I)^+$ or $V(w,x,k) = V_e(w,x,k)$. Therefore, it is optimal to exercise the real option if $X_k \geqslant \bar{x}_k$ at time t_k. Hence, $\bar{x}_k$ is the optimal exercise boundary of the real option at time t_k.

Now we are left to derive the optimal policy. Assume that $X_k = x$ and $W_k = w$ at time t_k before deciding to exercise the real option. Let $\bar{c}_k(w,x)$ and $\bar{\pi}_k(x)$ be the optimal consumption and the optimal risky investment at time t_k, respectively. If $x < \bar{x}_k$, the real option is not exercised and $\bar{c}_k(w,x) = \tilde{c}_k(w,x)$ with $\tilde{c}_k$ in (A6) and $\bar{\pi}_k(x) = \tilde{\pi}_k(x)$ where $\tilde{\pi}_k(x)$ is given by (A5). However, if $x \geqslant \bar{x}_k$, the real option should be exercised and the wealth becomes $w + (x-I)^+$. Therefore, after the exercise $\bar{c}_k(w,x) = c_k^*(w+(x-I)^+)$ and $\bar{\pi}_k = \pi_k^*$, where (c_k^*, π_k^*) is given by (A2). In sum,

$$\bar{c}_k(w,x) = \begin{cases} \tilde{c}_k(w,x), & x < \bar{x}_k, \\ c_k^*(w+(x-I)^+), & x \geqslant \bar{x}_k, \end{cases} \quad \text{and} \quad \bar{\pi}_k(x) = \begin{cases} \tilde{\pi}_k(x), & x < \bar{x}_k, \\ \pi_k^*, & x \geqslant \bar{x}_k. \end{cases}$$

By the definition of $Y_k(x)$, the above equations are indeed equation (9) and (10). □

Proof of Proposition 1

Proof. Without loss of generality we consider the case when $x < \bar{x}_k$ and $\rho = 0$. The case with nonzero ρ only requires a bit more calculation. Note first that $Y_k(x) \geqslant (x - I)^+$ and $0 < Y_k'(x) \leqslant 1$. $Y_k'(x) < 1$ for $x < \bar{x}_k$ and $Y_k'(x) = 1$ for $x \geqslant \bar{x}_k$. If $\rho = 0$, we have $u_1 = d_1 = \alpha_x h + \sigma_x \sqrt{h}$ and $u_2 = d_2 = \alpha_x h - \sigma_x \sqrt{h}$. Then, we have

$$Y_k'(x) = \max\left(1, \frac{1}{1+r}\left[A_{k+1}(x)Y_{k+1}'(x+u_1) + B_{k+1}(x)Y_{k+1}'(x+u_2)\right]\right) \tag{A7}$$

where $A_{k+1}(x) = 1 - B_{k+1}(x)$ and

$$\frac{1}{2} < B_{k+1}(x) := \frac{\exp\left(-\frac{\gamma r}{1+r}Y_{k+1}(x+u_2)\right)}{\exp\left(-\frac{\gamma r}{1+r}Y_{k+1}(x+u_1)\right) + \exp\left(-\frac{\gamma r}{1+r}Y_{k+1}(x+u_2)\right)} < 1.$$

Note that $Y_{k+1}'(x+u_2) < Y_{k+1}'(x+u_1)$. (A7) implies that if $x < \bar{x}_k$, $Y_k'(x)$ is the linear combination of $Y_{k+1}'(x+u_1)$ and $Y_{k+1}'(x+u_2)$. Now the proof follows from the following three facts: (i) the monotone decreasing property of the threshold $\bar{x}_k$ in k with $\bar{x}_N = I$ (see Remark 1), (ii) the smooth-pasting condition,i,e, $Y_k'(\bar{x}_k) = 1$, (iii) the convexity of $Y_k(x)$ with respect to x.

□

Proof of Theorem 3

Proof. For $k = N$, it is obvious from (19) that $V_1(w,a,x,k)$ can be written as (20), where $B_N(a,x)$ is defined by (21). For $k = N-1, N-2, \cdots, 0$, we use backward induction, which is similar to the proof of Theorem 1. Suppose that $V_1(w,a,x,k+1) = \frac{1+r}{r}U\left(\frac{r}{1+r}(w + B_{k+1}(a,x))\right)$. Then, it is enough to show that $V_1(w,a,x,k) = \frac{1+r}{r}U\left(\frac{r}{1+r}(w + B_k(a,x))\right)$ as a consequence, and this is done by comparing the continuation value and the intrinsic value. Let $V_1^e(w,a,x,k)$ be the maximized expected utility at time t_k if the first option is exercised at t_k and w is the wealth before the exercise of the first option. Then $V_1^e(w,a,x,k)$ becomes

$$V_1^e(w,a,x,k) = V_2(w + (a-I_1)^+, x, k) = \frac{1+r}{r}U\left(\frac{r}{1+r}(w + (a-I_1)^+ + Y_k(x))\right). \tag{A8}$$

On the other hand, let $V_1^c(w,a,x,k)$ be the maximized expected utility at time t_k if the first option is not exercised at t_k. Then we have

$$\begin{aligned} V_1^c(w,a,x,k) = \max_{c_{t_k}} \Big[U(c_k) &+ \frac{1}{4}\beta V_1((1+r)(w - c_{t_k}), a+u_1, x+u_x, k+1) \\ &+ \frac{1}{4}\beta V_1((1+r)(w - c_{t_k}), a+u_2, x+u_x, k+1) \\ &+ \frac{1}{4}\beta V_1((1+r)(w - c_{t_k}), a+d_1, x+d_x, k+1) \\ &+ \frac{1}{4}\beta V_1((1+r)(w - c_{t_k}), a+d_2, x+d_x, k+1)\Big]. \end{aligned}$$

In this case, we can easily show that

$$V_1^c(w,a,x,k) = \frac{1+r}{r}U\left(\frac{r}{1+r}(w + B_k^c(a,x))\right), \tag{A9}$$

where $B^c_k(a,x)$ is given as (23). If $V^e_1(w,a,x,k)$ is greater than $V^c_1(w,a,x,k)$, it is optimal to exercise the first option immediately at t_k, and $V_1(w,a,x,k) = V^e_1(w,a,x,k)$. However, if $V^e_1(w,a,x,k)$ is less than $V^c_1(w,a,x,k)$, then it means that it is optimal not to exercise the first option at t_k to move to the next time step t_{k+1}. Thus, in this case, $V_1(w,a,x,k) = V^c_1(w,a,x,k)$. Therefore, $V_1(w,a,x,k)$ can be written as (20), where $B_k(a,x)$ is defined as (22). ☐

Proof of Proposition 2

Proof. The idea is basically the same as that in proof of Proposition 1. We can obtain similar formulas for $\frac{\partial B_k(a,x)}{\partial a}$ and $\frac{\partial B_x(a,x)}{\partial a}$, which are omitted here. In addition, there are two threshold dynamics for each option: $\bar{a}_k$ and $\bar{x}_k$ for option A and X, respectively. It is straightforward to see that both $\bar{a}_k$ and $\bar{x}_k$ decreases in k and $\bar{a}_N = I$ and $\bar{x}_N = I$ and $\frac{\partial B_k(a,x)}{\partial a}$ and $\frac{\partial B_x(a,x)}{\partial a}$ are less than 1 when $a < \bar{a}_k$ and $x < \bar{x}_k$. Furthermore, $\frac{\partial B_k(\bar{a}_k,x)}{\partial a} = 1$ and $\frac{\partial B_k(a,\bar{x}_k)}{\partial x} = 1$, which completes the proof. ☐

B Robustness and Extension

B.1 Exercise Threshold

An option value is the sum of its intrinsic value and time value. The intrinsic value is determined just by the current value of the underlying asset regardless of the time to maturity and the idiosyncratic risk, while the time value depends on them. The time to maturity and the idiosyncratic risk affect the time value via the exercise threshold because a high exercise threshold reflects a high time value. In summary, for a given underlying asset value, the intrinsic value is fixed regardless of the time to maturity and the idiosyncratic risk, while the option value has a positive relationship with the exercise threshold, which depends on the time to maturity and the idiosyncratic risk. Thus, the exercise threshold of the real option has dynamics similar to those of the implied option value, which we will show in this subsection.

Consider Figures 9 and 10. By comparing three curves in each figure, we can observe that the exercise threshold decreases much more sharply as the time approaches maturity when the idiosyncratic risk is high than when the risk is small. At time zero, the curve with higher σ_x has a higher value than the curve with lower σ_x, but the former becomes smaller than the one with lower σ_x as time approaches maturity. Each curve crosses each other at some time before maturity. Notice that the right panel of Figure 10 does not have such a crossover with $T = 20$, but it will also show a similar pattern if we draw the graph with a larger T. Thus, the threshold increases (decreases) in the idiosyncratic risk if the time to maturity is long (short), similar to that of the implied option value in the previous subsection.

Notice that classical models of real options in the complete market show that an increase in project risk increases the investment threshold (Brennan and Schwartz 1985; McDonald and Siegel 1986; Dixit and Pindyck 1994). In the incomplete market, however, the threshold of the project value is lower as the idiosyncratic volatility of the project becomes higher because of the precautionary savings motive if the agent is sufficiently risk-averse (Henderson 2007; Hugonnier and Morellec 2007; Miao and Wang 2007). We find that these two standard cases also occur in our model as special cases in terms of the threshold dynamics. First, if the risk aversion is very high, then the exercise threshold decreases at any point of time as idiosyncratic risk increases (see the right panel of Figure 10), and this result is highlighted in the existing incomplete market real option literature. Second, if risk aversion is sufficiently low, the exercise threshold with higher idiosyncratic risk is much higher than that with lower idiosyncratic risk (see the left panel of Figure 9), which is consistent with the

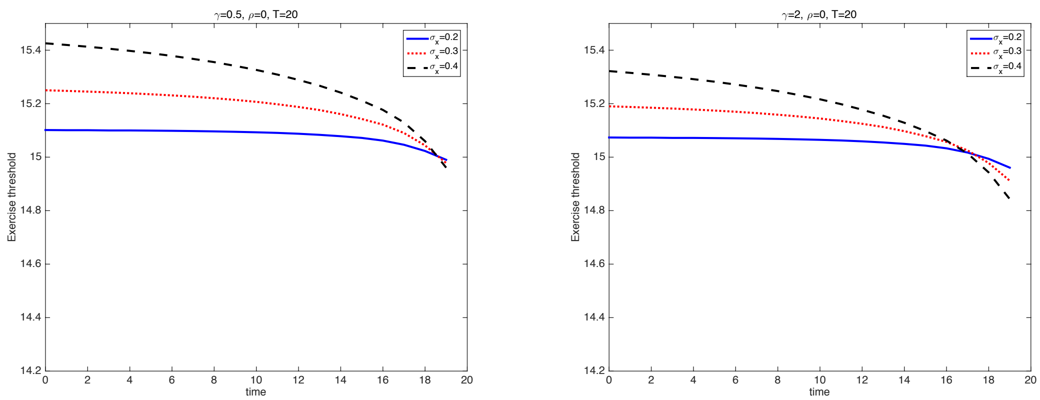


Figure 9: Time horizon effect on the exercise threshold: The parameters used in the left and right panels are the same except for risk aversion. The left panel has $\gamma = 0.5$ and the right panel has $\gamma = 2$.

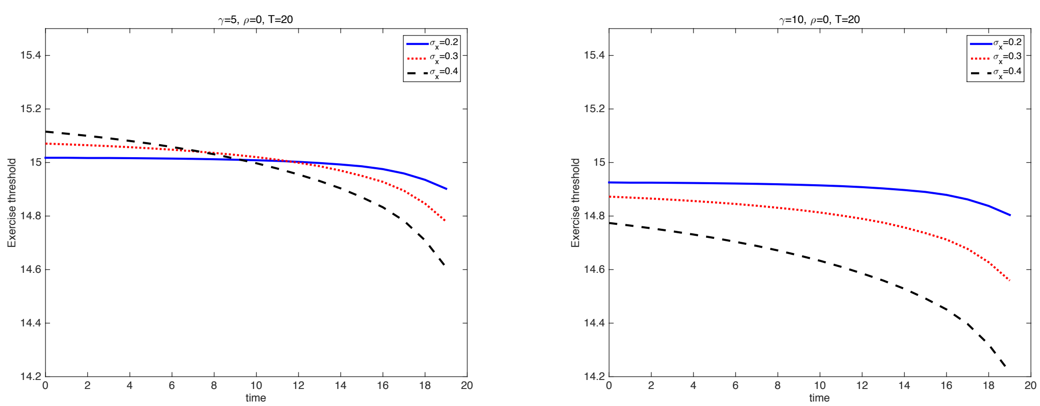


Figure 10: Time horizon effect on the exercise threshold: The parameters used in the left and right panels are the same except for risk aversion. The left panel has $\gamma = 5$ and the right panel has $\gamma = 10$.

result of the complete market real option literature. Lastly, our new finding is that an increase in the idiosyncratic volatility increases the threshold level of exercising the real option if the time to maturity is long enough, while the result is overturned if the time to maturity is short enough. This typical pattern can be seen in Figure 9 and the left panel of Figure 10. However, this pattern will definitely be shown in the right panel of Figure 10 if we extend the time to maturity. In this sense, our contribution to the literature is to show that the precautionary savings motive becomes stronger as the remaining time to maturity becomes shorter.

Before concluding this subsection, we discuss the effect of the correlation (ρ) on the exercise threshold. In the above numerical illustrations of the crossover of the exercise threshold by the horizon effect, we assume $\rho = 0$. Indeed, if $\rho \geqslant 0$, then the exercise threshold decreases with the idiosyncratic volatility, at least at the very last node, which is shown in Proposition 3.

Proposition 3. *Suppose that* $\bar{x}_{N-1} + u_2 - I > 0$ *and* $\bar{x}_{N-1} + d_2 - I > 0$*. Then we have*

$$\frac{\partial \bar{x}_{N-1}}{\partial \sigma_x} = \frac{\sqrt{h}}{r}(A_1\rho + A_2\sqrt{1-\rho^2} + A_3\sqrt{1-\rho^2}), \tag{A10}$$

with

$$\begin{aligned} A_1 &\triangleq \frac{2(1+r) - (\alpha_u + \alpha_d)}{\alpha_u - \alpha_d} < 0, \\ A_2 &\triangleq \frac{(1+r) - \alpha_d}{\alpha_u - \alpha_d} \frac{e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+u_1-I)} - e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+u_2-I)}}{e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+u_1-I)} + e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+u_2-I)}} < 0, \\ A_3 &\triangleq \frac{\alpha_u - (1+r)}{\alpha_u - \alpha_d} \frac{e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+d_1-I)} - e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+d_2-I)}}{e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+d_1-I)} + e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+d_2-I)}} < 0. \end{aligned} \tag{A11}$$

Moreover, we have

$$\frac{\partial \bar{x}_{N-1}}{\partial \sigma_x} < 0 \quad \textit{for } \rho \geqslant 0. \tag{A12}$$

Proof. Since $u_1 > u_2$ and $d_1 > d_2$, the equation (8) with $k = N-1$, can be written as

$$\begin{aligned} (\bar{x}_{N-1} - I) = &-\frac{1}{\gamma r}\frac{(1+r)-\alpha_d}{\alpha_u - \alpha_d} \ln\left(\frac{e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+u_1-I)} + e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+u_2-I)}}{2}\right) \\ &-\frac{1}{\gamma r}\frac{\alpha_u-(1+r)}{\alpha_u - \alpha_d} \ln\left(\frac{e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+d_1-I)} + e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+d_2-I)}}{2}\right) \end{aligned} \tag{A13}$$

with the assumptions $\bar{x}_{N-1} + d_2 - I > 0$ and $\bar{x}_{N-1} + u_2 - I > 0$. Notice that

$$\begin{aligned} \frac{\partial u_1}{\partial \sigma_x} &= (\rho + \sqrt{1-\rho^2})\sqrt{h}, \quad \frac{\partial u_2}{\partial \sigma_x} = (\rho - \sqrt{1-\rho^2})\sqrt{h}, \\ \frac{\partial d_1}{\partial \sigma_x} &= (-\rho + \sqrt{1-\rho^2})\sqrt{h}, \quad \frac{\partial d_2}{\partial \sigma_x} = (-\rho - \sqrt{1-\rho^2})\sqrt{h}. \end{aligned}$$

Therefore, by differentiating both sides of (A13) with respect to σ_x, we can obtain (A10) where A_1, A_2 and A_3 are defined in (A11). Notice that we have $2(1+r) - (\alpha_u + \alpha_d) < 0$, $\alpha_u - \alpha_d > 0$, $(1+r) - \alpha_d > 0$ and $\alpha_u - (1+r) > 0$ by the (3). Moreover, we have

$$e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+u_1-I)} - e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+u_2-I)} < 0 \text{ and } e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+d_1-I)} - e^{-\frac{\gamma r}{1+r}(\bar{x}_{N-1}+d_2-I)} < 0$$

because $u_1 > u_2$ and $d_1 > d_2$. As a consequence, $A_1 < 0$, $A_2 < 0$ and $A_3 < 0$. Then, the inequality (A12) is straightforward. □

However, if $\rho < 0$, then inequality (A12) in Proposition 3 may not hold and the exercise threshold may increase in the idiosyncratic risk uniformly across time without crossover, as in classical models of real options in the complete market, which is illustrated in Figure 11.

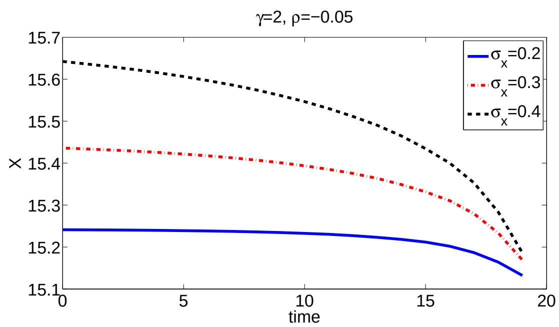


Figure 11: Time horizon effect on the exercise threshold: $T = 20$ years, $N = 20$, $\alpha = 0.07$, $\Sigma = 0.2$, $\alpha_x = 0.1$, $r = 0.02$, $I = 10$.

Figure 12 illustrates that the exercise threshold and the correlation have a negative relationship uniformly across time. Furthermore, we can observe, by comparing the left and right panels of Figure 12, that the negative relationship becomes stronger as the idiosyncratic risk becomes larger. By the same token, two different threshold curves in Figure 13 cross each other at an earlier time period with a larger ρ.

The basic intuition behind the negative relationship between the exercise threshold and the correlation can be more easily understood if we consider how the optimal hedging component $H(x, k, \rho)$ of (11) changes in ρ. The hedging demand is negatively related with correlation ρ (at least near maturity). As seen in Figure 14, given a positive excess return from the market portfolio, an increase in ρ decreases the risky investment. Therefore, the implied value of the option decreases in ρ and thus the exercise threshold decreases.

To be more specific, assume that the excess return of the risky asset is positive as usual. In this case, an increase in the correlation decreases the hedging component, preventing the agent from taking on enough risk in the market. Therefore, a high correlation signifies the role of idiosyncratic

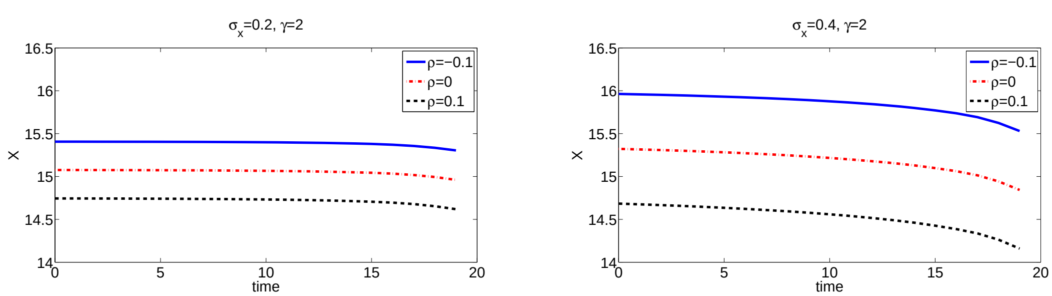


Figure 12: The left panel plots the threshold at each ρ when $\sigma_x = 0.2$ and $\gamma = 2$, and the right panel plots when$\sigma_x = 0.4$ and $\gamma = 2$. All the other parameter values are the same as in Figure 11.

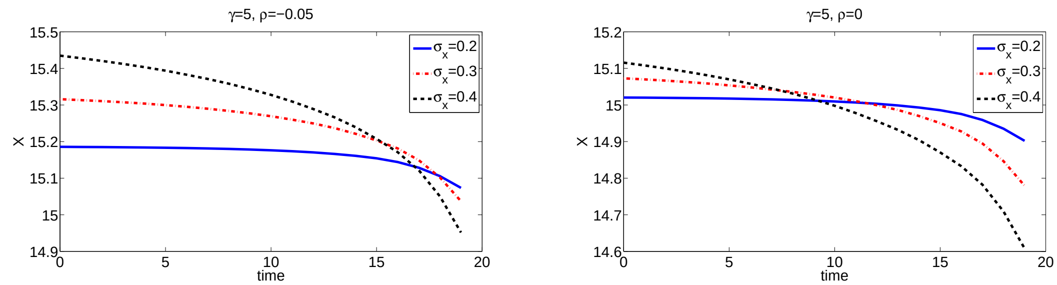


Figure 13: Case III: The left panel plots the threshold when $(\gamma, \rho) = (5, -0.05)$, and the right panel plots when $(\gamma, \rho) = (5, 0)$. All the other parameter values are the same as in Figure 11.

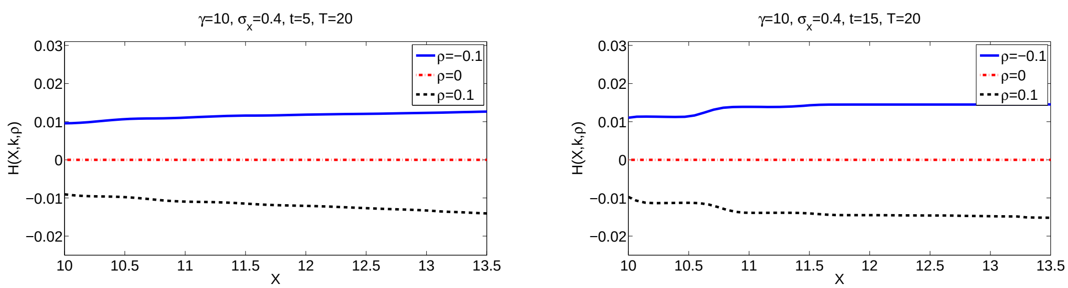


Figure 14: The left panel plots the hedging component $H(x, k, \rho)$ before the exercise of the real option when $t = 5$, $\gamma = 10$ and $\sigma_x = 0.4$, and the right panel plots when $t = 15$, $\gamma = 10$ and $\sigma_x = 0.4$. All the other parameter values are the same as in Figure 11.

risk, strengthens the precautionary savings motive, and thus decreases the implied option value. In other words, the crossover occurs earlier for a high correlation between the real option payoff and the market asset.

B.2 Geometric Browninan Motion

Our results are based on the assumption that the payoff process follows the equation (2), whose continuous time analogue is the arithmetic Brownian motion. However, instead of (2), we also can consider a geometric Brownian motion payoff process as follows.

$$X_{k+1} = X_k \exp\left(\alpha_x h + \rho\sigma_x\sqrt{h}\Delta b^1_{t_k} + \sqrt{1-\rho^2}\sigma_x\sqrt{h}\Delta b^2_{t_k}\right), \quad X_0 = x > 0, \tag{A14}$$

where $\alpha_x := \mu_x - \sigma_x^2/2$. In addition, we set

$$\begin{aligned}&\mathbb{P}\left(\frac{P_{k+1}}{P_k} = \alpha_u,\ \frac{X_{k+1}}{X_k} = u_1\right) = \mathbb{P}\left(\frac{P_{k+1}}{P_k} = \alpha_u,\ \frac{X_{k+1}}{X_k} = u_2\right)\\ &= \mathbb{P}\left(\frac{P_{k+1}}{P_k} = \alpha_d,\ \frac{X_{k+1}}{X_k} = d_1\right) = \mathbb{P}\left(\frac{P_{k+1}}{P_k} = \alpha_d,\ \frac{X_{k+1}}{X_k} = d_2\right) = 1/4\end{aligned}$$

with

$$u_1 \triangleq \exp\left(\alpha_x h + \rho\sigma_x\sqrt{h} + \sqrt{1-\rho^2}\sigma_x\sqrt{h}\right), \quad u_2 \triangleq \exp\left(\alpha_x h + \rho\sigma_x\sqrt{h} - \sqrt{1-\rho^2}\sigma_x\sqrt{h}\right),$$
$$d_1 \triangleq \exp\left(\alpha_x h - \rho\sigma_x\sqrt{h} + \sqrt{1-\rho^2}\sigma_x\sqrt{h}\right), \quad d_2 \triangleq \exp\left(\alpha_x h - \rho\sigma_x\sqrt{h} - \sqrt{1-\rho^2}\sigma_x\sqrt{h}\right).$$

In this case, the solution to the value function and the optimal strategies can be derived by using a similar technique; we only need to replace $x+u_1$, $x+u_2$, $x+d_1$ and $x+d_2$ by u_1x, u_2x, d_1x and d_2x, respectively. More importantly, our main results still hold with the payoff process in (A14). This is not surprising since the intuition of our results does not hinge on the type of payoff process.

B.3 Continuous-time Model

Let us consider a continuous time model with a risky asset

$$dS_t/S_t = \mu dt + \sigma dB_t,$$

and a single real option whose payoff is $(X_\tau - I)^+$, where τ is the exercise time of the real option and X_t follows

$$dX_t = \mu_x dt + \sigma_x(\rho dB_t + \sqrt{1-\rho^2}d\widetilde{B}_t). \tag{A15}$$

μ, σ, μ_x, σ_x, and $\rho \in (-1,1)$ are constant, and B_t and $\widetilde{B}_t$ are independent standard Brownian motions. The real option expires on $T > 0$ and it is assumed that the exercise decision of the real option can be made only on discrete dates $\{t_0, t_1, \cdots, t_N\}$, where $t_k = kh$ for $h := T/N$ and $k = 0, 1, \cdots, N$. Let c_t and π_t be the consumption rate and the amount invested in the risky asset at time t, respectively. Then, the agent's wealth process W_t follows

$$dW_t = [rW_t + (\mu - r)\pi_t - c_t]dt + \sigma\pi_t dB_t, \tag{A16}$$

where r is the risk-free rate.

Since the real option can only be exercised on t_k's, we focus on the optimal decisions on t_k's for $k = 0, 1, \cdots, N$, and the optimization problem of the agent at time t_k is given as follows:

$$V(w,x,t_k) := \max_{c_t,\pi_t,\tau\in\{t_k,t_{k+1},\cdots,t_N\}} \mathbb{E}\left[\int_{t_k}^{\tau} e^{-\beta t}u(c_t)dt + e^{-\beta\tau}V_M(W_\tau + (X_\tau - I)^+)\Big| W_{t_k} = w,\ X_{t_k} = x\right], \tag{A17}$$

subject to (A15) and (A16), where $V_M(\cdot)$ is defined as the value function of Merton's problem in infinite horizon.

Remark 3. *If $u(c) = -\exp(-\gamma c)/\gamma$, then $V_M(w)$ becomes*

$$V_M(w) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + \frac{\beta - r + \theta^2}{2\gamma r^2}\right)\right], \tag{A18}$$

where $\theta = (\mu - r)/\sigma$.

Theorem 4. *Assume that $u(c) = -\exp(-\gamma c)/\gamma$ and $\rho = 0$. For $j = 0, 1, \cdots, N$, define $K_{N-j}(x)$*

and $\bar{X}_{N-j}$ *recursively as follows:*

$$\begin{aligned}
j = 0: \quad & K_N(x) = 1, \quad \bar{X}_N = I, \\
j \geqslant 1: \quad & K_{N-j}(x) = \int_{\frac{\bar{X}_{N-j+1}-x-\mu_x h}{\sigma_x\sqrt{h}}}^{\infty} e^{-\gamma r(x+\mu_x h+\sigma\sqrt{h}z-I)}\phi(z)dz \\
& \qquad + \int_{-\infty}^{\frac{\bar{X}_{N-j+1}-x-\mu_x h}{\sigma_x\sqrt{h}}} K_{N-j+1}(x+\mu_x h+\sigma\sqrt{h}z)^{e^{-rh}}\phi(z)dz,
\end{aligned}$$

where $\phi(\cdot)$ *is the probability density function of standard Normal distribution, and* $\bar{X}_{N-j}$ *is the unique root of the following equation:*

$$(\bar{X}_{N-j} - I)^+ = -\frac{1}{\gamma r}\ln\left[K_{N-j}(\bar{X}_{N-j})^{e^{-rh}}\right]. \tag{A19}$$

Then, for $j = 0, 1, \cdots, N$, *we have*

$$V(w, x, t_{N-j}) = V_M(w)e^{-\gamma r \max\left[-\frac{1}{\gamma r}\ln\left(K_{N-j}(x)^{e^{-rh}}\right),(x-I)^+\right]} \tag{A20}$$

and the optimal exercise time of the real option is

$$\tau = \inf\{t \in \{t_0, t_1, \cdots, t_N\} | X_t \geqslant \bar{X}_t\}.$$

Figure 15 illustrates the exercise thresholds for the continuous time model. Although the time horizon is only 4 because of the computational cost, we can observe from Figure 15 that the exercise threshold is increasing (decreasing) in the idiosyncratic risk σ_x if the remaining time to maturity is long (short). This shows that our results, derived by using a discrete time model, still hold with a continuous time model.

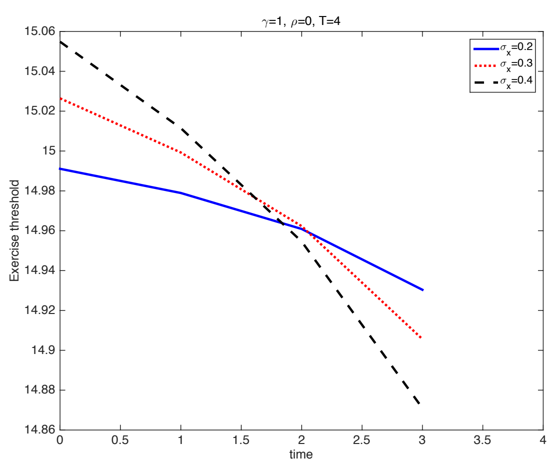


Figure 15: Exercise threshold for the continuous time model