Investing in a Random Start American Option Under Competition*

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Abstract

In this paper we develop a model to determine the value of the opportunity to invest in a random start American (real) option. In contrast to a typical American option, the random start option only exists if an exogenous event occurs materializing the (true) American option to invest. In addition, the effect of competition is also considered in the model. A higher risk of competition and a higher probability of the exogenous event promotes investment. Uncertainty has a non-monotonic effect on investment timing.

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1 Introduction

In this paper we develop a model capable of determining the value of the opportunity to invest in a random start American (real) option. In contrast to a typical American option, the random start option[1] (RSO) only exists if some exogenous event occurs. The random (exogenous) event is assumed to be outside of the investor's control, and only after it occurs the (true) American option to invest materializes.

Several examples fit with this setting. An investment opportunity that depends on the authorization of a public entity, which may eventually arrive in the future (e.g., the license to transform a rural land, with construction limitations, into an urban one); an R&D race where the discovery arrives randomly (Lint and Pennings, 1998); or a project that depends on a technology developed by a third-party firm (e.g., the iPad was dependent on an efficient multi-touch screen technology, developed outside Apple).

Additionally, we introduce competition over the random start American option, which means that, at the beginning, the firm has no proprietary rights on the RSO. The only possibility the firm has to eliminate competition is to spend an initial irreversible investment.

In the context of our examples, the initial capital investment could correspond to the acquisition of the rural land with the expectation that it will be later transformed into urban by the authorities (by acquiring the land the investor becomes proprietary of the random start option). Similarly, the firm can invest in patenting the potential discovery that may randomly arrive during the R&D process (the alternative that does not eliminate competition would be to patent the discover only if and when it occurs). Finally, for the last example, the firm can pay a third-party to secure exclusive rights in the case the technology arrives, ensuring monopolistic rents.

Our paper closely relates to Armerin (2017). The author also considers a similar American option that can only be exercised after a random period of time has passed. The value of this option is derived in detail, both for the case of a call (invest) and a put (abandon). The author also determines the expected time for the random American option be optimally exercised. We differ from Armerin (2017) in two major ways. Firstly, in contrast to Armerin's work, that considers that the firm already owns the random start American option, we go one step back and consider the decision to acquire the RSO. In other words, we depart from the assumption that the firm is, *ex ante*, endowed with the random start option, modeling, instead, the decision to acquire it. Secondly, we

[1] We interchangeably use "random start American option", "random start option", or simply RSO.

consider that the firm has no proprietary rights on the RSO, incorporating competition for the acquisition of the option. By accounting for competition, we make the model more complete and we allow it to fit important real world situations. For deriving the value functions and the solutions we follow the contingent claims approach, as presented by Dixit and Pindyck (1994).

The model unfolds as follows. Section 2 develops the model for investing in a random start American options under competition. Section 3 presents a numerical example with a comparative statics, highlighting the main insights of the model. Section 4 concludes.

2 The Model

Consider a real asset able to produce a stream of cash flows. The present value of these cash flows, X, is assumed to follow a geometric Brownian motion:

$$dX(t) = \alpha X(t)dt + \sigma X(t)dB \tag{1}$$

where $X(0) > 0$, $\alpha < r$ is the risk-neutral expected drift, r is the risk-free rate, σ the instantaneous volatility, and dB is the increment of a Wiener process.

The investment in this project has two stages. The first stage, in which K_1 is invested, allows the firm to become a monopolist over the second stage of the project, eliminating any possible competitive damage. However, the investment in this second stage, depends on some exogenous event without which the project is noneffective. In other words, it is impossible to implement the second (and last) stage of the project unless the exogenous event occurs. After this event, the firm is entitled with a perpetual American option to invest, which requires a lump sum investment of K_2. However, notice that if the exogenous event happens to occur before the firm invests K_1 (i.e., before securing monopolistic rights over the second stage), the option to invest in the project is shared with competitors.

This model considers three types of uncertainty. First, the cash flows of the project evolve randomly over time. Secondly, the effectiveness of the project depends on some exogenous event. Lastly, competition is also considered by including the existence of hidden rivals (Armada et al., 2011; Pereira and Armada, 2013; Lavrutich et al., 2016).

Figure 1 exhibits all possible states. In the beginning, the firm holds $F(X)$. This is a non-proprietary option to invest K_1 and receive $G(X)$, becoming a monopolist over the next stage. Two possible events may occur while the firm holds $F(X)$: the exogenous event occurs (transforming $F(X)$ into $H_C(X)$) or a (hidden) competitor moves in and invests K_1, and $F(X)$ becomes worthless for the company. After investing K_1 the firm is entitled with the monopolistic option $G(X)$. This option ends-up to be $H(X)$ if the exogenous event occurs. $H(X)$ is the perpetual American option to invest K_2 and receive X. Additionally, if the exogenous event occurs before the firm makes the first investment (before investing

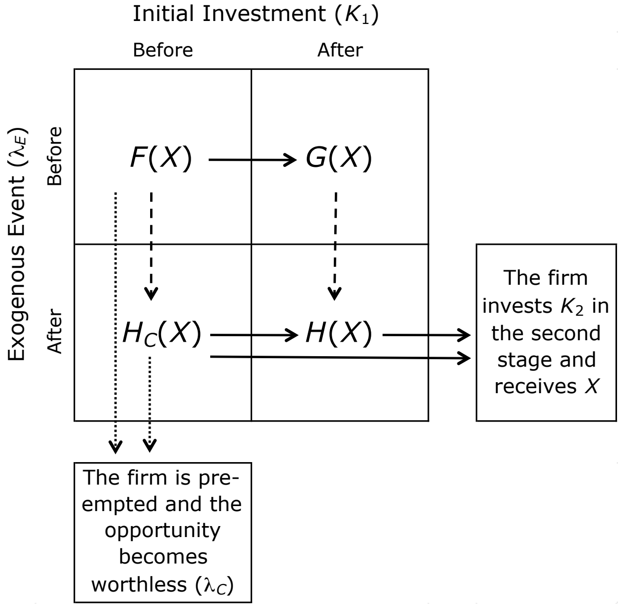


Figure 1: The solid lines represent the changes in the value functions as a result of firm's decisions (first stage and second stage investment). The dashed and the dotted lines represent, respectively, the change in the value functions if the exogenous event occurs or if the firm is preempted by a competitor.

K_1), $F(X)$ is transformed into $H_C(X)$, which corresponds to the non-monopolistic option to invest in the second stage. Given that $H_C(X)$ can suddenly disappear if a competitor preempts the firm, K_1 can be paid in order to secure the position of monopolist of the project ($H(X)$).

For solving the model we proceed backwards, starting with the last option $H(X)$, and then moving to the earlier stages.

2.1 The value of the project after the exogenous event

After the exogenous event that allows the firm to invest in the last stage, the firm can either have secured the monopolistic option to invest (by having invested K_1) or is still waiting to secure the investment and faces the hidden competition.

The monopolistic right to invest in the last stage

Let $H(X)$ be the value of the proprietary option to invest in the last stage, under which the firm receives X in exchange for the sunk investment cost K_2. Following the standard procedures, $H(X)$ is the solution to the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2 X^2 H''(X) + \alpha X H'(X) - rH(X) = 0 \tag{2}$$

The solution is the well known option to invest value (McDonald and Siegel, 1986; Dixit and Pindyck, 1994):

$$H(X) = \begin{cases} a_1 X^{\beta_1} & \text{for } X < X_2 \\ X - K_2 & \text{for } X \geqslant X_2 \end{cases} \tag{3}$$

where

$$a_1 = (X_2 - K_2)\left(\frac{1}{X_2}\right)^{\beta_1} = \frac{K_2}{\beta_1 - 1}\left(\frac{1}{X_2}\right)^{\beta_1} \tag{4}$$

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{5}$$

and X_2 is the threshold for investment:

$$X_2 = \frac{\beta_1}{\beta_1 - 1} K_2 \tag{6}$$

The shared option to invest in the last stage

Let $H_C(X)$ be the value function of the option to invest when the firm may be preempted by a hidden competitor destroying the option value. That event is modeled as a Poisson event with intensity λ_C. $H_C(X)$ is the solution to the following ODE (Dixit and Pindyck, 1994):

$$\frac{1}{2}\sigma^2 X^2 H_C''(X) + \alpha X H_C'(X) - r H_C(X) + \lambda_C(0 - H_C(X)) = 0 \tag{7}$$

and, considering the boundary at $X = 0$, is given by:

$$H_C(X) = b X^{\eta_1} \tag{8}$$

where

$$\eta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r + \lambda_C)}{\sigma^2}} \tag{9}$$

The firm can choose between two alternative strategies to kill competition: (1) to stage the investment, investing K_1 to secure a monopolistic position over the project, or (2) invest immediately in the last stage K_2. The optimal strategy will be the most valuable and not necessarily that with the earliest threshold.

Case 1 Staged investment

Under this strategy the firm will choose to secure the option to invest in the last stage by paying K_1 in the first stage and not pre-committing to the second stage investment. By paying K_1 the firm acquires the exclusive option to invest $H(X)$. It only makes economic sense to stage the investment if the threshold of the second stage X_2 has bot been reached. Therefore, the value-matching and smooth-pasting boundary conditions, at the threshold $X_{11}^c < X_2$, are:

$$b_1 {X_{11}^c}^{\eta_1} = a_1 {X_{11}^c}^{\beta_1} - K_1 \tag{10}$$

$$\eta_1 b_1 {X_{11}^c}^{\eta_1 - 1} = \beta_1 a_1 {X_{11}^c}^{\beta_1 - 1} \tag{11}$$

These boundary conditions produce the following solution for the option value:

$$H_C(X) = \begin{cases} b_1 X^{\eta_1} & \text{for } X < X_{11}^c \\ a_1 X^{\beta_1} - K_1 & \text{for } X_{11}^c \leqslant X < X_2 \\ X - K_2 - K_1 & \text{for } X \geqslant X_2 \end{cases} \tag{12}$$

where

$$b_1 = \left(a_1 {X_{11}^c}^{\beta_1} - K_1\right)\left(\frac{1}{X_{11}^c}\right)^{\eta_1} = \frac{\beta_1 K_1}{\eta_1 - \beta_1}\left(\frac{1}{X_{11}^c}\right)^{\eta_1} \tag{13}$$

and X_{11}^c is the threshold:

$$X_{11}^c = X_2 \left(\frac{\eta_1(\beta_1 - 1)}{\eta_1 - \beta_1}\frac{K_1}{K_2}\right)^{\frac{1}{\beta_1}} \tag{14}$$

The condition that the threshold X_2 must be greater that X_{11}^c implies that the initial investment K_1 must be sufficiently smaller that K_2:

$$K_1 < \frac{\eta_1 - \beta_1}{\eta_1(\beta_1 - 1)} K_2 \tag{15}$$

For the limiting cases where competition is absent ($\lambda_C = 0$) or is imminent ($\lambda_C \to \infty$), the condition becomes $K_1 < 0$ and $K_1 < K_2/(\beta_1 - 1)$, respectively. When there are no potential competitors, staging the investment is excluded because the firm holds an exclusive option on the second stage investment, while when the competitor is about to make the investment, the firm can pay the maximum amount $K_2/(\beta_1 - 1)$ to secure the investment. A higher risk of competition (higher λ_C or equivalently a higher η_1) induces the firm to be available to pay a larger K_1. Notice that the higher the market uncertainty (lower β_1), the larger the amount a firm is willing to pay to secure the exclusive right to later invest in the second stage.

Case 2 Investment in a single stage

Under this strategy the firm will choose the two investments ($K_1 + K_2$) in a single stage, eliminating competition.

The following value-matching and smooth-pasting boundary conditions, at the threshold X_{12}^c:

$$b{X_{12}^c}^{\eta_1} = X - (K_1 + K_2) \tag{16}$$

$$\eta_1 b{X_{12}^c}^{\eta_1 - 1} = 1 \tag{17}$$

produce the following solution:

$$H_C(X) = \begin{cases} b_2 X^{\eta_1} & \text{for } X < X_{12}^c \\ X - (K_1 + K_2) & \text{for } X \geqslant X_{12}^c \end{cases} \tag{18}$$

where

$$b_2 = (X_{12}^c - (K_1 + K_2)) \left(\frac{1}{X_{12}^c}\right)^{\eta_1} = \frac{K_1 + K_2}{\eta_1 - 1} \left(\frac{1}{X_{12}^c}\right)^{\eta_1} \tag{19}$$

and X_{12}^c is the threshold:

$$X_{12}^c = \frac{\eta_1}{\eta_1 - 1}(K_1 + K_2) \tag{20}$$

A higher risk of competition (higher λ_C or higher η_1) hastens investment ($\partial X_{12}^c / \partial \lambda_C < 0$). On the other hand, a higher market uncertainty (lower η_1) deters investment.[2]

Optimal strategy

A firm will prefer to stage the investment if the value of that strategy is higher than that of the alternative single stage investment ($b_1 > b_2$), even if the threshold of the latter (X_{12}^c) is reached before the threshold of the former (X_{11}^c). The following condition must hold for a staged investment to be preferred:

$$\left(\frac{X_{12}^c}{X_{11}^c}\right)^{\eta_1} > \frac{\eta_1 - \beta_1}{\beta_1(\eta_1 - 1)} \left(\frac{K_1 + K_2}{K_1}\right) \tag{21}$$

Appendix A proves that this condition always holds. Therefore, we need only condition (15) to define the optimal strategy. There is a value of K_1 that separates the two regions, in which one strategy is preferred over the other. K_1 must be sufficiently small to make the staged investment the preferred strategy.

2.2 The value of the project before the exogenous event

Before the exogenous event that allows the firm to invest in the last stage, the firm can choose between securing the monopolistic option to invest or waiting and sharing the option with hidden competitors. It is reasonable to assume that securing the investment before the exogenous event can be less costly. For instance, buying a piece of land without a construction permit is less costly than if construction has been already permitted.

[2]Notice that $\partial \eta_1 / \partial \sigma < 0$.

Therefore, we assume that the cost is θK_1 $(0 < \theta \leqslant 1)$.

The value of the monopolistic option invest in the first stage

After paying θK_1, the firm secures the investment opportunity $H(X)$ killing competition and waits for the occurrence of the exogenous event that will allow the investment in the last stage. This event arrives according to a Poisson process with an intensity rate λ_E. Let $G(X)$ be the value of the monopolistic option, which must be the solution to the following ODE:

$$\frac{1}{2}\sigma^2 X^2 G''(X) + \alpha X G'(X) - rG(X) + \lambda_E(H(X) - G(X)) = 0 \tag{22}$$

The exogenous event can occur either before or after the threshold X_2 has been reached. The two regions of $H(X)$ shown in Equation (18) produce the following solution to the above ODE:[3]

$$G(X) = \begin{cases} c_1 X^{\gamma_1} + a_1 X^{\beta_1} & \text{for } X < X_2 \\ c_4 X^{\gamma_2} + \Lambda_1 X - \Lambda_2 K_2 & \text{for } X \geqslant X_2 \end{cases} \tag{23}$$

where

$$\Lambda_1 = \frac{\lambda_E}{r - \alpha + \lambda_E} \tag{24}$$

$$\Lambda_2 = \frac{\lambda_E}{r + \lambda_E} \tag{25}$$

$$\gamma_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r + \lambda_E)}{\sigma^2}} \tag{26}$$

$$\gamma_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r + \lambda_E)}{\sigma^2}} \tag{27}$$

and a_1 and X_2 are as in Equations (4) and (6), respectively, and the constants a_3 and a_4 ensure that $G(X)$ is continuous and differentiable along X:[4]

$$c_1 = \frac{(\beta_1 - \gamma_2)(X_2 - K_2) + \Lambda_1(\gamma_2 - 1)X_2 - \Lambda_2\gamma_2 K_2}{\gamma_2 - \gamma_1}\left(\frac{1}{X_2}\right)^{\gamma_1} \tag{28}$$

$$c_4 = \frac{(\beta_1 - \gamma_1)(X_2 - K_2) + \Lambda_1(\gamma_1 - 1)X_2 - \Lambda_2\gamma_1 K_2}{\gamma_2 - \gamma_1}\left(\frac{1}{X_2}\right)^{\gamma_2} \tag{29}$$

[3] After considering the boundary condition when $X \to 0$ and $X \to \infty$.

[4] Using the value-matching and smooth-pasting conditions at X_2.

The value of the shared option to invest in the first stage

Let $F(X)$ be the value of the shared option to invest prior to the exogenous event, whose value must be the solution to the following ODE:

$$\frac{1}{2}\sigma^2 X^2 F''(X) + \alpha X F'(X) - rF(X) + \lambda_E(H_C(X) - F(X)) + \lambda_C(0 - F(X)) = 0 \quad (30)$$

where, λ_E is the arrival rate of the exogenous event, and λ_C corresponds to the arrival rate of a competitor that preempts the firm, killing the option value.

Depending on condition (15), $H_C(X)$ is given by Equation (12) or Equation (18), each of them with more than one branch. Let X_1 be the threshold for investment in the first stage. The following cases emerge:

Table 1: Investment strategy cases

	1. Staged investment		2. Single stage investment	
	$X_1 < X_{11}^c$	$X_1 \geqslant X_{11}^c$	$X_1 < X_{12}^c$	$X_1 \geqslant X_{12}^c$
$X_1 < X_2$	A	C	E	G
$X_1 \geqslant X_2$	B	D	F	H

Case 1 Staged investment

Considering the boundary condition when $X \to 0$, the solution to the ODE (30) has three branches:

$$F(X) = \begin{cases} d_1 X^{\psi_1} + b_1 X^{\eta_1} & \text{for } X < X_{11}^c \\ d_3 X^{\psi_1} + d_4 X^{\psi_2} + \Lambda_4 a_1 X^{\beta_1} - \Lambda_3 K_1 & \text{for } X_{11}^c \leqslant X < X_2 \\ d_5 X^{\psi_1} + d_6 X^{\psi_2} + \Lambda_1 (X - K_2 - K_1) & \text{for } X \geqslant X_2 \end{cases} \quad (31)$$

where

$$\Lambda_3 = \frac{\lambda_E}{r + \lambda_C + \lambda_E} \quad (32)$$

$$\Lambda_4 = \frac{\lambda_E}{\lambda_C + \lambda_E} \quad (33)$$

$$\psi_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r + \lambda_E + \lambda_C)}{\sigma^2}} \quad (34)$$

$$\psi_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r + \lambda_E + \lambda_C)}{\sigma^2}} \quad (35)$$

Let us now present the four sub-cases.

Case A $X_{11}^c < X_2$ and $X_1 < X_2$

The following boundary conditions are used to find the solution:

$$d_{11}{X_1}^{\psi_1} + b_1{X_1}^{\eta_1} = c_1{X_1}^{\gamma_1} + a_1{X_1}^{\beta_1} - \theta K_1 \tag{36}$$
$$\psi_1 d_{11}{X_1}^{\psi_1-1} + \eta_1 b_1{X_1}^{\eta_1-1} = \gamma_1 c_1{X_1}^{\gamma_1-1} + \beta_1 a_1{X_1}^{\beta_1-1} \tag{37}$$

Solving these two equations, we obtain the following value for the investment opportunity:

$$F(X) = \begin{cases} d_{11}X^{\psi_1} + b_1X^{\eta_1} & \text{for } X < X_1 \\ G(X) - \theta K_1 & \text{for } X \geqslant X_1 \end{cases} \tag{38}$$

where

$$d_{11} = \left(c_1{X_1}^{\gamma_1} + a_1{X_1}^{\beta_1} - \theta K_1 - b_1{X_1}^{\eta_1}\right)\left(\frac{1}{X_1}\right)^{\psi_1} \tag{39}$$

and the trigger, X_1, is numerically obtained by solving the following equation:

$$(\psi_1 - \gamma_1)c_1{X_1}^{\gamma_1} - (\psi_1 - \eta_1)b_1{X_1}^{\eta_1} + (\psi_1 - \beta_1)a_1{X_1}^{\beta_1} - \psi_1\theta K_1 = 0 \tag{40}$$

Case B $X_1 < X_{11}^c$ and $X_1 \geqslant X_2$

The following boundary conditions apply:

$$d_{12}{X_1}^{\psi_1} + b_1{X_1}^{\eta_1} = c_4{X_1}^{\gamma_2} + \Lambda_1 X_1 - \Lambda_2 K_2 - \theta K_1 \tag{41}$$
$$\psi_1 d_{12}{X_1}^{\psi_1-1} + \eta_1 b_1{X_1}^{\eta_1-1} = \gamma_2 c_4{X_1}^{\gamma_2-1} + \Lambda_1 \tag{42}$$

producing the following solution for the option value:

$$F(X) = \begin{cases} d_{12}X^{\psi_1} + b_1X^{\eta_1} & \text{for } X < X_1 \\ G(X) - \theta K_1 & \text{for } X \geqslant X_1 \end{cases} \tag{43}$$

where

$$d_{12} = (c_4X_1^{\gamma_2} + \Lambda_1 X_1 - \Lambda_2 K_2 - \theta K_1 - b_1{X_1}^{\eta_1})\left(\frac{1}{X_1}\right)^{\psi_1} \tag{44}$$

and the trigger, X_1, is numerically obtained by solving the following equation:

$$(\psi_1 - \gamma_2)c_4{X_1}^{\gamma_2} - (\psi_1 - \eta_1)b_2{X_1}^{\eta_1} + (\psi_1 - 1)X_1 - \psi_1\left(\Lambda_2 K_2 + \theta K_1\right) = 0 \tag{45}$$

Case C $X_1 \geqslant X_{11}^c$ and $X_1 < X_2$

The following value-matching and smooth-pasting conditions:

$$d_{33}{X_1}^{\psi_1} + d_{43}{X_1}^{\psi_2} + \Lambda_3(X_1 - (K_1 + K_2)) = c_1{X_1}^{\gamma_1} + a_1{X_1}^{\beta_1} - \theta K_1 \tag{46}$$
$$\psi_1 d_{33}{X_1}^{\psi_1-1} + \psi_2 d_{43}{X_1}^{\psi_2-1} + \Lambda_3 = \gamma_1 c_1{X_1}^{\gamma_1-1} + \beta_1 a_1{X_1}^{\beta_1-1} \tag{47}$$

produce the following solution for the option value:

$$F(X) = \begin{cases} d_{13}X^{\psi_1} + b_1 X^{\eta_1} & \text{for } X < X_{11}^c \\ d_{33}X^{\psi_1} + d_{43}X^{\psi_2} + \Lambda_4 a_1 X^{\beta_1} - \Lambda_3 K_1 & \text{for } X_{11}^c \leqslant X < X_1 \\ G(X) - \theta K_1 & \text{for } X \geqslant X_1 \end{cases} \tag{48}$$

where

$$d_{13} = d_{33} + \left(d_{43}{X_{11}^c}^{\psi_2} - (1-\Lambda_4)\, a_1 {X_{11}^c}^{\beta_1} + (1-\Lambda_3)\, K_1\right)\left(\frac{1}{X_{11}^c}\right)^{\psi_1} \tag{49}$$

$$d_{33} = \left(c_1 {X_1}^{\gamma_1} + (1-\Lambda_4)\, a_1 {X_1}^{\beta_1} - (\theta-\Lambda_3)\, K_1 - d_{41}{X_1}^{\psi_2}\right)\left(\frac{1}{X_1}\right)^{\psi_1} \tag{50}$$

$$d_{43} = \frac{(\psi_1-\beta_1)\,(1-\Lambda_4)\, a_1 {X_{11}^c}^{\beta_1} - \psi_1\,(1-\Lambda_3)\, K_1}{\psi_1-\psi_2}\left(\frac{1}{X_{11}^c}\right)^{\psi_2} \tag{51}$$

and the trigger, X_1, is numerically obtained by solving the following equation:

$$-(\psi_1-\psi_2)d_{43}{X_1}^{\psi_2} + (\psi_1-\gamma_1)c_1{X_1}^{\gamma_1} + (\psi_1-\beta_1)\,(1-\Lambda_4)\, a_1{X_1}^{\beta_1} - \psi_1\,(\theta-\Lambda_3)\, K_1 = 0 \tag{52}$$

Case D $X_1 \geqslant X_{11}^c$ and $X_1 > X_2$

The following boundary conditions:

$$d_{54}{X_1}^{\psi_1} + d_{64}{X_1}^{\psi_2} + \Lambda_1\,(X_1 - K_2) = c_4{X_1}^{\gamma_2} + \Lambda_1 X_1 - \Lambda_2 K_2 - \theta K_1 \tag{53}$$

$$\psi_1 d_{54}{X_1}^{\psi_1-1} + \psi_2 d_{64}{X_1}^{\psi_2-1} + \Lambda_1 = \gamma_2 c_4 {X_1}^{\gamma_2-1} + \Lambda_1 \tag{54}$$

produce the following solution for the option value:

$$F(X) = \begin{cases} d_{14}X^{\psi_1} + b_1 X^{\eta_1} & \text{for } X < X_{11}^c \\ d_{34}X^{\psi_1} + d_{44}X^{\psi_2} + \Lambda_4 a_1 X^{\beta_1} - \Lambda_3 K_1 & \text{for } X_{11}^c \leqslant X < X_2 \\ d_{54}X^{\psi_1} + d_{64}X^{\psi_2} + \Lambda_1\,(X - K_2 - K_1) & \text{for } X_2 \leqslant X < X_1 \\ G(X) - \theta K_1 & \text{for } X \geqslant X_1 \end{cases} \tag{55}$$

where

$$d_{14} = d_{34} + \left(d_{44}{X_{11}^c}^{\psi_2} + (\Lambda_4 - 1)\, a_1 {X_{11}^c}^{\beta_1} + (1 - \Lambda_3)\, K_1\right) \left(\frac{1}{X_{11}^c}\right)^{\psi_1} \tag{56}$$

$$d_{44} = d_{43} \tag{57}$$

$$d_{34} = d_{54} + \left((d_{64} - d_{44}){X_2}^{\psi_2} - (\Lambda_4 - \Lambda_3)\,(X_2 - K_2)\right) \left(\frac{1}{X_2}\right)^{\psi_1} \tag{58}$$

$$d_{54} = \left(c_4 {X_1}^{\gamma_2} + (\Lambda_1 - \Lambda_3)\, X_1 - (\Lambda_2 - \Lambda_3)\, K_2 - (\theta - \Lambda_3)\, K_1 - d_{64}{X_1}^{\psi_2}\right) \left(\frac{1}{X_1}\right)^{\psi_1} \tag{59}$$

$$d_{64} = d_{44} + (\Lambda_4 - \Lambda_3)\, \frac{(\psi_1 - 1)\, X_2 - \psi_1 K_2}{\psi_1 - \psi_2} \left(\frac{1}{X_2}\right)^{\psi_2} \tag{60}$$

and the trigger, X_1, is numerically obtained by solving the following equation:

$$\begin{aligned} -(\psi_1 - \psi_2) d_{64}{X_1}^{\psi_2} + (\psi_1 - \gamma_2) c_4 {X_1}^{\gamma_2} + (\psi_1 - 1)\,(\Lambda_1 - \Lambda_3)\, X_1 \\ -\psi_1\left((\Lambda_2 - \Lambda_3)\, K_2 + (\theta - \Lambda_3)\, K_1\right) = 0 \end{aligned} \tag{61}$$

Case 2 Single stage investment

For the case of a single stage investment Equation (18) is used to find the solution for the the ODE (30). The solution with two branches and considering the boundary condition when $X \to 0$, is the following:

$$F(X) = \begin{cases} e_1 X^{\psi_1} + b_2 X^{\eta_1} & \text{for } X < X_{12}^c \\ e_3 X^{\psi_1} + e_4 X^{\psi_2} + \Lambda_3(X - (K_1 + K_2)) & \text{for } X \geqslant X_{12}^c \end{cases} \tag{62}$$

Case E $X_1 < X_{12}^c$ and $X_1 < X_2$

The following value-matching and smooth-pasting boundary conditions apply:

$$e_{11}{X_1}^{\psi_1} + b_2 {X_1}^{\eta_1} = c_1 {X_1}^{\gamma_1} + a_1 {X_1}^{\beta_1} - \theta K_1 \tag{63}$$

$$\psi_1 e_{11}{X_1}^{\psi_1 - 1} + \eta_1 b_2 {X_1}^{\eta_1 - 1} = \gamma_1 c_1 {X_1}^{\gamma_1 - 1} + \beta_1 a_1 {X_1}^{\beta_1 - 1} \tag{64}$$

producing the following solution for the investment opportunity:

$$F(X) = \begin{cases} e_{11} X^{\psi_1} + b_2 X^{\eta_1} & \text{for } X < X_1 \\ G(X) - \theta K_1 & \text{for } X \geqslant X_1 \end{cases} \tag{65}$$

where

$$e_{11} = \left(c_1 {X_1}^{\gamma_1} + a_1 {X_1}^{\beta_1} - \theta K_1 - b_2 {X_1}^{\eta_1}\right) \left(\frac{1}{X_1}\right)^{\psi_1} \tag{66}$$

and the trigger, X_1, is numerically obtained by solving the following equation:

$$(\psi_1 - \gamma_1)c_1 {X_1}^{\gamma_1} - (\psi_1 - \eta_1)b_2 {X_1}^{\eta_1} + (\psi_1 - \beta_1)a_1 {X_1}^{\beta_1} - \psi_1 \theta K_1 = 0 \tag{67}$$

Case F $X_1 < X_{12}^c$ and $X_1 \geqslant X_2$

For this case the following boundary conditions apply:

$$e_{12} {X_1}^{\psi_1} + b_2 {X_1}^{\eta_1} = c_4 {X_1}^{\gamma_2} + \Lambda_1 X_1 - \Lambda_2 K_2 - \theta K_1 \tag{68}$$

$$\psi_1 e_{12} {X_1}^{\psi_1 - 1} + \eta_1 b_2 {X_1}^{\eta_1 - 1} = \gamma_2 c_4 {X_1}^{\gamma_2 - 1} + \Lambda_1 \tag{69}$$

and produce the following solution for the option value:

$$F(X) = \begin{cases} e_{12} X^{\psi_1} + b_2 X^{\eta_1} & \text{for } X < X_1 \\ G(X) - \theta K_1 & \text{for } X \geqslant X_1 \end{cases} \tag{70}$$

where

$$e_{12} = (c_4 X_1^{\gamma_2} + \Lambda_1 X_1 - \Lambda_2 K_2 - \theta K_1 - b_2 {X_1}^{\eta_1}) \left(\frac{1}{X_1} \right)^{\psi_1} \tag{71}$$

and the trigger, X_1, is numerically obtained by solving the following equation:

$$(\psi_1 - \gamma_2)c_4 {X_1}^{\gamma_2} - (\psi_1 - \eta_1)b_2 {X_1}^{\eta_1} + (\psi_1 - 1)X_1 - \psi_1 (\Lambda_2 K_2 + \theta K_1) = 0 \tag{72}$$

Case G $X_1 \geqslant X_{12}^c$ and $X_1 < X_2$

The following boundary conditions:

$$e_{33} {X_1}^{\psi_1} + e_{43} {X_1}^{\psi_2} + \Lambda_3 (X_1 - (K_1 + K_2)) = c_1 {X_1}^{\gamma_1} + a_1 {X_1}^{\beta_1} - \theta K_1 \tag{73}$$

$$\psi_1 e_{33} {X_1}^{\psi_1 - 1} + \psi_2 e_{43} {X_1}^{\psi_2 - 1} + \Lambda_3 = \gamma_1 c_1 {X_1}^{\gamma_1 - 1} + \beta_1 a_1 {X_1}^{\beta_1 - 1} \tag{74}$$

produce the following solution for the option value:

$$F(X) = \begin{cases} e_{13} X^{\psi_1} + b_2 X^{\eta_1} & \text{for } X < X_{12}^c \\ e_{33} X^{\psi_1} + e_{43} X^{\psi_2} + \Lambda_1 (X - (\theta K_1 + K_2)) & \text{for } X_{12}^c \leqslant X < X_1 \\ G(X) - \theta K_1 & \text{for } X \geqslant X_1 \end{cases} \tag{75}$$

where

$$e_{13} = e_{33} + \left(e_{43}{X^c_{12}}^{\psi_2} - (1-\Lambda_3)\left(X^c_{12} - (K_1+K_2)\right)\right)\left(\frac{1}{X^c_{12}}\right)^{\psi_1} \tag{76}$$

$$e_{33} = \left(c_1{X_1}^{\gamma_1} + a_1{X_1}^{\beta_1} - \Lambda_3(X_1-(K_1+K_2)) - \theta K_1 - e_{43}{X_1}^{\psi_2}\right)\left(\frac{1}{X_1}\right)^{\psi_1} \tag{77}$$

$$e_{43} = (1-\Lambda_3)\,\frac{(\psi_1-1)\,X^c_{12} - \psi_1(K_1+K_2)}{\psi_1-\psi_2}\left(\frac{1}{X^c_{12}}\right)^{\psi_2} \tag{78}$$

and the trigger, X_1, is numerically obtained by solving the following equation:

$$\begin{aligned} -(\psi_1-\psi_2)e_{43}{X_1}^{\psi_2} + (\psi_1-\gamma_1)c_1{X_1}^{\gamma_1} + (\psi_1-\beta_1)a_1{X_1}^{\beta_1} + (\psi_1-1)\Lambda_3 X_1 \\ -\psi_1\left(\theta K_1 - \Lambda_3(K_1+K_2)\right) = 0 \end{aligned} \tag{79}$$

Case H $X_1 \geqslant X^c_{12}$ and $X_1 \geqslant X_2$

The following boundary conditions apply:

$$e_{34}{X_1}^{\psi_1} + e_{44}{X_1}^{\psi_2} + \Lambda_3(X_1-(K_1+K_2)) = c_4{X_1}^{\gamma_2} + \Lambda_1 X_1 - \Lambda_2 K_2 - \theta K_1 \tag{80}$$

$$\psi_1 e_{34}{X_1}^{\psi_1} + \psi_2 e_{44}{X_1}^{\psi_2} + \Lambda_1 X_1 = \gamma_2 c_4{X_1}^{\gamma_2} + \Lambda_1 X_1 \tag{81}$$

producing the following solution:

$$F(X) = \begin{cases} e_{14}X^{\psi_1} + b_2 X^{\eta_1} & \text{for } X < X^c_{12} \\ e_{34}X^{\psi_1} + e_{44}X^{\psi_2} + \Lambda_3(X-(K_1+K_2)) & \text{for } X^c_{12} \leqslant X < X_1 \\ G(X) - \theta K_1 & \text{for } X \geqslant X_1 \end{cases} \tag{82}$$

where

$$e_{14} = e_{34} + \left(e_{44}{X^c_{12}}^{\psi_2} + (\Lambda_3-1)\left(X^c_{12} - (K_1+K_2)\right)\right)\left(\frac{1}{X^c_{12}}\right)^{\psi_1} \tag{83}$$

$$e_{34} = \left(c_4{X_1}^{\gamma_2} + (\Lambda_1-\Lambda_3)\,X_1 - (\Lambda_2-\Lambda_3)\,K_2 - (\theta-\Lambda_3)\,K_1 - e_{44}{X_1}^{\psi_2}\right)\left(\frac{1}{X_1}\right)^{\psi_1} \tag{84}$$

$$e_{44} = e_{43} \tag{85}$$

and the trigger, X_1, is numerically obtained by solving the following equation:

$$\begin{aligned} -(\psi_1-\psi_2)e_{44}{X_1}^{\psi_2} + (\psi_1-\gamma_2)c_4{X_1}^{\gamma_2} + (\psi_1-1)(\Lambda_1-\Lambda_3)X_1 \\ -\psi_1((\Lambda_2-\Lambda_3)K_2 + (\theta-\Lambda_3)K_1) = 0 \end{aligned} \tag{86}$$

3 Numerical Example and Comparative Statics

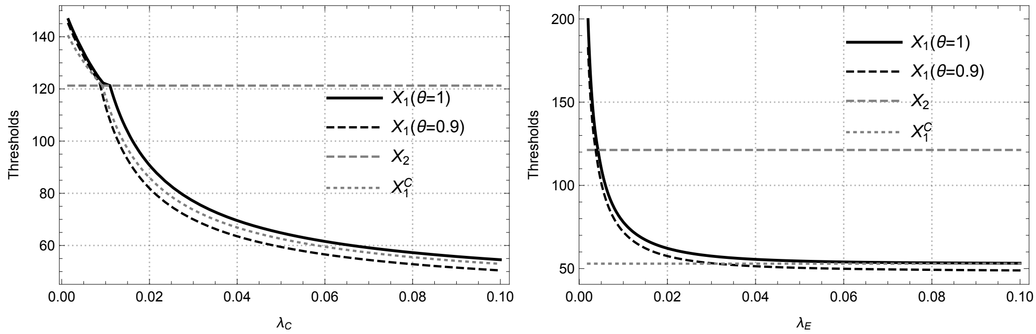
Given that the the solution for the investment thresholds and option values are found numerically for the six cases, we study the features of the solution using a numerical example. Table 2 presents the base-case parameters for the comparative statics.

Table 2: The base case parameters

Parameter	Description	Value
σ	Volatility	0.1
r	Risk-free rate	0.04
α	Risk-neutral drift	0.02
K_1	Stage 1 investment cost	10
K_2	Stage 2 investment cost	50
θ	Stage 1 investment cost discount	1
λ_E	Arrival rate of the exogenous event	0.05
λ_C	Arrival rate of the competitor	0.1

Figure 2 shows that a higher risk of a competitor arrival (λ_C) or a higher likelihood of the arrival of the exogenous event that allows the investment on the second stage (λ_E) induce an earlier investment. When these events have a low probability of occurrence the investment is delayed and can even occur later than when the second stage investment would become optimal if it were allowed ($X_1 > X_2$). The figure also shows that a higher discount (lower θ) on the first stage investment cost, when the second stage investment is not yet permitted, accelerates investment. When we compare the thresholds for the first stage investment before and after the exogenous event (X_1 and X_1^c) it is possible to conclude that the level of θ determines if the investment in the first stage occurs later or sooner that it would occur after the exogenous event occurrence. In particular when there is no discount ($\theta = 1$) it is always optimal to invest later if the exogenous event has not occurred ($X_1 > X_1^c$).

The firm also faces another source of risk - the cash flows risk measured by the volatility parameter σ. Figure 3 shows an unusual effect of uncertainty. Usually uncertainty deters investment in real options models. In the current model, it first deters investment, then, for intermediate levels of uncertainty, investment is hastened, and, finally, high levels of uncertainty deter investment again. This effect seems to be channeled through the threshold X_{11}^c (Equations (14)). This is the threshold for investment in the first stage, in a staged investment strategy, after the occurrence of the exogenous event. The effect of uncertainty on X_{11}^c is twofold: (i) on the one hand a higher uncertainty (lower η_1) increases the threshold and (ii) on the other hand it makes the option to invest in the second stage more valuable (increasing $a_1 X^{\beta_1}$), which promotes investment. These two effects dominate for different levels of uncertainty. The figure also shows, as in the previous

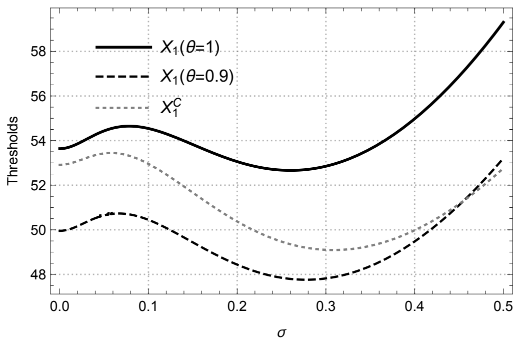


(a) (b)

$\sigma = 0.1$, $r = 0.04$, $\alpha = 0.02$, $\lambda_C = 0.1$, $\lambda_E = 0.05$, $K_1 = 10$, $K_2 = 50$, $\theta = 1$

Figure 2: Sensitivity of the investment thresholds to λ_C and λ_E

figure, that a discount in the investment cost can induce investment sooner before than after the allowance for the second stage investment is issued. This effect is higher for low levels of uncertainty. A high uncertainty decreases the incentive to secure the investment opportunity before the exogenous event occurrence.

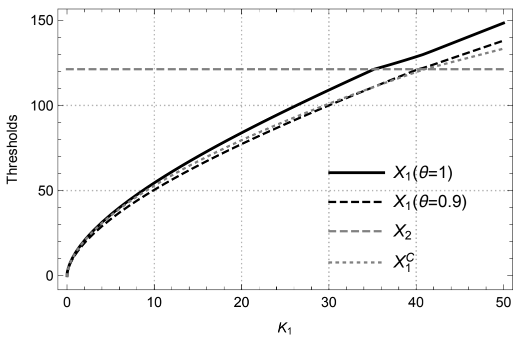


$\sigma = 0.1$, $r = 0.04$, $\alpha = 0.02$, $\lambda_C = 0.1$, $\lambda_E = 0.05$, $K_1 = 10$, $K_2 = 50$

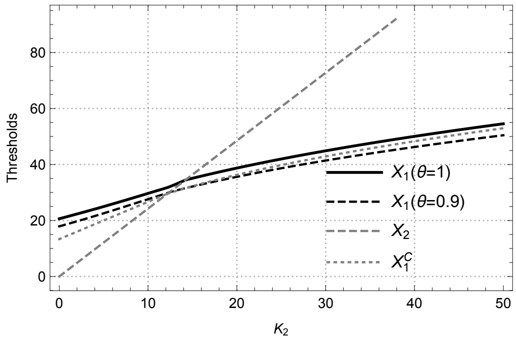
Figure 3: Sensitivity of the investment thresholds to σ

The effect of the investment costs are depicted in Figure 4. Higher investment costs delay investment. When the investment cost of the first stage (K_1) is not sufficiently smaller than that of the second stage (K_2), the firm invests in the first stage and waits for the exogenous event that allows the investment of the second stage, that will occur immediately after.

Finally, the effect of a discount in the first stage investment cost before the exogenous event is shown is Figure 5. A higher discount hastens investment. For the limiting case when there is no cost of investment, the firm invests immediately. A small discount induces the firm to invest sooner before the exogenous event than it would invest after the event ($X_1 < X_1^c$).



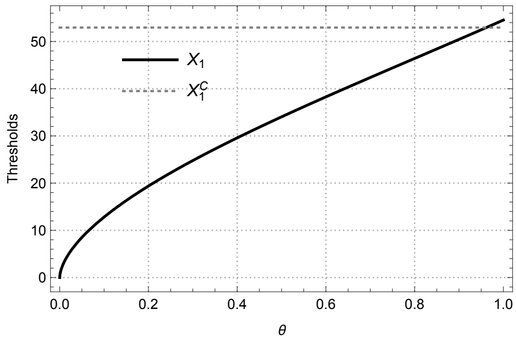
(a)



(b)

$\sigma = 0.1$, $r = 0.04$, $\alpha = 0.02$, $\lambda_C = 0.1$, $\lambda_E = 0.05$, $K_1 = 10$, $K_2 = 50$, $\theta = 1$

Figure 4: Sensitivity of the investment thresholds to K_1 and K_2



$\sigma = 0.1$, $r = 0.04$, $\alpha = 0.02$, $\lambda_C = 0.1$, $\lambda_E = 0.05$, $K_1 = 10$, $K_2 = 50$, $\theta = 1$

Figure 5: Sensitivity of the investment thresholds to θ

4 Conclusion

This paper develops a model to determine the value and optimal timing of an opportunity to invest in a random start American real option. A random start American option materializes into an American option only after an exogenous event, such as a permit or a discovery, occurs. While waiting to invest the firm faces the risk of a hidden competitor destroying the value of the opportunity to invest.

Investment is assumed to take place in two stages: in the first stage the investor shares with an hidden competitor the option to acquire the exclusive right to develop the second stage. The second stage investment is contingent on an exogenous event that permits it.

A comparative statics shows that investment is deterred when the risk of competition is low or the probability of arrival of a permission to invest in the development stage is also low. Investment is also deterred for high investment costs in both stages. When both investment costs are similar, investment, when permitted, takes places in a single stage.

The effect of uncertainty is shown to be non-monotonic. For low and high uncertainty levels an increase in uncertainty deters investment, and for intermediate uncertainty levels the effect is the reverse. When investment is optimally done in stages, a higher uncertainty increases the investment thresholds delaying investment, but, simultaneously, increases the option value of the the second stage, hastening investment.

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A Proof of Equation 21

We start by acknowledging that X_{12}^c is always greater than X_2 for any $K_1 > 0$. If fact using equations (20) and (6), the condition that $X_{12}^c > X_2$ is equivalent to $K_1 > -\frac{K_2}{\eta_1}$. Since $\eta_1 > 1$, K_1 needs to be negative to make $X_{12}^c < X_2$.

Given that $X_{11}^c < X_2$ and $X_{12}^c > X_2$, $X_{12}^c > X_{11}^c$. Given that $\eta_1 > \beta_1$,

$$\left(\frac{X_{12}^c}{X_{11}^c}\right)^{\eta_1} > \left(\frac{X_{12}^c}{X_{11}^c}\right)^{\beta_1} \tag{87}$$

Using equations (14), (20) and (6), and simplifying:

$$\left(\frac{X_{12}^c}{X_{11}^c}\right)^{\beta_1} = \left(\frac{X_{12}^c}{X_2}\right)^{\beta_1} \frac{\eta_1 - \beta_1}{\eta_1(\beta_1 - 1)} \frac{K_2}{K_1} \tag{88}$$

Given that $\beta_1 > 1$ and $X_{12}^c > X_2$,

$$\left(\frac{X_{12}^c}{X_{11}^c}\right)^{\beta_1} > \frac{X_{12}^c}{X_2} \frac{\eta_1 - \beta_1}{\eta_1(\beta_1 - 1)} \frac{K_2}{K_1} \tag{89}$$

Using equations (20) and (6), and simplifying:

$$\left(\frac{X_{12}^c}{X_{11}^c}\right)^{\eta_1} > \left(\frac{X_{12}^c}{X_{11}^c}\right)^{\beta_1} > \frac{\eta_1 - \beta_1}{\beta_1(\eta_1 - 1)} \left(\frac{K_1 + K_2}{K_1}\right) \tag{90}$$