STRATEGIC MARKET ENTRY WITH PRODUCTION FLEXIBILITY IN OLIGOPOLY

Abstract. We model an oligopoly whereby several firms may enter a market in the future competing with invested rivals in quantity. In this two-stage dynamic Cournot model, firms have production flexibility to set output given stochastic demand while facing capacity constraints. Early market entry decisions by rivals give rise to a coordination problem among would-be entrants. We characterize the Markov Perfect Equilibrium and derive the value of the investment showing that value is no longer monotone increasing and convex but exhibits "competitive waves."

This work is still "work in progress" and should not be cited or referred to without explicit authorization from the authors.

1. Introduction

The analysis of oligopoly under uncertainty is important in today's uncertain business environment. Typically duopoly models are used to generate prescriptions for strategic competition, but in many industries more than two rivals compete. For example, the automotive sector is dominated by Toyota, Volkswagen, Ford, BMW, Daimler and General Motors. Faced with disruptive technologies development (e.g., electric vehicles), firms in the sector need to reassess their strategy for both entry (investment in R&D and production facilities) and subsequent operating (output and pricing) decisions in light of future entrants. Such business initiatives should anticipate industry developments, especially the number of rivals and competitive reactions.

In today's turbulent markets, firms have trouble anticipating how markets will develop and how rivals will behave. In the academic literature, these issues have typically been addressed separately. The first issue regarding market developments is typically dealt with by assuming market variables (such as demand) follow stochastic processes whose parameters are estimated using standard econometric methods. Fluctuating market demand typically drives operational flexibility characterizing flexible manufacturing systems in Operations Management (see e.g., Sethi and Sethi, 1990). Adaptability to market fluctuations enables firms to limit the downside risk of their operations while fully tapping into favorable development. Such initiatives often depend on firms' operational hedging capabilities (Van Mieghem, 2003). Demand uncertainty influences a firm's operating strategy and thus affects its valuation. Real options (RO) analysis allows to quantify a firm's ability to adapt to exogenous market changes and to properly assess the value of a flexible firm under uncertainty (e.g., Dixit and Pindyck, 1994; Smith and Nau, 1995; Trigeorgis, 1996).[1] Non-cooperative game theory (e.g., Fudenberg and Tirole, 1991) provides a standard framework for addressing the second

[1]RO capitalizes on an analogy between cash flow claims generated in business situations and the contigent payoff structure of financial options. With strategic interactions having played a less significant role in determining equilibrium outcomes in competitive capital markets, analysis of RO initially disregarded competitive interactions.

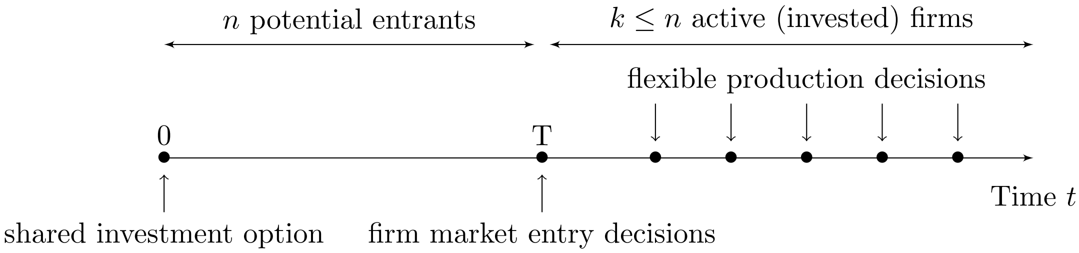

Figure 1. Timing structure of market entry and flexible production decisions

challenge concerning strategic interactions among parties with conflicting objectives. The theory on industrial organization (see, e.g., Tirole, 1988) provides additional prescriptive guidance into how firms should behave when faced with competition. These two pillar approaches, industrial organization and real options analysis, have recently been considered concurrently obtaining better insights into industry dynamics via option games (e.g., Chevalier-Roignant et al., 2011). The problems of market entry under competition and the optimal management of operational flexibility under uncertainty are in fact intertwined.

We model an oligopoly situation whereby several firms (e.g., car manufacturers) can decide whether to launch a new product (e.g., electric vehicles) or enter a new geographic market (e.g., China) at a certain future date. Once they do so, they compete in output quantity (a la Cournot) in a market subject to fluctuating demand. This situation is modeled in two stages. Figure 1 illustrates the timing structure of the market entry and flexible production decisions. In the first stage, rivals decide whether to enter the market (exercising their shared investment option) given threat from rivals, i.e., they play against strategic rival uncertainty. A firm will decide to enter the market at the specified time when disruptive technology appears or it forfeits the entry opportunity. This situation is characteristic of industries where standards and technologies evolve so rapidly that firms face a narrow time window to adopt the disruptive technology without missing out on the opportunity. This is also sometimes the case when there are licensing requirements or when markets provide a higher premium to innovators and early entrants. In the second stage, firms face two sources of uncertainty: future demand remains imperfectly predictable, and rivals compete a la Cournot in setting production output. We focus on the strategic interactions among would-be entrants at the future entry date as well as the optimal operating policy. This market-entry stage is analyzed analogous to a shared European-type call option on the value of firm assets, with subsequent embedded operational flexibility.

To solve this two-stage dynamic marekt entry game, we proceed backwards. We first characterize output decisions and rival firm profits under general Cournot competition involving k active (invested) firms. We adopt the assumption of a linear demand function with random intercept (additive shock), in-line with Kulatilaka and Perotti (1998) and Van Mieghem and Dada (1999). The demand level evolves over time following a geometric Brownian motion (GBM). Competing firms adjust their output quantity in view of realized demand given their capacity constraints. Under the Cournot structure, firm profits are non-linear in output and so the optimal operating policy involves a continuum from zero production at low demand to full capacity utilization at high demand levels. This contrasts with work on operational flexibility where firms switch from one operating mode to another (at a positive switching cost). We solve for equilibrium profit streams and derive firm value with embedded production flexibility under demand uncertainty and strategic competition. We then determine the demand levels at which firms invest in Markov Perfect Equilibrium (MPE). A central issue addressed is the coordination problem the in market-entry game. We finally determine the value of a firm's shared investment option at the outset, generalizing the Black-Scholes-Merton (BSM) option pricing formula (see Black and Scholes, 1973; Merton, 1973) to allow for strategic interactions among would-be investors at option maturity.

We show that production costs, while being irrelevant in standard Cournot quantity competition, they have a major influence on whether firms enter the market in the first place. Depending on the magnitude of fixed entry costs, firms may end up investing even at very low demand and staying put for a while if the prospects of future demand upsurge are substantial. Firm market entry decisions and flexible production strategies are essentially decoupled. Market entry at low demand may create valuable access to large future profit potential at higher demand levels. In low or intermediate demand scenarios with the firm producing below full capacity, expectation of future streams of profits might justify early market entry. Unlike in standard European call options, the resulting shared investment option values are not convex or monotone increasing in demand but exhibit "competitive waves".

2. Literature review

The competitive situation at hand involves two types of risk: (i) strategic risk arising from endogenous competitive interaction (*strategic uncertainty*) and (ii) operating risk due to demand fluctuation (*market uncertainty*). Our research agenda lies at the intersection of the literature on production flexibility (e.g., Kulatilaka and Perotti, 1998; Van Mieghem, 2003) and the option games literature on investment under uncertainty involving strategic interactions (e.g., Smit and Trigeorgis, 2004; Chevalier-Roignant et al., 2011). Our work explores a new territory involving the interactions between strategic market entry and operational flexibility under general oligopolistic competition.

Our setting shares basic similarity with Kulatilaka and Perotti (1998), who consider two firms facing a decision to enter a market at a given future time.[2] Our general approach, however, differs in several respects. First, rather than a duopoly, we consider a general oligopoly with k

[2]If the two firms both enter, they compete over quantity in a one-shot game receiving Cournot duopoly profits. One of the firms can invest early reducing variable cost c. The first-stage cost-reducing investment results in greater "capability" (convexity) of second-stage Cournot profits, enabling the firm to take better advantage of the shared investment option. Assuming a log-normal demand distribution, Kulatilaka and Perotti (1998) obtain a closed-form expression for the shared investment option.

active (invested) firms and n potential entrants (option holders). Second, once firms enter the market they engage in repeated Cournot competition (in an infinitely repeated continuous-time game rather than facing a one-shot game). Therefore, in our setting, the value upon entry is not merely a discounted sum of expected Cournot profits but reflects firms' flexibility to adapt their production regime to realized demand subject to capacity constraints. Finally, we analyze the coordination problem arising at the future market entry time among multiple firms. Thirdly, we derive a more general closed-form expression for the shared investment option in oligopoly adopting a more general demand distribution (see also discussion in Section 6).

Van Mieghem and Dada (1999) consider a two-stage model where a monopolist firm decides on installed capacity, output and price. They consider different postponement strategies differing in the timing of actions relative to demand realization. We use a similar notion of output postponement with hold-back to model production flexibility. We follow Kulatilaka and Perotti (1998) and Van Mieghem and Dada (1999) assuming a linear demand function with an uncertain intercept.

Many models dealing with the timing of lumpy investment follow the early literature on technology adoption that focuses on preemption and rent-equalization (e.g., see Reinganum, 1981a,b; Fudenberg and Tirole, 1985) extending the modeling to stochastic environments. A number of articles address the problem of timing rivalry among two firms in a stochastic setting in line with Fudenberg and Tirole's (1985) model on preemption (e.g., Grenadier, 1996; Huisman, 2001; Chevalier-Roignant and Trigeorgis, 2011). The standard assumption in these models is that firms have an infinitely lived opportunity to enter, which is somewhat restrictive. Although this modeling assumption eases the derivation of trigger policies it disregards the possibility that the market may suddenly disappear.

Aguerrevere (2003) considers two interlinked forms of flexibility. Oligopolistic firms decide at each time on the quantity supplied to the market and on whether to invest in extra capacity. Firms are assumed symmetric and their number is fixed and known at the outset, thus competition over market entry is not considered. Firms can increase capacity by any amount at any time. While we do also model flexible output decisions we do not allow capacity adjustments but a single entry decision for each firm. We thus focus on the industry structure that arises endogenously for different demand realizations, whereas Aguerrevere (2003) focuses on the dynamics and value of industry capacity.

3. Model Setup and Solution Approach

We consider a setting where n firms may launch a new product or potentially enter a market at a given future time, T. The industry structure is unknown at the outset but remains unchanged once firms have entered the marketplace. At the time firms make their investment decisions, future profits are stochastic influenced both by market demand realization and the future strategic firm choices in the new industry. Process $(X_t; t \geq 0)$ describes the uncertainty about market demand which follows a geometric Brownian motion (GBM) of the form:

$$\mathrm{d}X_t = \mu\ X_t\ \mathrm{d}t + \sigma X_t\ \mathrm{d}B_t, \quad X_0 = x\ (> 0), \tag{3.1}$$

where $\mu = r - \delta\ (> 0)$ is the risk-neutral drift or growth rate in demand, $\delta\ (> 0)$ is a "dividend" yield, $\sigma\ (> 0)$ is the constant volatility term, and B is a standard Brownian motion.[3]

We consider two stages (see Figure 1): (i) the market-entry game between n option holders taking place at future time T, and (ii) the ensuing repeated Cournot competition game involving flexible production decisions under demand uncertainty and capacity constraints. In the first stage, each potential entrant (firm $i = 1, \dots, n$) decides whether to enter the market at future time T, with the entry decision noted as $e_i \in \{0, 1\}$. If firm i enters ($e_i = 1$), it incurs a one-time lump-sum fixed cost I_i to install capacity thereafter facing a capacity constraint with upper production bound $\bar{q}_i$. I_i also incorporates financing costs which can differ according to whether the potential entrant has ready access to the capital markets and relatively affordable external financing sources. When later the firm operates in the second stage, it additionally incurs diverse (fixed and variable) production costs. Assuming a homogeneous production technology, variable production cost, c, is the same for all firms. This situation is more applicable when products are fairly standardized and differences in variable production costs are small or negligible. Depending on economies of scope, firms may differ in their fixed production costs, f_i, $i = 1, \dots, n$. For instance, conglomeration may reduce project-specific fixed costs by spreading administrative costs over several businesses. In the second stage, assuming firm i has entered, its decision variable is the quantity amount produced, with rivals competing in a Cournot fashion. At time $t\ (\geq T)$, invested firm i selects individual quantity $q_i(t)$, with all firms collectively supplying industry output $Q_t = \sum_{i=1}^{n} q_i(t)$. Firm i's output decision, $q_i(t)$, is subject to capacity constraints:

$$0 \leq q_i(t) \leq \bar{q}_i e_i,\ \forall t \in [T, \infty),\ \forall i = 1, \dots, n. \tag{3.2}$$

Firm i's strategy, noted γ_i with $\gamma_i = \{e_i, q_i(\cdot)\}$, consists in choosing whether to enter ($e_i = 1$) at the future time T (e.g., when the market opens or is deregulated) and in selecting the appropriate output $q_i(t)$ under capacity-constrained Cournot oligopolistic competition. Since future demand

[3]The process X_t is defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The augmented Brownian filtration $(\mathscr{F}_t; t \geq 0)$ captures the historical path of the process, $\mathscr{F}_t = \sigma(B_s, 0 \leq s \leq t)$ being the information set at time t and $\mathscr{F}_\infty = \mathcal{F}$. As standard in the real options literature, the firm can perform financial hedging to mitigate certain business risks; this allows considering a stochastic process with a drift adjusted for the underlying riskiness and therefore discounting future risk-adjusted cash-flows at the risk-free rate, $r (> 0)$, under the risk-neutral probability measure, $\mathbb{P}$. According to Birge (2000), option-pricing theory offers a rigorous way to incorporate risk aversion in linear cash-flow expressions without relying on (strictly concave) utility functions. Finally, δ may represent some form of convenience yield, anticipated competitive erosion, opportunity cost or below-equilibrium growth or return shortfall (e.g., see McDonald and Siegel, 1986; Trigeorgis, 1991).

is stochastic, the strategy must be non-anticipative and allow for flexibility in managerial decision making.[4]

As in Kulatilaka and Perotti (1998) and Van Mieghem and Dada (1999), we assume an inverse linear demand of the form

$$p(X_t; Q_t) = X_t - b\, Q_t, \quad b > 0. \tag{3.3}$$

The future market price, $p(X_t; Q_t)$, is driven by future demand realizations X_t and the firms' collective output choices captured by aggregate industry supply Q_t. Given demand realization X_t, firm i's gross profit is

$$\pi_i(X_t; Q_t) = \pi_i\left(X_t; q_i(t), Q_{-i}(t)\right) = p(X_t; Q_t)\; q_i(t) - c\; q_i(t), \tag{3.4}$$

where $Q_{-i}(t)$ is the quantity produced collectively by all other supplying firms except firm i at time t, with $Q_t \equiv q_i(t) + Q_{-i}(t)$. Firm i discounts future (risk-adjusted) cash flows at the (risk-free) rate r. At the outset, the value of firm i if it follows arbitrary strategy choice γ_i, with its rivals pursuing strategies γ_{-i}, is

$$J_i^x\left(\gamma_i, \gamma_{-i}\right) = \mathbb{E}_x\left[e^{-rT}\left\{\int_T^{\infty} e^{-rt}\left[\pi_i\left(X_t; Q_t\right) - f_i e_i\right] \mathrm{d}t - I_i e_i\right\}\right],$$

subject to stochastic demand (3.1) and capacity constraints (3.2). Here, $\mathbb{E}_x[\cdot] \equiv \mathbb{E}[\cdot \mid X_0 = x]$ denotes the conditional expectation under the risk-neutral probability measure. Expression $J_i^x\left(\gamma_i, \gamma_{-i}\right)$ gives the expanded net present value of future cash flows for strategy profile (γ_i, γ_{-i}). This differs from the standard notion of net present value considered in corporate finance since here firm i's management has flexibility in deciding whether to enter at T and what production strategy $q_i(t)$ to follow in each future period.

In the above dynamic game, we solve for a Markov Perfect Equilibrium (MPE), i.e., a profile of optimal strategies $\left(\gamma_i^*, \gamma_{-i}^*\right)$ in the class of Markov or feedback policies that yields a Markov Nash equilibrium (MNE) in each demand state x.[5] In MPE, firm i's investment option is worth

$$C_i(x) = \max_{\gamma_i}\; J_i^x(\gamma_i, \gamma_{-i}^*). \tag{3.5}$$

The solution approach proceeds backwards. First we obtain the MNE output decisions for demand level X_t, obtain the equilibrium Cournot oligopoly profits and derive the value of flexible firm i when it invests along with rivals. We then analyze the market entry game and identify the Pareto-optimal entry MNE. Since the n rival firms face a single possible entry time when the market opens, our initial investment assessment problem bears analogy with a European call option

[4]The entry decision, e_i, is made based on the information set at time T (e_i is $\mathscr{F}_T$-measurable), while the output decisions $(q_i(t); t \geq T)$ are $(\mathscr{F}_t; t \geq T)$-measurable. The output strategy $q_i(\cdot)$ is degenerate with $q_i(t) = 0$ for $t \geq T$ if $e_i = 0$.

[5]See Fudenberg and Tirole (1991, Chapter 13) for a discussion on MPE. The perfectness of a MNE strategy profile is easily shown.

shared by the n firms. Leveraging on this analogy, we then derive a general expression for the value of the shared investment option in the most likely MPE. For sake of exposition, we first assess the value of a flexible firm and investigate the market entry game as if it was taking place at time 0, with (initial) demand x. We later consider the initial option valuation problem, transposing the market-entry decision from time 0 to time T, with future realized demand X_T being random.

4. Operational Flexibility in Production Stage

In this section, we consider the problem of determining the equilibrium output of k firms in a Cournot oligopoly facing capacity constraints and determine their Cournot profits. We then derive the value for an oligopolist firm with production flexibility under demand uncertainty and capacity constraints.

4.1. **Cournot Competition in Production with Capacity Constraints.** We wish to identify a Markov Nash equilibrium, i.e., a profile of quantity choices such that no firm has an incentive to unilaterally deviate from its equilibrium output decision. Suppose k firms have entered at future time T and let $(q_i^C, Q_{-i}^C) \equiv (q_i^C(X_t), Q_{-i}^C(X_t))$ be the (k-tuple of) MNE output decisions in demand state X_t.[6] Here, the fixed cost components I_i and f_i play no role in the determination of the MNE once the firm has invested. The equilibrium Cournot (gross) profit, $\pi_i^C(X_t; k)$, of firm $i = 1, \ldots, k$ satisfies

$$\pi_i^C(X_t; k) \equiv \pi(X_t; q_i^C, Q_{-i}^C) \geq \pi_i(X_t; q_i(t), Q_{-i}^C)$$

for all output decisions $q_i(t)$ satisfying the capacity constraints, $0 \leq q_i(t) \leq \bar{q}_i, i = 1, ..., k$.

Suppose now firms are ranked and indexed by *increasing capacity*, namely

$$\bar{q}_0 < \bar{q}_1 \leq \cdots \leq \bar{q}_m \leq \cdots \leq \bar{q}_k < \bar{q}_{k+1},$$

with $\bar{q}_0 = 0$ and $\bar{q}_{k+1} = \infty$. Proposition 1 below establishes the MNE for the static Cournot game with capacity constraints.

Proposition 1. *In MNE, each firm i will produce output $q_i^C(X_t)$ in demand state X_t, with $q_i^C(X_t)$ given by*

$$q_i^C(X_t) = \begin{cases} 0 & \text{if } X_t \in (0, c), & \textit{(no production)} \\ \frac{X_t - \Sigma_m}{b(k-m+2)} & \text{if } X_t \in [x_{m-1}^C, x_m^C),\ m = 1, \ldots, i & \textit{(unconstrained production)} \\ \bar{q}_i & \text{if } X_t \in [x_i^C, \infty) & \textit{(constrained production)} \end{cases} \tag{4.1}$$

where $\Sigma_m \equiv c + b \sum_{j=0}^{m-1} \bar{q}_j$ and $x_m^C \equiv b\bar{q}_m (k - m + 2) + \Sigma_m$ for $m = 1, \ldots, i$.

Proof. See Appendix A. □

[6] Superscript C corresponds to Cournot-Nash equilibrium outcomes.

Similar to the known unconstrained Cournot-Nash equilibrium, a firm will not produce at low demand $X_t \leq c$ as it would not sell at a price lower than its marginal production cost c. For large demand $X_t \geq x_i^C$, firm i will produce at full capacity $\bar{q}_i$ (becoming constrained). When no capacity constraint is binding, $c \leq X_t < x_1^C$, firm i produces an unconstrained quantity

$$q_i^C(X_t) = \frac{X_t - c}{b(k+1)}, \ X_t \in [c, x_1^C). \tag{4.2}$$

The above solution resembles the equilibrium Cournot quantity in an unconstrained k-firm oligopoly (see, e.g., Tirole, 1988). However, in the capacity-constrained case above, firms' capacity limits $\bar{q}_m$, $m = 1, \ldots, i$, influence the quantity-setting behaviors for large demand. In demand region $[x_{m-1}^C, x_m^C)$, $m = 1, \ldots, k$, firm m is the smallest firm that is not constrained by its production capacity. In other words, for $X_t \in [x_{m-1}^C, x_m^C)$, $m-1$ firms are capacity-constrained, collectively producing $\sum_{j=0}^{m-1} \bar{q}_j$, while $k-m+1$ firms are not. Demand threshold x_m^C is exogenous through the distribution of firm capacities, but endogenous through the quantity decisions of the largest non-constrained firms. Unconstrained firms face a standard Cournot quantity game where excess demand X_t is adjusted downwards to $X_t - b\sum_{j=0}^{m-1} \bar{q}_j$ to account for the impact of the constrained firms' output decisions on the equilibrium price, as verified by employing $k' = k - m + 1$ and $X_t' = X_t - b\sum_{j=0}^{m-1} \bar{q}_j$. Effectively, constrained firms are absorbed as competitive fringe and become non-strategic. The remaining strategic unconstrained firms internalize the collective actions of constrained firms as an exogenous reduction in demand. We refer to the production pattern followed by firm i at intermediate demand $X_t \in \left[x_{m-1}^C, x_m^C\right)$ as "production regime" m.

Having determined the optimal quantity we next derive the corresponding equilibrium profits. Since the profit expression in (3.4) is quadratic in output choice q_i, the equilibrium output choice $q_i^C(X_t)$ is linear in X_t at intermediate demand $[c, x_i^C)$. Therefore, optimal output is continuous in the intermediary demand region. Proposition 2 below gives the Cournot profits earned by invested firm i in MNE under capacity constraints.

Proposition 2. *In Cournot oligopoly with k firms facing homogeneous capacity constraints, the (gross) profit of firm i at demand level X_t is given by*

$$\pi_i^C(X_t; k) = \begin{cases} 0, & \text{if } X_t \in (0, c), & \text{(no production)} \\ \frac{(X_t - \Sigma_m)^2}{b(k-m+2)^2}, & \text{if } X_t \in \left[x_{m-1}^C, x_m^C\right), m = 1, \ldots, i, & \text{(unconstrained production)} \\ \bar{q}_i \frac{(X_t - \Sigma_m)}{b(k-m+2)}, & \text{if } X_t \in \left[x_{m-1}^C, x_m^C\right), m = i+1, \ldots, k+1, & \text{(constrained production)} \end{cases}$$

where $\Sigma_m \equiv c + b\sum_{j=0}^{m-1} \bar{q}_j$, $x_m^C \equiv b\bar{q}_m(k-m+2) + \Sigma_m$ and $x_{k+1}^C = \infty$ by convention. The Cournot profit of firm i is monotone increasing in demand X_t and decreasing in variable cost c and in the number of invested firms k.

Proof. The profit expressions obtain directly using the MNE quantities from Proposition 1 and the demand function in (3.3). ☐

For low demand $X_t < c$, the firm does not produce earning zero profit. For intermediary demand, it is unconstrained as it produces below its capacity, with profits being quadratic (convex) in demand, as in Kulatilaka and Perotti (1998). For large demand, $X_t \geq x_i^C$, firm i's capacity constraint becomes binding, though this is not necessarily the case for its larger rivals; firm i's profit here increases in a linear fashion as the firm, faced with capacity constraints, is not in a position to expand output to tap on increased demand.

4.2. **Value of Production Flexibility in Constrained Cournot Oligopoly.** We next turn to determining the value of a firm with production flexibility. Assuming output decisions are independent over time, firm value is the expected discounted sum of Cournot profits, $\pi_i^C(X_t; k)$, with firms following the MNE strategies as in Proposition 1:

$$\begin{aligned} W_i(x;k) &\equiv \max_{q_i(\cdot)} \mathbb{E}_x \left[\int_0^\infty e^{-rt} \pi_i \left(X_t; q_i(t), Q_{-i}^C(t) \right) \, \mathrm{d}t \right] \\ &= \mathbb{E}_x \left[\int_0^\infty e^{-rt} \pi_i^C(X_t; k) \, \mathrm{d}t \right], \end{aligned}$$

where the demand process $(X_t; t \geq 0)$ follows the GBM of equation (3.1).

Proposition 4 below provides a closed-form expression for firm i's shared investment opportunity value, $W_i(x; k)$, when k firms have invested.

Proposition 3. *The value of flexible firm i, $W_i(x;k)$, in a k firm Cournot oligopoly with capacity constraints is:*

$$W_i(x;k) = \begin{cases} A_0 x^{\beta_1} & \text{if } x \in (0, c), \\ NPV_i^U(x;k,m) + A_m x^{\beta_1} + B_m x^{\beta_2} & \text{if } x \in [x_{m-1}^C, x_m^C),\ m = 1, \ldots, i, \\ NPV_i^C(x;k,m) + A_m x^{\beta_1} + B_m x^{\beta_2} & \text{if } x \in [x_{m-1}^C, x_m),\ m = i+1, \ldots, k+1, \end{cases} \tag{4.3}$$

where

$$NPV_i^U(x;k,m) \equiv \frac{1}{b(k-m+2)^2} \left[\frac{x^2}{2\delta - r - \sigma^2} - \frac{2\Sigma_m x}{r - \mu} + \frac{\Sigma_m^2}{r} \right], \qquad m = 1, \ldots, i,$$

$$NPV_i^C(x;k,m) \equiv \frac{\bar{q}_i}{b(k-m+2)} \left[\frac{x}{r-\mu} - \frac{\Sigma_m}{r} \right], \qquad m = i+1, \ldots, k+1,$$

$$\beta_1, \beta_2 \equiv -\frac{r - \delta - \sigma^2/2}{\sigma^2} \pm \sqrt{\left(\frac{r - \delta - \sigma^2/2}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2}}$$

with $\beta_2 < 0 < 2 < \beta_1$, $\Sigma_m \equiv c + b \sum_{j=0}^{m-1} \bar{q}_j$, and $x_m^C \equiv b\bar{q}_m(k-m+2) + \Sigma_m$. The A_m and B_m terms are uniquely defined and obtained recursively with $x \mapsto W_i(x;k)$ being continuously differentiable.[7] *The above firm value is monotone increasing in demand x with $W_i(0;k) = 0$ and $W_i(\infty;k) = \infty$. $x \mapsto W_i(x;k)$ decreases in the number of firms. Above assumes $\delta > [r + \sigma^2]/2$.*

Proof. See Appendix C. □

[7] Superscripts U and C stand for "Unconstrained" and "Constrained," respectively.

The value of flexible firm i, $W_i(x;k)$, depends on current demand x. Effectively, this information incorporates current market predictions about future demand developments (e.g., large or low expected demand). This value will adjust as revised or updated demand state x shows larger or smaller future demand. For low current demand, $x \in (0,c)$, production flexibility is not worthless; it reflects option value $A_0 x^{\beta_1} (> 0)$ since demand may potentially grow sufficiently to justify activating production at a later stage.

For intermediate demand, $[x^C_{m-1}, x^C_m)$, with $m = 1,\ldots,i$, unconstrained firm i produces quantity $(x - \Sigma_m)/[b(k-m+2)]$ (Proposition 1) and earns profit $(x-\Sigma_m)^2/[b(k-m+2)^2]$ (Proposition 3), which is quadratic or convex in demand x. If the unconstrained firm would produce at this rate forever, its net present value from a committed investment now would be

$$NPV^U_i(x;k,m) = \mathbb{E}_x\left[\int_T^\infty e^{-rt}\frac{(X_t-\Sigma_m)^2}{b(k-m+2)^2}\,\mathrm{d}t\right], \quad x \in [x^C_{m-1}, x^C_m),\ m = 1,\ldots,i,$$

as given in Proposition 4.[8] This is the first term appearing in the second line in the expression for $W_i(x;k)$ in Proposition 4. As demand fluctuates, however, a flexible firm will adjust production in view of demand realizations and rivals' actions given its capacity constraints. The second term, $A_m x^{\beta_1}$, on the second line for $W_i(x;k)$ in Proposition 4 captures the option-value adjustment for scenarios in which demand rises and the firm scales up production (up to its capacity limit $\bar{q}_i$). The third term, $B_m x^{\beta_2}$, captures the option-value adjustment for scenarios in which demand declines and the firm contracts production.

In the high demand region $[x^C_{m-1}, x_m)$ for $m = i+1,\ldots,k+1$, firm i is constrained producing at full capacity $\bar{q}_i$ (Proposition 1), earning only linear Cournot profit, $\bar{q}_i\,(x-\Sigma_m)/(k-m+2)$ (Proposition 3). The committed net present value for a constrained firm is then

$$NPV^C_i(x;k,m) = \mathbb{E}_x\left[\int_T^\infty \frac{\bar{q}_i\,(X_t-\Sigma_m)}{k-m+2}\,\mathrm{d}t\right], \quad X_T = x \in [x^C_{m-1}, x^C_m),\ m = i+1,\ldots,k+1$$

as given in Proposition 4. Again, for a flexible firm that will not follow this committed stand, value adjustments are needed. $A_m x^{\beta_1}$ corresponds to adjustments linked to flexible expansion strategies in case of future demand increases, while $B_m x^{\beta_2}$ captures the flexibility value of production curtailments in face of low demand.

This equilibrium production regime will adjust each time future demand X_t reaches a successively higher demand threshold x^C_m, since more rivals will face binding capacity constraints while the remaining firms do not. The terms $A_m x^{\beta_1}$ in effect relate to a series of distinct perpetual American call (expansion) options, each corresponding to an industry-wide production regime change arising because yet another previously-unconstrained firm now faces a binding capacity constraint

[8]For $X_t \in [x^C_{m-1}, x^C_m)$, $m = 1,\ldots,i$, the quadratic Cournot profit expression (see Proposition 3) has terms in X_t^2, X_t and a constant, Σ_m^2. The perpetuity value of the constant amount is Σ_m^2/r. For terms that trend upwards, we employ the (continuous-time) Gordon formula: the present value of receiving a cash flow starting at y and growing in perpetuity at a rate g $(< r)$ is $y/(r-g)$. The growth rate for $(X_t; t \geq 0)$ is $r - \delta$; it is $2\delta - r - \sigma^2$ for $(X_t^2; t \geq 0)$. This justifies the NPV^U_i expression in Proposition 4.

at new demand threshold x_m^C ($> x$). The terms $B_m x^{\beta_2}$ are similarly linked to a series of perpetual American put (contraction) options; each time a demand threshold x_m^C ($< x$) is "hit" from above, the capacity constraint of yet another firm is relaxed, with rivals subsequently facing fiercer competition in a less attractive marketplace, adapting their production regime accordingly.

In reality, different historical paths, resource accumulation policies or heterogeneous risk attitudes may lead firms to have different capacities at the future market entry time. While asymmetry among firms can be observed in a number of industries, it is difficult to explain it a priori. The assumption of symmetric capacity, $\bar{q}_i = \bar{q}$ for all $i = 1, ..., k$, hence seems mild as it avoids cherry picking firms.[9] In the symmetric capacity case, Propositions 1 to 3 reduce to simpler expressions summarized in Proposition 4 below. In the remainder of the paper, we focus on this case where invested firms have the same capacity.

Proposition 4. *In Cournot oligopoly with k invested/ active firms with homogeneous capacity $\bar{q}$, a firm's output $q^C(X_t; k)$ and profit $\pi^C(X_t; k)$ in MNE are given by*

$$q^C(X_t;k) = \begin{cases} 0 & \text{if } X_t \in (0,c) \\ \frac{X_t - c}{b(k+1)} & \text{if } X_t \in [c, \bar{x}_k), \\ \bar{q} & \text{if } X_t \in [\bar{x}_k, \infty), \end{cases} \quad \text{and} \quad \pi^C(X_t;k) = \begin{cases} 0 & \text{if } X_t \in (0,c), \\ \frac{(X_t - c)^2}{b(k+1)^2} & \text{if } X_t \in [c, \bar{x}_k), \\ \bar{q}\frac{(X_t - c - bk\bar{q})}{b(k+1)} & \text{if } X_t \in [\bar{x}_k, \infty), \end{cases}$$

with $\bar{x}_k = b(k+1)\bar{q} + c$.[10] *The (gross) value of flexible firm in a symmetric k-firm Cournot oligopoly is*

$$W(x;k) = \begin{cases} A_0' x^{\beta_1} & \text{if } x \in (0,c), \\ NPV^2(x;k) + A_1' \, x^{\beta_1} + B_1' \, x^{\beta_2} & \text{if } x \in [c, \bar{x}_k), \\ NPV^3(x;k) + B_2' \, x^{\beta_2} & \text{if } x \in [\bar{x}_k, \infty), \end{cases} \tag{4.4}$$

where

$$NPV^2(x;k) \equiv NPV_i^U(x;k,1) \equiv \frac{1}{b(k+1)^2}\left[\frac{x^2}{2\delta - r - \sigma^2} - \frac{2cx}{r-\mu} + \frac{c^2}{r}\right],$$
$$NPV^3(x;k) \equiv NPV_i^C(x;k,1) \equiv \frac{\bar{q}}{b(k+1)}\left[\frac{x}{r-\mu} - \frac{c + bk\bar{q}}{r}\right],$$

with A_0', A_1', B_1', and B_2' given in the Appendix. The value of a flexible firm is decreasing in the number of firms k. Above assumes $\delta > [r + \sigma^2]/2$.

Proof. See Appendix D. □

[9]In Proposition 3, capacity asymmetry is a *fait accompli* at time T in obtaining the value of a flexible firm. The original option valuation process takes the valuation perspective as of the outset. Pinning down the Markov Nash entry equilibria at the future market entry time T is a formidable task in this asymmetric capacity case as entry decisions depend on the number of incumbents, firm capacities, and their specific distribution among active firms.

[10]In Cournot oligopoly with symmetric capacity constraints demand thresholds x_k^C and $\bar{x}_k$ are identical.

The quantity q^C and profit expressions π^C in Proposition 4 obtain from those in Propositions 1 and 2, respectively, and are interpreted in a similar fashion. The economic interpretation of terms A'_0, A'_1, A'_2, B'_0, B'_1 and B'_2, is a bit simpler. $A'_0x^{\beta_1}$, $A'_1x^{\beta_1}$ and A'_2 correspond to perpetual American call (expansion) options. A'_2 vanishes as for large demand levels the firm's capacity constraint becomes binding and it cannot adjust its production regime to accommodate increased output (further expansion option becomes worthless in high demand regions). Suppose now $x \in [c, \bar{x}_k)$. At future time $\tau_A(x, \bar{x}_k) \equiv \inf\{t \geq 0 \mid X_t \geq \bar{x}_k\}$ when demand X_t exceeds demand threshold $\bar{x}_k$ for the first time, a symmetric firm together with all (until then unconstrained) invested firms will stop producing unconstrained output $(X_t - c)/[b(k+1)]$, which yields Cournot profit $(X_t - c)^2/[b(k+1)^2]$, and will instead produce constrained output $\bar{q}$ and earn $\bar{q}(X_t - c - bk\bar{q})/b$. Term $A'_1x^{\beta_1}$ thus reflects the future "value exchange" from switching from the industry-wide production regime with unconstrained firms to a production regime where firms fully utilize their capacities. Since this value exchange will take place in the future at stochastic time $\tau_A(x, \bar{x}_k)$, it is discounted to present with use of a (stochastic) discount factor that accounts for the likelihood of the exchange occuring sooner or later. Suppose now $x \in (0, c)$, with the firm at first producing nothing. At future time $\tau_A(x, \bar{x}_k)$, a similar exchange will take place, such that $A'_0x^{\beta_1}$ embeds this perpetual American call option value. In addition, at an earlier future time $\tau_A(x, c)$, the firm will start producing and earn $(X_t - c)^2/[b(k+1)^2]$. Expression $A'_0x^{\beta_1}$ thus captures the present value of producing in this additional production regime.

B'_0, $B'_1x^{\beta_2}$, and $B'_2x^{\beta_2}$ correspond to perpetual American put (contraction) options. B'_0 vanishes because, when demand is low and the firm does not produce, there is no room for a production regime change on the downside. In case of intermediary demand $x \in [c, \bar{x}_k)$, $B'_1x^{\beta_2}$ captures the realization that firm i will stop production when demand falls below the marginal production cost c at time $\tau_B(x, c) \equiv \inf\{t \geq 0 \mid X_t \leq c\}$. When firm i stops producing, it no longer earns profits but it avoids further losses; this is discounted to present time with the appropriate discount factor. For large demand $x \in [\bar{x}_k, \infty)$ with the firm producing at full capacity, expression $B'_2x^{\beta_2}$ captures the value change occurring at time $\tau_B(x, c)$ as well as the change in production regime arising before — at time $\tau_B(x, \bar{x}_k)$ — when demand falls in the intermediary demand region $x \in [c, \bar{x}_k)$. We note that $B'_1 = B_1$; indeed, in both the symmetric and asymmetric cases, all firms stop producing at low demand X_t $(< c)$.

Figure 2 depicts the value of a flexible firm in a symmetric constrained Cournot oligopoly with $k = 1, 2$ and 3 firms.

5. Shared Investment Option with Coordination Problem

We now proceed to analyze the earlier market-entry game among n entry option holders. The market entry decision depends in part on whether acquiring subsequent production flexibility is worth paying a sunk entry cost and incurring fixed per-period production costs f_i. Let $V_i(x; k)$ be

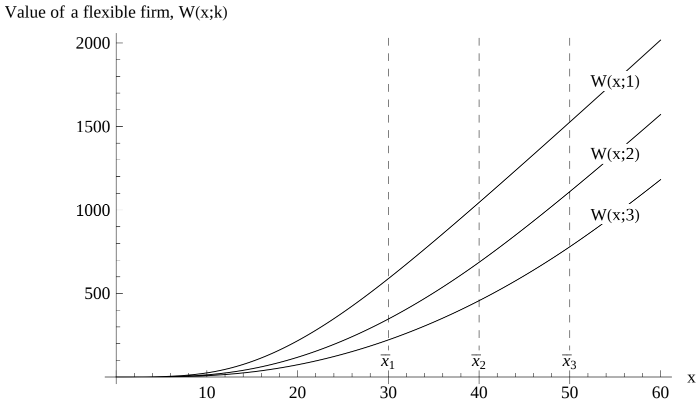

FIGURE 2. **Value of a flexible firm in monopoly** ($k = 1$)**, duopoly** ($k = 2$) **and triopoly** ($k = 3$) **industry structures**. We assume $c = 10, \bar{q} = 2, b = 5, I = 10, f_i = \sqrt{i}, \sigma = 0.1, r = \delta = 0.04$.

firm i's (expanded) net present value of investing and operating in a $k-$firm oligopoly given initial demand x. This value (net of entry cost E_i) equals

$$V_i(x;k) \equiv W(x;k) - E_i,$$

where $E_i \equiv I_i + f_i/r$ is the present value of sunk market entry costs I_i plus all fixed production costs when the firm commits to an investment plan at entry time T. As the firm incurs fixed production cost f_i per period in perpetuity, the value of such fixed production costs, f_i/r, is added besides the fixed entry cost. Even though, owing to the firm's production flexibility, the (gross) value of a flexible firm, $W(x;k)$, is necessarily positive, the (expanded) net present value, $V_i(x;k)$, may be negative if the firm incurs high entry costs E_i. In the following, we consider without loss of generality potential entrants to be weakly ranked in terms of increasing aggregate investment costs, $E_1 \leq E_2 \leq \cdots \leq E_n$.

5.1. **Market-Entry Game and Coordination Problem.** We next examine the entry decision of the n potential market entrants. Since firm i incurs fixed entry and operating costs, E_i, the firm faces a negative net value at zero demand, i.e., $V_i(0;k) < 0$, independent of the arising industry structure ($k = 1, \ldots, n$). Firm i's net present value strictly increases with demand x, all other factors (including the number of invested rivals) remaining constant. Rivals' market entries represent negative externalities as the gross value of flexible firm i, $W(x;k)$, decreases with the number of active firms, k. In case of capacity symmetry firms are indifferent concerning which of the n potential entrants will eventually operate in the marketplace at T. The number of investing firms k at market entry time T is actually unknown ex ante but can be predicted as the equilibrium outcome of strategic interactions when demand is realized and observed by all firms at time T. A potential entrant can make an inference on the equilibrium number of market entrants given realized demand. To pin-point this equilibrium outcome, we determine the net present value obtained by firm i in each possible future strategic scenario.

Using the symmetric duopoly ($n = 2$) as a simple example, we want to illustrate this situation. Consider the symmetric strategic-form game (with $E_1 = E_2 = E$) of Figure 3. If demand x is large and duopolist values are positive, $V(x; 2) \geq 0$, each firm will invest regardless of the rival action. The resulting MNE is (Invest, Invest) or $(e_1^\star, e_2^\star) = (1, 1)$ with identical payoffs $(V(x; 2), V(x; 2))$. If demand is so low that no firm (not even a monopolist) can turn a profit, namely if $V(x; 2) < V(x; 1) < 0$, both firms will stay out and not enter this market. The resulting MNE is (Stay out, Stay out) or $(e_1^\star, e_2^\star) = (0, 0)$ with payoffs $(0, 0)$. In the intermediate region, i.e., if demand is sufficiently large to accommodate one firm but not both, $V(x; 2) < 0 < V(x; 1)$, the situation is more involved. If one firm stays out, its rival should invest as the monopolist value is positive, with $V(x; 1) > 0$. But if the rival enters, the firm should stay out as it cannot operate profitably in a duopoly, with $V(x; 2) < 0$. Two pure-strategy MNEs result along the diagonal. These are (Invest, Stay out) or $(e_1^\star, e_2^\star) = (1, 0)$ and (Stay out, Invest) or $(e_1^\star, e_2^\star) = (0, 1)$, with payoffs $(V(x; 1), 0)$ and $(0, V(x; 1))$, respectively. A third MNE in mixed strategies also exists.

		Firm 2: Invest	Firm 2: Stay out
Firm 1	Invest	$V(x;2)$, $V(x;2)$	0, $V(x;1)$
	Stay out	$V(x;1)$, 0	0, 0

FIGURE 3. Symmetric Duopoly Market Entry Game

Returning to the general, asymmetric n-firm case at entry time T, when a firm faces a take-it-or-leave-it entry opportunity, the standard NPV rule applies: the firm invests if its (expanded) NPV is positive. If demand x is sufficiently large such that $V_i(x; k) \geq 0$, firm i can profitably enter the market jointly with $k - 1$ other firms. $X_i^k \equiv \min\{x \in \mathbb{R}_+ \mid V_i(x; k) \geq 0\}$ and $x_k \equiv X_k^k$ denotes the lowest demand level at which the i-ranked firm (respectively the k-ranked firm) is profitable in a k-firm asymmetric Cournot oligopoly. By convention, we set $X_n^0 = -\infty$ and $X_1^{n+1} = \infty$. Since $x \mapsto V_i(x; k)$ is monotone increasing from $V_i(0; k) < 0$ to $V_i(\infty; k) > 0$ for all $i = 1, ..., n$, we have $X_1^k \leq ... \leq X_n^k$. These demand thresholds are crucial in pinning down the Markov Nash equilibria. If condition $V_i(x; 1) > \cdots > V_i(x; n) \geq 0$ or $x \geq X_i^n$ is satisfied, firm i has a dominant strategy to invest regardless of its rivals' decisions. Firm i is in this case always better off receiving positive NPV because $V_i(x; n)$ is the lowest net value achievable in the asymmetric n-firm oligopoly. If $V_i(x; n) < \cdots < V_i(x; 1) < 0$ or $x < X_i^1$, firm i has a dominant strategy not to invest at demand level x. In the intermediate demand region $\left[X_i^1, X_i^n\right)$, firm i has no dominant strategy. If no rival decides to enter the market, firm i's optimal response is to enter the market since it obtains a positive value as a monopolist, with $V_i(x; 1) \geq 0$. However, if $n - 1$ other firms also decide to enter the market, firm i's best response is to stay out since it cannot operate profitably in a n-firm oligopoly, given $V_i(x; n) < 0$. Hence, firm i's optimal response to demand realization $x \in \left[X_i^1; X_i^n\right)$ is in fact contingent on its rivals' best-response actions.

To determine the resulting Nash equilibria, we first obtain the reaction functions of each firm. The reaction function of firm i if k *other* firms enter in demand state x is $R_i(x; k)$. Proposition 5 below derives the reaction function of potential market entrant i, providing guidance on the nature of the industry structure obtained at future market entry time T. It posits that the state space can be partitioned into (disjoint) regions in which the lower trigger boundaries obtain from the observation whether or not the k-th ranked firm can operate profitably in a k-firm oligopoly: $x_k = \min\{x \in \mathbb{R}_+ \mid V_k(x; k) \geq 0\}$ Intuitively, if the k-th ranked firm is profitable in such an oligopoly with $k-1$ other higher ranked firms, then at least k firms must be operating profitably. The same applies for the $k+1$-th firm in a $k+1$-firm oligopoly, yielding the upper boundary for this set.

Proposition 5. *Given current demand x the reaction function for firm i in the market entry game is given by:*

$$R_i(x; k) = \begin{cases} \text{“Stay out”} & \text{if } V_i(x; k+1) < 0, \\ \text{“Enter”} & \text{if } V_i(x; k+1) \geq 0. \end{cases}$$

Moreover, if $x \in [x_k, x_{k+1})$ such that $V_{k+1}(x; k+1) < 0 \leq V_k(x; k)$, then exactly k firms will enter the market.

Proof. See Appendix E. □

We next determine the number of Nash equilibria in each demand region. We denote by $\binom{n}{k} = n!/\left[(n-k)!k!\right]$, $k \leq n$, the number of distinct possibilities for k firms to enter the market among a “pool” of n potential entrants.

Proposition 6. *The number of Markov Nash equilibria (MNE) in pure strategies varies depending on the demand region reached at entry time T as follows*

(i) $\binom{n-i}{k-1-i}$ *equilbria for demand region* $x \in \left(X_i^k, X_{i+1}^k\right)$, $i < k \leq n$;
(ii) $\binom{i}{k}$ *equilbria for demand region* $x \in \left(X_i^k, X_{i+1}^k\right)$, $n-1 \geq i \geq k$;
(iii) $\binom{n}{k}$ *equilbria for demand region* $x \in \left(X_n^k, X_1^{k+1}\right)$, $k \leq n$.

Proof. See Appendix E. □

The intuition is as follows. In case (i), firms 1 to i ($\leq k-1$) can operate profitably in a k-firm oligopoly but not firms $i+1, \ldots, n$. As the market can only accommodate k out of n firms (see Proposition 5), firms $i+1, \ldots, n$ cannot preempt firms $1, \ldots, i$ as the latter have a dominant strategy to invest anyway, so firms $i+1, \ldots, n$ would have to back off. Hence, in oligopoly, the i first firms will enter. It remains to select the additional participants in the k-firm oligopoly, namely to select $k-i$ firms among the $n-i$ remaining potential entrants. In case (ii), the oligopoly consists of k firms. Firms 1 to i can operate profitably in this marketplace. As $i \geq k$, k firms need to be selected among i potential entrants that would operate profitably. Finally, in case (iii), all n firms could operate profitably in a k-firm oligopoly but none could risk competing alongside k rivals (in a $k+1$-firm oligopoly). As $k \leq n$, k firms will enter among n firms.

5.2. **Equilibrium Selection and Investment Triggers.** As stated in Proposition 6, the uniqueness of a Nash equilibrium cannot be assured in certain demand regions. Unless we refine our solution concept, there is no obvious way to overcome the corresponding equilibrium selection problem. To this end, we employ the "focal-point argument" introduced by Schelling (1960) to select among several pure-strategy equilibria. The idea is that some equilibria are more likely to occur owing to some common sense or psychological reasons. Here we consider the Pareto-dominating MNE as being "focal" among all possible Markov Nash equilibria.[11] Common sense suggests that advantaged firms, in our case lower-cost firms, are more likely to invest first in case of a coordination problem. If firm entry costs are strictly ranked, the Nash equilibrium where the most cost-advantaged firms enter first strictly Pareto-dominates all other. Proposition 7 characterizes this equilibrium.

Proposition 7. *If potential entrant firms are strictly ranked in terms of increasing entry and fixed production costs, E_i, the Pareto-dominating Markov Nash equilibrium is unique. In such Pareto-dominating MNE, firm i invests (enters the market) if demand $x \geq x_i$ $(\equiv X_i^i)$.*

Proof. See Appendix G. □

The above result can be interpreted as follows. Firm i will invest if and only if $V_i(x; i) \geq 0$ or if $x \geq x_i$. Essentially, each entrant adopts a myopic stance when determining its investment/ entry strategy in the Pareto-dominating Nash equilibrium, disregarding all rivals' investment policies since eventually they can affect only the overall value of its investment but not their own optimal investment strategy. This rests on the resolution of the coordination problem in the intermediate demand regions by use of the above focal-point argument.[12]

[11]Schelling (1960) introduced the notion of "focal point" to support the use of Nash equilibrium as a solution concept, not so much to justify one specific Nash equilibrium among multiple Nash equilibria. Many notable researchers (e.g., Fudenberg and Tirole, 1985) have argued that the Pareto-dominating Nash equilibrium is more likely to occur among a set of distinct Nash equilibria. We follow this reasoning.

[12]An alternative approach is to assume that oligopolists decide to leave their market-entry decision to chance using mixed strategies in each demand state x. In the intermediate region, each firm would then invest with a positive probability (non-degenerate mixed Markov strategy). However, this approach does not solve the coordination problem. It rather identifies yet another Nash equilibrium in an augmented strategy space. Since in certain demand regions there is a unique Nash equilibrium with fully-mixed strategies, some authors suggest that this specific equilibrium will be followed by other players. Mixed strategies seem problematic in the context of strategic investment decision-making as real market-entry decisions are left to chance. Cachon and Netessine (2006) argue that such managerial decisions should not be guided by chance. In our setting mixed strategies are furthermore less analytically tractable and do not lead to value expressions in closed form. Pure strategies based on trigger policies provide clearer guidance on whether a firm should invest at maturity. The existence of a trigger policy, with investment beyond a certain demand point and inaction below, is unwarranted in cases where several Nash equilibria exist. The choice of the Pareto-dominating Nash equilibrium strategy profile as a focal equilibrium allows an analysis based on these demand levels.

Given the value expression in Proposition 4, we can readily derive the investment trigger values in the Pareto-dominating Markov Nash equilibrium from Proposition 7. As part of this equilibrium, firm i invests if realized demand x exceeds a specific demand threshold x_i, and does not invest otherwise. The proposition below characterizes the investment threshold in the case studied in Section 4.[13]

Proposition 8. *In the Pareto-dominating Markov Nash equilibrium, firm i invests if its expanded net present value $W(x;k)$ is positive or if $x \geq x_i$, with x_i being the threshold such that $W(x_i;i) = E_i$. Specifically,*

(i) $x_i \in (0, c)$ *if firm i's fixed costs are low or $E_i \in (0, W(c, i))$;*
(ii) $x_i \in [c, \bar{x}_i)$ *if fixed costs are moderate or $E_i \in [W(c, i), W(\bar{x}_i, i))$;*
(iii) $x_i \in [\bar{x}_i, \infty)$ *if fixed costs are large or $E_i \in [W(\bar{x}_i, i), \infty)$.*

Firm i will produce

(i) *nothing [production regime 1] if $E_i \in (0, W(c, i))$ and $x \in (x_i, c)$;*
(ii) *below capacity [production regime 2] if $E_i \in (0, W(\bar{x}_i, i))$ and $x \in (x_i, \bar{x}_i)$;*
(iii) *at full capacity [production regime 3] if $x \geq \max\{x_i, \bar{x}_i\}$.*

Proof. Since $x \mapsto V_k(x;k)$ is continuous and monotone increasing on $\mathbb{R}_+$with $V_k(0;k) < 0$ and $V_k(\infty;k) > 0$, there exists a unique root $x_k > 0$ for $x \mapsto V_k(x;k)$. Other properties obtain from Proposition 3. □

Following Proposition 8, the industry is made up of $k^\star(x) = i$ active (invested) firms when demand is in region (x_i, x_{i+1}) with x_i and x_{i+1} obtained from Proposition 8. Firm i's trigger value, x_i, in Proposition 8 increases in variable production cost c, in aggregate fixed cost E_i, and decreases in "dividend yield" or opportunity cost δ. The market entry decision and flexible production strategy are decoupled: The first decision is long-term and strategic in nature, while the second relates to short-term production operations. Indeed, a firm might decide to enter the market but initially not produce if current demand is low but future prospects are high. In the long run, adaptability in the firm's production strategy helps increases firm value upon entry and therefore enhances the attractiveness of entering the market at a given demand level x. Figure 4 illustrates these market-entry dynamics. Demand thresholds x_i are obtained at the intersections of (gross) firm value $W(x;k)$ and aggregate fixed entry costs E_i (points 1, 2 and 3 on the graph). Firm gross value $W(x;k)$ decreases with the number of active firms k (i.e., $W(x;3) \leq W(x;2) \leq W(x;1)$), whereas entry thresholds increase in firm index i (i.e., $x_3 > x_2 > x_1$). The market can accommodate

[13]In Section 5.1, we made no assumption on the smoothness of the net present value function $x \mapsto V_i(x;k)$ or on the distribution of demand in $\mathbb{R}_+$. In other words, the previous analysis applies beyond the problem of production flexibility discussed in Section 4. As we seek to value the option based on the equivalent martingale measure, the triggers obtained do not necessarily match the ones in the physical world. Since these trigger expressions are subsequently used to value the shared investment option, we here present the risk-neutral version. See Birge (2000) for a related discussion.

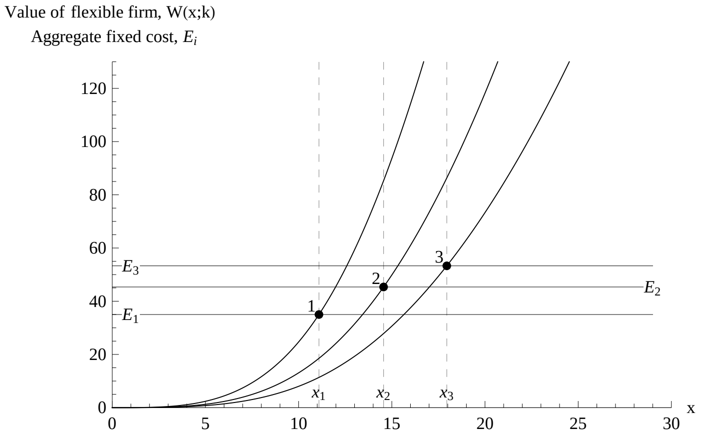

FIGURE 4. **Firm values and entry costs**. We assume $c = 10, \bar{q} = 2, b = 5, I = 10, f_i = \sqrt{i}, \sigma = 0.1, r = \delta = 0.04$.

at least i firms for demand levels larger than demand threshold x_i. For $W(x;1) < E_1$ no firm invests.

6. Shared Investment Option Value in Constrained Oligopoly

In earlier sections, we obtained the value of a firm with production flexibility in oligopoly with capacity constraints and derived the investment triggers in the Pareto-dominating MPE. We next seek to obtain a closed-form expression for the shared investment option, value $C_i(x)$, of equation (3.5).[14] Once we determine the industry structure arising in the various demand regions and assess the value of production flexibility in a k-firm Cournot oligopoly, we are in a position to obtain a closed-form expression for the shared option value of would-be entrant firm i.

Let $(e_i^\star(x), e_{-i}^\star(x))$ be the (n-tuple of) MNE market-entry decisions in given demand state x. Define $k^\star(x) = \sum_{i=1}^n e_i^\star(x)$ as the total number of market entrants in industry equilibrium when realized demand is x. In demand region (x_k, x_{k+1}), the industry will be made up of exactly $k^\star(x) = k$ operating firms (see Proposition 5) with thresholds x_k as defined in Proposition 6.

Compared to the problems analyzed in Sections 4 and 5, here entry decisions are delayed until future time T at which firms decide based on demand X_T, which is random from the time-0 perspective. By the law of iterated expectation

$$C_i(x) = \mathbb{E}_x \left[e^{-rT} \left(W(X_T; k^\star(X_T)) - E_i \right)^+ \right], \tag{6.1}$$

with $a^+ \equiv \max\{0, a\}$ for $a \in \mathbb{R}$ and $W(x;k)$ in the symmetric case given in (4.4). The above expression resembles the standard BSM formulation for the value of a European call option with strike price E_i. Similar to a European-type call option, firm i's payoff at maturity T depends

[14]The notion of optimality is more meaningful in monopolistic settings, as even if there exist several optimal policies each yields the same value. In multiplayer game-theoretic settings, however, this notion is not well-defined and so modelers employ an alternative notion of solution concepts. If the employed solution concept is not stringent enough and yields several equilibrium solutions, then the option may admit several distinct values, each corresponding to a specific equilibrium. If such a situation arises, one might want to refine the solution concept to narrow down the equilibrium set to a single strategy profile to obtain a unique option value. This is what we basically do by selecting the Pareto-dominating MPE.

on the realization of random variable X_T. However, here the realization of random demand may also trigger rivals' exercise strategies and thus indirectly affect firm i's payoff. That is, the MPE entry decisions, aggregated in $k^\star(X_T)$, are jointly driven by the realization of the same random variable, namely market demand x. Another marked difference from standard European call option valuation is that, in our competitive oligopolistic setting, the payoff function $x \mapsto W(x;k)$ is quadratic in the intermediate demand region $(x_k, x_{k+1}) \subseteq (c, \bar{x})$, while it is affine or linear in $(x_k, x_{k+1}) \subseteq (\bar{x}, \infty)$. The standard option value is linear in the profits over the entire exercise region (E_i, ∞). The additional second-order terms motivate the presence of terms noted d_2^k in the shared option value expression. We will proceed in steps to obtain a closed-form expression for the current value of the shared market-entry option with subsequent production flexibility. Proofs are presented in Appendix H.

6.1. **Cases with analytical option value expressions.** The present value of receiving a (non-contingent) payoff X_T^β at future time T, given that the current process value at time 0 is x, is

$$\mathbb{E}_x\left[e^{-rT}X_T^\beta\right] = x^\beta e^{-\mathscr{Q}(\beta)T}, \quad \beta \in \mathbb{R}, \tag{6.2}$$

where $\mathscr{Q}$ is given by

$$\mathscr{Q}(\beta) = r - (r - \delta - \sigma^2/2)\beta - \frac{1}{2}\sigma^2\beta^2. \tag{6.3}$$

Since the firm will not necessarily invest at time T but only contingent on future demand X_T being in a certain demand region, the present value of a contingent claim must also reflect the probability of this occurrence. This conditional probability given current demand x is noted $\mathbb{P}_x$. The conditional probability that future demand X_T will exceed a specific level E at time T given current (time-0) demand x is $\mathbb{P}_x\left[\{X_T \geq E\}\right] = N(\mathrm{d}_0^E)$, where

$$\mathrm{d}_0^E \equiv \frac{\ln(x/E) + (r - \delta - \sigma^2/2)T}{\sigma\sqrt{T}}, \tag{6.4}$$

with $N(\cdot)$ being the standard normal distribution. The conditional expected option value of receiving a contigent payoff X_T^β at future time T only if demand X_T is larger than E and zero otherwise is

$$\mathbb{E}_x\left[e^{-rT}X_T^\beta \mid X_T \geq E\right] = x^\beta e^{-\mathscr{Q}(\beta)T}\ N(\mathrm{d}_\beta^E), \tag{6.5}$$

where $\mathrm{d}_\beta^E \equiv \mathrm{d}_0^E + \beta\sigma\sqrt{T}$ for $\beta \in \mathbb{R}_+$. We let $d_\beta^0 = \infty$ and $d_\beta^\infty = -\infty$ by convention noting $\mathrm{d}_\beta^{x_k} = \mathrm{d}_\beta^k$ for $k = 1, \ldots, n$. The value expression in (6.2) is multiplied by $N(\mathrm{d}_\beta^E) \in (0, 1)$ to account for the probability that the option is exercised in a certain region (E, ∞) of the entire state space $\mathbb{R}_+$. Based on expression (6.5) with $\beta = 1$ or $\beta = 0$, we readily obtain the Black-Scholes-Merton formula giving the price of a European call option with exercise price E on an underlying asset $(X_t; t \geq 0)$ that pays a dividend at a rate δ.[15]

Similarly, using expression (6.5), we confirm the results obtained by Kulatilaka and Perotti (1998) in the special case of a Cournot duopoly (k=2) without capacity constraints.[16] This obtains as a special case of our symmetric unconstrained profit expression (second line) in Proposition 2, $\pi_i^C(x;k) = (x-c)^2/[b(k+1)^2]$ with inverse demand slope $b = 1$ and $k = 2$ (duopoly). In this case, the duopolists compete in the marketplace only once. This contrasts with our more general setting where k invested firms have repeated non-cooperative strategic interactions (in an infinitely repeated Cournot game). Upon market entry at time T each firm receives an infinite stream of Cournot oligopoly profits while enjoying production flexibility. Demand at maturity $(T = 1)$ follows a log-normal distribution as if assuming no time discounting with $r = \delta = 0$ in the GBM of equation (3.1). The quadratic function in (6.3) then becomes $\mathscr{Q}(\beta) = -\frac{1}{2}\beta(\beta-1)\sigma^2$. Expression (6.5) then yields

$$\mathbb{E}_x\left[\tilde{\pi}^C(x;2)\right] = \frac{x^2 e^{\sigma^2} N\left(\mathrm{d}_2^c\right) - 2xcN\left(\mathrm{d}_1^c\right) + c^2 N\left(\mathrm{d}_0^c\right)}{9}, \tag{6.6}$$

with $\mathrm{d}_2^c = [2\ln(x/c) + 3\sigma^2]/[2\sigma^2]$, $\mathrm{d}_1^c = \mathrm{d}_2^c - \sigma$ and $\mathrm{d}_0^c = \mathrm{d}_1^c - \sigma$. The above confirms a special case result in Kulatilaka and Perotti (1998, p. 1026).[17]

6.2. **General Option Value.** Based on a similar approach, we next derive an analytic expression for the value of strategic market entry with production flexibility and capacity constraints under oligopolistic competition based on equation (6.1).

Proposition 9. *In the Pareto-dominating MPE, firm i's investment option shared with n potential market entrants, $C_i(x)$, when time-0 demand is x, is given by*

$$C_i(x) = \sum_{k=i}^{n}\sum_{m=1}^{3} C_i^m(x;k), \tag{6.7}$$

where $C_i^m(x;k)$ is the option value of investing in a k-firm oligopoly (with $i \leq k \leq n$) while initially adopting production regime $m = 1, 2$ or 3. These option values are given by

$$C_i^1(x;k) = A_0' x^{\beta_1} N_0^c\left(\mathrm{d}_{\beta_1}^k, \mathrm{d}_{\beta_1}^{k+1}\right) - E_i e^{-rT} N_0^c\left(\mathrm{d}_0^k, \mathrm{d}_0^{k+1}\right), \tag{6.8a}$$

$$C_i^2(x;k) = C_{NPV}^U(x;k) + A_1' x^{\beta_1} N_c^{\bar{x}_k}\left(\mathrm{d}_{\beta_1}^k, \mathrm{d}_{\beta_1}^{k+1}\right) + B_1' x^{\beta_2} N_c^{\bar{x}_k}\left(\mathrm{d}_{\beta_2}^k, \mathrm{d}_{\beta_2}^{k+1}\right) - E_i e^{-rT} N_c^{\bar{x}_k}\left(\mathrm{d}_0^k, \mathrm{d}_0^{k+1}\right), \tag{6.8b}$$

$$C_i^3(x;k) = C_{NPV}^C(x;k) + B_2' x^{\beta_2} N_{\bar{x}_k}^{\infty}\left(\mathrm{d}_{\beta_2}^k, \mathrm{d}_{\beta_2}^{k+1}\right) - E_i e^{-rT} N_{\bar{x}_k}^{\infty}\left(\mathrm{d}_0^k, \mathrm{d}_0^{k+1}\right), \tag{6.8c}$$

[15] $\mathbb{E}_x\left[e^{-rT}(X_T - E)^+\right] = xe^{-\delta T}N(\mathrm{d}_1^E) - Ee^{-rT}N(\mathrm{d}_0^E)$, where d_0^E is given by (6.4) with $\mathrm{d}_1^E = \mathrm{d}_0^E + \sigma\sqrt{T}$.

[16] In the special setting where unconstrained symmetric duopolist firms face linear demand (with $b = 1$) each firm earns a positive Cournot profit, $\tilde{\pi}_i^C(x;2)$, when demand x exceeds marginal cost c, with $\tilde{\pi}_i^C(x;2) = \begin{cases} 0, & \text{if } x \in (0,c), \\ \frac{(x-c)^2}{9}, & \text{if } x \in [c,\infty). \end{cases}$

[17] Our result in Proposition 5 is more general, accounting for more firms in an oligopoly ($k \geq 3$) and allowing for repeated strategic Cournot interactions as well as capacity constraints, potentially leading to a linear (rather than quadratic) dependence of Cournot profit on demand x at large demand.

where demand thresholds x_k are those obtained in Proposition 7 and

$$C^U_{NPV}(x;k) \equiv \frac{1}{b(k+1)^2}\left[\frac{x^2 e^{-(2\delta-r-\sigma^2)T}}{2\delta-r-\sigma^2} N_c^{\bar{x}_k}\left(\mathrm{d}_2^k, \mathrm{d}_2^{k+1}\right) - \frac{2cxe^{-\delta T}}{r-\mu} N_c^{\bar{x}_k}\left(\mathrm{d}_1^k, \mathrm{d}_1^{k+1}\right) + \frac{c^2 e^{-rT}}{r} N_c^{\bar{x}_k}\left(\mathrm{d}_0^k, \mathrm{d}_0^{k+1}\right)\right]$$

$$C^C_{NPV}(x;k) \equiv \frac{\bar{q}}{b(k+1)}\left[\frac{xe^{-\delta T}}{r-\mu} N_{\bar{x}_k}^{\infty}\left(\mathrm{d}_1^k, \mathrm{d}_1^{k+1}\right) - \frac{c+bk\bar{q}}{r} N_{\bar{x}_k}^{\infty}\left(\mathrm{d}_0^k, \mathrm{d}_0^{k+1}\right)\right].$$

In the above,

$$N(\mathrm{d}_\beta^k, \mathrm{d}_\beta^{k+1}) \equiv N\left(\mathrm{d}_\beta^k\right) - N\left(\mathrm{d}_\beta^{k+1}\right), \quad \beta \in \mathbb{R}_+, \tag{6.9a}$$

is used to accommodate expression (6.5) *for an upper demand trigger x_{k+1}, while*

$$N_a^b(\mathrm{d}_\beta^k, \mathrm{d}_\beta^{k+1}) \equiv N\Big(\max\left\{\min\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^k\right\}, \mathrm{d}_\beta^b\right\}, \min\left\{\max\left\{\mathrm{d}_\beta^b, \mathrm{d}_\beta^{k+1}\right\}, \mathrm{d}_\beta^a\right\}\Big), \quad 0 \le a \le b \le \infty, \tag{6.9b}$$

enables assessing the value of a contingent claim that is exercised only if demand region (x_k, x_{k+1}) *is included in* (a,b)*, e.g.,* $(0,c)$*,* $(c,\bar{x}_k)$ *or* $(\bar{x}_k,\infty)$*. Above assumes* $\delta > \left[r+\sigma^2\right]/2$.

Proof. See Appendix G. □

To interpret expression (6.7), it is necessary to first interpret option value expressions $C_i^m(x;k)$. Essentially, we have to consider overlays of demand regions:

(i) the industry will consist of k operating firms if future demand X_T is in region (x_k, x_{k+1});

(ii) an active firm with production flexibility will initially adopt distinct production regimes depending on realized demand X_T. If $X_T \in (0,c)$, the firm initially will not produce (regime $m=1$). If $X_T \in (c,\bar{x}_k)$, the firm will produce but below full capacity (regime $m=2$). If $X_T \in (\bar{x}_k,\infty)$ the capacity constraint becomes binding, with the firm producing at full capacity (regime $m=3$).

These demand region overlays explain the need to decompose option value $C_i(x)$ into $C_i^m(x;k)$ under three distinct production regimes ($m=1,2,3$) and various industry structures ($k=i,...n$). For each option value component, $C_i^m(x;k)$, adjustments are needed compared to the simpler problems in (15) and (6.6) that involve unique exercise thresholds E and c. The state space for our more involved problem is partitioned by more than one exercise threshold; hence the need for expression (6.9a) that reflects the probability that realized demand lies within two given demand thresholds. Moreover, because we do not know a priori the ranking of these thresholds, especially thresholds c and $\bar{x}_k$, relative to the entry triggers $x_1,\ldots,x_n$, we employ "min" and "max" operators in (6.9b). Expression $C_i^m(x,k)$ in (6.7) and (6.8) corresponds to the value of an option that pays off only if the firm enters a k-firm Cournot oligopoly and initially adopts production regime $m=1,2,$ or 3. The terms $C_i^m(x,k)$ are all mutually exclusive.

The option value expressions in (6.8a)–(6.8c) reflect the value of a flexible firm in Equation 4.4, $W(x;k)$, derived in Proposition 4. The first component results from the NPV expression of a committed production strategy, while the other terms relate to the option values of adjusting the production regime over time. Because in a k-firm oligopoly three initial production regimes are feasible, we consider the sum $\sum_{m=1}^{3} C_i^m(x;k)$. Because firm i will only invest in the Pareto-dominating MPE if future realized demand $X_T \geq x_i$, we must also consider all demand regions (x_k, x_{k+1}) for $k = i, ...n$ from the outset, yielding the aggregated value expression for the shared investment option shown in (6.7).

Figure 5 illustrates the shared investment option values for firms 1, 2 and 3 in a Cournot oligopoly with 3 potential entrants (for our specified set of base parameters and two different values for σ and T). Contrary to standard European call options, shared investment option values here are not necessarily monotone increasing and convex. They rather exhibit "competitive waves" of alternating convex and concave segments. These phenomena are irrespective of the capacity-linked waves discussed in our earlier analysis. Here, increasing demand inevitably leads to an enhanced option value for firm i, $W(x;k)$, but it also encourages rival entry with an increased number of incumbents resulting in a value drop from $W(x;k)$ to $W(x;k+1)$.[18] In the face of heightened uncertainty, these value drops caused by rival entry are smoothed out as seen in Figure 5. As shown in panel (a) for low σ and T, option values fail to be monotone increasing or convex in demand x. The value drop from $W(x;k)$ to $W(x;k+1)$ is reflected in the general "wavy" shape of option value, $C_i(x)$. In situations with close maturity dates (e.g., $T = 0.25$), swift changes in competitive structure may occur. With increased volatility and/or entry time (maturity), as in panel (d), the effect of competitive arrivals erodes and gets smoothed out. Asymptotically, $C_i(x)$ then resembles more a classical European call option with a nearly smooth monotone increasing and convex shape. This is more clear for firm 3 who faces no subsequent rival entry threat.

The existence of such competitive waves is a bit reminiscent of shared option results in the seminal duopoly model of Dixit and Pindyck (1994, Chapter 9). There, the follower's entry at a given threshold causes an inflection on the option value of the leader. A marked difference is that in that simpler duopoly setting the value of the leader's investment option is still monotone increasing. In an American option setting a follower will enter when the market is "deep in the money" (not simply "in the money"), resulting in a delayed investment.

7. Conclusion

Option games analysis can give valuable guidance as to when to pursue certain investment strategies and how to value firm flexibility in competitive settings. We considered a setting where several firms have an option to enter a new market at a future time. Once they enter, the invested firms enjoy production flexibility to adjust their output decisions to realized demand alongside their Cournot rivals. We analyse the general coordination problem potentially arising at the entry time, pin down the Markov Perfect Equilibria and articulate why the Pareto-dominating MPE is more likely to arise. We further determine the investment entry triggers in the Pareto-dominating MPE. We finally obtain a closed-form expression for the shared investment option value. This

[18]Naturally, this is not the case for firm 3 since being the last entrant it faces no subsequent rival entries.

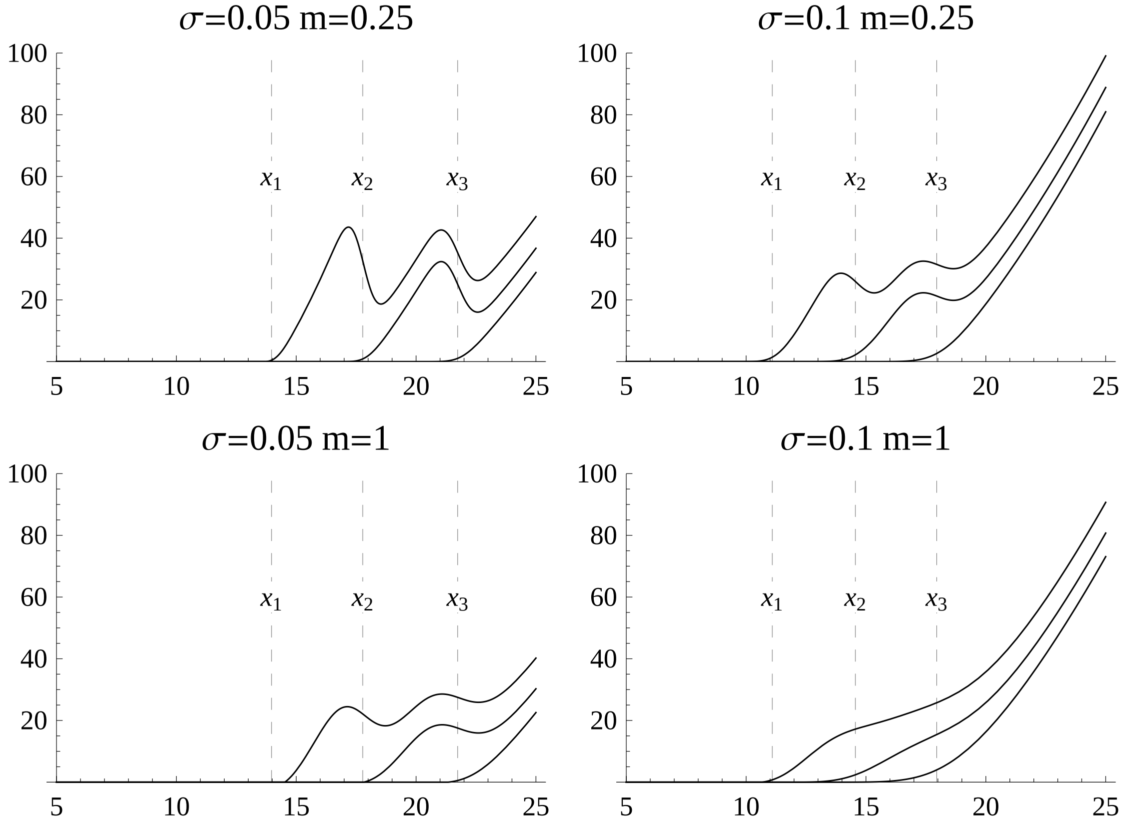

FIGURE 5. **Option values for different maturity dates and volatility values.** We assume $c = 10, \bar{q} = 2, b = 5, I = 10, f_i = \sqrt{i}, r = \delta = 0.04$.

expression bears some resemblance to the Black-Scholes-Merton formula for European call options, but is adjusted and generalized to accommodate strategic interactions, capacity constraints and production flexibility in distinct demand regions. Due to strategic interactions in the market, firms cannot simply set their investment triggers in a monopolist fashion as they have to collectively anticipate whether rivals will also exercise their investment options in future demand scenarios.

Our model extends and generalizes much of extant literature, most notably the work of Kulatilaka and Perotti (1998). Our model is more general in that it accounts for more firms in oligopoly and allows for capacity constraints. Capacity constraints potentially lead to a linear (rather than quadratic or convex) dependance of Cournot profits on demand in high demand regions. The value achieved by an investing firm at the future market entry time also involves additional option value terms. The aggregate option value extends over three distinct production regimes and across various demand states, being the result of optimizing behavior in Cournot oligopoly under capacity constraints and equilibrium industry structures. In contrast to standard models, we find that the value of the shared investment option is not monotone increasing and convex in demand. Option values exhibit "competitive waves" as the underlying market value drops with new rival entrants. We examine comparative statics illustrating the importance of this competitive effect under uncertainty across different parameter ranges.

Acknowledgements: The authors would like to thank Avinash K. Dixit, Kuno Huisman, and Jacco Thijssen for various insightful comments on earlier versions of this work. Any remaining errors are ours.

References

Aguerrevere, F. L. (2003). Equilibrium investment strategies and output price behavior: A real-options approach. *Review of Financial Studies 16*(4), 1239–1272.

Birge, J. R. (2000). Option methods for incorporating risk into linear capacity planning models. *Manufacturing & Service Operations Management 2*(1), 19–31.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy 81*(3), 637–654.

Cachon, G. P. and S. Netessine (2006). Game theory in supply chain analysis. In S. D. W. David Simchi-Levi and Z.-J. Shen (Eds.), *Handbook of Quantitative Supply Chain Analysis: Modeling in the eBusiness Era*. Kluwer.

Chevalier-Roignant, B., C. M. Flath, A. Huchzermeier, and L. Trigeorgis (2011). Strategic investment under uncertainty: A synthesis. *European Journal of Operational Research 215*(3), 639–650.

Chevalier-Roignant, B. and L. Trigeorgis (2011). *Competitive Strategy: Options and Games*. The MIT Press.

Dixit, A. K. and R. S. Pindyck (1994). *Investment under Uncertainty*. Princeton University Press.

Fudenberg, D. and J. Tirole (1985). Preemption and rent equalization in the adoption of new technology. *Review of Economic Studies 52*(3), 383–401.

Fudenberg, D. and J. Tirole (1991). *Game Theory*. The MIT Press.

Grenadier, S. R. (1996). The strategic exercise of options: Development cascades and overbuilding in real estate markets. *Journal of Finance 51*(5), 1653–1679.

Huisman, K. J. M. (2001). *Technology Investment: A Game Theoretic Real Options Approach*. Kluwer Academic Publishers.

Kulatilaka, N. and E. C. Perotti (1998). Strategic growth options. *Management Science 44*(8), 1021–1031.

McDonald, R. L. and D. Siegel (1986). The value of waiting to invest. *The Quarterly Journal of Economics 101*(4), 707–728.

Merton, R. C. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science 4*(1), 141–183.

Reinganum, J. F. (1981a). Market structure and the diffusion of new technology. *Bell Journal of Economics 12*(2), 618–624.

Reinganum, J. F. (1981b). On the diffusion of new technology: A game theoretic approach. *Review of Economic Studies 48*(3), 395–405.

Schelling, T. C. (1960). *The strategy of conflict.* Harvard University Press.

Sethi, A. K. and S. P. Sethi (1990). Flexibility in manufacturing: A survey. *International Journal of Flexible Manufacturing Systems 2*(4), 289–328.

Smit, H. T. J. and L. Trigeorgis (2004). *Strategic Investment: Real options and Games.* Princeton University Press.

Smith, J. E. and R. F. Nau (1995). Valuing risky projects: Option pricing theory and decision analysis. *Management Science 41*(5), 795–816.

Tirole, J. (1988). *The Theory of Industrial Organization.* The MIT Press.

Trigeorgis, L. (1991). A log-transformed binomial numerical analysis method for valuing complex multi-option investments. *Journal of Financial and Quantitative Analysis 26*(3), 309–326.

Trigeorgis, L. (1996). *Real options: Managerial flexibility and strategy in resource allocation.* The MIT Press.

Van Mieghem, J. A. (2003). Capacity management, investment, and hedging: Review and recent developments. *Manufacturing & Service Operations Management 5*(4), 269–302.

Van Mieghem, J. A. and M. Dada (1999). Price versus production postponement: Capacity and competition. *Management Science 45*(12), 1631–1649.

Appendices

Appendix A: Markov Nash Equilibrium (MNE) in Cournot Competition under Capacity Constraints. When no confusion arises, Q denotes the vector of production outputs or the total industry output. In MNE, the quantity-setting problem is

$$\max_{q_i} \quad \pi_i\left(x, q_i, Q_{-i}^C\right) = \left(x - b\left(q_i + Q_{-i}^C\right)\right) q_i - c q_i, \tag{7.1}$$

subject to non-negativity constraint, $q_i \geq 0$, and the production capacity constraint, $\bar{q}_i - q_i \geq 0$. The objective function in (7.1) is concave. Besides, the feasible region is convex. The Karush-Kuhn-Tucker (KKT) conditions are thus both necessary and sufficient. Denote the KKT multipliers by μ_1 and μ_2. In MNE, the Lagrangian is given by

$$\mathcal{L}\left(q_i, \mu_1, \mu_2\right) = \left(x - b\left(q_i + Q_{-i}^C\right)\right) q_i - c q_i + \mu_1 q_i + \mu_2\left(\bar{q}_i - q_i\right).$$

Solving the optimization problem then yields individual quantity as a function of industry output:

$$q_i^C(x;k) = \begin{cases} 0 & \text{if } x \leq bQ_{-i}^C + c \\ \frac{x - bQ_{-i}^C - c}{2b} & \text{if } bQ_{-i}^C + c < x < 2b\bar{q}_i + bQ_{-i}^C + c \\ \bar{q}_i & \text{if } x \geq 2b\bar{q}_i + bQ_{-i}^C + c \end{cases} \tag{7.2}$$

Equation (4.1) obtains by pinning down the MNE, i.e., by verifying that each firm is playing its best response, for each case in (7.2). We do so by expressing Q_{-i}^C as the sum of the q_{-i}^C.

Appendix C: Value of Operational Flexibility in MPE. Suppose $x \mapsto W_i(x;k)$ is sufficiently smooth, with W_x and W_{xx} denoting the first and second-order derivatives of the perpetuity value W_i. By the Feynman-Kac Theorem, function $x \mapsto W_i(x;k)$ solves the second-order differential equation (ODE)

$$rW_i(x;k) = \pi_i^C(x;k) + \mu x\ W_x(x;k) + \frac{1}{2}\sigma^2 x^2\ W_{xx}(x;k), \tag{7.3}$$

where $\pi_i^C(x;k)$ is given in Proposition 2 and $W_i(0;k) = 0$. ODE (7.3) is defined without ambiguity if $x \mapsto W_i(x;k)$ is continuously differentiable everywhere (C^1) and twice continuously differentiable almost everywhere (piecewise C^2). For notational conciseness, we drop the superscript C in threshold values $x_m^C = x_m$, $m = 0, \ldots, k+1$ in the Appendix.

Functions $x \mapsto x^{\beta_1}$ and $x \mapsto x^{\beta_2}$ are two independent solutions of the homogenous ODE

$$rf(x) = \mu x\ f'(x) + \frac{1}{2}\sigma^2 x^2\ f''(x), \tag{7.4}$$

where β_1 and β_2, given in Proposition 3, are the positive and negative roots of the quadratic function

$$\mathscr{Q}(\beta) = r - \beta\mu - \frac{1}{2}\beta(\beta-1)\sigma^2, \quad \beta \in \mathbb{R}. \tag{7.5}$$

We have $\mathscr{Q}(1) = r - \mu$ (> 0) and $\mathscr{Q}(0) = r$ (> 0).

Consider first a contingent claim θ with payoff structure

$$\theta(x) = \begin{cases} 0 & \text{if } x < z, \\ (x-c)^2 & \text{if } x \geq z, \end{cases}$$

and $z \geq c$. Note that θ is not C^1 if $z > c$. If one finds a solution Θ of

$$r\Theta(x) = \theta(x) + \mu x\ \Theta'(x) + \frac{1}{2}\sigma^2 x^2\ \Theta''(x) \tag{7.6}$$

that is C^1 and piecewise C^2, then it corresponds to the discounted stream of contingent claims θ,

$$\Theta(x) = \mathbb{E}_x\left[\int_0^\infty e^{-rt}\theta(X_t)\ \mathrm{d}t\right].$$

Suppose $\mathscr{Q}(2) > 0$. We can easily verify that

$$\mathbb{E}_x\left[\int_0^\infty e^{-rt}(X_t - c)^2\ \mathrm{d}t\right] = \frac{x^2}{\mathscr{Q}(2)} - \frac{2cx}{r-\mu} + \frac{c^2}{r}, \tag{7.7}$$

is a particular solution of (7.6) for $x \geq z$. The general solution of (7.6) is

$$\Theta(x) = \begin{cases} g_1(x,z,c) & \text{if } x < z, \\ \frac{x^2}{\mathscr{Q}(2)} - \frac{2cx}{r-\mu} + \frac{c^2}{r} + g_2(x,z,c) & \text{if } x \geq z, \end{cases} \tag{7.8}$$

where $g_1(x,z,c) = x^{\beta_1}g_1(z,c)$ and $g_2(x,z,c) = x^{\beta_2}g_2(z,c)$. Θ in (7.8) is obviously piecewise C^2. We are looking for a solution of (7.6) such that Θ is C^1. The smoothness conditions at z read:

$$\begin{aligned} z^{\beta_1}g_1(z,c) &= \frac{z^2}{\mathscr{Q}(2)} - \frac{2zc}{r-\mu} + \frac{c^2}{r} + z^{\beta_2}g_2(z,c) \\ \beta_1 z^{\beta_1 - 1}g_1(z,c) &= \frac{2z}{\mathscr{Q}(2)} - \frac{2c}{r-\mu} + \beta_2 z^{\beta_2 - 1}g_2(z,c). \end{aligned}$$

It easily obtains

$$\begin{aligned} g_1(x,z,c) &= \left(\frac{x}{z}\right)^{\beta_1}\left\{\frac{2-\beta_2}{\beta_1-\beta_2}\frac{z^2}{\mathscr{Q}(2)} - \frac{1-\beta_2}{\beta_1-\beta_2}\frac{2cz}{r-\mu} - \frac{\beta_2}{\beta_1-\beta_2}\frac{c^2}{r}\right\} \\ g_2(x,z,c) &= -\left(\frac{x}{z}\right)^{\beta_2}\left\{\frac{\beta_1-2}{\beta_1-\beta_2}\frac{z^2}{\mathscr{Q}(2)} - \frac{\beta_1-1}{\beta_1-\beta_2}\frac{2cz}{r-\mu} + \frac{\beta_1}{\beta_1-\beta_2}\frac{c^2}{r}\right\}. \end{aligned}$$

We know that $\theta(x) \geq 0$ for all $x \in \mathbb{R}_+$ and $\theta(x) > 0$ for $x \geq z$. Thus by probabilistic arguments, $\Theta(x) > 0$ for all $x \in \mathbb{R}_+$. In particular, $g_1(x,z,c) > 0$ for $x < z$. Besides, $\theta(x) \leq (x-c)^2$ for all

$x \in \mathbb{R}_+$ and $\theta(x) < (x-c)^2$ for $0 < x < z$. Hence,

$$\Theta(x) < \frac{x^2}{\mathscr{Q}(2)} - \frac{2cx}{r-\mu} + \frac{c^2}{r}$$

for all $x \in \mathbb{R}_+$. In particular, $g_2(x,z,c) < 0$ for $x \geq z$. Here $g_1(x,z,c)$ is the value of the adjustment made to the (zero) present value to account for the optionality to receive positive profit $(X_t - c)^2$ when future demand X_t exceeds a certain demand threshold z (perpetual American call option). The term $g_2(x,z,c)$ is the (downward) value adjustment needed to account for the fact that the firm will not receive profit $(X_t - c)^2 \geq 0$ for low demand $X_t < z$ but instead receive zero. The term $(x/z)^{\beta_1}$ in the expression for $g_1(x,z,c)$ corresponds to the present value of a bond that pays \$1 when demand X_t exceeds demand level z ($> x$) for the first time at (stopping) time $\tau_A(x,z) \equiv \inf\{t \geq 0 \mid X_t \geq z\}$. The second term in $\{\cdot\}$ corresponds to the positive forward value received when the firm earns positive profit for the first time at $\tau_B(x,z) \equiv \inf\{t \geq 0 \mid X_t \leq z\}$. $(x/z)^{\beta_2}$ is the present value of a bond that pays \$1 at time $\tau_B(x,z)$ with $z < x$, while the term in $\{\cdot\}$ is the forward value of losing the positive profit stream from time $\tau_B(x,z)$ onwards.

Similarly, we consider a simple contingent claim with payoff structure

$$\psi(x) = \begin{cases} 0 & \text{if } x < z, \\ x - c & \text{if } x \geq z, \end{cases}$$

with $z \geq c$. $x \mapsto x/[r-\mu] - c/r$ is a particular solution of

$$r\Psi(x) = \psi(x) + \mu x\ \Psi'(x) + \frac{1}{2}\sigma^2 x^2 \Psi''(x)$$

on $[z, \infty)$. The general solution of the second-order ODE is

$$\Psi(x) = \begin{cases} h_1(x,z,c) & \text{if } x < z, \\ \frac{x}{r-\mu} - \frac{c}{r} + h_2(x,z,c) & \text{if } x \geq z, \end{cases}$$

where $h_1(x,z,c) = x^{\beta_1} h_1(z,c)$ and $h_2(x,z,c) = x^{\beta_2} h_2(z,c)$, obtained by appropriate smoothness conditions, are given by

$$h_1(x,z,c) = \left(\frac{x}{z}\right)^{\beta_1} \left\{ \frac{1-\beta_2}{\beta_1-\beta_2}\frac{z}{r-\mu} + \frac{\beta_2}{\beta_1-\beta_2}\frac{c}{r} \right\}$$

$$h_2(x,z,c) = -\left(\frac{x}{z}\right)^{\beta_2} \left\{ \frac{\beta_1-1}{\beta_1-\beta_2}\frac{z}{r-\mu} - \frac{\beta_1}{\beta_1-\beta_2}\frac{c}{r} \right\}.$$

We know that $\psi(x) \geq 0$ for all $x \in \mathbb{R}_+$ and $\psi(x) > 0$ for $x > z$. Thus, $\Psi(x) > 0$ for all $x \in \mathbb{R}_+$. In particular, $h_1(x,z,c) > 0$ for $x < z$. Besides, $\psi(x) \geq x - c$ for all $x \in \mathbb{R}_+$ and $\psi(x) > x - c$ for $x < c$. Hence, $\Psi(x) > x/[r-\mu] - c/r$ for all $x \in \mathbb{R}_+$. In particular, $h_2(x,z,c) > 0$. Here, $h_1(x,z,c)$ corresponds to the value of a call option to earn $X_t - c$ when current demand X_t is sufficiently high, i.e., when $X_t \geq z$, while $h_2(x,z,c)$ is the value of the option to stop producing, earning zero profit, when demand is not sufficiently high, i.e., when $X_t < z$.

We now consider the nonhomogenous ODE (7.3). The general solution is

(7.9)

$$W_i(x;k) = \begin{cases} A_0 x^{\beta_1} + B_0 x^{\beta_2}, & \text{if } x \in (0, x_0), \\ \frac{1}{b(k-m+2)^2}\left[\frac{x^2}{\mathscr{Q}(2)} - \frac{2\Sigma_m x}{r-\mu} + \frac{\Sigma_m^2}{r}\right] + A_m x^{\beta_1} + B_m x^{\beta_2}, & \text{if } x \in [x_{m-1}, x_m), m = 1, \ldots, i, \\ \frac{\bar{q}_i}{k-m+2}\left[\frac{x}{r-\mu} - \frac{\Sigma_m}{r}\right] + A_m x^{\beta_1} + B_m x^{\beta_2}, & \text{if } x \in [x_{m-1}, x_m), m = i+1, \ldots, k+1, \end{cases}$$

where A_m and B_m, $m = 0, \ldots, k+1$ are constants to be determined by appropriate smoothness conditions. Note that $x \mapsto W_i(x;k)$ given in (7.9) is piecewise C^2 in $(-\infty, x_0)$ and $[x_{m-1}, x_m)$ for $m = 1, \ldots, k+1$. It remains to select values for A_m and B_m, $m = 0, \ldots, k+1$, such that $x \mapsto W_i(x;k)$ is C^1 at x_m, $m = 0, \ldots, k$.

To ensure that $W_i(0;k) = 0$ and avoid bubble solutions, we set $B_0 = A_{k+1} = 0$. The smoothness conditions at $x_0 = c$ read

$$\begin{aligned} (A_0 - A_1)\, x_0^{\beta_1} + (B_0 - B_1)\, x_0^{\beta_2} &= \frac{1}{b(k+1)^2}\left[\frac{x_0^2}{\mathscr{Q}(2)} - \frac{2cx_0}{r-\mu} + \frac{c^2}{r}\right] \\ \beta_1(A_0 - A_1)x_0^{\beta_1-1} + \beta_2(B_0 - B_1)x_0^{\beta_2-1} &= \frac{1}{b(k+1)^2}\left[\frac{2x_0}{\mathscr{Q}(2)} - \frac{2c}{r-\mu}\right]. \end{aligned}$$

It obtains

$$A_0 = A_1 + \frac{g_1(x_0, c)}{b(k+1)^2}\ (> A_1) \quad \text{and} \quad B_1 = B_0 + \frac{g_2(x_0, c)}{b(k+1)^2}\ (< B_0).$$

The smoothness conditions at x_m, $m = 1, \ldots, i-1$, are

$$\begin{aligned} (A_m - A_{m+1})x_m^{\beta_1} + (B_m - B_{m+1})x_m^{\beta_2} =& \frac{1}{b(k-m+1)^2}\left[\frac{x_m^2}{\mathscr{Q}(2)} - \frac{2\Sigma_{m+1}x_m}{r-\mu} + \frac{\Sigma_{m+1}^2}{r}\right] \\ &- \frac{1}{b(k-m+2)^2}\left[\frac{x_m^2}{\mathscr{Q}(2)} - \frac{2\Sigma_m x_m}{r-\mu} + \frac{\Sigma_m^2}{r}\right] \\ \beta_1(A_m - A_{m+1})x_m^{\beta_1-1} + \beta_2(B_m - B_{m+1})x_m^{\beta_2-1} =& \frac{1}{b(k-m+1)^2}\left[\frac{2x_m}{\mathscr{Q}(2)} - \frac{2\Sigma_{m+1}}{r-\mu}\right] - \frac{1}{b(k-m+2)^2}\left[\frac{2x_m}{\mathscr{Q}(2)} - \frac{2\Sigma_m}{r-\mu}\right] \end{aligned}$$

We thus have

$$A_m = A_{m+1} + \frac{g_1(x_m, \Sigma_{m+1})}{b(k-m+1)^2} - \frac{g_1(x_m, \Sigma_m)}{b(k-m+2)^2}, \quad \text{and} \quad B_{m+1} = B_m + \frac{g_2(x_m, \Sigma_{m+1})}{b(k-m+1)^2} - \frac{g_2(x_m, \Sigma_m)}{b(k-m+2)^2},\ m = 1, \ldots, i-1.$$

The smoothness conditions at x_i are

$$\begin{aligned} (A_i - A_{i+1})\, x_i^{\beta_1} + (B_i - B_{i+1})\, x_i^{\beta_2} &= \frac{\bar{q}_i}{b(k-i+1)}\left[\frac{x_i}{r-\mu} - \frac{\Sigma_{i+1}}{r}\right] - \frac{1}{b(k-i+2)^2}\left[\frac{x_i^2}{\mathscr{Q}(2)} - \frac{2\Sigma_i x_i}{r-\mu} + \frac{\Sigma_i^2}{r}\right] \\ \beta_1\,(A_i - A_{i+1})\, x_i^{\beta_1-1} + \beta_2\,(B_i - B_{i+1})\, x_i^{\beta_2-1} &= \frac{\bar{q}_i}{b(k-i+1)}\frac{1}{r-\mu} - \frac{1}{b(k-i+2)^2}\left[\frac{2x_i}{\mathscr{Q}(2)} - \frac{2\Sigma_i}{r-\mu}\right]. \end{aligned}$$

Similarly, we obtain

$$A_i = A_{i+1} + \bar{q}_i\frac{h_1(x_i, \Sigma_{i+1})}{b(k-i+1)} - \frac{g_1(x_i, \Sigma_i)}{b(k-i+2)^2} \quad \text{and} \quad B_{i+1} = B_i + \bar{q}_i\frac{h_2(x_i, \Sigma_{i+1})}{b(k-i+1)} - \frac{g_2(x_i, \Sigma_i)}{b(k-i+2)^2}.$$

Finally, the smoothness conditions at x_m, $m = i+1, \ldots, k$, read

$$\begin{aligned} (A_m - A_{m+1})\, x_m^{\beta_1} + (B_m - B_{m+1})\, x_m^{\beta_2} &= \frac{\bar{q}_i}{b(k-m+1)}\left[\frac{x_m}{r-\mu} - \frac{\Sigma_{m+1}}{r}\right] - \frac{\bar{q}_i}{b(k-m+2)}\left[\frac{x_m}{r-\mu} - \frac{\Sigma_m}{r}\right] \\ \beta_1\,(A_m - A_{m+1})\, x_m^{\beta_1-1} + \beta_2\,(B_m - B_{m+1})\, x_m^{\beta_2-1} &= \frac{\bar{q}_i}{b(k-m+1)}\frac{1}{r-\mu} - \frac{\bar{q}_i}{b(k-m+2)}\frac{1}{r-\mu}. \end{aligned}$$

It obtains

$$A_m = A_{m+1} + \frac{\bar{q}_i}{b}\left[\frac{h_1(x_m, \Sigma_{m+1})}{k-m+1} - \frac{h_1(x_m, \Sigma_m)}{k-m+2}\right]$$

$$B_{m+1} = B_m + \frac{\bar{q}_i}{b}\left[\frac{h_2(x_m, \Sigma_{m+1})}{k-m+1} - \frac{h_2(x_m, \Sigma_m)}{k-m+2}\right],\ m = i+1, \ldots, k.$$

Appendix D: Quantity, Profit and Firm Value in Symmetric Capacity MNE/MPE. The assumption of symmetric capacity greatly simplifies the analysis. Following the same solution approach as for Proposition 1, we prove that firm i produces Cournot quantity

$$q_i^C(x) = \begin{cases} 0 & \text{if } x \in (0, c), \\ \frac{x-c}{b(k+1)} & \text{if } x \in [c, \bar{x}_k), \\ \bar{q} & \text{if } x \in [\bar{x}_k, \infty), \end{cases}$$

in MNE, where $\bar{x} \equiv b(k+1)\bar{q} + c$. The Cournot profit expression readily obtains.

We now turn to the value expression. We are now looking for a solution of the second-order ODE

$$rW_i(x;k) = \pi_i^C(x;k) + \mu x\, W_x(x;k) + \frac{1}{2}\sigma^2 x^2\, W_{xx}(x;k), \quad W_i(0;k) = 0.$$

that is continuously differentiable and piecewise twice continuously differentiable. Obviously, the candidate solution $x \mapsto W_i(x,k)$ given in Proposition 4 is piecewise C^2.

A first smoothness condition is

$$(A_0' - A_1')x_0^{\beta_1} + (B_0' - B_1')x_0^{\beta_2} = \frac{1}{b(k+1)^2}\left[\frac{x_0^2}{\mathscr{Q}(2)} - \frac{2cx_0}{r-\mu} + \frac{c^2}{r}\right]$$

$$\beta_1(A_0' - A_1')x_0^{\beta_1 - 1} + \beta_2(B_0' - B_1')x_0^{\beta_2 - 1} = \frac{1}{b(k+1)^2}\left[\frac{2x_0}{\mathscr{Q}(2)} - \frac{2c}{r-\mu}\right].$$

We obtain

$$A_0' = A_1' + \frac{g_1(x_0, c)}{b(k+1)^2} \; (> A_1')$$

$$B_1' = B_0' + \frac{g_2(x_0, c)}{b(k+1)^2} \; (< B_0').$$

The smoothness condition at $\bar{x}_k$ is:

$$(A_1' - A_2')\bar{x}_k^{\beta_1} + (B_1' - B_2')\,\bar{x}_k^{\beta_2} = \frac{\bar{q}}{b(k+1)}\left[\frac{\bar{x}_k}{r-\mu} - \frac{c+bk\bar{q}}{r}\right] - \frac{1}{b(k+1)^2}\left[\frac{\bar{x}_k^2}{\mathscr{Q}(2)} - \frac{2c\bar{x}_k}{r-\mu} + \frac{c^2}{r}\right]$$

$$\beta_1(A_1' - A_2')\bar{x}^{\beta_1 - 1} + \beta_2(B_1' - B_2')\,\bar{x}_k^{\beta_2 - 1} = \frac{\bar{q}}{b(k+1)}\frac{1}{r-\mu} - \frac{1}{b(k+1)^2}\left[\frac{2\bar{x}_k}{\mathscr{Q}(2)} - \frac{2c}{r-\mu}\right].$$

It obtains

$$A_1' = A_2' + \frac{\bar{q}}{b(k+1)}h_1(\bar{x}_k, c+bk\bar{q}) - \frac{g_1(\bar{x}_k, c)}{b(k+1)^2}$$

$$B_2' = B_1' + \frac{\bar{q}}{b(k+1)}h_2(\bar{x}_k, c+bk\bar{q}) - \frac{g_2(\bar{x}_k, c)}{b(k+1)^2}.$$

As usual, $A_2' = B_0' = 0$. Hence,

$$A_0' = \frac{\bar{q}}{b(k+1)}h_1(\bar{x}_k, c+bk\bar{q}) + \frac{g_1(x_0, c) - g_1(\bar{x}_k, c)}{b(k+1)^2},$$

$$A_1' = \frac{\bar{q}}{b(k+1)}h_1(\bar{x}_k, c+bk\bar{q}) - \frac{g_1(\bar{x}_k, c)}{b(k+1)^2},$$

$$B_1' = \frac{g_2(x_0, c)}{b(k+1)^2},$$

$$B_2' = \frac{\bar{q}}{b(k+1)}h_2(\bar{x}_k, c+bk\bar{q}) + \frac{g_2(x_0, c) - g_2(\bar{x}_k, c)}{b(k+1)^2}.$$

Appendix E: Industry structure. Suppose first $x < X_1^k$. We thus have $V_n(x;k) < \cdots < V_1(x;k) < 0$. Therefore, $R_i(x;k) =$ "Stay out" for all $i = 1, \ldots, n$, with no Nash equilibrium involving k operating firms.

Suppose now $x \in \left[X_1^k, X_k^k\right)$. Here,

$$\begin{aligned} R_i(x;k-1) &= \begin{cases} \text{“Enter,”} & i = 1, \ldots, k-1 \\ \text{“Stay out,”} & i = k, \ldots, n, \end{cases} \\ R_i(x;k-2) &= \text{“Enter,”} \quad i = 1, \ldots, n. \end{aligned}$$

Thus, if $x \in \left[X_1^k, X_k^k\right)$, all Nash equilibria involve at least $k-1$ firms. Suppose a strategy profile with k invested firms, say without loss of generality firms $1, \ldots, k$. This strategy profile is not a Nash equilibrium since $R_k(x;k-1) =$ 'Stay out." Hence, if $x \in \left[X_1^k, X_k^k\right)$, all Nash equilibria involve exactly $k-1$ operating firms.

Therefore, if $x \in \left(-\infty, X_k^k\right)$, respectively $x \in \left(-\infty, X_{k+1}^{k+1}\right)$, no Nash equilibrium involves k, respectively $k+1$, operating firms. Therefore, if $x \in \left[X_k^k, X_{k+1}^{k+1}\right)$, all Nash equilibria involve exactly k investing firms.

Appendix F: Pure-strategy Markov Nash equilibria. If $x \in \left(-\infty, X_i^1\right)$, then i has a dominant strategy not to invest. All firms have a dominant strategy not to invest if $x \in \cap_{i=1}^n \left(-\infty, X_i^1\right)$. Given firms' ranking, $\cap_{i=1}^n \left(-\infty, X_i^1\right) = \left(-\infty, X_n^1\right)$. Hence, if $x \in \left(-\infty, X_n^1\right)$, there is a unique Nash equilibrium ("Stay out,"..., "Stay out"). Setting $X_n^0 = -\infty$, this obtains as a special case of (iii) with $\binom{n}{0}$. If $x \in (X_i^n, \infty)$, firm i has a dominant strategy to invest. All firms have a dominant strategy to invest if $x \in \cap_{i=1}^n (X_i^n, \infty)$. As $\cap_{i=1}^n (X_i^n, \infty) = \left(X_n^n, X_1^{n+1}\right)$ where we set $X_1^{n+1} = \infty$, this case obtains a special case of (iii) with $\binom{n}{n} = 1$ and a unique Nash equilibrium ("Enter,"...,"Enter"). If $x \in \left(X_n^k, X_1^{k+1}\right)$, then all n firms can profitably operate and invest; there are $\binom{n}{k}$ possible Nash equilibria, proving (iii).

Consider state region $\left[X_i^k, X_{i+1}^k\right]$. As $V_n(x;k) < \cdots < V_{i+1}(x;k) < 0 < V_i(x;i+1) < \cdots < V_1(x;k)$, we have

$$\begin{aligned} R_i(x;k-1) &= \begin{cases} \text{"Enter,"} & \text{for firms} 1, \ldots, k, \\ \text{"Stay out,"} & \text{for firms} k+1, \ldots, n, \end{cases} \\ R_i(x;k-2) &= \text{"Enter,"} \quad \text{for firms} 1, \ldots, n. \end{aligned}$$

If $k \leq i$ and $i+1 \leq n$, i.e., if $k \leq i \leq n-1$, then exactly k firms enter the market. Hence, exactly k firms out of the i ($\geq k$) possible entrants will enter the market, leading to $\binom{i}{k}$ distinct Nash equilibria, as stated in (ii).

If $i < k$ or $i \leq k-1$, then $\left[X_i^k, X_{i+1}^k\right] \subseteq \left[X_{k-1}^{k-1}, X_k^k\right]$. Then, all Nash equilibria involve exactly $k-1$ operating firms. Suppose a strategy profile where $k-1$ firms operate but firms $1, \ldots, i$ do not. This strategy profile is not a Nash equilibrium because, as $R_i(x; k-1) =$ "Enter," firms $1, \ldots, i$ are not reacting optimally. Hence, firms labeled $1, \ldots, i$ are invested in all Nash equilibria. Of the remaining $n-i$ firms, $k-1-i$ will also be profitably operating in Nash equilibrium, leading to $\binom{n-i}{k-1-i}$ Nash equilibria, which proves (i).

Appendix G: Pareto-dominating Nash equilibrium. From Proposition 6, k firms will operate in demand region (x_k, x_{k+1}) in MNE. Let $V(x;k) = (V_j(x;k))_{j=1}^{n}$ denote the column vector of firm values and $\mathbf{1}$ be a n-dimensional vector of 1. Vector $e(k)$ denotes an arbitrary MNE in demand region (x_k, x_{k+1}), with $e(k) \cdot \mathbf{1} = k$; is the set of MNE. $\hat{e}(k) \in \mathscr{E}_k$ is the MNE with $e_j = 1$ for $j = 1, \ldots, k$ and $e_j = 0$ for $j = k+1, \ldots, n$. As firms are strongly ranked in terms of increasing values, $V_1(x;k) > \cdots > V_i(x;k) \geq \cdots \geq V_n(x;k)$, there exists no MNE $e(k) \in \mathscr{E}_k \setminus \{\hat{e}(k)\}$ such that $e(k) \cdot V(x;k) > \hat{e}(k) \cdot V(x;k)$. This proves that the MNE with firms 1 to k investing Pareto-dominates all other MNE in $[x_k, x_{k+1})$. Similarly, in the Pareto-optimal MNE, firms 1 to $k+1$ operates in $[x_{k+1}, x_{k+1})$, and so on. In summary, firm k operates for $x \geq x_k$.

Appendix H: Shared Investment Option Value in Cournot Oligopoly. For conciseness, we write $z \equiv X_T$ in the appendix. We derive first the following lemma.

Lemma. *For $\beta \in \mathbb{R}$ and $y, y_1, y_2 \in \mathbb{R}_{++}$ with $y_1 \leq y_2$ and $a \leq b$, we have*

$$\mathbb{E}_x\left[e^{-rT} z^{\beta} \mid z \geq y\right] = x^{\beta} e^{-\mathscr{Q}(\beta)T} N(\mathrm{d}_{\beta}^{y}), \tag{7.10a}$$

$$\mathbb{E}_x\left[e^{-rT} z^{\beta} \mid y_1 \leq y \leq y_2\right] = x^{\beta} e^{-\mathscr{Q}(\beta)T} N(\mathrm{d}_{\beta}^{y_1}, \mathrm{d}_{\beta}^{y_2}) \tag{7.10b}$$

$$\mathbb{E}_x\left[e^{-rT} z^{\beta} \mid y \in (y_1, y_2) \cap (a, b)\right] = x^{\beta} e^{-\mathscr{Q}(\beta)T} N_a^b(\mathrm{d}_{\beta}^{y_1}, \mathrm{d}_{\beta}^{y_2}), \tag{7.10c}$$

where

$$N(\mathrm{d}_{\beta}^{y_1}, \mathrm{d}_{\beta}^{y_2}) \equiv N(\mathrm{d}_{\beta}^{y_1}) - N(\mathrm{d}_{\beta}^{y_2})$$

$$N_a^b(\mathrm{d}_{\beta}^{y_1}, \mathrm{d}_{\beta}^{y_2}) \equiv N\left(\max\left\{\min\left\{\mathrm{d}_{\beta}^{a}, \mathrm{d}_{\beta}^{y_1}\right\}, \mathrm{d}_{\beta}^{b}\right\}, \min\left\{\max\left\{\mathrm{d}_{\beta}^{b}, \mathrm{d}_{\beta}^{y_2}\right\}, \mathrm{d}_{\beta}^{a}\right\}\right).$$

Proof. We know that $\ln z$ is normally distributed with mean $\gamma \equiv \ln x + (r - \delta - \sigma^2/2)T$ and standard deviation $s \equiv \sigma\sqrt{T}$. Let ε denote the realization of a standard normal random variable with density $\left(1/\sqrt{2\pi}\right)\exp\left\{-\varepsilon^2/2\right\}$. It obtains $\mathrm{d}_0^y = (\gamma - \ln y)/s$. Since $x \mapsto \exp(x)$ is monotone increasing and invertible, $\exp\{\gamma + s\varepsilon\} \geq y$ is equivalent to $\varepsilon \geq -\mathrm{d}_0^y$. Define $\mathrm{d}_{\beta}^y \equiv \mathrm{d}_0^y + \beta s$ and $\beta \in \mathbb{R}_+$. We have

$$\begin{aligned}
\mathbb{E}_x\left[z^{\beta} \mid z \geq y\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\mathrm{d}_0^y}^{\infty} \exp\{\beta\gamma + \beta s\varepsilon\} \exp\left\{-\varepsilon^2/2\right\} \mathrm{d}\varepsilon \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{\beta\gamma + \beta^2 s^2/2\right\} \int_{-\mathrm{d}_0^y}^{\infty} \exp\left\{-\left(\varepsilon^2 - 2\beta s\varepsilon + \beta^2 s^2\right)/2\right\} \mathrm{d}\varepsilon.
\end{aligned}$$

Set $\epsilon = \varepsilon - \beta s$ and note that $\varepsilon \geq -\mathrm{d}_0^y$ implies $\epsilon \geq -\mathrm{d}_{\beta}^y$. Therefore,

$$\begin{aligned}
\mathbb{E}_x\left[z^{\beta} \mid z \geq y\right] &= \exp\left\{\beta\gamma + \beta^2 s^2/2\right\} \int_{-\mathrm{d}_{\beta}^y}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\epsilon^2/2\right\} \mathrm{d}\epsilon \\
&= \exp\left\{\beta\gamma + \beta^2 s^2/2\right\} N\left(\mathrm{d}_{\beta}^y\right) \qquad (7.11)
\end{aligned}$$

by symmetry of the standard normal distribution (of ϵ) at zero. Substituting $\gamma \equiv \ln x + (r - \delta - \sigma^2/2)T$ and $s \equiv \sigma\sqrt{T}$ in (7.11) yields (7.10a).

Besides, we have $\mathbb{E}_x\left[z^\beta \mid z \geq y\right] + \mathbb{E}_x\left[z^\beta \mid z \leq y\right] = \mathbb{E}_x\left[z^\beta\right]$. Since $\mathbb{E}_x\left[e^{-rT} z^\beta\right] = x^\beta e^{-\mathscr{Q}(\beta)T}$, it obtains from (7.10a)

$$\mathbb{E}_x\left[e^{-rT} z^\beta \mid z \leq y\right] = x^\beta e^{-\mathscr{Q}(\beta)T}\left[1 - N(d_\beta^y)\right].$$

We thus have

$$\begin{aligned}\mathbb{E}_x\left[e^{-rT} z^\beta \mid y_1 \leq z \leq y_2\right] &= \mathbb{E}_x\left[e^{-rT} z^\beta \mid z \leq y_2\right] - \mathbb{E}_x\left[e^{-rT} z^\beta \mid z \leq y_1\right] \\ &= x^\beta e^{-\mathscr{Q}(\beta)T}\left[N(\mathrm{d}_\beta^{y_1}) - N(\mathrm{d}_\beta^{y_2})\right],\end{aligned}$$

proving (7.10b)

Suppose now $y_1 \leq y_2$ and $a \leq b$ and define $A = (y_1, y_2) \cap (a, b)$. We consider five cases:

(1) $y_1 \leq y_2 \leq a \leq b$. Since $x \mapsto 1/x$ and $x \mapsto \ln(x)$ are respectively monotone decreasing and monotone increasing, we have $\mathrm{d}_\beta^{y_1} \geq \mathrm{d}_\beta^{y_2} \geq \mathrm{d}_0^a \geq \mathrm{d}_0^b$. On the one hand, we have $\mathbb{P}\left[z \in A\right] = 0$. On the other hand, we have

$$\begin{aligned}\max\left\{\min\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^{y_1}\right\}, \mathrm{d}_\beta^b\right\} &= \max\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^b\right\} = \mathrm{d}_\beta^a, \\ \min\left\{\max\left\{\mathrm{d}_\beta^b, \mathrm{d}_\beta^{y_2}\right\}, \mathrm{d}_\beta^a\right\} &= \min\left\{\mathrm{d}_\beta^{y_2}, \mathrm{d}_\beta^a\right\} = \mathrm{d}_\beta^a,\end{aligned}$$

or $N_a^b(\mathrm{d}_\beta^{y_1}, \mathrm{d}_\beta^{y_2}) = 0$.

(2) $y_1 \leq a \leq y_2 \leq b$. Here, $\mathrm{d}_\beta^{y_1} \geq \mathrm{d}_\beta^a \geq \mathrm{d}_\beta^{y_2} \geq \mathrm{d}_\beta^b$. We have $\mathbb{P}_x\left[z \in A\right] = \mathbb{P}_x\left[a \leq z \leq y_2\right] = N(\mathrm{d}_0^a, \mathrm{d}_0^{y_2})$ and

$$\begin{aligned}\max\left\{\min\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^{y_1}\right\}, \mathrm{d}_\beta^b\right\} &= \max\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^b\right\} = \mathrm{d}_\beta^a, \\ \min\left\{\max\left\{\mathrm{d}_\beta^b, \mathrm{d}_\beta^{y_2}\right\}, \mathrm{d}_\beta^a\right\} &= \min\left\{\mathrm{d}_\beta^{y_2}, \mathrm{d}_\beta^a\right\} = \mathrm{d}_\beta^{y_2}.\end{aligned}$$

Thus, $N_a^b(\mathrm{d}_\beta^{y_1}, \mathrm{d}_\beta^{y_2}) = N(\mathrm{d}_\beta^a, \mathrm{d}_\beta^{y_2})$.

(3) $y_1 \leq a \leq b \leq y_2$. Here, $\mathrm{d}_\beta^{y_1} \geq \mathrm{d}_\beta^a \geq \mathrm{d}_\beta^b \geq \mathrm{d}_\beta^{y_2}$, $\mathbb{P}_x\left[z \in A\right] = \mathbb{P}_x\left[a \leq z \leq b\right] = N\big(\mathrm{d}_0^a, \mathrm{d}_0^b\big)$, and

$$\begin{aligned}\max\left\{\min\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^{y_1}\right\}, \mathrm{d}_\beta^b\right\} &= \max\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^b\right\} = \mathrm{d}_\beta^a, \\ \min\left\{\max\left\{\mathrm{d}_\beta^b, \mathrm{d}_\beta^{y_2}\right\}, \mathrm{d}_\beta^a\right\} &= \min\left\{\mathrm{d}_\beta^b, \mathrm{d}_\beta^a\right\} = \mathrm{d}_\beta^b.\end{aligned}$$

(4) $a \leq b \leq y_1 \leq y_2$. Here, $\mathrm{d}_\beta^a \geq \mathrm{d}_\beta^b \geq \mathrm{d}_\beta^{y_1} \geq \mathrm{d}_\beta^{y_2}$, $\mathbb{P}_x\left[z \in A\right] = 0$, and

$$\begin{aligned}\max\left\{\min\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^{y_1}\right\}, \mathrm{d}_\beta^b\right\} &= \max\left\{\mathrm{d}_\beta^{y_1}, \mathrm{d}_\beta^b\right\} = \mathrm{d}_\beta^b, \\ \min\left\{\max\left\{\mathrm{d}_\beta^b, \mathrm{d}_\beta^{y_2}\right\}, \mathrm{d}_\beta^a\right\} &= \min\left\{\mathrm{d}_\beta^b, \mathrm{d}_\beta^a\right\} = \mathrm{d}_\beta^b.\end{aligned}$$

(5) $a \leq y_1 \leq y_2 \leq b$. Here, $\mathrm{d}_\beta^a \geq \mathrm{d}_\beta^{y_1} \geq \mathrm{d}_\beta^{y_2} \geq \mathrm{d}_\beta^b$, $\mathbb{P}_x\left[z \in A\right] = \mathbb{P}_x\left[y_1 \leq z \leq y_2\right] = N(\mathrm{d}_0^{y_1}, \mathrm{d}_0^{y_2})$, and

$$\begin{aligned}\max\left\{\min\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^{y_1}\right\}, \mathrm{d}_\beta^b\right\} &= \max\left\{\mathrm{d}_\beta^{y_1}, \mathrm{d}_\beta^b\right\} = \mathrm{d}_\beta^{y_1}, \\ \min\left\{\max\left\{\mathrm{d}_\beta^b, \mathrm{d}_\beta^{y_2}\right\}, \mathrm{d}_\beta^a\right\} &= \min\left\{\mathrm{d}_\beta^{y_2}, \mathrm{d}_\beta^a\right\} = \mathrm{d}_\beta^{y_2}.\end{aligned}$$

This completes the proof of (7.10c). Finally, we note from (7.10c) that

$$N_0^b\left(\mathrm{d}_\beta^{y_1}, \mathrm{d}_\beta^{y_2}\right) = N\left(\max\left\{\mathrm{d}_\beta^{y_1}, \mathrm{d}_\beta^b\right\}, \max\left\{\mathrm{d}_\beta^b, \mathrm{d}_\beta^{y_2}\right\}\right),$$
$$N_a^\infty\left(\mathrm{d}_\beta^{y_1}, \mathrm{d}_\beta^{y_2}\right) = N\left(\min\left\{\mathrm{d}_\beta^a, \mathrm{d}_\beta^{y_1}\right\}, \min\left\{\mathrm{d}_\beta^{y_2}, \mathrm{d}_\beta^a\right\}\right).$$

This proves the lemma. The remaining of the proof should be obvious given Proposition 5. □