The Odd Notion of "Reversible Investment"

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Abstract: Irreversible investment, with its notion of option value, has been well discussed. Complete reversibility has been less studied. We examine a simple lumpy stopping problem for the full range, from completely irreversible to completely reversible investment, but with a focus on the latter. The optimal stopping rule under complete reversibility is to invest when the project generates enough net cash flow to cover Jorgenson's opportunity cost of investment, and to disinvest when it does not. Given the static nature of this rule, net present value as a timing rule under reversibility is not pertinent, despite suggestions to the contrary. We find that investments that are even slightly irreversible have much in common with completely irreversible investments but nothing in common with completely reversible investments. The case of reversible investment provides a foil for understanding that the distinguishing feature of investment compared with other inputs is that it entails some irreversibility.

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1. Introduction

Real options theory has had much to say about the timing of irreversible investment under uncertainty. To understand irreversibility fully, one must also understand reversibility. In this note we investigate the case of a deferrable, fully reversible, lumpy investment. Lumpiness is a frequent assumption in engineering economic analysis of investment and appears to be an empirical regularity (Doms and Dunne 1998). *Reversible* lumpy investment is the mainstay of the microeconomic theory of the firm in the short run: the flow of instantaneous profit is $pq - wl - ck$, where ck is a quasi-fixed opportunity cost associated with the reversible use of a lump of capital of size K costing $k = PK$. Reversible lumpy investment is also the only situation where the orthodox net present value (NPV) rule has been suggested to appropriately adjudicate deferrable capital timing decisions under uncertainty (Dixit and Pindyck 1994).

Our point is brought out using a modification of the Brennan and Schwartz (1985) model of deferrable, irreversible investment. The modification produces the most transparent model of investment reversibility that we can devise: the degree of reversibility or irreversibility is summarized in a single parameter. Timing decisions are relatively easily analyzed. Valuation is also transparent. Statics, certainty, variability, and reversibility are intimately related in the analysis.

We demonstrate that the orthodox NPV rule is not applicable to the special case of reversible lumpy investment. Instead, a simple static comparison of current net revenues and the current opportunity cost of capital is appropriate. Under reversibility, uncertainty does not affect timing. Indeed, reversibility removes all of the usual attributes associated with capital. Irreversibility is thus a central feature of what economists mean when distinguishing capital inputs from current inputs.

2. The literature

A brief review of the relevant literature is warranted so as to place our paper within this space. Analyses tend to assume either continuously divisible capital or discrete, lumpy capital. In the foundation of the modern day microeconomic treatment of a firm's decisions, Marshall treated capital as a continuous, reversible input to production. The reversibility in his analysis, though not made explicit, is inherent in the attachment of a user cost to capital. Jorgenson's (1963) formalized treatment of capital as a continuous, reversible input proposed what is known as Jorgenson's user cost of capital, $c = (r + \delta)P$, where δ is the rate of depreciation of capital and P is the real unit price of capital. Irreversible and partially reversible continuous investment under uncertainty have been studied by Abel and Eberly (1996), Hartman and Hendrickson (2002), Dixit and Pindyck (2000), Guo and Pham (2005), and Merhi and Zervos (2007). As with Jorgenson, these analyses yield rules for intensity of investment that equate present value benefits and costs at the margin. Completely reversible continuous investment under uncertainty has been treated as a limiting case by Bertola and Caballero (1994), Abel and Eberly (1996), Hartman and Hendrickson (2002), and Guo and Pham (2005). Here the rule for intensity of investment degenerates to Jorgenson's rule equating the instantaneous user cost of capital with its now stochastic instantaneous marginal revenue product.

The main distinction for exogenously lumpy, irreversible actions is the focus on investment timing and hysteresis, rather than investment intensity.[1] Bernanke (1983) provides a summary of

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[1] The focus on timing, rather than intensity, is because in the applications where these analyses were first applied (mining and energy extraction problems) the discreteness of the lumps was relatively coarse, making the intensity decision easier than the timing decision. Exceptions are Bar-Ilan and Strange (1999), who allows the investment of a

the early work, and in his own analysis of irreversible lumpy investment points out that the distinctions between (a) reversibility and irreversibility and (b) lumpiness and divisibility are important in an uncertain environment. Dixit (1989) emphasizes irreversible investments that immediately rust upon disuse. Brennan and Schwartz (1985) and Dixit and Pindyck (1994) consider irreversible and partially reversible lumpy investment that can be mothballed. Unlike continuous investment, comparisons of marginal investment conditions are not made in these analyses. Rather, the emphasis is that investment timing is different from that suggested by the orthodox NPV rule or even optimal timing under certainty due to what Davis and Cairns (2012) call *quasi-option flow*, the information gains from waiting.

Brennan and Schwartz (1985), Dixit (1989) and Dixit and Pindyck (1994) all provide the modeling framework to examine the limiting case of completely reversible lumpy investments, but do not extend their analysis to this special case.[2] To our knowledge the treatment of completely reversible lumpy investments has been limited to a short paper by Shackleton and Wojakowski (2001). In what follows we examine the timing of completely reversible, lumpy capital using an adaptation of Brennan and Schwartz's model. Once we derive our result we will compare it with Shackleton and Wojakowski's model.

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single lump that is continuously divisible, and Guthrie (2012), who allows for repeated continuously divisible lumpy investments.

[2] For the purpose of comparison to the original works we note that, in their notation, Brennan and Schwartz would set $k_1 = -k_2$, Dixit would set $l = -k$, and Dixit and Pindyck (Chapter 7) would set $E = -I$.

3. The model

The simplest version of the Brennan and Schwartz model, in Section II of their paper, studies the production of a homogeneous good with repeated options to invest and disinvest. The essence of their model is as follows:

1. A firm can produce a good or service at exogenous rate $q > 0$ or it can remain idle. In order to move from producing nothing to producing at q an exogenous real cash cost $k = PK$ is incurred, where P is a constant real unit cost indicative of set-up costs or investment in capital or capacity of size K. The effects of the cash cost persist, in the sense that there is no depreciation of the effect over time.
2. There is no constraint on how long the firm can remain idle before investing.
3. The real unit price of the good, $p > 0$, follows a risk-neutral geometric Brownian motion $dp = \eta p dt + \sigma p dz$. The real risk-neutral rate of drift, η , is less than the real risk-free rate r such that there is a rate-of-return shortfall $\kappa = r - \eta > 0$. Equivalently, $\kappa = \rho - \mu > 0$, where ρ is the real risk-adjusted discount rate appropriate for this asset and μ is the asset's real rate of drift.[3]
4. Should the firm elect to switch from positive production to being idle it incurs an instantaneous cost (e.g., dismantling costs, reclamation costs). Idle capital cannot be held in stock by the firm.

[3] Brennan and Schwartz (1985) use the concept of a storable commodity that produces an observable net rate of convenience, κ. McDonald and Siegel (1984) introduce the less restrictive notion of rate-of-return shortfall, which can apply to non-storable commodities.

5. The firm maximizes the expected present value of production by optimally switching production on and off. It may switch between producing and being closed as often as it likes, incurring the relevant cash flow in item 4 each time it changes policy.

We emphasize reversible investment timing decisions by making the following modifications to the model:

1. There are no operating costs. Capital is the only factor input.[4] It can decay at rate $\delta > 0$ while in use, though such decay is matched by continuous real reinvestment of δk to maintain the capital stock at level K and keep production at level q.[5] Jorgenson (1963) calls this replacement investment.
2. The rate of inflation of the nominal unit price of capital, $\pi = \frac{\dot{I}}{I}$, is the index by which the nominal interest rate and spot price are deflated.
3. The instantaneous real cash flows for changes of policy are $-k < 0$ for opening the project and $\phi k \geq 0$ for closing the project, where $\phi \in [0,1]$ is a time invariant constant reflecting the degree of reversibility.[6] Full irreversibility is induced via $\phi = 0$, and full

[4] This is also the approach taken in Dixit and Pindyck's (2000) analysis of the reversibility of divisible capital. Abstracting from operating cost allows for a stronger focus on the properties of capital.

[5] We abstract from the possibility that the desired level of capital stock could be less than K and that the firm could get there by allowing capital to depreciate. Instead, as in Jorgenson (1963), we assume that replacement investment must occur and that capital sales or investments are the only way to adjust the capital stock.

[6] The constancy of ϕ and k simplifies the analysis greatly. Dixit and Pindyck (2000) consider the case in which the wedge between disinvesting and investing is growing over time, though they model continuous investment and

reversibility via $\phi = 1$. Partial reversibility obtains for $0 < \phi < 1$. When $\phi = 1$, reversing the investment resets the program with no diminishment of the number of investment or reversal options at hand.[7]

4. There are no holding costs while the project is in an idle state. This obviates the need for the separate option to permanently scrap the project.

These modifications produce a very simple lumpy reversible investment problem.

4. The optimal timing solution

The solution to this problem yields two unit price triggers, p_{open}, at which it is optimal to open a closed project, and p_{close}, at which it is optimal to close an open project, $0 \leq p_{\text{close}} \leq p_{\text{open}}$. There are also two value functions, one for an open project and one for a closed project. The solution technique is well known and is presented in Chapter 7 of Dixit and Pindyck (1994). The ordinary differential equations for project value, combined with boundary conditions, produce the two analytic value functions for the project. The project value when closed is

disinvestment. The constant wedge that we use can represent a lemons problem with capital, endogenous capital market prices, or fixed set-up and dismantling costs.

[7] Discussions of the reversibility of lumpy investment are sometimes confined to considerations of the absolute amount of investment recovered by disinvesting. Wong (2010) allows for complete recovery of the investment amount at the end of the planning period, and Abel et al. (1996) allow for recovery and reinvestment over two periods. Arguably, full reversibility requires an infinite set of options, such that the action to invest does not reduce the remaining opportunity set. In that sense, a finite number of options to fully reverse an investment is partial reversibility. In Dixit and Pindyck's (2000) vernacular our case is complete reversibility, complete expandability, and no initial lemons effect.

$$w(p) = \beta_1 p^{\gamma_1} > 0, \quad \beta_1 > 0 \tag{1}$$

where

$$\gamma_1 = \alpha_1 + \alpha_2 > 1,\ \alpha_1 = \tfrac{1}{2} - \frac{r-\kappa}{\sigma^2},\ \alpha_2 = \left[\alpha_1{}^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}},$$

and β_1 is a constant to be determined by the optimality conditions. The project value when open is

$$v(p) = \beta_2 p^{\gamma_2} + \frac{qp}{\kappa} - \frac{\delta k}{r},\ \ \beta_2 > 0,\ \gamma_2 = \alpha_1 - \alpha_2 < 0\,, \tag{2}$$

where β_2 is another constant to be determined by the optimality conditions.[8] The expected value of the open project at current unit price *p* in the absence of the option to close is the present value of a perpetual income stream, $\frac{qp}{\kappa} - \frac{\delta k}{r}$.[9] Given this, the term $\beta_2 p^{\gamma_2}$ in (2) is the value of reversibility of the investment decision. It reflects an infinitely repeated compound put option, since the put has embedded within it a call to reopen, which in turn contains a put to reclose, and so on.

An optimal solution for p_{open} and p_{close} includes the value-matching conditions

$$w(p_{\text{open}}) = \beta_1 p_{\text{open}}^{\gamma_1} = v(p_{\text{open}}) - k = \beta_2 p_{\text{open}}^{\gamma_2} + \frac{qp_{\text{open}}}{\kappa} - \frac{\delta k}{r} - k, \tag{3}$$

$$v(p_{\text{close}}) = \beta_2 p_{\text{close}}^{\gamma_2} + \frac{qp_{\text{close}}}{\kappa} - \frac{\delta k}{r} = w(p_{\text{close}}) + \phi k = \beta_1 p_{\text{close}}^{\gamma_1} + \phi k\,. \tag{4}$$

It also includes the so-called smooth-pasting conditions derived from (3) and (4) using the derivatives of (1) and (2):

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[8] Dixit and Pindyck set $q = 1$ and $\delta = 0$ in their analysis of the problem.

[9] The replacement capital is discounted at the risk-free discount rate because it is a certain cash flow.

$$v'(p_{\text{open}}) = \gamma_2\beta_2 p_{\text{open}}{}^{\gamma_2-1} + \frac{q}{\kappa} = w'(p_{\text{open}}) = \gamma_1\beta_1 p_{\text{open}}{}^{\gamma_1-1}, \tag{5}$$

$$w'(p_{\text{close}}) = \gamma_1\beta_1 p_{\text{close}}{}^{\gamma_1-1} = v'(p_{\text{close}}) = \gamma_2\beta_2 p_{\text{close}}{}^{\gamma_2-1} + \frac{q}{\kappa}. \tag{6}$$

Rearranging these four equations,

$$-\beta_1 p_{\text{open}}{}^{\gamma_1} + \beta_2 p_{\text{open}}{}^{\gamma_2} + \frac{qp_{\text{open}}}{\kappa} - \frac{\delta k}{r} = k, \tag{7}$$

$$-\gamma_1\beta_1 p_{\text{open}}{}^{\gamma_1-1} + \gamma_2\beta_2 p_{\text{open}}{}^{\gamma_2-1} + \frac{q}{\kappa} = 0, \tag{8}$$

$$-\beta_1 p_{\text{close}}{}^{\gamma_1} + \beta_2 p_{\text{close}}{}^{\gamma_2} + \frac{qp_{\text{close}}}{\kappa} - \frac{\delta k}{r} = \phi k\ , \tag{9}$$

$$\gamma_1\beta_1 p_{\text{close}}{}^{\gamma_1-1} - \gamma_2\beta_2 p_{\text{close}}{}^{\gamma_2-1} - \frac{q}{\kappa} = 0. \tag{10}$$

These four equations can be solved iteratively for the four unknowns β_1, β_2, p_{open}, and p_{close}. The model can be solved for any amount of reversibility $0 \leq \phi \leq 1$. Dixit (1989) shows in an unpublished Appendix that a solution for this functional form exists and is unique. Analytic solutions for p_{open}, and p_{close} are possible only when $\phi = 1$. For $\phi = 1$ the unique solution to (7) through (10) has $p_{\text{open}} = p_{\text{close}}$ by inspection. When $0 \leq \phi < 1$

$$p_{\text{open}} = \frac{\gamma_2\left[\left(\frac{\delta k}{r} + \phi k\right) - \left(\frac{\delta k}{r} + k\right)x^{\gamma_1}\right]}{\frac{q}{\kappa}(x - x^{\gamma_1})(\gamma_2 - 1)}, \tag{11}$$

where $x(\phi) = \frac{p_{close}}{p_{open}} < 1$ is the solution to the nonlinear equation

$$\frac{(x - x^{\gamma_2})(\gamma_1 - 1)}{\gamma_1\left[\left(\frac{\delta k}{r} + \phi k\right) - \left(\frac{\delta k}{r} + k\right)x^{\gamma_2}\right]} = \frac{(x - x^{\gamma_1})(\gamma_2 - 1)}{\gamma_2\left[\left(\frac{\delta k}{r} + \phi k\right) - \left(\frac{\delta k}{r} + k\right)x^{\gamma_1}\right]} \tag{12}$$

and inversely reflects the degree of investment hysteresis. Because the factor k could be eliminated from (12) $x(\phi)$ is invariant to scale for our model. We leave k in the denominator for comparability to the nonlinear equation presented by Brennan and Schwartz, which is scale dependent because of operating costs (1985, p. 146). The solutions for β_1 and β_2 are,[10]

$$\beta_1 = \frac{\frac{q}{\kappa} p_{\text{open}}(\gamma_2 - 1) - \gamma_2\left(\frac{\delta k}{r} + k\right)}{(\gamma_2 - \gamma_1)\, p_{\text{open}}^{\gamma_1}}, \tag{13}$$

and

$$\beta_2 = \frac{\frac{q}{\kappa} p_{\text{close}}(\gamma_1 - 1) - \gamma_1\left(\frac{\delta k}{r} + \phi k\right)}{(\gamma_2 - \gamma_1)\, p_{\text{close}}^{\gamma_2}}. \tag{14}$$

In the rest of the paper we use these results to contrast the case of completely reversible investment with partially reversible or completely irreversible investment.

5. Stopping under complete irreversibility

Though we are concerned with complete reversibility we first present an analytic solution for a special case of complete irreversibility to illustrate in a simple way the usual rejection of the orthodox NPV rule for timing investment under uncertainty. When the initial investment is irreversible, $\phi = 0$. For the case of $\delta = 0$ this problem becomes analytically solvable (Dixit and

[10] There is a typographical error in the Brennan and Schwartz paper; the correct general expression for their β_4, which we have labeled β_2, is $\frac{d\hat{s}_1(\gamma_1 - 1) + e\gamma_1}{(\gamma_2 - \gamma_1)\hat{s}_1^{\gamma_2}}$ (in their notation).

Pindyck 1994, pp. 182-184). The optimal rule is to never close the project (p_{close} = 0) and to open the project when[11]

$$p_{\text{open}} = \frac{\kappa k}{q} \frac{\gamma_1}{(\gamma_1 - 1)} . \tag{15}$$

The orthodox NPV calculates project value conditional on current price *p* and immediate irreversible investment,

$$V(p) = \frac{qp}{\rho - \mu} - k = \frac{qp}{\kappa} - k. \tag{16}$$

Our interpretation of the *orthodox NPV rule* is to invest when

$$\frac{qp}{\kappa} - k \geq 0 ; \tag{17}$$

therefore, under this rule

$$p_{\text{open}} = \frac{\kappa k}{q} . \tag{18}$$

Since γ_1 > 1 (cf. equation 15), the orthodox NPV rule causes premature initiation of the irreversible project, a well-publicized result.

[11] Under the assumption of no operating costs, no firm would disinvest from a positive, perpetual income stream if there was no compensation for doing so. This is proven formally by Merhi and Zervos (2007) for the continuous investment case. See their Example 1, with α = 1 and β = 0, along with n > 1: Corollary 10 (a) holds, and disinvestment never occurs. Dixit and Pindyck solve the problem for q = 1, and so q does not show up explicitly in their solution.

6. Stopping under complete reversibility

Investment is completely reversible when there is no intertemporal price difference between the initial investment cost, the net proceeds from dismantling that investment, and the cost of reinstalling it. This condition disallows switching costs, for example, and endogeneity of price in capital markets. The case of complete reversibility is clearly a fiction, the limit of the more reasonable case of partially irreversible investment where there is a wedge between disinvestment net proceeds and reinvestment cost.

When investment is completely reversible $\phi = 1$ and $x(1) = 1$. From (11) and given $x'(\phi) > 0$ (Dixit 1989, Appendix A), the solution for p_{open} becomes analytically solvable. Using l'Hôpital's rule in step 3 and the simplifications that $\gamma_1\gamma_2 = -\frac{2r}{\sigma^2}$ and $\gamma_1 + \gamma_2 = -\frac{r - \kappa - \frac{1}{2}\sigma^2}{\frac{1}{2}\sigma^2}$ in step 4, we find that

$$\begin{aligned}\lim_{\phi\to 1^-} p_{\text{open}} &= \lim_{\phi\to 1^-} \frac{\gamma_2\left[\left(\frac{\delta k}{r} + \phi k\right) - \left(\frac{\delta k}{r} + k\right)x(\phi)^{\gamma_1}\right]}{\frac{q}{\kappa}\left(x(\phi) - x(\phi)^{\gamma_1}\right)(\gamma_2 - 1)} = \lim_{x\to 1^-} \frac{\gamma_2\left[\left(\frac{\delta k}{r} + \phi k\right) - \left(\frac{\delta k}{r} + k\right)x(\phi)^{\gamma_1}\right]}{\frac{q}{\kappa}\left(x(\phi) - x(\phi)^{\gamma_1}\right)(\gamma_2 - 1)} \\ &= \frac{\kappa\left(\frac{\delta}{r} + 1\right)\gamma_1\gamma_2}{(\gamma_1 - 1)(\gamma_2 - 1)}\frac{k}{q} = \frac{(r+\delta)k}{q}.\end{aligned} \tag{19}$$

The economic results for complete reversibility are revealing. From (19) the rules for opening and closing collapse to

$$p_{\text{open}} = \frac{(r+\delta)k}{q} = p_{\text{close}}\,. \tag{20}$$

The only variables of relevance to the timing of the application of capital to production are the current spot price, the rate of depreciation, the real riskless interest rate and the capital-output ratio. The rule is independent of the rate of drift of price and of risk preferences.[12] Most notably, there is no relationship between investment and uncertainty, even under our assumption of risk aversion. Only irreversible capital is placed at risk upon investment. Nor does the rule require dynamic analysis: multiplication by q in equation (20) shows that the rule is tantamount to a direct comparison of current revenues pq against Jorgenson's opportunity cost of capital $(r + \delta)k$, or, if the interest rate is expressed in nominal terms, $\left(r_n + \delta - \frac{\dot{I}}{I} \right) k$.

When viewed as a comparison of current opportunity costs versus benefits, the intuition of (20) suggests that it is a general result that applies to other stochastic processes for price, to time varying interest rates, and to projects with instantaneous operating costs wl. Reversibility can thus be viewed as rendering the stopping problem generalizable and trivial. Decisions are taken point by point, as in the textbook microeconomic representation of the firm employing lumpy rental capital. In managing the project there is no need for dynamic planning, no need for present value analyses, and no need for a NPV timing rule.

In a related paper Shackleton and Wojakowski (2001) examine reversible real options through the analogy of a reversible American put-call collar on an underlying financial asset whose price S follows a geometric Brownian motion. They show that when the exercise prices are the same for each option, which is full reversibility, the exercise triggers for each option are the same. This common exercise trigger is

$$S_{\text{open}} = rX / \gamma = S_{\text{close}}, \tag{21}$$

[12] These factors continue to influence the function valuing the stream of cash flows generated by this timing rule: equations (1) and (13) for a closed operation or equations (2) and (14) for an open operation.

where X is the options' common exercise price and γ is the dividend yield on the stock. When mapped to a real option and compared with (20) the rule appears to require substantially more information: one is assumed to be able to track the realizable value of the underlying project, S, an asset that is not traded and for which valuation would likely require cash flow projections and risk adjusted discounting, as well as the asset's rate-of-return shortfall γ, another unobservable datum. Nor is the underlying leveraged project likely to follow a geometric Brownian motion given that we often take the output price to follow this process: our model, which makes this assumption, produces underlying project value $\frac{qp}{\kappa} - \frac{\delta k}{r}$ (see equation (2)), which is not a geometric Brownian motion when $\delta > 0$. Shackleton and Wojakowski interpret γS as the flow of benefits from holding the stock, and when this is equated to the cash flows from the realized project in our model, $qp - \delta k$, and substituting $X = k$, we obtain stopping rule (20). Hence, (20) is consistent with their result, though derived without the need to assume that the real option mirrors a financial put-call collar.

7. The implications of reversibility for the orthodox NPV rule of investment

The orthodox NPV rule of investment is presented by Dixit and Pindyck (1994, 4) as follows: "First, calculate the present value of the expected stream of profits that [the lumpy investment] will generate. Second, calculate the present value of the stream of expenditures required to [make the investment]. Finally, determine whether the difference between the two—the net present value (NPV) of the investment—is greater than zero. If it is, go ahead and invest."[13] This is the timing rule presented in (18) above.

The orthodox rule is the mainstay of analysis in engineering economy. The literature on real options, however, emphasizes the suboptimality of using the orthodox rule as a basis for decisions on investment because "…it assumes that either the investment is reversible, that is, it can somehow be undone and the expenditures recovered should market conditions turn out worse than anticipated, or, if the investment is irreversible, it is a now or never proposition…" (Dixit and Pindyck 1994, 6).

That the orthodox rule applies to now-or-never decision making is well accepted. In a now-or-never analysis the NPV rule holds for a sunk (irreversible) input whose opportunity cost is the full value *k* which, because of irreversibility, must be compared to the present value of benefits of investment.[14]

Although authors from a wide array of disciplines have repeated Dixit and Pindyck's claim that it is also applicable to deferrable, reversible investments,[15] we have found that the

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[13] The orthodox rule could also be interpreted as suggesting investment as soon as the present value index, the ratio of the present value of project net cash flows to the present value of investments, equals one.

[14] As predicted by the advantage of having a put option to reverse the investment, the stopping point is to the left of the stopping point under complete irreversibility for the analytically solvable case of $\delta = 0$, equation (15), since $0 < \frac{\gamma_2}{(\gamma_2 - 1)} < 1$ given $\gamma_2 < 0$.

[15] See, for example, Holland et al. (2000, 34); Adner and Levinthal (2004, 76); Demont, Wesseler, and Tollens (2005, 116); and Huisman (2010, 30). In 2003 testimony to the FCC Pindyck (2003) asserts that the orthodox NPV timing rule is optimal under complete reversibility. This advice has also appeared in articles aimed at the practitioner (e.g., Dixit and Pindyck 1995).

application of reversible capital does not require present value analysis. In contrast to the NPV timing rule for a now-or-never investment, (18), the rule under reversibility, (20), is degenerate: The opportunity cost of reversible capital, $(r+\delta)k$, applies to an instant of time.

Therefore, the situations represented in (18) and (20) are not manifestations of the same phenomenon or even comparable.

Further confusion on the role of NPV analysis arises from the frequent assumption that the risk-neutral drift in price is zero, implying that $\kappa = r$. In this case (18) can be written $qp_{\text{open}} = rk$. An artificial overlay of the dynamic with the static arises. As a result it appears that the benefits of operating are being compared to the instantaneous riskless opportunity cost of capital, as in (20), which expresses complete reversibility. Precisely, the NPV test is whether the present value of a perpetual stream of revenues is greater than the full opportunity cost of sinking the capital, where the capital has persistence that allows for continued production: $\frac{qp_{\text{open}}}{\rho-\mu} = \frac{qp_{\text{open}}}{\kappa} = k$. If written as $qp_{\text{open}} = \kappa k$, this test is revealed to be different from the test under complete reversibility.

8. The implications of reversibility for investment timing

While there is no need to calculate a net present value, or indeed conduct any dynamic analysis in managing a reversible stopping problem, the value functions in (1) and (2) nevertheless allow for additional insights as to why the orthodox NPV rule fails by inducing premature investment. We can present the solution to equations (7) through (10) as the notional equivalent to a general NPV timing rule, taking all available choices into account, i.e., when (a) mutually exclusive investments at alternative dates are considered to be mutually exclusive projects and (b) the rule prescribes investing in the project with the highest NPV as of the current date. This general rule is a simple extension of the rule used in cost-benefit analysis for analysis involving mutually exclusive projects at a given point in time, usually the current date.

Davis and Cairns (2012) show that application of the general NPV timing rule reveals that at optimal stopping for an irreversible investment,

$$\frac{E[dy(p_{\text{open}})]}{y(p_{\text{open}})dt} + \tilde{\alpha}(p_{\text{open}}) = r\,, \tag{22}$$

where y is the payoff received from sinking capital and $\tilde{\alpha}$ is quasi-option flow, the value of information from further delay.

We will now show that under reversible investments of lumpy capital there is no quasi-option flow.[16] Let $y(p_{\text{o}})$ denote the returns from immediate investment at price p_{o} given the potential for reversibility. From (2),

$$y(p_0) = \beta_2 p_0^{\gamma_2} + \frac{qp_0}{\kappa} - k. \tag{23}$$

From Davis and Cairns (2012), the quasi-option flow in our problem is,

$$\tilde{\alpha}(p_{\text{open}}) = \frac{\frac{1}{2}\sigma^2 p_{\text{open}}^2 \left(w''(p_{\text{open}}) - y''(p_{\text{open}})\right)}{y''(p_{\text{open}})}\,, \tag{24}$$

Using (1) and (3),

$$w''(p_{\text{open}}) = \frac{\gamma_1(\gamma_1 - 1)\beta_1 p_{\text{open}}^{\gamma_1}}{p_{\text{open}}^2} \tag{25}$$

and

[16] To simplify the notation we set $\delta = 0$.

$$y''(p_{\text{open}}) = \frac{\gamma_2(\gamma_2 - 1)\beta_2 p_{\text{open}}^{\gamma_2}}{p_{\text{open}}^2} \tag{26}$$

Some tedious algebra shows that at $p_{\text{open}} = \frac{rk}{q}$, $w''(p_{\text{open}}) = y''(p_{\text{open}})$. From equation (24) when $w''(p_{\text{open}}) = y''(p_{\text{open}})$ there is no quasi-option flow from waiting and the myopic r-percent stopping rule for investment timing obtains;

$$\frac{E[dy(p_{\text{open}})]}{y(p_{\text{open}})dt} = r\ . \tag{27}$$

This is the rule for stopping irreversible investments under certainty (Davis and Cairns 2012). It is intuitive and devoid of the complications normally associated with stopping a stochastic process. When price is below p_{open} the project's NPV is rising at greater than the discount rate. Waiting as price rises above p_{open} would cause the rate of growth in NPV to fall below the interest rate. Our result above reveals that condition (27) holds when stopping fully reversible investments under uncertainty: from (22) and making use of Ito calculus,

$$\frac{E[dy(p_{\text{open}})]}{y(p_{\text{open}})dt} = \frac{\eta\left(\gamma_2\beta_2 p_{\text{open}}^{\gamma_2} + \frac{qp_{\text{open}}}{\kappa}\right) + \frac{1}{2}\sigma^2\gamma_2\left(\gamma_2 - 1\right)\beta_2 p_{\text{open}}^{\gamma_2}}{\beta_2 p_{\text{open}}^{\gamma_2} + \frac{qp_{\text{open}}}{\kappa} - k}\ . \tag{28}$$

More tedious algebra shows that the right-hand side of (28) equals the riskless rate r at the stopping point $p_{\text{open}} = \frac{rk}{q}$.

In sum, when capital is completely reversible one notionally invests once the risk-neutral expected rate of gain in project value falls to the riskless interest rate, not as soon as the project value is positive, which is the orthodox NPV rule. Moreover, the timing rule that holds for optimal reversible investment under uncertainty is the same as that for optimal irreversible investment under certainty.

9. A numerical example

We use the parameter values in Brennan and Schwartz, Table 1, and the simplifications in our model (no operating costs, no taxes, etc.) to show a numerical example for a fully reversible cash flow of k = -\$1 million yielding 10 million units of output per period, shut-down cash flow of -k = \$1 million, average production cost a = 0, shut-down maintenance cost f = 0, capital inflation rate $\pi = 0$, depreciation rate $\delta = 0$, unit price rate of net convenience $\kappa = 1\%$, price volatility σ^2 = 8%, no taxes, and real risk-free rate of return r = 10%. Stopping rule (20) yields p_{open} = 0.01. Substituting this into (26) shows that at stopping the risk-neutral expected rate of growth in net realizable project value is r = 10%. The orthodox NPV stopping rule would suggest investment at p_{open} = 0.001, at which point the risk-neutral expected rate of growth of project value is greater than 10%.

Using equations (1), (2), (13), and (14), the value of the project conditional on spot price is shown in Figure 1. Despite complete reversibility of capital, project value is finite due to the price of time. The project value on each side of the opening price is strictly convex in price, as indicated in equations (1) and (2). The value of a project with any degree of irreversibility would be everywhere below the value function in Figure 1 (e.g., Keswani and Shackleton 2006). Of note, at the optimal investment trigger the open project value, at \$10.1 million, is 10.1 times the investment cost, giving a present value index of 10.1. Via equation (22), of the \$9.1 million

Figure 1: The value of the project ($ million). The optimal open/shut trigger is at unit price $p_{\text{open}} = p_{\text{close}} = 0.01$.

present value at entry, only $0.1 million arises from the ability to costlessly reverse that decision.[17]

Introducing even slight irreversibility into this example changes the unit price triggers quite substantially from that associated with reversibility: for $\phi = 0.98$, $p_{\text{open}} = 0.012$ and $p_{\text{close}} = 0.008$. Bazdrecsch (2013) estimates that US manufacturing has a reversibility parameter of 0.87. In a more specialized sector, aerospace, Ramey and Shapiro (2001) estimate the parameter to be 0.28.

[17] Reversibility has relatively little value due to the risk-neutral expectation that revenues will rise at close to the interest rate, making the probability of exercising the disinvestment/reinvestment options low and their timing distant. The risk-free discount rate valuing the incremental cash flows from those distant options is also relatively high in this example.

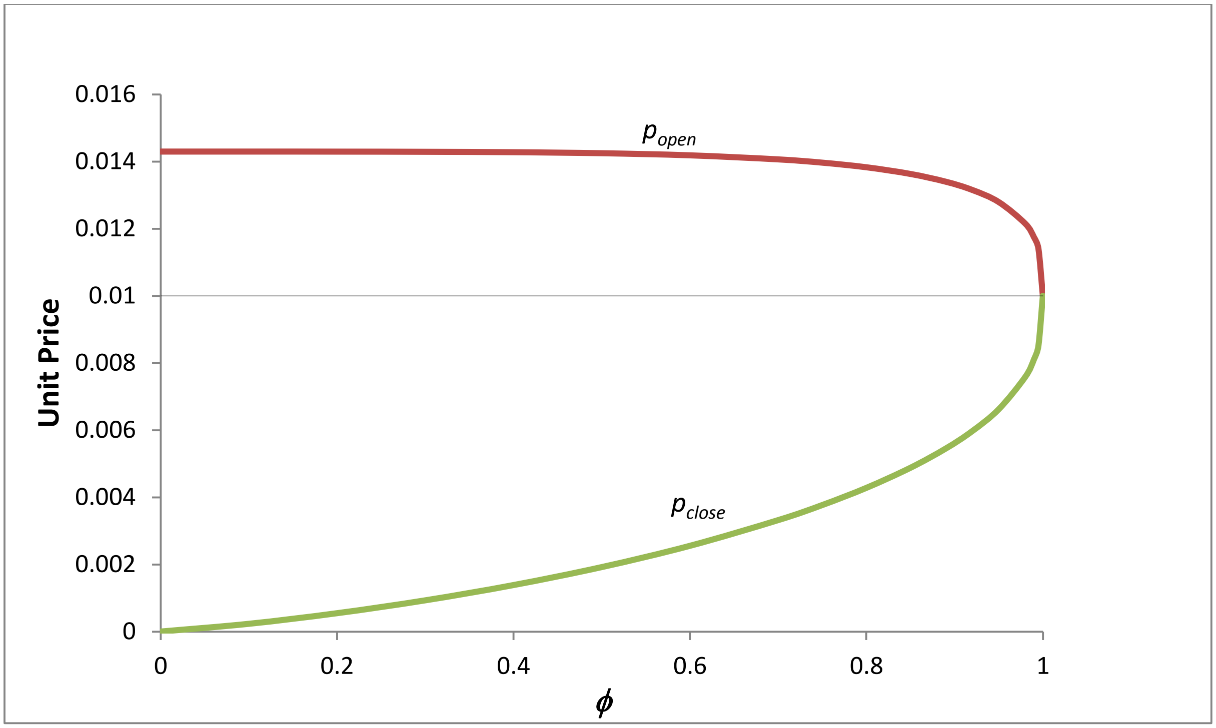

Figure 2: Open and close price triggers for varying degrees of reversibility, ϕ, for the parameter values given in Figure 1. $\phi = 0$ is complete irreversibility and $\phi = 1$ is complete reversibility.

Figure 2 provides the opening and closing unit price triggers for the range of investment reversibility.[18] The nonlinearity of price triggers to degree of reversibility has previously been noted by Abel and Eberly (1996, 587). Applying the method in Dixit (1989, Appendix B) to

[18] Via (12), the ratio of open to close price for a given level of irreversibility is independent of the size of the set-up costs *k* and production level *q*, though it will be dependent on *r*, δ, κ, and σ. Dixit (1989) produces a similarly shaped graph where the horizontal axis is the ratio of sunk investment costs to operating costs. The upper price trigger initiates sinking of the irreversible capital, while the lower trigger initiates exit to avoid operating costs. As in our model, exit causes immediate loss of the sunk capital. The price triggers widen the greater the ratio of sunk capital to operating costs.

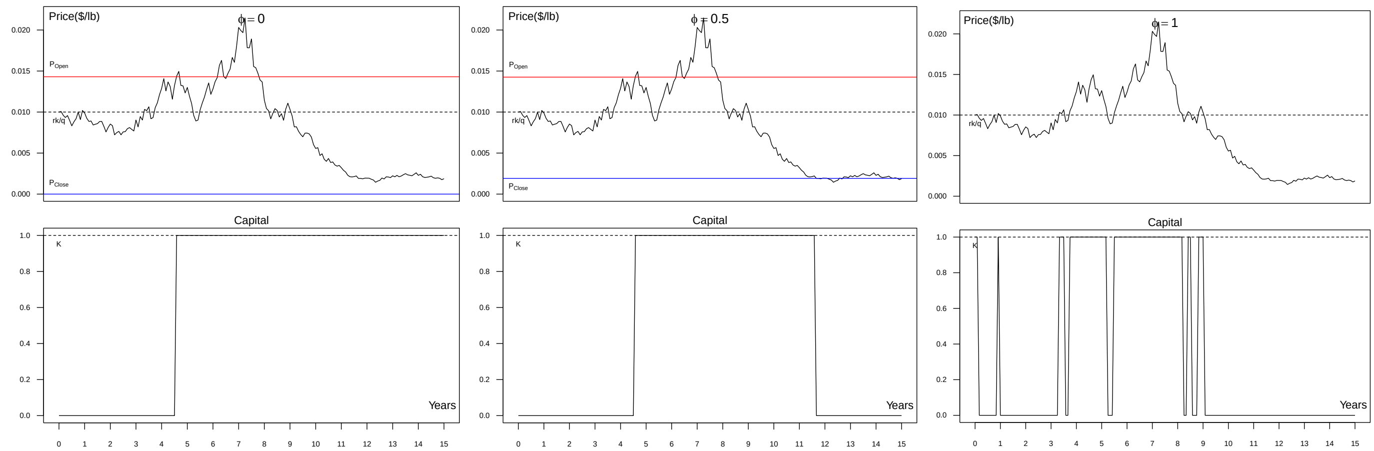

Figure 3: The time path of capital for a particular random price draw for cases $\phi = 0$ (complete irreversibility), $\phi = 0.5$ (partial reversibility) and $\phi = 1$ (complete reversibility).

equations (3) through (6), the slopes of the upper and lower price triggers in Figure 2 can be shown to converge to negative and positive infinity respectively at $\phi = 1.00$. Of interest in this numerical case is the greater sensitivity of the closing trigger than the opening trigger to the degree of reversibility. This is in contrast to a numerical example shown in Dixit and Pindyck (1994, 226), where the opening and closing price triggers “do not rise and fall by very much” as reversibility varies. In both their case and ours the opening trigger is almost invariant to the degree of irreversibility for $\phi > 0.5$. At the point where the project is initiated the option to close is out of the money and its value relatively invariant to the irreversibility of closure. The decision to terminate is, on the other hand, impacted by the degree of irreversibility, which represents the portion of investment that can be recovered. When there are operating costs, as in Dixit and Pindyck’s example the decision becomes less sensitive to the amount of investment recovery. The general rule is that the closing trigger will be more sensitive to irreversibility the higher the proportion of capital to present value operating expenses.

Figure 3 shows a sample price path given the parameters of the problem and the optimal response of capital investment over time for three different degrees of capital reversibility. The price path starts at 0.01, which induces immediate reversible investment but not irreversible investment. For this particular price path reversible capital is volatile, while partially irreversible capital switches in and out only once. In line with our comments on Figure 2 the price trigger for completely irreversible investment is imperceptibly higher than the partial reversibility case. The most noted difference between these two cases is the inability to disinvest irreversible capital as price drops near the end of the series, though such disinvestment would be desirable.

10. Conclusions

Reversibility is not a benchmark or idealization in the theory of capital or in the measurement of capital. It is a fiction, assuming, at a minimum, that there are no transactions costs and that there is no endogeneity in the price of capital; all atomistic producers using the same type of capital can sell it to agents in a secondary market during a downturn for the same price at which they bought it from these agents in the preceding boom. These assumptions are common in partial equilibrium microeconomic analyses of the firm and with the resultant projection of capital as reversible, or equivalently, as rented at a fixed price with no contract termination fees. In asserting this fiction, however, the distinguishing attributes of capital are purged:

1. Investing in capital is often said to entail a sacrifice, and its effect to have persistence. Its opportunity cost is reflected as being the portion that is sunk, an outflow of cash at the beginning of a project. Reversible capital does not entail a sacrifice beyond the extent to which any variable input entails an opportunity cost. Because of the lack of persistence decisions are static, not dynamic.

2. Information required for investment decisions is reduced to the observation of current economic conditions. There is no need to compute project present value.

3. There is no impact of uncertainty or risk aversion on the investment decision when capital is reversible, since nothing is placed at risk.

The stopping problem under reversibility is the limit of the traditional problem as the level of irreversibility goes to zero. This limit is qualitatively different: it is static, not dynamic. The conceptual discontinuity is attested to by the infinite slopes of the curves for the trigger prices in Figure 2.

The purging of the three properties through the assumption of reversibility indicates that irreversibility is the defining quality of capital under both certainty and uncertainty. Reversibility is a limit representing the nullity, the non-substantiality of capital. With even an infinitesimal level of irreversibility, as most easily brought out through the reality of non-zero transaction costs or endogenous prices in secondary capital markets, the usual attributes of capital re-assert themselves – in particular, timing, risk, and net present value reemerge. Reversibility is the grin on the Cheshire cat of investment.

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