A Closed-Form Model for Valuing Real Options Using Managerial Cash-Flow Estimates - Draft Abstract for ROC2013 *

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Abstract

In this work, we build on a previous real options approach that utilizes managerial cash-flow estimates to value early stage project investments. Through a simplifying assumption, where we assume that the managerial cash-flow estimates are normally distributed, we derive a closed-form solution to the real option problem. The model is developed through the introduction of a market sector indicator, which is assumed to be correlated to a tradeable market index and drives the project's sales estimates. Another indicator, assumed partially correlated to the sales indicator, drives the gross margin percent estimates. In this way we can model a cash-flow process that is partially correlated to a traded market index. This provides the mechanism for valuing real options of the cash-flow in a financially consistent manner under the risk-neutral minimum martingale measure. The method requires minimal subjective input of model parameters and is very easy to implement.

Keywords: Real Options; Managerial Estimates; Closed-Form Solution; Cash-Flow Replication; Project Valuation.

1 Introduction

Real option analysis (ROA) is recognized as a superior method to quantify the value of real-world investment opportunities where managerial flexibility can influence their worth, as compared to standard net present value (NPV) and discounted cash-flow (DCF) analysis. ROA stems from the work of Black and Scholes (1973) on financial option valuation. Myers (1977) recognized that both financial options and project decisions are exercised after uncertainties are resolved. Early techniques therefore applied the Black-Scholes equation directly to value put and call options on tangible assets (see, for example, Brennan and Schwartz (1985)). Since then, ROA has gained significant attention in academic and business publications, as well as textbooks (Copeland and Tufano (2004), Trigeorgis (1996)).

While a number of practical and theoretical approaches for real option valuation have been proposed in the literature, industry's adoption of real option valuation is limited, primarily due to the inherent complexity of the models (Block (2007)). A number of leading practical approaches, some of which have been embraced by industry, lack financial rigor, while many theoretical approaches are not practically implementable. The work presented in this paper is an extension of an earlier real options method co-developed by the author (see Jaimungal and Lawryshyn (2010)). In our

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previous work we assumed that future cash-flow estimates are provided by the manager in the form of a probability density function (PDF) at each time period. As was discussed, the PDF can simply be triangular (representing typical, optimistic and pessimistic scenarios), normal, log-normal, or any other continuous density. Second, we assumed that there exists a *market sector indicator* that uniquely determines the cash-flow for each time period and that this indicator is a Markov process. The market sector indicator can be thought of as market size or other such value. Third, we assumed that there exists a tradable asset whose returns are correlated to the market sector indicator. While this assumption may seem somewhat restrictive, it is likely that in many market sectors it is possible to identify some form of market sector indicator for which historical data exists and whose correlation to a traded asset/index could readily be determined. One of the key ingredients of our original approach is that the process for the market sector indicator determines the managerial estimated cash-flows, thus ensuring that the cash-flows from one time period to the next are consistently correlated. A second key ingredient is that an appropriate risk-neutral measure is introduced through the minimal martingale measure[1] (MMM) (Föllmer and Schweizer (1991)), thus ensuring consistency with financial theory in dealing with market and private risk, and eliminating the need for subjective estimates of the appropriate discount factor typically required in a DCF calculation. We then expanded our methodology to be able to account for the fact that when managers build their cash-flow estimates, the estimates are usually dependent on revenue and cost estimates (Jaimungal and Lawryshyn (2011)).

In this work, we show how our method can lead to an analytical solution, if it is assumed that the possible cash-flows are normally distributed. While this assumption may seem somewhat restrictive, very often, when managers estimate cash-flows (or revenues or expenses) the estimate, itself, is uncertain enough that assumptions regarding the distribution type ultimately only add "noise" to already "noisy" assumptions. Using the model from our previous work (Jaimungal and Lawryshyn (2011)), we show the impact of assuming the cash-flows are normally distributed, versus, triangularly, with specific emphasis to skewed distributions.

2 Real Options in Practice

2.1 Project Valuation in Practice

According to Ryan and Ryan (2002), 96% of Fortune 1000 companies surveyed indicated that NPV/DCF was the preferred tool for valuing capital budget decisions, and 83% chose the WACC (weighted average cost of capital) as the appropriate discount factor. While the methodology is well understood, we review a few important details to highlight the main assumptions regarding the DCF approach that are often glossed over or not fully appreciated by practitioners. As discussed in most elementary corporate finance texts (e.g. Berk and Stangeland (2010)) the WACC (equivalently the return on assets) for a company does not change with the debt to equity ratio when there are no tax advantages for carrying debt. Therefore, without loss of generality, in this discussion, we will assume that the company is fully financed through equity and thus, the cost of equity, r_E, will be the WACC.

The standard approach to estimating r_E is through the Capital Asset Pricing Model (CAPM), which has a significant list of assumptions (again, the reader is referred to any standard text). The

[1]As we discuss in Section 3, the risk-neutral MMM is a particular risk-neutral measure which produces variance minimizing hedges.

CAPM expected return on equity for a given i-th company is estimated as follows,

$$\mathbb{E}[r_{E_i}] = r_f + \beta_i(\mathbb{E}[r_{MP}] - r) \tag{1}$$

where r is the risk-free rate, r_{MP} is the return on the market portfolio and

$$\beta_i = \frac{\rho_{i,MP}\sigma_i}{\sigma_{MP}} \tag{2}$$

where $\rho_{i,MP}$ is the correlation of the returns between the i-th company and the market portfolio, σ_i is the standard deviation of the returns of the i-th company, and σ_{MP} is the standard deviation of the returns of the market portfolio. The point of introducing this well known result is to highlight the fact that by applying the WACC for the valuation of a project, by definition, the manager is assuming that the project has a similar risk profile, compared to the market, as does the company, i.e. that the ρ and σ for the project are identical to that of the company. Clearly, this will often not be the case and a simple thought experiment can illustrate the point.

Consider the case where a manager needs to make two decision: the first is whether to buy higher quality furniture with a longer lifetime versus lesser quality with a shorter lifetime; the second is whether to invest in the development of an enhancement to an existing product line or the development of a new product. The standard DCF approach would use the WACC to discount each of the alternative projected future cash-flows. Clearly, this is wrong. The *value* of the furniture cash-flows likely has little volatility and might, only slightly, be correlated to the market. The product enhancement is likely more closely aligned with the risk profile of the company, while the new product development may, or may not be, highly correlated to the market, but will likely have a high volatility. Our proposed approach decouples the ρ and σ for each project being valued. For the case of valuing the project as of time 0, or, as will be discussed further, for the case of a European real option, the effective σ of the project is determined by the uncertainty in the managerial cash-flow estimates. The ρ for the project does require estimation, however, we provide a mechanism to estimate it as well.

2.2 Real Options in Practice

As has been well documented (Dixit and Pindyck (1994), Trigeorgis (1996), Copeland and Antikarov (2001)), the DCF approach assumes that all future cash-flows are static, and no provision for the value associated with managerial flexibility is made. To account for the value in this flexibility numerous practical real options approaches have been proposed. Borison (2005) categorizes the five main approaches used in practice, namely, the Classical, the Subjective, the Market Asset Disclaimer (MAD), the Revised Classical and the Integrated. As discussed by Borison (2005), each method has its strengths and weaknesses. As Borison points out, all of the five approaches at that time, had issues with respect to implementation.

Arguably, one of the promising approaches is the MAD method of Copeland and Antikarov (2001). A number of practitioners have utilized the MAD method (see, for example, Pendharkar (2010)), however, the theoretical basis for the model is debated. Brando, Dyer, and Hahn (2012) have highlighted issues with the way the volatility term is treated in the MAD approach and have recommended an adjustment.

Both the Revised Classical (see Dixit and Pindyck (1994)) and Integrated (see Smith and Nau (1995)) approaches recognize that most projects consist of a combination of market (systematic /

exogenous) and private (idiosyncratic / endogenous) risk factors. The latter provides a mechanism to value the market risk of a project through hedging with appropriate tradeable assets while private risk is valued by discounting expected values at the risk-free rate. Properly applied, the integrated approach is consistent with financial theory. As Borison (2005) notes, the integrated approach requires more work and is more difficult to explain, but is the only approach that accounts for the fact that most corporate investments have both market and private risk. As we will show, our approach also accounts for both market and private risks, yet is easy to implement, and, in its simplest form, easy to understand as well.

3 Model Development

A key assumption of the MAD approach, as well as the methods proposed by others, such as Datar and Mathews (2004) and Collan, Fullér, and Mezei (2009), is that the risk profile of the project is reflected in the distribution of uncertainty provided by managerial cash-flow estimates. In Jaimungal and Lawryshyn (2010), we introduced a "Matching Method", where we assumed that there exists a *market sector indicator* Markov process that ultimately drives the managerial-supplied cash-flow estimates. In Jaimungal and Lawryshyn (2011), we extended our matching approach for the more practical case where managers provide uncertain revenue and gross margin percent (GM%) estimates. While both models provide a mechanism to value cash-flows and real options using a simple spreadsheet, because we assumed that the cash-flows, or revenues and GM%, estimates were triangularly distributed, numerical methods were required to solve the project cash-flow and real option valuations. In our work with managers in implementing our models, we realized that very often managers prefer to provide simple estimates consisting of a particular value with a given level of uncertainty. For example, often, managers are just as comfortable to provide a revenue estimate of, say $\$100 \pm \$20$, as they are to provide low, medium and high estimates, such as \$70, \$100, and \$120. In the former case, we can assume that the distribution of the estimate is normal. This assumption leads to an analytical solution for our methodology.

In the following subsections we develop our methodology for the case of normally distributed managerial estimates. First, we briefly review our Matching Method for the general case, where we derive a function, φ, that links the market sector indicator process to any normally distributed managerial cash-flow, revenue or GM% estimate. Then we utilize this function in the risk-neutral minimum martingale measure to develop a very simple expression for the valuation of cash-flows. Next, we develop an analytical formula to value a real option based on the cash-flow estimates, which is essentially a simplification of the model presented in Jaimungal and Lawryshyn (2010). Finally, we develop an analytical real options formula for the case where managers provide uncertain revenue and GM% estimates; a simplification of Jaimungal and Lawryshyn (2011).

3.1 Matching Managerial Estimates

Our objective is to provide a consistent dynamic model that can replicate cash-flow, revenue or GM% distribution estimates provided by managers. By applying the minimum martingale measure, we are assured of a financially consistent valuation. As such, we assume that there exists a traded market index I_t which the manager can invest in, and we assume that the price of the index follows geometric Brownian motion (GBM),

$$\frac{dI_t}{I_t} = \mu dt + \sigma dB_t, \tag{3}$$

where B_t is a standard Brownian motion under the real-world measure $\mathbb{P}$. Here, we will assume that the manager has provided cash-flow estimates at times T_k, where $k = 1, 2, ..., n$, however the formulation for sales and GM% is similar. Specifically, we assume the manager has supplied a cash-flow estimate of the form $\mu_k^F \pm 2\sigma_k^F$ and we assume the resulting distribution to be normal, i.e., $N(\mu_k^F, (\sigma_k^F)^2)$. We assume the cash-flow process to be F_t.

We introduce an observable, but not tradable, *market sector indicator* process A_t which we use to drive the cash-flow estimates provided by the manager. We assume the process to be a standard Brownian motion (BM)

$$dA_t = \rho_{FI} dB_t + \sqrt{1 - \rho_{FI}^2} dW_t^F, \tag{4}$$

where dW_t^F is a standard Brownian motion under the real-world measure $\mathbb{P}$ independent from B_t, and ρ_{FI} is a constant $(-1 \leq \rho_{FI} \leq 1)$. As discussed in Jaimungal and Lawryshyn (2011), choosing a BM instead of a GBM will have no bearing on the results of the valuation from a European real option perspective.

Next, we introduce the collection of functions $\varphi_k^F(A_t)$ such that at each T_k, $F_k = \varphi_k^F(A_{T_k})$. Furthermore, at each cash-flow date T_k we match the the distribution F_k to the cash-flow distribution supplied by the manager, namely, $\Phi\left(\frac{a-\mu_k^F}{\sigma_k^F}\right)$, where $\Phi(\bullet)$ is the standard normal distribution. Thus, we require

$$\mathbb{P}(F_T < a) = \Phi\left(\frac{a - \mu_k^F}{\sigma_k^F}\right). \tag{5}$$

Proposition 1. The Replicating Cash-Flow Payoff. *The cash-flow payoff function $\varphi_k^F(a)$ which produces the managerial specified distribution $\Phi\left(\frac{a-\mu_k^F}{\sigma_k^F}\right)$ for the cash-flows at time T_k, when the underlying driving uncertainty A_t is a BM, and $A_0 = 0$, is given by*

$$\varphi_k^F(a) = \frac{\sigma_k^F}{\sqrt{T_k}} a + \mu_k^F. \tag{6}$$

We now have a very simple expression for φ which makes the valuation of risky cash-flows very simple.

3.2 Valuation of Risky Cash-Flows

As discussed in Jaimungal and Lawryshyn (2011), the real-world pricing measure should not be used, and instead, we propose the risk-neutral measure $\mathbb{Q}$, corresponding to a variance minimizing hedge. Under this measure, we have the following dynamics

$$\frac{dI_t}{I_t} = r\, dt + \sigma\, d\widehat{B}_t, \tag{7}$$

$$dA_t = \widehat{\kappa}\, dt + \rho_{FI}\, d\widehat{B}_t + \sqrt{1 - \rho_{FI}^2}\, d\widehat{W}_t^F, \tag{8}$$

where $\widehat{B}_t$ and $\widehat{W}_t^F$ are standard uncorrelated Brownian motions under the risk-neutral measure $\mathbb{Q}$ and the risk-neutral drift of the indicator is

$$\widehat{\kappa} = -\rho_{FI}\,\frac{\mu - r}{\sigma}. \tag{9}$$

We emphasize that the drift of the indicator is precisely the CAPM drift of an asset correlated to the market index and is a reflection of a deeper connection between the MMM and the CAPM as demonstrated in (Cerny 1999). Given this connection, our reliance on parameter estimation is similar to those invoked by standard DCF analysis when the WACC is used to discount cash-flows and the cost of equity is estimated using CAPM. In DCF analysis the CAPM drift is, however, estimated based on the company's beta, while in our approach, the CAPM drift derives from historical estimates of the sector indicator and traded index dynamics. Furthermore, the *riskiness* of the project is appropriately captured by the distribution of the cash-flows. Consequently, our approach is more robust. Given the risk measure $\mathbb{Q}$, the values of the cash-flows can now be computed.

Proposition 2. Value of the Cash-Flows. *For a given set of cash-flow estimates, normally distributed with mean μ_k^F and standard deviation σ_k^F, given at times T_k, where $k = 1, 2, ..., n$, the value of these cash-flows at time $t < T_1$ is given by*

$$V_t(A_t) = \sum_{k=1}^{n} e^{-r(T_k - t)} \left(\frac{\sigma_k^F}{\sqrt{T_k}} \left(A_t + \widehat{\kappa}(T_k - t)\right) + \mu_k^F \right), \tag{10}$$

and for the case where $t = 0$,

$$V_0 = \sum_{k=1}^{n} e^{-rT_k} \left(\widehat{\kappa}\sigma_k^F \sqrt{T_k} + \mu_k^F \right). \tag{11}$$

Equations (10) and (11) provide a very simple alternative to valuing cash-flows using standard DCF approaches. The inherent risk of the project is captured by the cash-flow uncertainty estimates, rather than relying on an exogenously determined hurdle rate. We emphasize, that, while at first glance it may appear to be difficult to estimate ρ_{FI}, historical data of the market sector indicator to a traded asset can be utilized. Furthermore, if an assumption of ρ_{FI} is made, this is only a single assumption compared to using the β in the CAPM formulation. If one wanted to be consistent with the DCF CAPM based valuation, the ρ from the β could be backed out. A major advantage of our proposed approach is that two projects with the same expected cash-flow estimates will necessarily be valued differently if their estimated distributions are different. In our experience while working with managers, this was a very favorable aspect of our model.

3.3 Real Option Valuation of Risky Cash-Flows Estimates

In the real options context, similar to the scenarios proposed by many others (see for example Copeland and Antikarov (2001), Datar and Mathews (2004), Collan, Fullér, and Mezei (2009)), we assume that at some time T_0, managers can invest an amount K in a project and receive the uncertain managerial estimated cash flows at times T_k. The value of the real option at time $t < T_0$ can be computed under the risk-neutral measure $\mathbb{Q}$ as follows

$$RO_t(A) = e^{-r(T_0 - t)}\,\mathbb{E}^{\mathbb{Q}}\left[\left(V_{T_0}(A_{T_0}) - K\right)_+ \middle| \, A_t = A \right], \tag{12}$$

leading to the following Proposition.

Proposition 3. Real Option Value of Risky Cash-Flows Estimates. *For a given set of cash-flow estimates, normally distributed with mean μ_k^F and standard deviation σ_k^F, given at times T_k, where $k = 1, 2, ..., n$, the value of the option at time $t < T_0$ to invest the amount K at time $T_0 < T_k$ to receive these cash flows is given by*

$$RO_t(A_t) = e^{-r(T_0-t)} \left[(\xi_1(A_t) - K)\, \Phi\left(\frac{\xi_1(A_t) - K}{\xi_2}\right) + \xi_2\, \phi\left(\frac{\xi_1(A_t) - K}{\xi_2}\right) \right] \tag{13}$$

where $\Phi(\bullet)$ and $\phi(\bullet)$ are the standard normal distribution and density functions, respectively, and

$$\xi_1(A_t) = \sum_{k=1}^{n} e^{-r(T_k - T_0)} \left(\frac{\sigma_k^F}{\sqrt{T_k}} (A_t + \widehat{\kappa}(T_k - t)) + \mu_k^F \right), \tag{14}$$

$$\xi_2 = \sqrt{T_0 - t}\ \sum_{k=1}^{n} e^{-r(T_k - T_0)} \frac{\sigma_k^F}{\sqrt{T_k}}. \tag{15}$$

At $t = 0$, ξ_1 reduces to

$$\xi_1 = \sum_{k=1}^{n} e^{-r(T_k - T_0)} \left(\widehat{\kappa} \sigma_k^F \sqrt{T_k} + \mu_k^F \right). \tag{16}$$

The result of Proposition 3 is significant for it provides a simple analytical real options valuation method that is easy to apply and is consistent with financial theory. Arguably, it meets the requirements of reducing solution complexity, minimizing parameter estimation and being intuitive (see Block (2007)). We believe that this result can easily be embraced by practitioners.

3.4 Real Option Valuation of Risky Revenues and GM% Estimates

Following Jaimungal and Lawryshyn (2011), we introduce an underlying observable, but not tradable, process X_t which drives the sales S_t generated from the project. This underlying process can be thought of as a *market sales sector indicator*. As above, we assume that the sector indicator is a standard BM

$$dX_t = \rho_{SI}\, dB_t + \sqrt{1 - \rho_{SI}^2}\, dW_t^S \tag{17}$$

where W_t^S is a standard Brownian motion under the real-world measure $\mathbb{P}$ independent of B_t of equation (3) and ρ_{SI} is a constant ($-1 \le \rho_{SI} \le 1$). As well, we introduce a second underlying observable, but not tradable, process Y_t which drives the GM%, M_t. We assume that this process is directly correlated to the traded index and only, indirectly, to the sales index, such that

$$dY_t = \rho_{MI}\, dB_t + \sqrt{1 - \rho_{MI}^2}\, dW_t^M \tag{18}$$

where W_t^M is a standard BM under the real-world measure $\mathbb{P}$ independent of B_t and W_t^S, and ρ_{MI} is a constant ($-1 \le \rho_{MI} \le 1$). We note that the correlation between the sales process and the GM% process becomes $\rho_{SM} = \rho_{SI}\rho_{MI}$.

As was the case for the cash-flow analysis, the sales and GM% are determined by

$$S_k = \varphi_k^S(X_{T_k}) \qquad \text{and} \qquad M_k = \varphi_k^M(Y_{T_k}). \tag{19}$$

Using Proposition 1 we can directly write

$$\varphi_k^S(x) = \frac{\sigma_k^S}{\sqrt{(T_k)}} x + \mu_k^S, \tag{20}$$

and

$$\varphi_k^M(y) = \frac{\sigma_k^M}{\sqrt{(T_k)}} y + \mu_k^M, \tag{21}$$

where, similar to the cash-flow analysis, the managerial supplied sales and GM% distributions are $\Phi\left(\frac{x-\mu_k^S}{\sigma_k^S}\right)$ and $\Phi\left(\frac{y-\mu_k^M}{\sigma_k^M}\right)$, respectively. The cash-flow at T_k is given as

$$v_k = \varphi_k^S(\varphi_k^C - \kappa_k) - \alpha_k, \tag{22}$$

where α_k and κ_k represent fixed and variable costs, also supplied by management.[2] We emphasize that the resulting cash-flow is driven by the respective sales and GM% indicators. This provides a natural correlation between cash-flows induced by the path dependence of the indicators. If the indicators are high at the time of one cash-flow, resulting in a large cash-flow, then the probability of a large cash-flow at the next time-step is also high. This is a very desirable feature which has clear economic grounding. Contrastingly, in a number of practical approaches, the cash-flow distributions are typically assumed to be independent or correlation is introduced in a rather ad hoc manner.

As above, we propose the risk-neutral measure $\mathbb{Q}$, such that

$$dX_t = \widehat{\nu}\, dt + \rho_{SI}\, d\widehat{B}_t + \sqrt{1-\rho_{SI}^2}\, d\widehat{W}_t^S, \tag{23}$$

$$dY_t = \widehat{\gamma}\, dt + \rho_{MI} d\widehat{B}_t + \sqrt{1-\rho_{MI}^2}\, d\widehat{W}_t^M, \tag{24}$$

where $\widehat{B}_t$, $\widehat{W}_t^S$ and $\widehat{W}_t^C$ are standard uncorrelated Brownian motions under the risk-neutral measure $\mathbb{Q}$ and the risk-neutral drift of the indicators are

$$\widehat{\nu} = -\rho_{SI}\,\frac{\mu - r}{\sigma} \qquad \text{and} \qquad \widehat{\gamma} = -\rho_{MI}\,\frac{\mu - r}{\sigma}. \tag{25}$$

The value of the cash-flows can now be computed.

Proposition 4. Value of the Cash-Flows for Managerial Estimated Sales and GM%. *For a given set of sales and GM% estimates, each normally distributed and uncorrelated with means μ_k^S and μ_k^M, respectively, and standard deviations σ_k^S and σ_k^M, respectively, given at times T_k, where $k = 1, 2, ..., n$, the value of these cash-flows at time $t < T_1$ is given by*

$$V_t^{SM}(X_t, Y_t) = \sum_{k=1}^{n} e^{-r(T_k-t)} \left(\left(\frac{\sigma_k^S}{\sqrt{T_k}}(X_t + \widehat{\nu}(T_k - t)) + \mu_k^S \right) \left(\frac{\sigma_k^M}{\sqrt{T_k}}(Y_t + \widehat{\gamma}(T_k - t)) + \mu_k^M - \kappa_k \right) + \rho_{SM}\sigma_k^S\sigma_k^M \frac{T_k - t}{T_k} - \alpha_k \right) \tag{26}$$

[2]We note that the formulation given in equation (22) is slightly different than what we proposed in Jaimungal and Lawryshyn (2011).

which can also be written as

$$V_t^{SM}(X_t, Y_t) = C_t^{XY} X_t Y_t + C_t^X X_t + C_t^Y Y_t + C_t^0 \tag{27}$$

where the coefficients C_t^{XY}, C_t^X, C_t^Y and C_t^0 are given in the Appendix. For the the case where $t = 0$, equation (26) reduces to

$$V_0 = \sum_{k=1}^{n} e^{-rT_k} \left(\left(\sigma_k^S \widehat{\nu} \sqrt{T_k} + \mu_k^S \right) \left(\sigma_k^M \widehat{\gamma} \sqrt{T_k} + \mu_k^M - \kappa_k \right) + \rho_{SM} \sigma_k^S \sigma_k^M - \alpha_k \right). \tag{28}$$

Similar to Proposition 2, Proposition 4 provides a very simple result for the valuation of cash-flows. In this case, the inherent risk of the project is captured through managerial supplied sales and GM% estimates. Again, parameter estimation has been minimized and the only required parameters are ρ_{SI} and ρ_{MI}. Often, sales are highly correlated to a market index, whereas GM% values are likely low. In any event, historical analysis of data should provide reliable estimates for both of these parameters.

Following a similar analysis as above (see equation (12)), the real option value at time $t < T_0$ is given by

$$RO_t(X, Y) = e^{-r(T_0 - t)} \, \mathbb{E}^{\mathbb{Q}} \left[\left(V_{T_0}^{SM}(X_{T_0}, Y_{T_0}) - K \right)_+ \middle| X_t = X, \, Y_t = Y \right]. \tag{29}$$

Unfortunately, no analytical solution exists for the solution to equation (29), however, it is possible to reduce the equation to a single integral which can be easily solved using the Gauss-Hermite quadrature. We present the following proposition.

Proposition 5. Real Option Value of Risky Cash-Flows Estimates. *For a given set of sales and GM% estimates, each normally distributed and correlated with $\rho = \rho_{SI}\rho_{MI}$, with means μ_k^S and μ_k^M, respectively, and standard deviations σ_k^S and σ_k^M, respectively, given at times T_k, where $k = 1, 2, ..., n$, the value of the option at time $t < T_0$ to invest the amount K at time $T_0 < T_k$ to receive these cash flows is given by*

$$RO_t(X_t, Y_t) = e^{-r(T_0 - t)} \int_{-\infty}^{\infty} \left(\left(B_1 x^2 + B_3 x + B_0 \right) \Phi(\widehat{B}) + \left(B_2 x + B_4 \right) \phi(\widehat{B}) \right) \phi(x) dx \tag{30}$$

where $\Phi(\bullet)$ and $\phi(\bullet)$ are the standard normal distribution and density functions, respectively, and we have the following,

$$\begin{aligned}
\widehat{B} &= \frac{B_1 x^2 + B_3 x + B_0}{B_2 x + B_4} \\
B_0 &= C_{T_0}^{XY} \left(X_t Y_t + \widehat{\gamma}(T_0 - t) X_t + \widehat{\nu}(T_0 - t) Y_t + \widehat{\nu}\widehat{\gamma}(T_0 - t)^2 \right) + C_{T_0}^X (X_t + \widehat{\nu}(T_0 - t)) \\
&\quad + C_{T_0}^Y (Y_t + \widehat{\gamma}(T_0 - t)) + C_{T_0}^0 - K \\
B_1 &= \rho \, C_{T_0}^{XY} (T_0 - t) \\
B_2 &= \sqrt{1 - \rho^2} \, C_{T_0}^{XY} (T_0 - t) \\
B_3 &= \left(C_{T_0}^{XY} (Y_t + \widehat{\gamma}(T_0 - t)) + C_{T_0}^X + \rho \, (C_{T_0}^{XY} (X_t + \widehat{\nu}(T_0 - t)) + C_{T_0}^Y) \right) \sqrt{T_0 - t} \\
B_4 &= \sqrt{1 - \rho^2} \, (C_{T_0}^{XY} (X_t + \widehat{\nu}(T_0 - t)) + C_{T_0}^Y) \sqrt{T_0 - t}
\end{aligned}$$

and $\rho \equiv \rho_{SI}\rho_{MI}$.

We emphasize that equation (30) consists of a single integral and all coefficients in Proposition 5 are deterministic. In the following Section we present practical examples.

4 Practical Examples

The theoretical foundation was established in Section 3; here we provide a practical implementation of the methodology. We assume that a company is interested in investing in an early stage R&D project. The project will require a substantial investment of $50 million 2 years from now, at which point sales will begin and cash-flows will be realized in years 3 to 10. The market parameters are assumed to be as follows:

- Risk-free rate: $r = 3\%$
- Expected market growth: $\mu = 9\%$
- Market volatility: $\sigma = 10\%$.

Managers have estimated the cash-flows to be as depicted in Table 1 and the correlation of the cash-flows to the traded index are estimated to be 0.5. Furthermore, the managers have supplied standard deviation estimates for each of sales, cost of goods sold (COGS) and capital expenditures (CAPEX) as a percentage of each, as presented in Table 2. As can be seen uncertainty increases farther out in the future. Furthermore, COGS are estimated to be 60% correlated to sales and CAPEX 50%. If we assume that the correlation between COGS and CAPEX is driven solely through their correlations to sales, then the standard deviation of the cash-flows can be calculated as follows,

$$\sigma_{CF} = \sqrt{\sigma_S^2 + \sigma_C^2 + \sigma_{EX}^2 - 2\rho_{S,C}\sigma_S\sigma_C - 2\rho_{S,EX}\sigma_S\sigma_{EX} + 2\rho_{C,EX}\sigma_C\sigma_{EX}} \tag{31}$$

where σ denotes the standard deviation, ρ denotes the correlation and the subscripts "S", "C" and "EX" represent sales, COGS and CAPEX, respectively. The standard deviation values are provided in Table 3.

Table 1: Managerial Supplied Cash-Flow (Millions $).

	3	4	5	6	7	8	9	10
Sales	10.00	30.00	50.00	100.00	100.00	80.00	50.00	30.00
COGS	6.00	18.00	30.00	60.00	60.00	48.00	30.00	18.00
GM	4.00	12.00	20.00	40.00	40.00	32.00	20.00	12.00
SG&A	0.50	1.50	2.50	5.00	5.00	4.00	2.50	1.50
EBITDA	3.50	10.50	17.50	35.00	35.00	28.00	17.50	10.50
CAPEX	1.00	3.00	5.00	10.00	10.00	8.00	5.00	3.00
Cash-Flow	2.50	7.50	12.50	25.00	25.00	20.00	12.50	7.50

Using equation (11), the present value of the cash-flows is calculated to be $58.8 million and using equation (16) in equation (30) with $t = 0$ gives a value of $16.1 million for the real option.

Table 2: Managerial Supplied Standard Deviation Percent Estimates.

	3	4	5	6	7	8	9	10
σ_S (Sales) %	10%	11%	12%	13%	15%	16%	18%	19%
σ_C (COGS) %	10%	11%	12%	13%	15%	16%	18%	19%
σ_{EX} (CAPEX) %	5%	6%	6%	7%	7%	8%	9%	10%

Table 3: Estimated Standard Deviations (Millions $).

	3	4	5	6	7	8	9	10
σ_S (Sales) 1.00	3.30	6.05	13.31	14.64	12.88	8.86	5.85	
σ_C (COGS)	0.60	1.98	3.63	7.99	8.78	7.73	5.31	3.51
σ_{EX} (CAPEX)	0.05	0.17	0.30	0.67	0.73	0.64	0.44	0.29
σ_{CF} (Cash-Flow)	0.78	2.58	4.73	10.40	11.44	10.07	6.92	4.57

5 Conclusions

In this work we developed a method to value cash-flows and the real option value of the cash-flows that requires only the solution of a simple analytical formula. The model is consistent with financial theory and properly accounts for both idiosyncratic and systematic risk. Through the introduction of a market sector indicator, which is used to drive the cash-flows, the methodology provides a natural mechanism to link the cash-flows from one period to the next. Furthermore, because the market sector indicator is assumed to be partially correlated to a traded index, we are able to develop a valuation under the risk-neutral minimum martingale measure. The method requires minimal parameter estimation, as only the correlation between the market sector indicator and the traded index may not be easily obtained, however, we believe, it can often be estimated based on historical data. Managers especially like the intuitive aspect of the model where the value of the cash-flows is linked to their estimated uncertainty. As was discussed in Jaimungal and Lawryshyn (2010), depending on the value of the correlation of the market sector indicator to the market index, and the strike cost, K, the method will lead to a decreasing or increasing real option value as cash-flow uncertainty is increased. This feature is realistic, however is not often properly accounted for in other real options methods, because many other real options approaches presented in the literature assume the value of the cash-flows can be perfectly hedged with a traded asset – i.e. they do not allow for idiosyncratic risk. Ultimately, we believe that our proposed method preserves the essential elements to make it consistent with financial theory, yet simple enough to implement so that it can be easily embraced by practitioners not versed in financial mathematics.

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Appendix: Proofs of the Propositions

Proof of Proposition 1

We seek $\varphi(.)$ such that $\mathbb{P}(\varphi(A_T) \leq s|\mathcal{F}_0) = \Phi\left(\frac{s-\mu}{\sigma}\right)$. Since,

$$A_{T|\mathcal{F}_0} \stackrel{d}{=} \sqrt{T}Z \quad \text{where} \quad Z \underset{\mathbb{P}}{\sim} \mathcal{N}(0,1),$$

we have that

$$\begin{aligned}\mathbb{P}(\varphi(A_T) \leq s|\mathcal{F}_0) &= \mathbb{P}\left(\sqrt{T}Z \leq \varphi^{-1}(s)\right)\\ &= \Phi\left(\frac{\varphi^{-1}(s)}{\sqrt{T}}\right) \triangleq \Phi\left(\frac{s-\mu}{\sigma}\right).\end{aligned}$$

Consequently,

$$\varphi(A_T) = \frac{\sigma}{\sqrt{T}}A_T + \mu$$

and the proof is complete. □

Proof of Proposition 2

The value of the cash-flows are given as

$$\begin{aligned}V_t(A_t) &= \sum_{k=1}^{n} e^{-r(T_k-t)}\mathbb{E}^Q\left[\varphi_k^F(A_{T_k})|A_t\right]\\ &= \sum_{k=1}^{n} e^{-r(T_k-t)}\mathbb{E}^Q\left[\frac{\sigma_k^F}{\sqrt{T_k}}A_{T_k} + \mu_k^F\middle| A_t\right]\\ &= \sum_{k=1}^{n} e^{-r(T_k-t)}\mathbb{E}^Q\left[\frac{\sigma_k^F}{\sqrt{T_k}}\left(A_t + \widehat{\kappa}(T_k-t) + \sqrt{T_k-t}Z\right) + \mu_k^F\right]\\ &= \sum_{k=1}^{n} e^{-r(T_k-t)}\left(\frac{\sigma_k^F}{\sqrt{T_k}}\left(A_t + \widehat{\kappa}(T_k-t)\right) + \mu_k^F\right).\end{aligned}$$

For the case where $t=0$, $A_0=0$, therefore

$$V_0 = \sum_{k=1}^{n} e^{-rT_k}\left(\widehat{\kappa}\sigma_k^F\sqrt{T_k} + \mu_k^F\right)$$

and the proof is complete. □

Proof of Proposition 3

We begin with equation (12), repeated here for convenience,

$$RO_t(A) = e^{-r(T_0-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\left(V_{T_0}(A_{T_0}) - K\right)_+\middle| A_t = A\right],$$

and using the interim result from Proposition 2, with $t = T_0$, we have

$$RO_t(A) = e^{-r(T_0-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{k=1}^{n} e^{-r(T_k-T_0)}\left(\frac{\sigma_k^F}{\sqrt{T_k}}\left(A_t + \widehat{\kappa}(T_k-t) + \sqrt{T_0-t}Z\right) + \mu_k^F\right) - K\right)_+\right].$$

Defining

$$\xi_1(A_t) \equiv \sum_{k=1}^{n} e^{-r(T_k - T_0)} \left(\frac{\sigma_k^F}{\sqrt{T_k}} (A_t + \widehat{\kappa}(T_k - t)) + \mu_k^F \right),$$

$$\xi_2 \equiv \sqrt{T_0 - t} \; \sum_{k=1}^{n} e^{-r(T_k - T_0)} \frac{\sigma_k^F}{\sqrt{T_k}}$$

we have

$$RO_t(A) = e^{-r(T_0 - t)} \, \mathbb{E}^{\mathbb{Q}} \left[(\xi_1(A_t) - K + \xi_2 Z)_+ \right],$$

which simplifies to

$$RO_t(A_t) = e^{-r(T_0 - t)} \left[(\xi_1(A_t) - K) \, \Phi \left(\frac{\xi_1(A_t) - K}{\xi_2} \right) + \xi_2 \, \phi \left(\frac{\xi_1(A_t) - K}{\xi_2} \right) \right]$$

completing the proof. □

Proof of Proposition 4

The value of the cash-flows are given as

$$\begin{aligned} V_t^{SM}(X_t, Y_t) &= \sum_{k=1}^{n} e^{-r(T_k - t)} \mathbb{E}^Q \left[v_k(X_{T_k}, Y_{T_k}) \mid X_t, Y_t \right] \\ &= \sum_{k=1}^{n} e^{-r(T_k - t)} \mathbb{E}^Q \left[\varphi_k^S(X_{T_k}) \left(\varphi_k^M(Y_{T_k}) - \kappa_k \right) - \alpha_k \mid X_t, Y_t \right]. \end{aligned}$$

Given that

$$X_{T|X_t} \stackrel{d}{=} X_t + \widehat{\nu}(T - t) + \sqrt{T - t} \, Z^S \tag{32}$$

$$Y_{T|Y_t} \stackrel{d}{=} Y_t + \widehat{\gamma}(T - t) + \sqrt{T - t} \, (\rho Z^S + \sqrt{1 - \rho^2} \, Z^M) \tag{33}$$

where $Z^S \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$, $Z^M \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$ and $\mathbb{E}^Q[Z^S Z^M] = 0$, with $\rho \equiv \rho_{SM} = \rho_{SI} \rho_{MI}$, and using equations (20) and (21) leads to the result (equation (27)), repeated here for convenience,

$$V_t^{SM}(X_t, Y_t) = C_t^{XY} X_t Y_t + C_t^X X_t + C_t^Y Y_t + C_t^0$$

where

$$\begin{aligned} C_t^0 &= \sum_{k=1}^{n} e^{-r(T_k - t)} \left(\frac{\sigma_k^S \sigma_k^M}{T_k} \widehat{\nu} \widehat{\gamma} (T_k - t)^2 + \frac{\sigma_k^S}{\sqrt{T_k}} \widehat{\nu}(T_k - t)(\mu_k^M - \kappa_k) + \frac{\sigma_k^M}{\sqrt{T_k}} \widehat{\gamma}(T_k - t) \mu_k^S \right. \\ &\qquad \left. + \mu_k^S (\mu_k^M - \kappa_k) + \rho \sigma_k^S \sigma_k^M \left(\frac{T_k - t}{T_k} \right) - \alpha_k \right) \\ C_t^{XY} &= \sum_{k=1}^{n} e^{-r(T_k - t)} \frac{\sigma_k^S \sigma_k^M}{T_k} \\ C_t^X &= \sum_{k=1}^{n} e^{-r(T_k - t)} \left(\frac{\sigma_k^S \sigma_k^M}{T_k} \widehat{\gamma}(T_k - t) + \frac{\sigma_k^S}{\sqrt{T_k}} (\mu_k^M - \kappa_k) \right) \\ C_t^Y &= \sum_{k=1}^{n} e^{-r(T_k - t)} \left(\frac{\sigma_k^S \sigma_k^M}{T_k} \widehat{\nu}(T_k - t) + \frac{\sigma_k^M}{\sqrt{T_k}} \mu_k^S \right). \quad \square \end{aligned}$$

Proof of Proposition 5

We begin with equation (29), repeated here for convenience,

$$RO_t(X,Y) = e^{-r(T_0-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\left(V_{T_0}^{SM}(X_{T_0},Y_{T_0}) - K\right)_+\Big|\, X_t = X,\, Y_t = Y\right]$$

and using the result from Proposition 4, with $t = T_0$, we have

$$RO_t(X_t,Y_t) = e^{-r(T_0-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\left(C_{T_0}^{XY}X_{T_0}Y_{T_0} + C_{T_0}^{X}X_{T_0} + C_{T_0}^{Y}Y_{T_0} + C_{T_0}^{0} - K\right)_+\Big|\, X_t, Y_t\right]. \tag{34}$$

We utilize equations (32) and (33) to substitute for X_{T_0} and Y_{T_0} in equation (34),

$$RO_t(X_t,Y_t) = e^{-r(T_0-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\left(B_1(Z^S)^2 + B_2Z^SZ^M + B_3Z^S + B_4Z^M + B_0\right)_+\right], \tag{35}$$

where the coefficients B_0, B_1, B_2, B_3 and B_4 are given in Proposition 5. Equation (35) can be written in integral form as,

$$RO_t(X_t,Y_t) = e^{-r(T_0-t)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(B_1x^2 + B_2xy + B_3x + B_4y + B_0\right)_+\,\phi(y)\phi(x)dy\,dx.$$

Now $B_1x^2 + B_2xy + B_3x + B_4y + B_0 > 0$ when $y > -\widehat{B}$ (see Proposition 5 for the definition of $\widehat{B}$), and therefore,

$$\begin{aligned}
RO_t(X_t,Y_t) &= e^{-r(T_0-t)}\int_{-\infty}^{\infty}\int_{-\widehat{B}}^{\infty}\left(B_1x^2 + B_2xy + B_3x + B_4y + B_0\right)\,\phi(y)\phi(x)dy\,dx \\
&= e^{-r(T_0-t)}\int_{-\infty}^{\infty}\int_{-\infty}^{\widehat{B}}\left(B_1x^2 - B_2xy + B_3x - B_4y + B_0\right)\,\phi(y)\phi(x)dy\,dx \\
&= e^{-r(T_0-t)}\int_{-\infty}^{\infty}\Big(\left(B_1x^2 + B_3x + B_0\right)\Phi(\widehat{B}) + \left(B_2x + B_4\right)\phi(\widehat{B})\Big)\,\phi(x)dx
\end{aligned}$$

and the proof is complete. □