The Choice of Stochastic Process in Real Option Valuation

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Abstract

A main issue in valuation modeling is the correct choice of the stochastic process that better describes the asset price performance. Particularly, in investment projects that show a high level of managerial flexibility in conditions of uncertainty – for which it would be proposed the real option valuation models – the assumption of a specific process can have an impact not only on the project value, but also on the investment rule. This work discusses the choice of stochastic process in real options valuation and the main useful tests and theoretical considerations to give support to this task.

Key-words

Stochastic Process, Real Options Valuation, Geometric Brownian Motion, Mean Reversion Model, Jump Diffusion Process, Multiple Factor Models.

1 Introduction

The investment decisions in stocks, financial derivatives and corporate projects are influenced by uncertainties of different types. One way to deal with these uncertainties is to research the stochastic process that better describes the random behavior of the assets prices in time.

Typically in the financial derivatives valuation the Geometric Brownian Motion (GBM) is assumed as appropriate to describe the behavior of stock prices and stock indexes, as in Black & Scholes (1973) and Cox, Ross & Rubinstein (1979). The GBM is also largely used to describe the uncertainties in corporate project valuation by real options analysis. As a contrast, in the valuation of commodities and derivatives related to them, it is common to use Mean Reversion Models (MRM) (Gibson & Schwartz, 1990; Dixit & Pindyck, 1994; Schwartz, 1997), assuming that the commodities price might wander randomly in short term, but that they tend to converge to an equilibrium level in the long run regarding their marginal cost of production. Nevertheless, commonly it is not so easy to determine which one – GBM or MRM – is the more applicable stochastic process. Besides statistical tests, some questions must be considered in the stochastic process choice, such as: the economical features and the asset lifetime, the difficulty in the parameters calibration of the selected stochastic model, the applicability of the chosen process in solutions (analytical or numerical) of the models used to valuation, among other factors.

This paper discusses the choice of stochastic process in real option valuation and the main useful tests and considerations to give support to this task. The work is structured as follows: after this introduction (i), in (ii) a bibliographical revision of stochastic processes applied to real option analysis is presented, in (iii) we describe some statistical tests that can be used to support the stochastic process choice, in (iv) we will present some theoretical considerations and (v) we conclude.

2 Stochastic Process and Real Options Theory Applications

We can define stochastic process as variables that move discretely or continuously in time unpredictably or, at least, partially randomly. Formally, be Ω a set that represents the randomness, where $w \in \Omega$ denotes a state of the world and f a function which represents a stochastic process. The function f depends on $x \in R$ e $w \in \Omega$: $R \times \Omega \rightarrow R$ or f(x,w), and it has the following property: given $w \in \Omega$, f (º ,w) becomes a function of only x. Thus, for different values of $w \in \Omega$ we get different functions of x. When x represents time, we can interpret $f(x,w_1)$ and $f(x,w_2)$ as two different trajectories that depend on different states of the world, as we can see in figure 1:

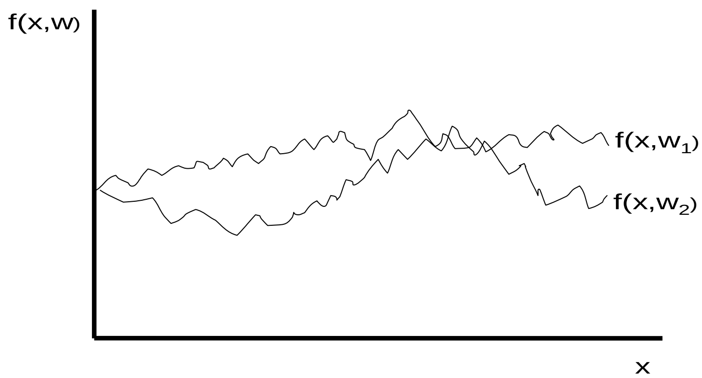

Figure 1 – Stochastic process trajectories.

With the aim at representing the uncertainties related to the investments, the choice of the stochastic process is an issue of great relevance in the assets valuation modeling. In the case of real options valuation – in which the uncertainties are straightforward considered in the future cash flow of the assets – the relevance is even greater.

A class of stochastic process that plays an important role in financial modeling is Markov Processes. In Markov Processes only the latest observed value is considered to forecast the future values, which is consistent with the Weak Form of Market Efficient. Among many types of Markov Processes, one of the most popular is Geometric Brownian Motion, which is the base case used in the modeling of financial options (Black & Scholes, 1973) and real option (Brennan & Schwartz, 1985; McDonald & Siegel, 1985, 1986; Paddock, Siegel, & Smith, 1988). GBM is usually defined by the equation:

$dx = \alpha x dt + \sigma x dz$

Where:

x is the asset price;

α is the *drift* parameter;

σ is the volatility parameter;

dz is a Wiener increment.

Among other pros, the main advantages of GBM are: its mathematical simplicity, the small number of parameters to be estimated and the fact that it is easy to obtain analytical solutions to asset valuation. In some way, these characteristics can be considered the main reason to explain its popularity. As a contrast, it has as major con the fact that the prices tend to diverge when the time goes to the infinite, which could create unrealistic scenarios and it is an undesirable property in cases of long run assets. In figure 2 we present a price projection supposing it follows a GBM.

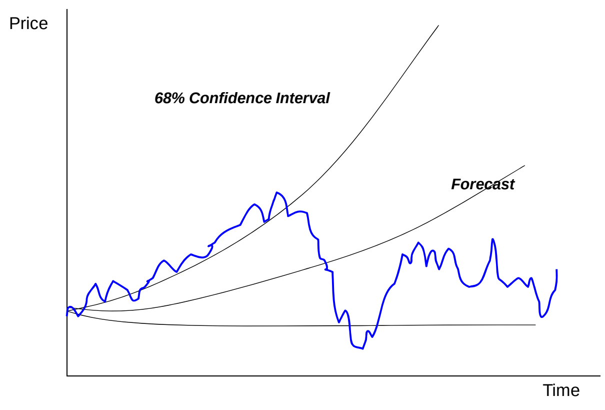

Figure 2 – Price forecast supposing prices follow a GBM.

In other situations, when the uncertainties in prices depend on an equilibrium level, such as in case of commodities and interest rates, it is debated if the use of GBM would be appropriate (AL-HARTHY, 2007, GEMAN, 2005, PINDYCK, 2001, 1999, METCALF & HASSET, 1995, SMITH & MCCARDLE, 1998, BRENNAN & SCHWARTZ, 1985, BHATTACHARYA, 1978). In case of commodities – such as oil, copper, sugar and ethanol – it is usual to assume that the price is driven, at least partially, by a mean reversion component, which makes the prices wander randomly in short term, but, in the long run tend to converge to the equilibrium level of the prices associated to the marginal cost of production. The most traditional MRM is called Ornstein Uhlenbeck, which is defined by the equation:

$$dx = \eta(\bar{x} - x)dt + \sigma dz$$

Where:

x is the price of commodity;

$\bar{x}$ is the equilibrium level to which the process reverts in the long run;

η is the speed of reversion parameter;

σ is the volatility parameter;

dz is a Wiener increment.

Although MRM is a Markov Process it does not have independent increments, considering that the expected changes of x are a function of the difference between the long run equilibrium level and the last observation of the process. The straightforward application of Ornstein-Uhlenbeck model in the prices may generate an inconvenience that is the appearance of negative values, which is an undesirable characteristic to price representation. An alternative that can be used to solve this problem is not directly applying the Ornstein-Uhlenbeck model in the prices but in the logarithm of the prices, as in model 1 of Schwatz (1997). In figure 3 we present a price projection supposing it follows a MRM.

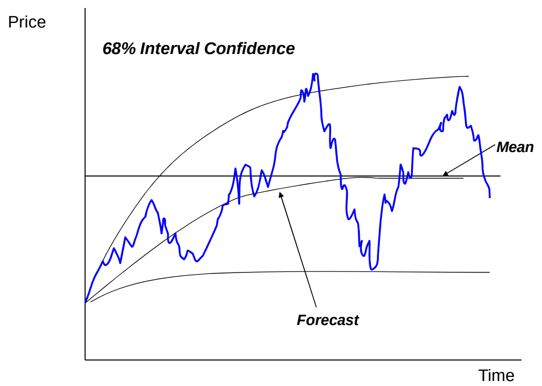

Figure 3 – Price forecast supposing prices follow a MRM.

A third type of process that is commonly used in finance is the Poisson Processes that is also known as Jump Diffusion Processes. Poisson Processes also belong to Markov Processes class and they are characterized by the occurrence of discrete and infrequent jumps in time. This type of process is frequently used in the modeling of rare events, such as the occurrence of accidents in insurance industry and the effect on crisis in the oil prices. In the Poisson Process the jump´s appearance follows a Poisson distribution and it may work with jumps of fixed or variable sizes. As we will see later, it is common to mix Poisson Process with other types of processes as GBM and MRM, in order to model the uncertainties in a more realistic way in the real options valuation.

As a formal definition: Be x a Poisson Process, in which all the randomly of the process is concentrated on the appearance of jumps that have their sizes determined by the function $g(x,t)$. The Poisson Process can be described by the differential equation:

$$dx = f(x,t)dt + g(x,t)dq$$

Where:

$f(x,t)$ and $g(x,t)$ are deterministic known functions;

dq is a Poisson increment.

The typical parameters of the Poisson Process are:

λ which corresponds to the average rate of the jump occurrence for a time period;

λdt corresponding to the probability of jump occurrence;

$1 - \lambda dt$ is the probability of the non-occurrence of the jump;

u indicating the size of the jump;

q representing the randomness of the Poisson Process.

There are different types of the Poisson Processes. They can be homogeneous, in which the events are random and the increments are independent and stationary or non-homogenous, in which the jumps are not stationary. There are also Poisson Processes in which the randomness is observed in the size of the jumps and they follow a specific probability distribution. Finally, among the mentioned processes, there are the Compensated Poisson Processes, which are obtained by subtracting the drift determining its conversion in a Martingal. In figure 4 it can be seen a Homogeneous Poisson Process with upward fixed size jumps.

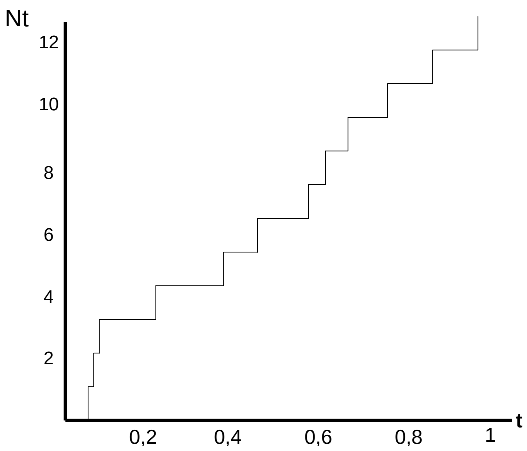

Figure 4 – Homogeneous Poisson Process with upward fixed size jumps.

The task of determining which is the most appropriated process in order to represent the main uncertainties involved in a valuation is usually not a trivial question and, in some cases, analysts realize that these uncertainties have elements of more than one type of process. As a result, in order to generate more realistic models, several authors presented papers in recent decades to propose models of multiple factors that mix different types of processes. One of the pioneer works that presented a multiple factors model was Merton (1976) in which is mixed GBM and Poisson Process. The author justified this model to stocks, in which the effect of common news in the stock prices would be represented by GBM, while in case of rare event occurrence there would be a Poisson jump. Using a Compensated Poisson Process, in which the jumps were considered non-systematic and using a lognormal distribution for the size of jumps, Merton

(1976) managed to find a close formula to European call options. The differential formula of the model is represented by the equation:

$dx/x = [\alpha - \lambda k]dt + \sigma dz + dq$

Where:

x is the stock price,

$E[dq]=E[\phi]\lambda dt=k\lambda dt$;

α is the drift parameter;

σ is the volatility parameter;

dz is a Wiener increment;

q is a Independent Poisson Process with non-systematic jumps.

Another work that presents a Multiple Factors Model with Jump Diffusion Process is Dias & Rocha (1999), in which the authors proposed the mix of Poisson Process and MRM to represent the stochastic behavior of oil prices in real options valuation, as it can be seen in the equation:

$dx/x = [\eta(\bar{x} - x)dt - \lambda k] + \sigma dz + dq$

Where:

x is the oil price;

dz is a Wiener increment;

η is a speed of reversion parameter;

$\bar{x}$ is the equilibrium level to which the process reverts in the long run;

$k=E[\phi-1]$;

dq is a Poisson Process increment which can assume value zero with $1-\lambda dt$ of probability and $\phi-1$ with λdt of probability.

Since $k=E[\phi-1]$ implies that $E[dx/x]= \eta(\bar{x} - x)dt$.

In this model, similar to Merton (1976), the common news would cause marginal adjusts in oil prices, while abnormal events – such as crisis, wars and economic booms – would cause discrete jumps on time. The uncertainty about the size and direction of the jump is represented by ϕ. The jumps can be systematic, which do not allow to obtain a risk neutral portfolio, or non-systematic, which allow the use of contingent claims.

Other papers (Gibson & Schwartz, 1990; Schwartz, 1997; Pindyck, 1999; Schwartz & Smith, 2000) are focused on the stochastic behavior of commodity prices. These works claim that besides MRM factor price processes of some commodities may also have a stochastic upward trend factor. In practical terms, this trend factor would tend to increase the equilibrium level to which the process reverts in the long run as time passes. These increases would have additional motivations to momentary mismatches of supply and demand (captured by MRM) and they would be caused by the progressive exhaustion of natural resources and incremental costs related to new requirements of environmental laws, among other issues. As a contrast, the improvements in the exploration and production technologies could imply in a downward trend of the commodity prices. Among other works that share the same concept, one that has a huge popularity is Schwartz & Smith (2000) which proposes a two stochastic factor model – GBM[1] and MRM – correlated and non-observable to describe the behavior of commodity prices. The sum of these two stochastic factor forms the logarithm of the asset price ($\ln S_t$), as it can be seen in the equation:

$$\ln S_t = \chi_t + \xi_t$$

Where:

S_t is the spot price of the commodity;

χ_t is the factor which represents the changes of the prices in short term;

ξ is the factor which represents the tendency of the prices in the long run.

The differential equations of the two stochastic processes are:

$$d\chi_t = -\kappa\chi dt + \sigma_\chi dz_\chi$$

$$d\xi_t = \mu_\xi dt + \sigma_\xi dz_\xi$$

$$dz\xi . dz\chi = \rho dt.$$

Where:

κ is the speedy of reversion parameter of MRM;

σ_χ is the volatility parameter of the short run changes in prices;

dz_χ is the Wiener increment of the short run changes in prices;

μ_ξ is the drift parameter of the long run price tendency;

σ_ξ is the volatility parameter of the long run price tendency;

dz_ξ is the Wiener increment of the long run price tendency;

ρ is the correlation parameter of the two factor increments.

In order to estimate the parameters of Schwartz & Smith (2000), the authors used future prices of commodities and applied the State-Space approach combined with Kalman Filtering[2].

An interesting way to summarize and categorize the stochastic models used in the real options is presented by Dias (2009), in which the processes are classified in three levels, as it can be seen in the table 1.

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[1] The model proposes an Arithmetic Brownian Motion (AMB) for the long run tendency of the price logarithm, which would be equivalent to a GBM for the prices.

[2] The Space-State approach is an adequate tool to deal with state variables that are not observable; nevertheless it is known that those are generated by a Markov process. When the model is placed in the Space-State approach, the Kalman Filter combined with maximum likelihood estimators can be used to estimate the parameters of unobservable state variables, which in the case of Schwartz & Smith (2000) would be the spot price of commodities.

Type of Stochastic Process	Model Name	References
Unpredictable Model	Geometric Brownian Motion (MGB)	Paddock, Siegel & Smith (1988)
Predictable Model	Pure Mean Reversion (MRM)	Dixit & Pindyck (1994), Schwartz (1997, model 1)
More Realistic Models	Two or Three factor models, and Mean reversion to uncertain long term mean	Gibson & Schwartz (1990), Schwartz (1997, models 2 & 3), Baker, Mayfield & Parsons (1998), Schwartz & Smith (2000)
	Mean Reversion with Jumps	Dias & Rocha (1999, 2001), Aiube, Tito & Baidya (2008)

Table 1 - More Usual Stochastic Processes

3 Tests for Determination of Stochastic Processes

Some statistical tools can be useful to research which stochastic process would be prevalent in the asset prices and other type of uncertainties. One of the most used approaches in this task is the Unit Root Test, also known as Dickey-Fuller Test. This test consists of the analysis of the hypothesis that the slope (b) of the regression between the log-returns and lagged log-returns of the prices is different from 1, as shown in the equation:

$$ln(x_t) = a + b\ ln(x_{t-1}) + \varepsilon_t$$

Where x_t is the asset price in the time t.

Failure to reject the null hypothesis would strengthen the idea of the presence of GBM. The critical values of Dickey-Fuller Test can be seen in the table 2.

Significance Level	1%	2.5%	5%	10%
Critical Values	-3.43	-3.12	-2.86	-2.57

Source: Wooldridge, 2000, p. 580

Table 2 - Asymptotic Critical Value of the t-test for the Unit Root with no trend.

In case of autocorrelation between the log-returns of prices and residues of the regression it is recommended the use of Augmented Dickey-Fuller Test. In this test it should be included sufficient lagged log-returns so that the residues become a white noise.

In case log-returns do not present stationarity, it is recommended the Dickey-Fuller Test with Tendency. This test consists of applying a regression between log-returns and lagged log-returns including a drift as can be seen in the equation:

$$ln(x_t) = a + b\ ln(x_{t-1}) + ct + \varepsilon_t$$

Where c is the coefficient of the tendency.

The critical values of Dickey-Fuller Test with Tendency can be seen in the table 3.

Significance Level	1%	2.5%	5%	10%
Critical Values	-3.96	-3.66	-3.41	-3.12

Source: Wooldridge, 2000, p. 583

Table 3 - Asymptotic Critical Value of the t-test for the Unit Root with trend

Generally, it is difficult to reject the hypothesis that the process follows a GBM, nevertheless it does not mean that there would be another process more suitable to describe the prices behavior. An interesting result is when b<1, which would indicate the possibility of MRM presence, even in the cases that GBM has not been rejected. In order to illustrate this difficulty, we can mention Dixit & Pindyck (1994), in which the authors expose that tests made with 30 and 40 year price series did not allow to reject the hypothesis that oil prices would follow GBM. It was necessary to make tests with 120 year series to manage the rejection of the unit root.

Other approach that can be used to support the choice of the stochastic process is to verify if the level of the shocks is persistent, which could be more relevant than the unit root research. In the autoregressive processes – such as MRM – the shocks tend to dissipate when there is permanent reversion strength. As a contrast, in case of GBM – which is not an autoregressive process – the shocks in prices are persistent. In order to verify this condition, Pindyck (1999) proposes a Variance Ratio Test, which consists of verifying if the log-return variance increases proportionally in time, that is one of the main important hypothesis of GBM. The test measures the level to where the variance converges with the increase of the lags in the log returns. The Variance Ratio can be measured by the equation:

$$R_k = \frac{1}{k}\frac{Var\left(P_{t+k} - P_t\right)}{Var\left(P_{t+1} - P_t\right)}$$

The term *Var* (.) in the formula represents the variance of the series of the differences between the log-returns of prices: P_t, with lag of *k* periods. In case of GBM, it would be expected that variance would increase proportionally and linearly to *k*, which implies that R_k should converge to 1 when k grows. As a contrast, in case of MRM the variance is limited to a certain level even considering the growth of *k* which implies that R_k should decrease when k grows.

In addition to Dickey-Fuller and Variance Ratio Tests, analysis can be made with Adherence Measures in sample, which compares one step ahead results estimated by the models and the observations of the price series correspondents. Among other measures, 3 approaches typically used in this analysis are: Pseudo R^2, Mean Quadratic Error (MQE) and Mean Absolute Percentage Error (MAPE).

The Pseudo R^2 consists of the square of the correlation between the values of the price series and the forecasts one step ahead, both related to the same period. Larger values (closer to 1) of the Pseudo R^2 indicate a higher adherence of the model. The Pseudo R^2 can be calculated by the equation:

$$Pseudo\, R^2 = \left[\rho(S_t, E(S_t|S_{t-1}))\right]^2$$

Where:

$\rho(a,b)$ is the correlation between a and b;

S_t is an observation of the price series;

$E(S_t|S_{t-1})$ is the estimated value one step ahead of the price series.

The Mean Quadratic Error is the average of the square of the difference between the values of the price series and the estimated prices one step ahead. Lower values for the MQE indicate a better predictive ability of the model. The MQE can be calculated by the equation:

$$MQE = Average\left\{\left[S_t - E(S_t|S_{t-1})\right]^2\right\}$$

Where:

S_t is an observation of the price series;

$E(S_t|S_{t-1})$ is the estimated value one step ahead of the price series.

Mean Absolute Percentage Error is the average of the modulus of the difference between the values of the price series and the estimated prices one step ahead, standardized by the values of the price series. Similarly to MQE, lower values of MAPE indicate a better predictive ability of the model. The MAPE can be calculated by the equation:

$$MAPE = \ Average\left\{ \frac{|S_t - E(S_t|S_{t-1})|}{S_t} \right\}$$

Where:

S_t is an observation of the price series;

$E(S_t|S_{t-1})$ is the estimated value one step ahead of the price series.

4 Theoretical Considerations about the Stochastic Process Choice in the Real Option Valuation

Beyond the statistical tests, the choice of the stochastic process to represent the uncertainties in the real options valuation can be supported by theoretical considerations referenced in the economic theory. An example would be the assumption of the equilibrium mechanism in the prices of commodities which would justify the use of MRM to represent the behavior of the price of these assets. In the same vein, the supposition of the gradual increase in the production marginal cost and the occurrence of rare events (such as crisis and wars) would support the mix of MRM with GBM and Poisson Process, respectively, in the search for more realistic models.

Another relevant issue is the lifetime of the assets. Generally, if the lifetime is relatively short, further research to determine the best stochastic process could be considered as a matter of minor relevance, allowing the choice of the process to be guided by such issues as the ease in calibration of parameters and construction of the valuation model. Dixit & Pindyck (1994) show that in short periods of time, price processes of GBM type are mainly guided by the stochastic shocks, while as time passes the drift component becomes more relevant. Thus, as in most models the randomness is represented by increments of Wiener – that is, treated similarly as GBM – the search for a more appropriate process to represent the stochastic behavior of prices could be considered an expensive task, taking into consideration the benefits to be obtained. As a contrast, in cases of long lifetime of the assets, the research to obtain a process with more adherence to the performance of the asset prices could be crucial to its valuation and the definition of investment rule. Bastian-Pinto, Brandão & Hahn (2009) show that in a switch option valuation in the sugar-ethanol industry, the difference in the option value could change from 20% to 70% in relation to the base case, when the uncertainties are modeled by MRM and GBM, respectively. Kerr, Martin, Pereira, Kimura & Lima (2009) estimated the optimum time to cut trees in the forest products investments considering that uncertainties could be modeled by MRM and GBM. The authors concluded that the critical prices to decide the cut in relation to the time, would change substantially when a different type of process is used. In the case studied by the authors, the use of MRM could anticipate the exercise decision of the option when the results are compared to GBM.

Although the mixing of processes can generate more realistic models, this implies in a greater difficulty in the parameters estimation. Usually, the multiple factor models require future price series of the assets for the parameters estimation, as in Schwartz (1997) and Schwartz & Smith (2000). In these works the authors used the Space-State approach mixed with the Kalman Filter applied in series of future prices of commodities to estimate jointly the parameters of the models. Nevertheless, in most cases of commodities prices and other kind of uncertainties, future prices are not available, such as ethanol prices and the volume of traffic on a toll road. In these cases, despite the advantages of using multiple factor models, the choice of stochastic process might be influenced by limitations related to the database availability.

Regarding the quality of the available databases to calibrate the parameters of the models it is important to consider the extension and periodicity of the price series. Generally, it is recommended to use long time series for estimating the drift parameters. Taking into account that the variance of the drift estimator is proportional to time, the longer the series the more efficient will be the estimator. Nevertheless, the information periodicity is relevant to calibrate the volatility parameters. The higher the information frequency the better the estimator quality.

Finally, an issue that should be considered in the choice of the stochastic process is its applicability on close formulas and numerical solutions used in real options valuation. Comparing GBM with other models, one of its biggest advantages is the small number of parameters to calibrate and the ease of obtaining analytical solutions, which are huge incentives to its use. Generally, the use of MRM does not allow the achievement of analytical solutions to the decision rule, which implies in the use of numerical solutions such as Monte Carlo Simulation (MCS) and Binomial Lattices[3]. Usually, it is possible to obtain solutions for multiple factor models using MCS and when there is more than one uncertainty to be considered in the analysis. It is important to observe that before 1993 MCS was only used in the solution of European options and since then, with the development of optimization methods pluggable to MCS, it became possible to value American options (Dias, 2008).

5 Conclusions

In this work a discussion is raised about the alternatives of stochastic processes for applying in the real options valuation. Besides the main types of pure stochastic models, more contemporary models were presented, which mix different kinds of processes in order to provide a more realistic in the characterization of the uncertainties involved in the analysis.

In several situations – mainly in projects with long lifetime – the choice of stochastic process can be relevant in the real options valuation, with influences on the value and optimal rule of the investment. Typically, it is recommended the Dickey-Fuller Test as a support to the choice of stochastic processes; nevertheless in most cases the results of the tests are inconclusive. In section 3 were presented some tools that can improve Dickey Fuller Test and other statistical approaches, in order to obtain more conclusive results in the analysis. In addition to statistical tests, some theoretical considerations based on microeconomics and some restrictions caused by the availability of the database were considered relevant to define the most appropriate stochastic model.

As further research in the same field, it is suggested the study of other types of statistical tests for the analysis of adherence of multiple factor models and the analysis of implications related to the use of these models on the several kinds of managerial flexibilities.

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