On the valuation and capital cost of project flexibility within sequential investment

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Abstract

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Valuing optimal investment policies requires choosing uncertainty processes. Having committed to a diffusion and differential equation, option solutions are customized to boundary conditions but generalizations to other processes or flexibility sequences can be difficult.

By placing investment within a mathematical graph (network), we separate diffusion from flexibility choices. Matrices containing discount factors and their elasticities (betas) are used to capture value matching and smooth pasting. For diffusions where a discount can be formed, this allows investment costs and values to be determined as a function of multiple policy thresholds. Insights into valuation and capital costs of flexibility follow.

Keywords: value matching, smooth pasting, flexibility discounting, real options, investment sequence and bi–partite directed graph. C02, C61, G31.

Great strides have been made in valuing financial optionality through the development of dynamic asset pricing. For a range of underlying stochastic processes delta hedging and risk neutrality in particular[1] allow options on stocks, bonds, rates and operational assets to be valued. If timing flexibility is present and capital expenditure must justify opportunity costs, then investment policies can benefit from the study of flexibility methods.[2] This is true for any economic policy, of which the investment problems are a part.

At the heart of dynamic valuation is the idea that assets with flexibility can be valued using similar means to those with financial optionality. Given diffusion specifications for stochastic variables and costs of control, this frequently reduces to a question of determining optimal policy trigger points. If the underlying risks are not traded, in some situations risk neutral valuation may not be appropriate but this assumption has facilitated progress with many articles solving investment and switching problems.[3]

Despite advances, the complexity of the policies that have been modelled is limited, often to a few sequential decisions after which the asset becomes inflexible. Even when the number of embedded option functions that require matching at boundaries is small, presenting policy thresholds as a function of investment quantities results in a non–linear system of equations, which cannot be solved analytically. Consequently for the numerical solutions that depend on a particular choice of parameters, it is often difficult to say if results would also apply to other parameter or diffusion choices and to find useful heuristics and generalizations.

It is also unrealistic to assume a simple sequence, especially one that ends with no flexibility. Most assets do not become passive and inflexible after

[1] Black and Scholes (1973), Merton (1973) & Cox, Ross and Rubinstein (1979).

[2] The term real options dates from Myers (1977), but also see Brennan and Schwartz (1985), McDonald and Siegel (1985) and the texts of Dixit and Pindyck (1994) and Trigeorgis (1996).

[3] Including capacity (Pindyck (1988)), land (Capozza and Li (1994)), costly reversibility (Abel and Eberly (1996)), Q theory, marginal cost of capital (Hayashi (1982) and Abel et al. (1996)).

early decision making has been exercised and managers must continually examine compound and value improving policies. It is undesirable to assume that the same uncertainty process pertains before and after each investment. To capture different market dynamics pre and post investment, it would be useful to vary diffusions across policy stages.

In this article we formalize a solution technique for policies with flexibility. It encompasses a wide range of multistage problems, arbitrary numbers of thresholds and different diffusion choices. This tractability is attained by presenting investment quantities and payoffs as functions of policy thresholds (rather than the other way round). Initially assuming thresholds and ensuring optimal values are consistent with these produces a set of equations for investment quantities and flexibilities that are linear in presumed factors.

At each policy threshold, we examine what flexibility is consumed and what is generated net of required investment. The careful identification and separation of flexibility before and after investment allows discounting techniques to optimize multiple threshold policies using matrices.

Discount factors are applicable to flexibility because, unlike the investment's current cash revenue and cost rate, the option to change flows in the future has no immediate flow or dividend. Commensurate with its use, flexibility is a discount instrument whose only value depends on a future payoff (or payoffs) even if these are linked in a convoluted sequence.

We use discount functions[4] in matrices to form flexibility chains in a valuation system that captures sequential policies. The passage of decision taking is marked on a mathematical graph (network) by transitions from one investment and flexibility state to another. Especially for systems that contain recursive, circular, policies that cannot be worked back from final end conditions, this facilitates solution of simultaneous equations. The particular choice of matrices is also driven by the need to encompass first order optimal-

[4] See Dixit, Pindyck and Sødal (1999) and Sødal (2006) for the discount factor approach.

ity (smooth pasting) conditions in the same framework. The formalization derived applies to frictional, perfectly reversible and terminal decisions across as many states and policy thresholds as desired. It also allows different stochastic processes to be embedded at each stage.

This development is important for several reasons. Much of the investment literature produces firm or policy values for one particular diffusion and parameter set where results depend heavily on that choice. If market price dynamics differ pre and post a firm's investment, having one diffusion throughout is unsatisfactory. Secondly, focusing on one diffusion restricts empirical estimation. A valuation technique that puts the flexibility structure ahead of the diffusion dynamics is more amenable to testing different structure and uncertainty specifications with data. Finally it allows a broader range of problems to be solved.

This article proceeds as follows, in Section 1 we examine the classical two stage cycle captured by hysteresis, or costly investment and divestment (Dixit (1989) etc. with two states and decision points). This allows notation and the main principles of the article to be laid out for a problem that has been well studied.

In Section 2 we discuss the discount factors that are at the heart of this method, indicating how they can be adapted for stochastic processes other than Geometric Brownian Motion (GBM). This allows restatement of the same problem as in Section 1 but with general notation. Next we utilize alternate diffusion processes.

In Section 3 we employ a serial hysteresis sequence with four stages and policy thresholds over two cycles. This allows hysteresis cycles to allocated different processes.

In Section 4 we represent two way flexibilities in a four stage investment ladder (the Appendix shows a worked example). This accommodates convoluted flexibility sequences.

The majority of equations fall into one of three types so we use consistent prefixes; Vm for value matching, Sp for smooth pasting and Di for discounting equations (Gr for growth and other less common types are spelt out). Their subsection number is appended as a suffix, e.g. Di1.0 in Section 1.0. Equations have a common interpretation across sections, but the matrices and vectors contents can differ. This is because within each application and section the number and structure of flexibility components vary. In order to distinguish these visually, each application has its own graph (or network of choices, within a Table). These display the separate flexibility values and their interplay where it is particularly useful to distinguish their timing using red and blue colors. Conclusions are offered in Section 5.

1 Classical hysteresis under GBM

Before working on more complex policies, to illustrate we apply the techniques to a well known problem with only two decision points. Investment hysteresis concerns a policy controlling dynamic stochastic underlying value P in only two ways; either launching or suspending a cashflow (with optimal timing) when its dynamic present value P coincides with known thresholds P_2 and P_1. The state variable P is stochastic whilst P_2, P_1 at which switching occurs are policy points subject to choice but treated as fixed.

The policy can launch P by incurring a lump sum cost X_2 (an exercise price equal to a risk free capitalized perpetual cost rate x/r plus a switching on cost $K_2 > 0$) when the value at that time P_2 is gained. Although the launch flexibility is consumed, in addition to the cash flow's present value on opening P_2, a further option to close is acquired.

This next flexibility is the option to suspend at a lower threshold P_1 later if it becomes unprofitable. In return for recouping a lower divestment sum X_1 (e.g. the capitalized cost saving x/r less a switch off cost $K_1 > 0$) P_1

can be relinquished. When costs and values are lower at divestment than investment ($X_1 < X_2$ and $P_1 \ll P_2$) economic hysteresis is said to occur (Dixit (1989)).

We label the value of flexibility in the idle state as $V_i(P)$ and $V_f(P)$ in the full, both a function of dynamic P. These flexibilities are discount instruments represented by the time value of call (from idle i) and put (from full f) options excluding the value and present value of cost (P, x/r). These are easiest to illustrate with a process for P that follows geometric Brownian motion (GBM – but the next section accommodates other processes). Flexibility values then rely on positive solution constants A, B as well as powers $a > 1, b < 0$ that satisfy a quadratic condition. The flexibility solutions that correspond to idle and full states are power functions $V_i(P) = AP^a$, $V_f(P) = BP^b$. Both functions $V(P)$ satisfy a second order Bellman equation in P but subject to boundary conditions at P_2 for the call and P_1 for the put. Solution constants A, B capture these conditions but here we explicitly interpret them through payoffs. In particular we identify the final flexibility value at each policy exercise point; one value is AP_2^a for the call at P_2 and for the put BP_1^b at P_1.

We subsume A, B into new constants, preferring to work with idle flex payoff $V_{i2} = AP_2^a$ at P_2 (opening) and full flex payoff $V_{f1} = BP_1^b$ at P_1 (closing). In Di1.0 full $V_f(P)$ and idle $V_i(P)$ flex functions depend on these constant payoffs V_{f1}, V_{i2} through well used discounting formulae $D_f(P, P_1)$ and $D_i(P, P_2)$ that depend on current P and boundaries P_1 or P_2.

$$\begin{aligned} V_f(P) &= D_f(P, P_1) V_{f1} \qquad D_f(P, P_1) = \left(\frac{P}{P_1}\right)^b \\ V_i(P) &= D_i(P, P_2) V_{i2} \qquad D_i(P, P_2) = \left(\frac{P}{P_2}\right)^a . \end{aligned} \tag{Di1.0}$$

For $P > P_1$ in the first and $P < P_2$ in the second of Di1.0, both discounts are greater than zero and less than or equal to one. The first argument of functions $D(.,.)$ is the current dynamic level P and the second the static boundary against which valuation occurs, P_1 or P_2. A value at fixed policy thresholds, e.g. V_{i2}, has its threshold P_2 carried in its subscript but functions $D(P,.)$ retain explicit dependence on P until evaluated at one particular policy point, e.g. $D_i\left(P,P_2\right)|_{P=P_2}=D_{i22}$ when subscripts indicate the use of an initial threshold too.

Both functions $D_i\left(P,P_2\right), D_f\left(P,P_1\right)$ solve the same Bellman equation in P but the first when evaluated at $P=P_1$ is subject to $D_{f11}=1$ and the second $D_{i22}=1$ at $P=P_2$.[5]

The step in Di1.0 has the advantage that values or payoffs at the policy points are separated from the functional forms of the discount factors. This means that the flexibility analysis can be performed before selecting processes and discount factors and that the latter can be easily changed. The slopes of discount functions are required next.

1.1 Local flexibility beta

Having captured the functional forms of the flexibility in two discount factors, we can evaluate their betas (relative to the unit beta of P). For GBM, the betas or elasticities $\beta_f\left(P\right), \beta_i\left(P\right)$ of $D_{f,i}\left(P,.\right)$ are constant at b,a i.e. within

[5] For a risk free rate of r and GBM for P with δ dividend yield and σ volatility, two claims $D(P,.)$ specialise powers a,b

$$\begin{aligned}\frac{dP}{P} &= \left(r-\delta\right)dt+\sigma dW \quad a,b=\frac{1}{2}-\frac{r-\delta}{\sigma^2}\pm\sqrt{\left(\frac{r-\delta}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}.\\ 0 &= \frac{1}{2}\sigma^2P^2\frac{\partial^2D\left(P,.\right)}{\partial P^2}+\left(r-\delta\right)P\frac{\partial D\left(P,.\right)}{\partial P}-rD\left(P,.\right)\end{aligned}$$

Claim $D_i\left(P,.\right)$ in state i (equivalently $V_i\left(P\right)$) and $D_f\left(P,.\right)$ in state f ($V_f\left(P\right)$) model GBM idle and full flex values subject to the Bellman equation in P but with boundary conditions captured by their second argument (i.e. P_2 in $D_i(P,P_2)$ and P_1 in $D_f(P,P_1)$). No bubble, zero limits for D_i as $P\to 0$ and $P\to\infty$ for D_f match a,b to i,f. Other processes are shown in Section 2.

Beta1.1 call and put flexibilities are iso–elastic and no longer depend on P

$$\begin{aligned} \beta_f(P) &= \frac{P}{D_f(P,P_1)}\frac{\partial D_f(P,P_1)}{\partial P} = b \\ \beta_i(P) &= \frac{P}{D_i(P,P_2)}\frac{\partial D_i(P,P_2)}{\partial P} = a. \end{aligned} \tag{Beta1.1}$$

In the idle state the call is active (with $a > 1$), this flexibility has a beta and rate of return higher than P itself. In the full state the put is active with negative beta ($b < 0$) which offers insurance at a cost.

Since the GBM solutions are iso–elastic, their betas do not change with threshold proximity and $\beta_f = b$, $\beta_i = a$ do not depend on P (for other processes this is not true and betas evaluated at levels corresponding to the end and beginning of their life i.e. $\beta_{f1} = \beta_f(P_1)$ and $\beta_{f2} = \beta_f(P_2)$ differ in Section 2).

Having dealt with the terminal boundary payoffs of each option V_{f1} or V_{i2} as well as discounts' betas, next we deal with their initial conditions. For sequential and circular problems, these are required as a link to the prior option's terminal conditions. By specializing the discount factors from P to those that pertain at the beginning of their lives we next determine V_{f2} and V_{i1}.

1.2 Fixed discounting from V_{f2} to V_{f1} and V_{i1} to V_{i2}

Over a hysteresis cycle, P may travel from policy point P_2 (launch) to P_1 (close) and back again, potentially many times. Current flex values depend on the local value of P but the only values relevant to optimality are those determined at these policy points. We have already identified two constant values V_{f1} or V_{i2} as the terminal conditions to which the discount factors in Di1.0 apply.

However the hysteresis policy is cyclical and the end of one flexibility cycle corresponds to the beginning of another. In order to track the transformation of value at policy triggers, we also need the policy constants at the beginning of each state. For the idle state this is V_{i1} at P_1 and for the full state it is V_{f2} at P_2. This introduces two flex constants in addition to V_{f1} and V_{i2} but these are compensated by the additional discounting relationships.

The idle or call flex initial value is a fixed proportion of its end value $V_{i1} = (P_1/P_2)^a V_{i2}$, so too for the full put state where the initial value (at P_2) is a proportion of its value at P_1 (the lower threshold) $V_{f2} = (P_2/P_1)^b V_{f1}$. These are shown in Di1.2 which is similar to Di1.0 but specialized at policy points P_1, P_2 (and no longer dynamic functions). Since P_1, P_2 are fixed (unlike P) in Di1.2 we use compact notation for D_{f21}, D_{i12} which are fixed discount factors evaluated at policy points

$$\begin{aligned} V_{f2} &= D_{f21} V_{f1} \qquad D_{f21} = D_f\left(P_2, P_1\right) \\ V_{i1} &= D_{i12} V_{i2} \qquad D_{i12} = D_i\left(P_1, P_2\right). \end{aligned} \tag{Di1.2}$$

Now we have four constant values (V_{f1}, V_{i2}, V_{f2} and V_{i1}). Although their number has doubled from two ($A = V_{i2}P_2^{-a}$, $B = V_{f1}P_1^{-b}$), extra discounting conditions Di1.2 (relating option values to their discounted later selves) provide additional equations that ensure solution. Firstly this brings greater economic meaning to the interpretation of flexibility values, secondly working with these at the beginning and at the end of the diffusion state (i.e. after a recent investment transition and just before the next) brings tractability.

We also have used a natural means to link these functionally, one that captures the discounting dynamics of the diffusion process within that state. Equations Di1.0 intuitively account for the evolution (in P) of flexibility value which has no current cashflow benefit or cost but which grows or shrinks like a discount bond with proximity to a threshold. Moreover, isolating the functional role of discounting makes it easy to replace one stochastic process with another. In order to form a network of flexibilities, we take values V_{i2}, V_{f1} etc. and place them in a mathematical graph. The links in this network are the discount functions $D_i(P,.)$, $D_f(P,.)$ that represent stochastic processes.

1.3 Value matching at P_1, P_2

Using the four specific values V_{i2}, V_{f2}, V_{i1}, V_{f1} we can describe the transitions of value from options that are exercised to those that are acquired. When launching from the idle state the call V_{i2} is used up in favor of gaining the value of revenue P_2 less its present value costs $X_2 = x/r + K_2$ but an additional put option V_{f2} is also gained ($V_{f2} < V_{f1}$). When closing from full at P_1, the put V_{f1} evaluated at P_1 is used in favor of recouping costs $X_1 = x/r - K_1$ and the original idle call is also regained, but with a lesser value V_{i1} ($V_{i1} < V_{i2}$)

$$\begin{aligned} V_{i2} &= V_{f2} + P_2 - X_2 \\ V_{f1} &= V_{i1} + X_1 - P_1. \end{aligned} \tag{Vm1.3}$$

In Vm1.3, functions $V_i\left(P\right)$, $V_f\left(P\right)$ have been evaluated at $P_{2,1}$ but it is understood that until a thresholds is reached, flex components rely on the diffusion P and have sensitivities linked to betas. This is important because the optimality condition used next requires that the limiting slopes in P also match within Vm1.3.

1.4 Smooth pasting as a beta condition

For the decision making at P_2, P_1 to be optimal, not only must the total values match either side of Vm1.3 but the derivatives of values with respect to P evaluated at thresholds must match too. This is the so called smooth pasting condition. Smooth pasting also has a rate of return interpretation (see Sødal (1998) and Shackleton and Sødal (2005)) and in this context a beta interpretation.

Matching slopes on either side of Vm1.3 at P_2 requires $\partial V_i(P)/\partial P|_{P=P_2} = \partial V_f(P)/\partial P|_{P=P_2} + 1$ and $\partial V_f(P)/\partial P|_{P=P_1} = \partial V_i(P)/\partial P|_{P=P_1} - 1$ at P_1. Since these options are functions of fixed payoffs, the derivatives carry into the discount factors, i.e. from Di1.0 $\partial V_i(P)/\partial P = V_{i2}\partial D_i(P,P_2)/\partial P$ and $\partial V_f(P)/\partial P = V_{f1}\partial D_f(P,P_1)/\partial P$.

Also from Beta1.1 the slopes of the discount factors, when scaled by P, yield their betas multiplied by the discount factor, i.e. $P\partial D_i(P,P_2)/\partial P = aD_i(P,P_2)$ and $P\partial D_f(P,P_1)/\partial P = bD_f(P,P_1)$. Coupled with the slope matching conditions (which contain $V_{i2}\partial D_i(P,P_2)/\partial P$ and $V_{f1}\partial D_f(P,P_1)$) smooth pasting at P_2, P_1 for the GBM case in Sp1.4 reduces to simple value weighted betas within Vm1.3

$$\begin{aligned} aV_{i2} &= bV_{f2} + P_2 \\ bV_{f1} &= aV_{i1} - P_1. \end{aligned} \tag{Sp1.4}$$

Sp1.4 contains betas (proportional to excess rates of return) of options in Vm1.3, evaluated at P_2, P_1. The GBM claims representing the flex values are iso–elastic so the elasticities in Beta1.1 for the call are the same at both thresholds (for the put too, i.e. V_{f2} and V_{f1} both have relative beta b).

Costs of capital are normally averaged across debt and equity but here it is averaged across asset and option. Smooth pasting thus has an interpretation as a Weighted Cost of Capital Matching (WCCM) across (unit beta) P, options V_i, V_f (betas a, b) and $X_{1,2}$ (zero beta). The betas determine the excess returns on each of the flexibility components V_i, V_f and the investment gain or divestment loss $\pm P$ and are weighted by the values of flexibility present in Vm1.3.

Smooth pasting equation Sp1.4 says that at the optimal thresholds, the underlying and its flexibility pre (left) and post (right) investment have weighted average costs of capital (local expected returns) that balance, i.e. the weighted beta of flex consumed is equal to that gained. When value P is between P_2 and P_1 neither value nor beta matching holds and no action is taken.

This third set of conditions Sp1.4 are as important as Di1.2 and Vm1.3. Together with discounting and value matching they provide six equations in total. These are enough linear constraints to solve for V_{i1}, V_{f1}, V_{i2}, V_{f2}, X_1, X_2 as functions of P_1, P_2, the discount factors D_{i12}, D_{f21} that depend on them and elasticities or betas $\beta_i = a$, $\beta_f = b$. In particular all the dynamics associated with GBM have been locked up in the discount factors and their betas.

Typically the physical attributes (cost rates and frictions) of the underlying P dictate investment and divestment quantities X_2, X_1. Then a solution would require finding values for the values V_{i1}, V_{f1}, V_{i2}, V_{f2} and thresholds P_1, P_2 that solve six equations. This is a non-linear system incapable of solution in P_1, P_2 because thresholds cannot be expressed as a linear set of functions of quantities X_2, X_1 etc. Even though the non-linear system can be solved numerically, solutions must follow a choice of process for P whereas solving for X_2, X_1 first as functions of P_1, P_2 defers this decision.

The separation within Di1.0 allows the diffusion dynamics to be summarized within the discount functions (and matrix) alone. This facilitates valuation because the only relevant dynamics from a particular stochastic process are the discount factor itself (conditional on thresholds) and its sensitivity to those assumed thresholds. Using discount factors and their derivatives (related to betas) stochastic processes can be manipulated separately to their payoffs.

Starting with given policy thresholds that are assumed, corresponding optimal investment and divestment costs can be calculated. Solutions can be formed ad hoc but simultaneous equations are best resolved using matrices.

action, threshold	idle V_i	investment payoff	fully active V_f
open, P_2	V_{i2}	$\overset{P_2-X_2}{\longrightarrow}$	V_{f2}
	$\uparrow$		$\downarrow$
close, P_1	V_{i1}	$\overset{X_1-P_1}{\longleftarrow}$	V_{f1}

Table 1: Hysteresis values $V_{i,f1,2}$ on left and right of value matching (horizontal arrows) occuring at $P_{2,1}$ with payoffs net of $X_{2,1}$. Vertical arrows represent diffusions and discounting.

The matrix structures we develop contain discount and scaled elasticities. These operate on vectors of flexibility values consistent with the economic interpretations of value matching, discounting and smooth pasting.

1.5 Vector and matrix notation

The value matching equations Vm1.3 can be stacked into compact vector form (vectors and matrices are capitalized in bold), each row being evaluated at threshold P_2 or P_1.

$$\begin{array}{ccccccc} \text{End flex} & & \text{Beg. flex} & & \text{Payoff} & & \text{Present value cost} \\ \mathbf{W} & = & \mathbf{U} & + & \mathbf{\Omega} & - & \mathbf{X} \\ \begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix} & = & \begin{bmatrix} V_{f2} \\ V_{i1} \end{bmatrix} & + & \begin{bmatrix} P_2 \\ -P_1 \end{bmatrix} & - & \begin{bmatrix} X_2 = K_2 + \frac{x}{r} \\ -X_1 = K_1 - \frac{x}{r} \end{bmatrix} \end{array} \tag{Vm1.5}$$

It makes sense to allocate flexibility values according to their timing. In Vm1.5 the vector on the left of the equality sign $\mathbf{W} = [V_{i2}, V_{f1}]^{\intercal}$ values[6] flexibility at the end of both states whilst the vector to the right of the equality sign $\mathbf{U} = [V_{f2}, V_{i1}]^{\intercal}$ values flex at the beginning of the next state just after the investment decision and policy payoffs have been achieved. For this reason (also in the investment network, or graph of Table 1) the values on the left of Vm1.5 have been colored in red and those on the right in blue.

[6] $\intercal$ represents the transpose of a vector.

The red beginning flex values are collated in a vector $\mathbf{W}$ and the end blue values in a vector labelled $\mathbf{U}$. For state s and threshold n, this is the most useful way to partition values V_{sn}.

The other two elements on the right of Vm1.5 comprise the net payoff $\boldsymbol{\Omega} - \mathbf{X}$ associated with value matching at transitions; the benefits and costs depend on value at that threshold and investment or divestment sums. These are grouped with P_n and X_n into a policy payoff vector $\boldsymbol{\Omega} = [P_2, -P_1]^\intercal$ and an investment cost vector $\mathbf{X} = [X_2, -X_1]^\intercal$ both of which track the change in (net) investment through their embedded signs. At P_2 the value P_2 is gained at a cost of X_2 and at P_1 the divestment X_1 gain is made net of foregoing value P_1.

With the discounted interpretation of flexibility we have another relationship between the same values at the beginning and the end of a state, that of discounting. This relationship Di1.5 (a compact version of Di1.2) between the two involves first a discount matrix $\mathbf{D}$ with off diagonal elements associated with functions from the GBM options (also its inverse $\mathbf{G}$ in Gr1.5)

$$\begin{array}{ccccc} \mathbf{U} & = & \mathbf{D} & \mathbf{W} \\ \begin{bmatrix} V_{f2} \\ V_{i1} \end{bmatrix} & = & \begin{bmatrix} 0 & D_{f21} \\ D_{i12} & 0 \end{bmatrix} & \begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix} \end{array} \tag{Di1.5}$$

$$\begin{array}{ccccc} \mathbf{W} & = & \mathbf{G} & \mathbf{U} \\ \begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix} & = & \begin{bmatrix} 0 & D_{i12}^{-1} \\ D_{f21}^{-1} & 0 \end{bmatrix} & \begin{bmatrix} V_{f2} \\ V_{i1} \end{bmatrix} \end{array}. \tag{Gr1.5}$$

At this stage, equations Vm1.5 and Di1.5 can be used to solve for X in functions of P_n but without ensuring optimality.[7] The smooth pasting or WCCM condition Sp1.4 in matrix form is given by beta matrices $\boldsymbol{\beta}_W, \boldsymbol{\beta}_U$

[7] $\mathbf{W} = \mathbf{U} + \boldsymbol{\Omega} - \mathbf{X}$ and $\mathbf{U} = \mathbf{DW}$ imply non–optimal $\mathbf{W} = [\mathbf{I} - \mathbf{D}]^{-1} (\boldsymbol{\Omega} - \mathbf{X})$. For given $\mathbf{X}$ a numerical maximization of $\mathbf{W}$ in P_n is possible through the dependence of $\mathbf{D}, \boldsymbol{\Omega}$.

that operate on $\mathbf{W}, \mathbf{U}$ respectively in Sp1.5.

$$\begin{array}{ccccccc} \boldsymbol{\beta}_W & \mathbf{W} & = & \boldsymbol{\beta}_U & \mathbf{U} & + & \boldsymbol{\Omega} \\ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} & \begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix} & = & \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} & \begin{bmatrix} V_{f2} \\ V_{i1} \end{bmatrix} & + & \begin{bmatrix} P_2 \\ -P_1 \end{bmatrix} \end{array} \tag{Sp1.5}$$

These beta matrices operate on value components and carry commensurate elasticities. For GBM it is convenient that idle flex has the same beta a, at the beginning (bottom right in $\boldsymbol{\beta}_U$) as the end (top left in $\boldsymbol{\beta}_W$ – similarly for full flex and b in the other locations).

From Sp1.5 and Di1.5 one of the flexibility vectors (beginning values $\mathbf{U}$) can be eliminated to solve for the other end flex values $\mathbf{W}$ (shown in $\mathbf{W}$1.5)

$$\mathbf{W} = \begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix} = \begin{bmatrix} a & -bD_{f21} \\ -aD_{i12} & b \end{bmatrix}^{-1} \begin{bmatrix} P_2 \\ -P_1 \end{bmatrix}. \tag{$\mathbf{W}$1.5}$$

Beginning flex values $\mathbf{U} = \mathbf{D}\mathbf{W}$ and finally investment quantities $X_{1,2}$ can be derived using Vm1.5 and Di1.5 to form $\mathbf{X} = \mathbf{D}\mathbf{W} + \boldsymbol{\Omega} - \mathbf{W}$.

Although this solution to the hysteresis smooth pasting using betas is only presented for the iso–elastic GBM case, flexibility valuation has been reduced to a system of equations linear in functions of $P_{1,2}$. The key ingredients used were the identification of flexibility elements both at the beginning and end of their lives and their betas. Especially for convoluted systems, this separation allows the discounting between elements to be captured. The next section moves away from GBM and generalizes the standard discount factors for other diffusions.

RN diffusion dP	Discount function by state i, f	a, b solve $q(.) = 0$
$(r-\delta) P dt$ $+\sigma P dW$	$D_i^{GBM}(P, P_2) = \left(\frac{P}{P_2}\right)^a$ $D_f^{GBM}(P, P_1) = \left(\frac{P}{P_1}\right)^b$	$\frac{\sigma^2}{2} a(a-1) +$ $(r-\delta) a - r = 0$
$\alpha dt + \sigma dW$	$D_i^{ABM}(P, P_2) = e^{a(P-P_2)}$ $D_f^{ABM}(P, P_1) = e^{b(P-P_1)}$	$\frac{\sigma^2}{2} a^2 +$ $\alpha a - r = 0$
$\eta\left(\overline{P} - P\right) P dt$ $+\sigma P dW$	$D_f^{HYP}(P, P_1) = \frac{\left(\frac{P}{P_1}\right)^b H\left(\frac{2\eta}{\sigma^2} P, b, 2\left(b + \frac{\eta \overline{P}}{\sigma^2}\right)\right)}{H\left(\frac{2\eta}{\sigma^2} P_1, b, 2\left(b + \frac{\eta \overline{P}}{\sigma^2}\right)\right)}$	$\frac{\sigma^2}{2} b(b-1) +$ $\eta \overline{P} b - r = 0$

Table 2: Discount factors for three different processes

2 Other discount factors

The discount factor approach rests on a future fixed amount paid at a random time associated with a diffusion achieving a pre–specified threshold. This stopping time approach has been studied in Dixit, Pindyck and Sødal (1999) and Sødal (2006). Due to time value and uncertainty, the present value of a future payoff involves discounting and risk, not over the quantity of the payoff (say a dollar) only over a time τ at which P_τ reaches a boundary. The current, dynamic, level is P and the boundary either P_1 (for the put) or P_2 (for the call). Taking risk neutral (RN) expectations E_P^{RN} conditional on P of the continuous discount factor to stochastic time τ yields

$$D_i(P, P_2) = E_P^{RN}\left[e^{-r\tau} \middle| P_\tau = P_2\right] \quad D_f(P, P_1) = E_P^{RN}\left[e^{-r\tau} \middle| P_\tau = P_1\right] \tag{Dfactor2.0}$$

As mentioned $D_i(P, P_2), D_f(P, P_1)$ satisfy an ordinary differential equation in P but subject to different conditions. Bellman equations have analytic solutions for several processes. Dixit, Pindyck and Sødal (1999) detail discount factors for the mean reverting Hypergeometric (HYP) process, which along with GBM, Arithmetic Brownian motion (ABM) and their fundamental quadratics are shown in Table 2.

For other processes even when analytic solutions are not available, numerical solutions and their derivatives can still be used (subject to precision).

The discount functions and betas carry the (dynamic) dependence between current and next threshold flex values whilst V_{i1} etc. are treated as constant. When discount factors are required between two policy thresholds, compact notation $D_{i12} = D_i(P_1, P_2)$ and $D_{f21} = D_f(P_2, P_1)$ is used. Before working with these in the next section, we evaluate the betas of more general claims as a function of dynamic P.

Discount factors are zero yield claims, supporting themselves only via capital gains. Under physical or real world[8] (not RN) probabilities their rates of return are linked to local betas or elasticities $\beta_i(P), \beta_f(P)$. To form these betas, the flex is evaluated at a given level P via discounting from a future payoff **W**. This can then be differentiated with respect to P and multiplied by P. Two versions of $P\frac{\partial V(P)}{\partial P} = \beta(P)V(P)$ in lines result for full f and idle i flexibilities

$$P\frac{\partial}{\partial P}\begin{bmatrix} V_f(P) \\ V_i(P) \end{bmatrix} = \begin{bmatrix} 0 & P\frac{\partial D_f(P,P_1)}{\partial P} \\ P\frac{\partial D_i(P,P_2)}{\partial P} & 0 \end{bmatrix}\begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix}$$
$$= \begin{bmatrix} \beta_f(P) & 0 \\ 0 & \beta_i(P) \end{bmatrix}\begin{bmatrix} V_f(P) \\ V_i(P) \end{bmatrix}$$

where the beta matrix at levels of P is

$$\begin{bmatrix} \beta_f(P) & 0 \\ 0 & \beta_i(P) \end{bmatrix} = \begin{bmatrix} \frac{P}{D_f(P,P_1)}\frac{\partial D_f(P,P_1)}{\partial P} & 0 \\ 0 & \frac{P}{D_i(P,P_2)}\frac{\partial D_i(P,P_2)}{\partial P} \end{bmatrix}. \qquad (\beta P2.0)$$

This is a version of Beta1.1 for general processes (a similar construction can be used for a growth matrix too[9]). Through the formation of new matrices

[8] Unlike risk neutral, using physical or real world expectations $E^{RW}[.]$ to determine local expected returns of dD generates a beta dependent risk premium

$$\frac{E^{RW}[dD(P,.)]}{D(P,.)} = (r + \beta(P) * \text{risk premium})\,dt \qquad \frac{E^{RN}[dD(P,.)]}{D(P,.)} = rdt.$$

[9] As well as a discount matrix for general P, a similar growth matrix and its differential

$\mathbf{D}', \mathbf{G}'$ the next subsection specializes these to two particular values of P namely $[P_2, P_1]^\intercal$ ($P = P_2$ in the first and $P = P_1$ in the second row).

2.1 Smooth pasting matrices

To achieve smooth pasting, differentials of discount factors must be evaluated. To perform this task, the beta sensitivities must be calculated at the thresholds. Discount and growth matrices $\mathbf{D}, \mathbf{G}$ are evaluated at fixed thresholds but we can still evaluate elasticities at these thresholds when understanding that they are taken in the limit as dynamic P hits a threshold.

Taking the value matching condition, we wish to evaluate its slope at thresholds P_2, P_1 through the betas there. As in Section 1, the key is to apply growth (Gr1.5 $\mathbf{W} = \mathbf{G}\mathbf{U}$) and discounting (Di1.5 $\mathbf{U} = \mathbf{D}\mathbf{W}$) projections to Vm1.3.

$$\begin{array}{ccccccc} \mathbf{G} & \mathbf{U} & = & \mathbf{D} & \mathbf{W} & + & \boldsymbol{\Omega} - \mathbf{X} \\ \begin{bmatrix} 0 & D_{i12}^{-1} \\ D_{f21}^{-1} & 0 \end{bmatrix} & \begin{bmatrix} V_{f2} \\ V_{i1} \end{bmatrix} & = & \begin{bmatrix} 0 & D_{f21} \\ D_{i12} & 0 \end{bmatrix} & \begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix} & + & \begin{bmatrix} P_2 - X_2 \\ X_1 - P_1 \end{bmatrix} \end{array} \tag{Vm2.1}$$

Vm2.1 is amenable to slope calculations at each threshold, i.e. at P_2 in the first and P_1 in the second row because the projections have separated the diffusion dynamics (in matrices) from the problem constants (in vectors). The matrix product results from the top row depend on threshold P_2 only through D_{i12}^{-1} and D_{f21} whilst in the second row they depend on threshold P_1 through D_{f21}^{-1} and D_{i12}. Because $\mathbf{D}, \mathbf{G}$ have no leading diagonal elements, when in the first and second rows the threshold slopes of matrix products $[\mathbf{G}\mathbf{U}]$ or $[\mathbf{D}\mathbf{W}]$ are required, the matrices alone bear the effects of differentiation at

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can be formed

$$P\frac{\partial}{\partial P}\begin{bmatrix} V_i(P) \\ V_f(P) \end{bmatrix} = \begin{bmatrix} 0 & P\frac{\partial D_i(P,P_2)^{-1}}{\partial P} \\ P\frac{\partial D_f(P,P_1)^{-1}}{\partial P} & 0 \end{bmatrix}\begin{bmatrix} V_{f2} \\ V_{i1} \end{bmatrix} \text{ etc.}$$

the threshold.

In order to conform with a beta interpretation (Beta1.1), the slope calculation in each row must be multiplied by its threshold value i.e. P_2 and P_1. The joint effect can be captured defining a partial elasticity operator with notation $[\mathbf{D}]', [\mathbf{G}]'$. This indicates row wise differentiation of the latent elements within $[\mathbf{D}]$ or $[\mathbf{G}]$, evaluation of this slope at the threshold, then multiplication by this threshold. The row by row scaled partial elasticities with respect to the thresholds P_2, P_1 required for smooth pasting are captured by two new matrices $\mathbf{D}', \mathbf{G}'$ which (for $\mathbf{D}, \mathbf{G}$ in Di1.5 and Gr1.5) are shown by $\mathbf{D}'$2.1

$$\mathbf{D}' = \begin{bmatrix} P_2 \frac{\partial}{\partial P}\Big|_{P=P_2} & 0 \\ 0 & P_1 \frac{\partial}{\partial P}\Big|_{P=P_1} \end{bmatrix} \mathbf{D} = \begin{bmatrix} 0 & P_2 \frac{\partial D_f(P,P_1)}{\partial P}\Big|_{P=P_2} \\ P_1 \frac{\partial D_i(P,P_2)}{\partial P}\Big|_{P=P_1} & 0 \end{bmatrix} \quad (\mathbf{D}'2.1)$$

and $\mathbf{G}'$2.1

$$\mathbf{G}' = \begin{bmatrix} P_2 \frac{\partial}{\partial P}\Big|_{P=P_2} & 0 \\ 0 & P_1 \frac{\partial}{\partial P}\Big|_{P=P_1} \end{bmatrix} \mathbf{G} = \begin{bmatrix} 0 & P_2 \frac{\partial D_i(P,P_2)^{-1}}{\partial P}\Big|_{P=P_2} \\ P_1 \frac{\partial D_f(P,P_1)^{-1}}{\partial P}\Big|_{P=P_1} & 0 \end{bmatrix}. \quad (\mathbf{G}'2.1)$$

From $\mathbf{D}'$ the beta matrix $\boldsymbol{\beta}_U$ for the beginning flexibilities $\mathbf{U}$

$$\begin{aligned} \mathbf{D}'\mathbf{G} &= \begin{bmatrix} 0 & P_2 \frac{\partial D_f(P,P_1)}{\partial P}\Big|_{P=P_2} \\ P_1 \frac{\partial D_i(P,P_2)}{\partial P}\Big|_{P=P_1} & 0 \end{bmatrix} \begin{bmatrix} 0 & D_{i12}^{-1} \\ D_{f21}^{-1} & 0 \end{bmatrix} \quad (\beta\mathbf{U}2.1) \\ &= \begin{bmatrix} \frac{P_2}{D_{f21}} \frac{\partial D_f(P,P_1)}{\partial P}\Big|_{P=P_2} & 0 \\ 0 & \frac{P_1}{D_{i12}} \frac{\partial D_i(P,P_2)}{\partial P}\Big|_{P=P_1} \end{bmatrix} = \begin{bmatrix} \beta_{f2} & 0 \\ 0 & \beta_{i1} \end{bmatrix} = \boldsymbol{\beta}_U \end{aligned}$$

Similarly, from $\mathbf{G}'$ the beta matrix $\boldsymbol{\beta}_W$ for the end flexibilities $\mathbf{W}$ is

$$\mathbf{G}'\mathbf{D} = \begin{bmatrix} 0 & P_2 \left. \frac{\partial D_i(P,P_2)^{-1}}{\partial P} \right|_{P=P_2} \\ P_1 \left. \frac{\partial D_f(P,P_1)^{-1}}{\partial P} \right|_{P=P_1} & 0 \end{bmatrix} \begin{bmatrix} 0 & D_{f21} \\ D_{i12} & 0 \end{bmatrix} \quad (\beta \mathbf{W}2.1)$$
$$= \begin{bmatrix} \frac{P_2}{D_{i12}^{-1}} \left. \frac{\partial D_i(P,P_2)^{-1}}{\partial P} \right|_{P=P_2} & 0 \\ 0 & \frac{P_1}{D_{f21}^{-1}} \left. \frac{\partial D_f(P,P_1)^{-1}}{\partial P} \right|_{P=P_1} \end{bmatrix} = \begin{bmatrix} \beta_{i2} & 0 \\ 0 & \beta_{f1} \end{bmatrix} = \boldsymbol{\beta}_W.$$

2.2 Smooth pasting with betas

As discussed in Section 1, it is easier to understand the smooth pasting conditions and flexibility solutions by relating them to weighted cost of capital and beta matching. This is possible because the scaled elasticities of the present value discount factors are relative betas ($\beta\mathbf{W}, \mathbf{U}2.1$).

Following value matching (Vm1.5) $\mathbf{W} = \mathbf{U} + \boldsymbol{\Omega} - \mathbf{X}$, and forward and backward projections $\mathbf{G}'\mathbf{U} = \mathbf{D}'\mathbf{W} + \boldsymbol{\Omega}'$ smooth pasting can be effected using matrices $\boldsymbol{\beta}_W, \boldsymbol{\beta}_U$ which operate on $\mathbf{W}, \mathbf{U}$ respectively (a more general version of Sp1.5)

$$\boldsymbol{\beta}_W \mathbf{W} = \boldsymbol{\beta}_U \mathbf{U} + \boldsymbol{\beta}_\Omega \boldsymbol{\Omega}. \quad \text{(Sp2.2)}$$

Since $\boldsymbol{\Omega}' = \left[P_2 \left. \frac{\partial \Omega_2}{\partial P} \right|_{P=P_2}, P_1 \left. \frac{\partial \Omega_1}{\partial P} \right|_{P=P_1} \right]^\intercal = [P_2, -P_1]^\intercal = \boldsymbol{\beta}_\Omega \boldsymbol{\Omega}$ for hysteresis the beta payoff $\boldsymbol{\beta}_\Omega = \mathbf{I}$ (payoffs are linear and $\boldsymbol{\Omega} = \boldsymbol{\Omega}'$ carry sign changes). Sp2.2 represents weighted cost of capital matching (WCCM) at thresholds[10]. Writing Sp2.2 out in full would, on the first line of the beta matrices show the idle (call) option value weighted by its beta $\beta_{i2} V_{i2}$ equating the full (put) option value times its beta $\beta_{f2} V_{f2}$ plus the payoff P_2 (times its unit beta). The sunk costs X_2, X_1 have zero beta so do not contribute to this equation or the excess cost of capital above the risk free rate.

[10] When $\boldsymbol{\beta}_U$ premultiplies $\mathbf{U}$ the relative costs of capital are attributed to the correct value components $\boldsymbol{\beta}_U \mathbf{U} = [\beta_{f2} V_{f2}, \beta_{i1} V_{i1}]^\intercal$. This is an essential part of forming the scaled cost of capital matching (WCCM) condition that works as smooth pasting. When $\boldsymbol{\beta}_W$ operates on $\mathbf{W}$ again the correct relative costs of capital are allocated to each component $\boldsymbol{\beta}_W \mathbf{W} = [\beta_{i2} V_{i2}, \beta_{f1} V_{f1}]^\intercal$.

In the second line, the beta weighted full (put) option at its exercise threshold $\beta_{f1}V_{f1}$ equates to the idling (opening) option weighted by its beta $\beta_{i1}V_{i1}$ less the unit beta times the loss of value $-P_1$. This representation of smooth pasting with betas is intuitive and allows weighted cost of capital matching to be derived for other diffusions (from Table 2).

2.3 Simultaneous matching, pasting and discounting

Now we have three matrix equations Di1.5, Vm2.1 and Sp2.2 for the vectors $\mathbf{W}, \mathbf{U}, \mathbf{X}$ (each has as many lines as thresholds). These equations contain matrices $\mathbf{D}$, $\mathbf{G}$, $\mathbf{D}'$, $\mathbf{G}'$ along with the payoff vector $\boldsymbol{\Omega}$, and its sensitivity $\boldsymbol{\Omega}' = \boldsymbol{\beta}_\Omega \boldsymbol{\Omega}$ all of which depend on P_2, P_1. The sensitivity matrices were converted into beta matrices $\boldsymbol{\beta}_W = \mathbf{G}'\mathbf{D}$ and $\boldsymbol{\beta}_U = \mathbf{D}'\mathbf{G}$. Three sets of equations can be stacked with (groups of) rows representing value matching, discounting and smooth pasting simultaneously ($\mathbf{I}$ is the identity matrix and $\mathbf{0}$ one of zeros).

$$\begin{bmatrix} \mathbf{I} & -\mathbf{I} & \mathbf{I} \\ \mathbf{D} & -\mathbf{I} & \mathbf{0} \\ \boldsymbol{\beta}_W & -\boldsymbol{\beta}_U & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{U} \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Omega} \\ \mathbf{0} \\ \boldsymbol{\beta}_\Omega \boldsymbol{\Omega} \end{bmatrix} \quad \begin{matrix} \text{Vm} \\ \text{Di} \\ \text{Sp} \end{matrix} \qquad \text{(System2.3)}$$

The first line is a vector equation (the only matrix involved is the identity, $\mathbf{I}$). Moreover since X_n are not involved in the elasticities or discounting they do not play a direct role in flex values solutions in P_n. The system determines $\mathbf{U}, \mathbf{W}$ from a block in the lower left hand corner. If we define the system matrix $\mathbf{S}$, the joint vector of values $\mathbf{V}$, can be solved

$$\mathbf{S} = \begin{bmatrix} \mathbf{D} & -\mathbf{I} \\ \boldsymbol{\beta}_W & -\boldsymbol{\beta}_U \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} \mathbf{W} \\ \mathbf{U} \end{bmatrix} = \mathbf{S}^{-1} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\beta}_\Omega \boldsymbol{\Omega} \end{bmatrix}. \qquad \text{(Sol2.3)}$$

2.4 Flexibility returns at investment

The value matching equations link flexibility values on either side of an investment decision. The beta matching equations link the local rates of return at these transitions too. The WCCM within Sp2.2 shows that prior to investment, the local rate of return and elasticity of flex is driven by $\boldsymbol{\beta}_W \mathbf{W}$ but after the transition is effected (and $\mathbf{X}$ is sunk/recouped) it is driven by $\boldsymbol{\beta}_U \mathbf{U}$, that of the new flex, plus that of the payoff $\boldsymbol{\beta}_\Omega \boldsymbol{\Omega}$. This is an intuitive condition on the linkage between, and constraints on, flexibility values. From this equalization and the discounting relationship, end flex values are isolated easily from Sp2.2. From $\boldsymbol{\beta}_W \mathbf{W} = \boldsymbol{\beta}_U \mathbf{U} + \boldsymbol{\beta}_\Omega \boldsymbol{\Omega}$ substituting for $\mathbf{U}$ and collecting terms $[\boldsymbol{\beta}_W - \boldsymbol{\beta}_U \mathbf{D}]\, \mathbf{W} = \boldsymbol{\beta}_\Omega \boldsymbol{\Omega}$ yields

$$\mathbf{W} = [\boldsymbol{\beta}_W - \boldsymbol{\beta}_U \mathbf{D}]^{-1} \boldsymbol{\beta}_\Omega \boldsymbol{\Omega}. \qquad (\mathbf{W}2.4)$$

The WCCM in Sp2.2 represents dollar rates of return or value weighted beta equalization, whilst **W**2.4 represents flexibility present value as a perpetuity (ratio) of $\boldsymbol{\beta}_\Omega \boldsymbol{\Omega}$ pre–multiplied by an (inverse) factor $[\boldsymbol{\beta}_W - \boldsymbol{\beta}_U \mathbf{D}]^{-1}$.

This last element $\boldsymbol{\beta}_W - \boldsymbol{\beta}_U \mathbf{D}$ is easy to interpret as a net beta, it is the immediate beta $\boldsymbol{\beta}_W$ of $\mathbf{W}$ less that derived from $\mathbf{U}$ which is expressed as a discounted product. The amount by which the immediate beta needs to be decreased, $-\boldsymbol{\beta}_U \mathbf{D}$ is a beta matrix on $\mathbf{U}$ discounted by the expected factors until its component betas are applied.

Furthermore, for a given risk free rate and risk premium, a beta equation maps into an excess return equation. This is true of the row calculations within $[\boldsymbol{\beta}_W - \boldsymbol{\beta}_U \mathbf{D}]$ meaning that its inverse can be treated as a perpetuity factor matrix.

Thus the implication for flexibility values $\mathbf{W}$ is that they depend on aperiodic perpetual payoffs proportional to $\boldsymbol{\beta}_\Omega \boldsymbol{\Omega}$ with their relative timing carried though a perpetuity factor matrix $[\boldsymbol{\beta}_W - \boldsymbol{\beta}_U \mathbf{D}]^{-1}$. Cycles are of different expected lengths for each component but the matrices carry the magnitude and timing of each through their embedded discount functions. A similar grown beta argument applies to $\mathbf{U}$ and $[\boldsymbol{\beta}_W \mathbf{G} - \boldsymbol{\beta}_U]^{-1}$ as can be seen in **U**2.4

$$\mathbf{U} = [\boldsymbol{\beta}_W \mathbf{G} - \boldsymbol{\beta}_U]^{-1} \boldsymbol{\beta}_\Omega \boldsymbol{\Omega}. \tag{U2.4}$$

To summarize, in a system where flexibility is sequential and its use begets another form, optimal timing is dictated by a net discounted condition applied to value betas. When optimally used, maximum flexibility values (elements of $\mathbf{U}$ and $\mathbf{W}$) satisfy relative beta or net sensitivity conditions on their interlinking. A discounted net beta operator $[\boldsymbol{\beta}_W \mathbf{G} - \boldsymbol{\beta}_U]$ or $[\boldsymbol{\beta}_W - \boldsymbol{\beta}_U \mathbf{D}]$ applied to their values $\mathbf{U}$ and $\mathbf{W}$ matches the sensitivities of their payoffs at exercise. Note that these conditions are matrices which can carry more complex interactions between each of the components than hysteresis.

Since the fixed costs have zero elasticity, the values of X_n that correspond to the assumed thresholds P_n are not determined through betas. Instead $\mathbf{X}$ is derived from value matching $\mathbf{X} = \mathbf{U} + \boldsymbol{\Omega} - \mathbf{W}$.

This solution method is robust to many different setups. These equations were demonstrated for simple hysteresis but they are much more general and the next sections show how to expand the matrix contents for different investment problems and different stochastic processes. Although the components of discount, growth matrices, payoff vector $\mathbf{D}, \mathbf{G}, \boldsymbol{\Omega}$ and scaled elasticities $\mathbf{D}'$,$\mathbf{G}'$, $\boldsymbol{\Omega}'$ (or more intuitively beta matrices $\boldsymbol{\beta}_W, \boldsymbol{\beta}_U, \boldsymbol{\beta}_\Omega$) change, the same solution technique and formulae can be used. Section 3 shows a longer sequence and how to apply this solution method to incorporate different stochastic processes.

action, thr.	idle V_i	p.off	full V_f	p.off	idle V_I	p.off	full V_F
open, P_4	V_{i4}	$\overset{P_4-X_4}{\longrightarrow}$	V_{f4}				
	$\uparrow$		$\downarrow$				
open, P_3					V_{I3}	$\overset{P_3-X_3}{\longrightarrow}$	V_{F3}
	$\uparrow$		$\downarrow$		$\uparrow$		$\downarrow$
close, P_2			V_{f2}	$\overset{X_2-P_2}{\longrightarrow}$	V_{I2}		
	$\uparrow$						$\downarrow$
close, P_1	V_{i1}			$\overset{X_1-P_1}{\longleftarrow}$			V_{F1}

Table 3: Serial hysteresis flexibility values $V_{i,f,I,F}$ red before and blue after value matching transitions (horizontal arrows) occuring at $P_{4,2,3,1}$ with payoffs net of $X_{4,2,3,1}$. Vertical arrows represent diffusions and discounting.

3 Serial hysteresis and changing dynamics

In this Section we employ an extended system for tracking switching decisions again at four different thresholds (random times). This is similar to Ekern (1993) who evaluates operational flexibility in a sequential investment/divestment situation but without cyclicality. The firm there has limited capacity to switch between idle and full status a finite number of times before losing all flexibility. Whilst such situations can be solved in reverse from a terminal condition, the aim here is to solve cyclical systems without being able to refer to terminal conditions. Having four operating cycles also allows up to four different stochastic processes to be accommodated within the investment stages.

What flexibility value applies depends on the number and nature of (potentially limited) switching opportunities. Here sequential and cyclical decisions at thresholds P_{1-4} are labelled in level order (not their sequence, in time order these are encountered in the sequence $4, 2, 3, 1$). Using colors again facilitates identification of different flexibilities within state (blue at the beginning and red at the end). The graph in Table 3 reflects the time line and usage of sequential timing flexibility.

Transitions (opening or closing across row) are labelled in the left hand column of Table 3. States (in columns) can either be idle (off) or full (on) with these labelled by subscript, e.g. $V_{i,f}$ whilst idle, full (e.g. P_1 to P_4 or P_4 to P_2). Idle and full states that occur later depend upon different transition costs ($X_3 <> X_1$) or different diffusion dynamics for P. This means they have different values and discount functions and must be labelled V_I, V_F. At the two thresholds P_4, P_3 opening occurs whilst at P_2, P_1 closing occurs. Note that although a full sequence is implied here, like hysteresis on occasion this could get stuck, either open at a very high price, or closed with a low one (this is indicated by a column having an open top or bottom).

By allowing for a difference between X_1 and X_3 (or X_4 and X_2), as well as Ekern (1993) this also generalizes single (Dixit (1989)) to serial hysteresis allowing for different cost rates with each cycle of operation. In full on state V_f for example, if rates and required capital were higher than those in state V_F, then $X_4 > X_3$. Similarly the present value of spared operational costs and recovered capital at the closing thresholds may be different $X_1 <> X_2$. These are all captured in Vm3.0.

$$\begin{array}{ccccccc} \mathbf{W} & = & \mathbf{U} & + & \mathbf{\Omega} & - & \mathbf{X} \\ \begin{bmatrix} V_{i4} \\ V_{I3} \\ V_{f2} \\ V_{F1} \end{bmatrix} & = & \begin{bmatrix} V_{f4} \\ V_{F3} \\ V_{I2} \\ V_{i1} \end{bmatrix} & + & \begin{bmatrix} P_4 \\ P_3 \\ -P_2 \\ -P_1 \end{bmatrix} & - & \begin{bmatrix} X_4 = \frac{x_f}{r} + K_4 \\ X_3 = \frac{x_F}{r} + K_3 \\ -X_2 = K_2 - \frac{x_f}{r} \\ -X_1 = K_1 - \frac{x_F}{r} \end{bmatrix} \end{array} \tag{Vm3.0}$$

where x_f, x_F are the operational cost rates associated with the policy in the different phases and r the risk free rate. Opening and closing frictions K_n are incurred at the end of all idle and full states. On opening the PV cost rate must be incurred plus an additional amount whilst on closing, the saving is less than the PV operational cost. These can be customized to the nature of each transition through x_f, x_F etc.

The other form of customization that can occur, is to presume that the market dynamics will be different in each operational stage. Different dynamics from Table 2 can be combined inside a customized **D**. This would be important when investments are large and their market impact needs to be reflected through different dynamics pre and post investment.

In the full on states, using D_{f42}^{ABM} for arithmetic (which has parameters α, σ_A, r) and D_{F31}^{HYP} for hypergeometric $\left(\eta, \overline{P}, \sigma_H, r\right)$ along with two GBMs in the idle states D_{i14}^{GBM}, D_{I23}^{GBM} but with potentially different convenience (dividend) yields δ_i, δ_I and volatilities σ_i, σ_I, forms a customized dynamic system **D** shown in Di3.0 ($\mathbf{G} = \mathbf{D}^{-1}$ is also required, parameters α, r, η etc. are shown to indicate which are relevant at each investment stage)

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & D_{f42}^{ABM} & 0 \\ 0 & 0 & 0 & D_{F31}^{HYP} \\ 0 & D_{I23}^{GBM} & 0 & 0 \\ D_{i14}^{GBM} & 0 & 0 & 0 \end{bmatrix}. \tag{Di3.0}$$

Both **D** and **G** are amenable to row wise differentiation and P scaling to facilitate smooth pasting. The required matrices are $\mathbf{D}' =$

$$\begin{bmatrix} 0 & 0 & P_4 \left.\frac{\partial D_f^{ABM}(P,P_2)}{\partial P}\right|_{P=P_4} & 0 \\ 0 & 0 & 0 & P_3 \left.\frac{\partial D_F^{HYP}(P,P_1)}{\partial P}\right|_{P=P_3} \\ 0 & P_2 \left.\frac{\partial D_I^{GBM}(P,P_3)}{\partial P}\right|_{P=P_2} & 0 & 0 \\ P_1 \left.\frac{\partial D_i^{GBM}(P,P_4)}{\partial P}\right|_{P=P_1} & 0 & 0 & 0 \end{bmatrix} \tag{$\mathbf{D}'3.0$}$$

and $\mathbf{G}' =$

$$\begin{bmatrix} 0 & 0 & 0 & P_4 \left. \frac{\partial D_i^{GBM}(P,P_4)^{-1}}{\partial P} \right|_{P=P_4} \\ 0 & 0 & P_3 \left. \frac{\partial D_I^{GBM}(P,P_3)^{-1}}{\partial P} \right|_{P=P_3} & 0 \\ P_2 \left. \frac{\partial D_f^{ABM}(P,P_2)^{-1}}{\partial P} \right|_{P=P_2} & 0 & 0 & 0 \\ 0 & P_1 \left. \frac{\partial D_F^{HYP}(P,P_1)^{-1}}{\partial P} \right|_{P=P_1} & 0 & 0 \end{bmatrix} \quad (\mathbf{G}'3.0)$$

These matrices are used to form the betas (as $\mathbf{G}'\mathbf{D} = \boldsymbol{\beta}_W$ by $\beta\mathbf{W}2.1$ and $\mathbf{D}'\mathbf{G} = \boldsymbol{\beta}_U$ by $\beta\mathbf{U}2.1$) required for WCCM in Sp2.2 as well as for the flex solutions in **W**2.4 and **U**2.4 as a function of the payoff vector $\boldsymbol{\Omega} = [P_4, P_3, -P_2, -P_1]^\intercal$ (and $\boldsymbol{\beta}_\Omega = \mathbf{I}$).

Traditionally, four inputs of cost rates (within **X**) would have been provided with sufficient conditions (eight, from Vm3.0 and a four line version of Sp2.2) to pin down four option constants and four output thresholds. We assumed four optimal thresholds, split flexibilities into two per state (eight in total) and used twelve conditions (in three matrix expressions) to retrieve their eight values along with the four investment and divestment costs. This worked because of the extra set of discounting conditions in **D**.

If solved $\mathbf{X} = \mathbf{U} + \boldsymbol{\Omega} - \mathbf{W}$ values do not match the underlying's properties, then it remains a numerical task to iterate on P_{1-4} until target X_{1-4} are reached (alternatively for fixed **X** the non–optimal equation in footnote 7 can be maximized by iteration on P_{1-4}). Whilst these searches remain numerical, considerable tractability has been generated for use in the computational process.

Moreover, after the design of the flexibility investment graph, diffusions can be combined in a modular fashion and tested. The flexibility determines where the entries occur within **D** but the processes in play during each state are captured functionally and economically through the discount timing (present values) and discount slopes (returns). Note that different processes support different ranges of P (GBM has $P > 0$) so care must be taken to make sure that transitions occur into acceptable regions.

For even longer sequences, if the matrix $\mathbf{D}$ that represents the investment graph can be inverted and differentials taken, any system can be solved via construction of flexibility value vectors $\mathbf{U}$,$\mathbf{W}$ whose components are identified by state and threshold. Next we look at a more convoluted investment graph.

4 Two way flexibility

In this Section we illustrate a more complex investment scenario involving a third state that offers either the chance to further invest if things go well or the chance to recoup some investment if not. First we need to construct discount factors that can accommodate two different diffusion outcomes. This is done with discount factors that are subject to knockout features.

4.1 Discount factors with knockouts

It is possible to construct discount factors which yield a unit payment at one threshold but are subject to a knockout possibility and zero value at a second, attainable, threshold. For example starting from $P = P_3$ there might be two thresholds of interest $P_4 > P_3$ and $P_1 < P_3$. At P_4 a payoff might be attained but at P_1 any flexibility could become worthless. This might occur when another flex causes the first to be knocked out and become worthless.

Knockout discount factors for state s are defined with three thresholds in their argument D_{s341}. The first threshold in the argument is the current level e.g. $P = P_3$, the second P_4 the target or knock in level and the third e.g. P_1 the level at which the flexibility dies or is knocked out. The complimentary knock out D_{s314} does the opposite. Both rely on one threshold being reached before the other, and have $1, 0$ boundary conditions in Dfactors4.1.

$$\begin{aligned} D_{s341} &= E_{P_3}^{RN}\left[e^{-r\tau}\middle| P_\tau = P_4, \min P > P_1\right] \\ D_{s314} &= E_{P_3}^{RN}\left[e^{-r\tau}\middle| P_\tau = P_1, \max P < P_4\right] \\ D_{s441} &= 1 \quad D_{s141} = 0 \quad D_{s114} = 1 \quad D_{s414} = 0 \end{aligned} \tag{Dfactors4.1}$$

They are complimentary in the sense that if one pays off the other dies and vice versa. These claims can be priced as linear combinations of simple factors D_{s34} and D_{s31}. Constructing c, d multiples of these, the boundary conditions can be met normalizing with D_{s41}, D_{s14}

$$\begin{aligned} D_{s341} &= cD_{s34} + dD_{s31} = \frac{D_{s34} - D_{s31}D_{s14}}{1 - D_{s41}D_{s14}} \\ 1 &= c + dD_{s41} \qquad 0 = cD_{s14} + d \\ D_{s314} &= \frac{D_{s31} - D_{s34}D_{s41}}{1 - D_{s14}D_{s41}}. \end{aligned} \tag{Dsoln4.1}$$

Although for GBM the component discount factors D_{s34}, D_{s31} have constant local betas, when mixed in this fashion D_{s314}, D_{s341} have dynamic betas because the numerator (but not the denominator) weights will change as P moves away from P_3. With respect to dynamic P, these discount factors also need differentiation and (scaled) slope evaluation at $P = P_3, P_4, P_1$.

4.2 Two way discounting

With knock in and knockout factors in place, a situation can now be evaluated where one of two remote (in time) payoffs can be achieved, say V_{s4} at P_4 or V_{s1} at P_1. From an initial threshold P_3 ($P_1 < P_3 < P_4$) another linear combination can be constructed to reflect two possible outcomes. This yields an expected discounted value of V_{s3} at the time and location of P_3 as a

from/to thr.	idle V_i	p.off	power V_p	p.off	full V_f
power/full P_4			V_{p4} $\uparrow$	$\overset{P_4-P_4^\gamma-X_4}{\longrightarrow}$	V_{f4} $\downarrow$
full/power P_3			$\uparrow$ V_{p3}	$\overset{P_3^\gamma-P_3+X_3}{\longleftarrow}$	V_{f3}
idle/power P_2	V_{i2} $\uparrow$	$\overset{P_2^\gamma-X_2}{\longrightarrow}$	V_{p2} $\downarrow$ $\downarrow$		
power/idle P_1	V_{i1}	$\overset{X_1-P_1^\gamma}{\longleftarrow}$	V_{p1}		

Table 4: Elasticity ladder with flexibility states Idle, Power and Full flow $V_{i,p,f}$; red before and blue after value matching transitions (horizontal arrows) occuring at $P_{1,2,3,4}$ with payoffs net of $X_{1,2,3,4}$. Vertical arrows represent diffusions and discounting.

function of two mutually exclusive payoffs V_{s4}, V_{s1}

$$V_{s3} = D_{s341}V_{s4} + D_{s314}V_{s1}. \tag{Di4.2}$$

This is a combined discounting equation, since we wish to treat V_{s4} and V_{s1} as end of state values in $\mathbf{W}$ any matrix representing two way discounting will have two entries on line P_3. Within state s and region P_1, P_4 the process for P could still follow any diffusion of choice so long as the one way discount factors can be found analytically or numerically and combined. Furthermore, because the flexibility and discount factors pertain to state s and region P_1, P_4 alone, in subsequent states and regions different processes can be used.

4.3 Investment ladder

We can now consider a system with the flexibility to ratchet up or down a "ladder" with at least three investment states; idle i, full f and introducing an intermediate power state p with elasticity γ of an underlying flow[11]. We

[11] E.g. $0 < \gamma < 1$ is a convergence condition for the PV of the power of a GBM diffusion for flow $p(t)$ in $P^\gamma = E_P^{RN}\left[\int_0^\infty p(t)^\gamma e^{-rt}dt\right]$. More ladder regions can be considered using different γ_1, γ_2 etc.

use the same algebra but vary the matrix and vector contents in order to examine this new flexibility setup.

Operational value is again nil in the idle state, but now varies with P^{γ} in a new power state and full (power 1) in the top state. This leads to non-flex values varying with level according to $0, P^{\gamma}$ and P but controlled by state. Starting with the transition at the highest threshold, from power to full, one control occurs at threshold P_4 with a payoff of $P_4 - P_4^{\gamma} - X_4$ net of exercise cost. That is to say that at the top threshold on going to full, P_4 is gained but P_4^{γ} is lost along with incremental investment X_4.

Reversion can occur at P_3 yielding $P_3^{\gamma} - P_3 + X_3$ a reverse payoff including a partial return of fixed investment cost $X_3 < X_4$. Similar transitions occur at P_2 into the power state from the idle and P_1 back to the idle from the power. These last two have investment exercise prices of X_2 and X_1 $(X_1 < X_2)$.

The value matching conditions within this system are shown in Vm4.3 (different contents to previous sections).

$$\begin{array}{ccccccc} \mathbf{W} & = & \mathbf{U} & + & \mathbf{\Omega} & - & \mathbf{X} \\ \begin{bmatrix} V_{p4} \\ V_{f3} \\ V_{i2} \\ V_{p1} \end{bmatrix} & = & \begin{bmatrix} V_{f4} \\ V_{p3} \\ V_{p2} \\ V_{i1} \end{bmatrix} & + & \begin{bmatrix} P_4 - P_4^{\gamma} \\ P_3^{\gamma} - P_3 \\ P_2^{\gamma} \\ -P_1^{\gamma} \end{bmatrix} & - & \begin{bmatrix} X_4 \\ -X_3 \\ X_2 \\ -X_1 \end{bmatrix} \end{array} \qquad \text{(Vm4.3)}$$

For the three states, four thresholds P_{1-4} and switching costs X_{1-4} the new investment/divestment graph is given in Table 4. This graph (and Tables 1 and 3) are bipartite and directed[12] but this one requires two way discount factors (discount matrix entries) within the power state. Note that the extra state notation V_p required for two way flexibility in the power state along

[12] Bipartite in the sense that investment nodes are either beginning (blue) or end (red) of states and from one you can only move to the other type. Directed in the sense that each link between nodes is only forward and does not offer the opportunity for immediate reversal (see Wilson (1985)).

with two new possible outcomes of entering the power state, either from the idle or full conditions (rows two and three) from which branching is possible.

From the power state, since reversion to the idle state is possible (at P_1) as well as elevation to the full state (at P_4), the discount matrix is populated with six elements, and in particular two rows now contain complementary discount factors with mutually exclusivity, i.e. conditional upon each other not paying off. For example in row three (within the power state) the element D_{p341} represents from the point of view of threshold P_3, the discounted chance of paying a dollar at P_4 knowing that if P_1 is reached the opportunity dies (zero value).

The second D_{p314} in row two is complementary and pays off at P_1 assuming P_4 is not hit. Other discount factors are interpreted similarly but D_{f43}, D_{i12} are one way factors in full and idle states respectively. The detailed discounting equations $\mathbf{U} = \mathbf{D}\mathbf{W}$ are thus given in Di4.3 (different contents to prior Di equations).

$$\begin{array}{ccc} \mathbf{U} & = & \mathbf{D} & \mathbf{W} \\ \begin{bmatrix} V_{f4} \\ V_{p3} \\ V_{p2} \\ V_{i1} \end{bmatrix} & = & \begin{bmatrix} 0 & D_{f43} & 0 & 0 \\ D_{p341} & 0 & 0 & D_{p314} \\ D_{p241} & 0 & 0 & D_{p214} \\ 0 & 0 & D_{i12} & 0 \end{bmatrix} & \begin{bmatrix} V_{p4} \\ V_{f3} \\ V_{i2} \\ V_{p1} \end{bmatrix} \end{array} \qquad \text{(Di4.3)}$$

The inverse discount matrix $\mathbf{D}^{-1} = \mathbf{G}$ exists[13] and is shown in Gr4.3 but it is harder to interpret since it has negative elements. These occur because within the power state, hitting one boundary and achieving a payoff is conditional on not hitting the other, thus it is not certain where each path originated

[13]Without the existence of $\mathbf{G}$ a one to one, unique, mapping between the elements in $\mathbf{U}, \mathbf{W}$ could not be assured.

(consistent with Markovian processes).

$$\mathbf{G} = \begin{bmatrix} 0 & \frac{D_{p214}}{D_{p214}D_{p341}-D_{p241}D_{p314}} & -\frac{D_{p314}}{D_{p214}D_{p341}-D_{p241}D_{p314}} & 0 \\ \frac{1}{D_{f43}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{D_{i12}} \\ 0 & -\frac{D_{p241}}{D_{p214}D_{p341}-D_{p241}D_{p314}} & \frac{D_{p341}}{D_{p214}D_{p341}-D_{p241}D_{p314}} & 0 \end{bmatrix} \quad \text{(Gr4.3)}$$

Finally along with the line by line derivatives of $\mathbf{D}, \mathbf{G}$ (with respect to $P_{4,3,2,1}$) the scaled sensitivity of the payoff vector with respect to the thresholds $\mathbf{\Omega}' = \boldsymbol{\beta}_{\Omega}\mathbf{\Omega} = [P_4 - \gamma P_4^{\gamma}, \gamma P_3^{\gamma} - P_3, \gamma P_2^{\gamma}, -\gamma P_1^{\gamma}]^{\intercal}$ is required[14] in order to construct the solution from smooth pasting $\mathbf{U} = [\boldsymbol{\beta}_W \mathbf{G} - \boldsymbol{\beta}_U]^{-1} \boldsymbol{\beta}_{\Omega}\mathbf{\Omega}$.

Numerical results for this system under GBM are shown in the Appendix for $P_{4,3,2,1} = 4, 3, 2, 1$ with $a, b, \gamma = 2, -1, 0.5$.[15] Figure 4 plots the relationship between the flexibility values (on the vertical) and underlying value (on the horizontal; note that due to the transposition, value matching occurs vertically at fixed P_n). The Appendix also shows values for certain limiting cases when flexibility disappears.

The advantage of this solution system is apparent. It is modular in the sense that cashflow components and their intertwined flexibility features can be placed within systems governed by a common framework and solution method. This is an important aspect of sequential investment. With two way discount factors a system that embraces different scenarios (i.e. 2,4,3,1 or 2,1,2,1 etc.) can be modeled.

[14] As before $\mathbf{G}'\mathbf{D} = \boldsymbol{\beta}_W$, $\mathbf{D}'\mathbf{G} = \boldsymbol{\beta}_U$ but now $\mathbf{\Omega}' \neq \mathbf{\Omega}$, so $\boldsymbol{\beta}_{\Omega} \neq \mathbf{I}$.

[15] In this example P_{4-1} were selected and optimal X_{4-1} calculated using System3.3. If these X_{4-1} do not conform to known policy costs, P_{4-1} can be varied until X_{4-1} match them. The second order conditions in the next subsection should be checked to ensure flex values are maxima. Alternatively using Vm and Di ($\mathbf{W} = \mathbf{U} + \mathbf{\Omega} - \mathbf{X}$, $\mathbf{U} = \mathbf{D}\mathbf{W}$ but not Sp) non optimal flex values can be mapped straight from X_n via $\mathbf{W} = [\mathbf{I} - \mathbf{D}]^{-1} [\mathbf{\Omega} - \mathbf{X}]$. Then P_{4-1} must still be varied numerically to find the turning point in values $\mathbf{W}$ etc.

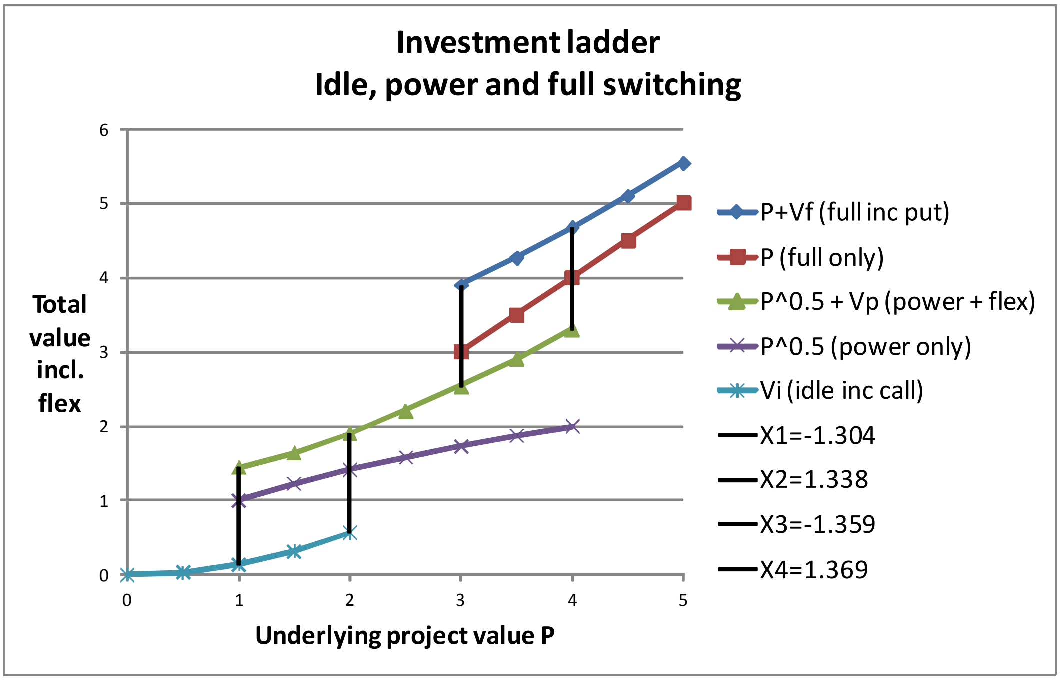

Figure 4: Investment ladder $P_{1..4} = 1..4$ example for GBM and power with $a, b, \gamma = 2, -1, 0.5$. Table 4 shows the investment graph whilst this plot shows the idle, power and full flex values with their investment quantities X_{1-4} with value mathching occuring vertically.

4.4 Maximum flexibility values

There are several different types of restriction on thresholds P_{1-4}. Firstly for GBM (and other diffusions but not arithmetic Brownian motion) thresholds must be positive. Secondly in order for the discount functions to make sense, pairs must satisfy inequalities such as $P_2 > P_1$ ($P_4 > P_3$ but $P_3 <> P_2$) This immediately constrains the flex values $\mathbf{U}$, $\mathbf{W}$ and outputs X_{1-4}. We would also like flexibility to be an asset (not a liability, negative) and therefore restrict solutions to $\mathbf{W} > \mathbf{0}, \mathbf{U} > \mathbf{0}$.

Smooth pasting is ensured by the equations Sp2.2, but these define a unique turning point only. Any proposed solution should also be confirmed as a maximum and neither a minimum nor a saddle point in value. Since we do not wish to construct more matrices of differentials than necessary, here for given $\mathbf{X}$ from the proposed solution, we recommend a numerical investigation of the second order conditions required to establish maximization.

For each threshold in turn, values of P_n can be perturbed a small amount. Given this fixed or target $\mathbf{X}$ (i.e. not allowing X_n to vary with P_n) the effect on flex values indicates if the proposed solutions are maxima, minima or saddle points. Bearing in mind that there are multiple thresholds and multiple flex values, we must take care to identify which flex values within $\mathbf{W}, \mathbf{U}$ we would expect to have zero slope as a function of each threshold change or non zero slope.

Smooth pasting means that, line by line, thresholds P_n are chosen to equalize slopes across value matchings. Although smooth pasting says that option pairs within each value matching line are optimally co–determined, it does not say that values (in rows of $\mathbf{W}, \mathbf{U}$) have turning points with respect to their threshold.

For example at P_1 in the last row of Vm4.3, V_{p1} transforms into V_{i1} less P_1^γ (plus X_1) For fixed $\mathbf{X}$ as P_1 moves from its local optimum, smooth pasted components will change in a manner that preserves total sensitivity across the transition $V_{p1} = V_{i1} - P_1^\gamma + X_1$ (X_1 is constant). When P_1 is varied, flex values at other thresholds however (e.g. V_{p3} to V_{p2} inclusive) will display turning points in P_1. This is because in order to maintain the smooth pasting status or each line, value changes must be the same on both sides of Vm4.3 (for fixed P_{4-2}). Consequently, it is a flexibility's sensitivity with respect to thresholds other than its own (smooth pasted one) that should be zero, with negative curvature for a maximum. A local perturbations analysis of the system in this section, showed that the values in Figure 4 (and the Appendix) were maxima.

There are also restrictions on P_{1-4} that arise from the economics of desired target, or input, values X_{1-4} for the target investment costs $\mathbf{X}$. With an analytic method for mapping from a (restricted) set of P_{1-4} to solved $\mathbf{X}$, the system can be used numerically (via search) to find the P_{1-4} that map to specific investment values X_n. If a match against these target values can be found and it is one that satisfies the numerical maximization test in this subsection, then the solution produced matches all criteria. The nature of the matrix inversions ensures uniqueness of solutions.

5 Conclusion

Flexibility to time the launch, closure or other transformations within a policy are valued as discount claims on an underlying process for uncertainty. Their values at the beginning and end of their lives can be separately identified on a mathematical graph or network which then lends itself to modular analysis through discounting.

Discounting has been used before (Dixit, Pindyck and Sødal (1999) and Sødal (2006)) and the factors (and values) that are appropriate in each flexibility state depends on diffusion dynamics. We captured these features using a matrix that carries the appropriate diffusion dynamics at each different point in the investment network.

The rate of return of each flexibility value depends on an elasticity that carries through from its discount factor. Not only must discounting hold between flex values, but for optimal smooth pasting, their rates of return net of payoff at exercise must match too. This can be captured using beta matrix conditions derived from, but complementary to, those carrying the discounts.

For a given set of threshold separations, optimal flexibility values at the beginning and end of each state are driven by two considerations; discounting and smooth pasting or beta matching. For given thresholds, the third and most natural condition of value matching only contributes to the solution of fixed investment costs (these have zero beta). To find thresholds that match policy costs, a numerical search must be undertaken but the equations presented here facilitate this task.

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6 Appendix

6.1 Investment ladder values

To illustrate Section 4 numerically, in this appendix we show worked examples for a GBM with $a, b = 2, -1$ (one of r, δ, σ still free) and power elasticity $\gamma = 0.5$. Table 4 shows the graph for the investment problem. For ease of exposition we set $P_4, P_3, P_2, P_1 = 4, 3, 2, 1$ and then calculate each of $\mathbf{X}$,$\mathbf{W}$,$\mathbf{U}$. In this case Di4.3 and Gr4.3 become (to three decimal places)

$$\mathbf{D} = \begin{bmatrix} 0 & 0.750 & 0 & 0 \\ 0.550 & 0 & 0 & 0.196 \\ 0.222 & 0 & 0 & 0.444 \\ 0 & 0 & 0.250 & 0 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 0 & 2.211 & -0.974 & 0 \\ 1.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & -1.105 & 2.737 & 0 \end{bmatrix}.$$

The first scaled differentials with respect to thresholds $P_{4,3,2,1}$ are

$$\mathbf{D}' = \begin{bmatrix} 0 & -0.750 & 0 & 0 \\ 1.164 & 0 & 0 & -0.624 \\ 0.540 & 0 & 0 & -0.635 \\ 0 & 0 & 0.5 & 0 \end{bmatrix} \qquad \mathbf{G}' = \begin{bmatrix} 0 & 5.368 & -4.079 & 0 \\ -1.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 1.579 & -3.053 & 0 \end{bmatrix}$$

and the beta matrices calculated from $\boldsymbol{\beta}_W = \mathbf{G}'\mathbf{D}$ and $\boldsymbol{\beta}_U = \mathbf{D}'\mathbf{G}$ are

$$\boldsymbol{\beta}_U = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3.263 & -2.842 & 0 \\ 0 & 1.895 & -2.263 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \boldsymbol{\beta}_W = \begin{bmatrix} 2.048 & 0 & 0 & -0.762 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0.190 & 0 & 0 & -1.048 \end{bmatrix}.$$

The payoff vector $\boldsymbol{\Omega}$, its scaled elasticity $\boldsymbol{\Omega}'$ and beta matrix $\boldsymbol{\beta}_\Omega$ are

$$\begin{bmatrix} P_4 - P_4^\gamma \\ P_3^\gamma - P_3 \\ P_2^\gamma \\ -P_1^\gamma \end{bmatrix} = \begin{bmatrix} 2.000 \\ -1.268 \\ 1.414 \\ -1.000 \end{bmatrix} \qquad \begin{bmatrix} P_4 - \gamma P_4^\gamma \\ \gamma P_3^\gamma - P_3 \\ \gamma P_2^\gamma \\ -\gamma P_1^\gamma \end{bmatrix} = \begin{bmatrix} 3.000 \\ -2.134 \\ 0.707 \\ -0.500 \end{bmatrix} \qquad \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 1.683 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

From these $\mathbf{U}$, $\mathbf{W}$, $\mathbf{X}$ can be solved, with $\mathbf{\Omega}$ they satisfy value matching Vm6.1

$$\begin{array}{ccccccc} \mathbf{W} & = & \mathbf{U} & + & \mathbf{\Omega} & - & \mathbf{X} \\ \begin{bmatrix} V_{p4} : 1.303 \\ V_{f3} : 0.895 \\ V_{i2} : 0.564 \\ V_{p1} : 0.445 \end{bmatrix} & = & \begin{bmatrix} V_{f4} : 0.672 \\ V_{p3} : 0.804 \\ V_{p2} : 0.487 \\ V_{i1} : 0.141 \end{bmatrix} & + & \begin{bmatrix} 2.000 \\ -1.268 \\ 1.414 \\ -1.000 \end{bmatrix} & - & \begin{bmatrix} X_4 : 1.369 \\ -X_3 : -1.359 \\ X_2 : 1.338 \\ -X_1 : -1.304 \end{bmatrix} \end{array}. \tag{Vm6.1}$$

and discounting $\mathbf{U} = \mathbf{DW}$ Di7.1

$$\begin{bmatrix} 0.672 \\ 0.804 \\ 0.487 \\ 0.141 \end{bmatrix} = \begin{bmatrix} 0 & 0.750 & 0 & 0 \\ 0.550 & 0 & 0 & 0.196 \\ 0.222 & 0 & 0 & 0.444 \\ 0 & 0 & 0.250 & 0 \end{bmatrix} \begin{bmatrix} 1.303 \\ 0.895 \\ 0.564 \\ 0.445 \end{bmatrix}. \tag{Di6.1}$$

These numerical results can be seen in the Figure 4 in Section 4, which represents these values plotted with underlying (full) value on the horizontal and component values on the vertical axes. Idle (zero cashflow), power (P^{γ}) and full (states) have values zero, $P^{0.5}$ (square root P in purple) and P (45 degree red line) respectively, however their switching options V_i, V_p and V_f must be added to these non flex values. These are displayed in the vertical separation between the non-flex and flexible values ($P+V_f$ is in blue, $P^{\gamma}+V_p$ in green and V_i in turquoise). The four quantities in $\mathbf{X}$ are represented by the vertical black lines at each of the four thresholds $4, 3, 2, 1$.

In the idle state, as potential cashflow picks up, so does the call option V_i and at an increasing rate until at $P_2 = 2$ it is worth sinking fixed PV operating costs of $X_2 = 1.338$ in order to launch the power mode which then has value $\Omega_2 = 1.414$. This is because although the launch option of $V_{i2} = 0.564$ was sacrificed, $V_{p2} = 0.487$ was gained. This last option is a discounted combination allowing either switching to full (at $P_4 = 4$) or reverting to idle (at $P_1 = 1$) but not both.

If cashflows continue to increase, at $P_4 = 4$, the switching option $V_{p4} = 1.303$ is sacrificed along with the power flow but in return the full flow 4 is gained ($\Omega_4 = 4 - 2 = 2$). This costs $X_4 = 1.369$ but compensation is made by the receipt of a closure, put, option $V_{f4} = 0.672$.

On the way down if cashflows fall, the closure put option increases in value and then pays off at $P_3 = 3$; at this point it is worth sacrificing its higher value $V_{f3} = 0.895$ along with the full flow value 3 in return for the power flow 1.732 (i.e. $\Omega_3 = 3^{0.5} - 3 = -1.268$) and cost rate savings with PV of 1.359 (i.e. $X_3 = -1.359$). If revenues were to continue to fall further, at $P_1 = 1$ even though $\Omega_1 = -1$ is lost, it is worthwhile cashing in the switching option for its closure value $V_{p1} = 0.445$ in return for cost savings of $X_1 = -1.304$ because the idling option is regained but at now at a value of $V_{i1} = 0.141$.

Due to two way switching in the power state, this sequence of transitions can occur in a different order to P_2, P_4, P_3, P_1 just presented; e.g. P_2, P_4, P_3, P_4 is also possible as is P_2, P_1, P_2, P_1 etc. Since it determines local rate of return matching, smooth pasting is best scaled by P_n to represent the beta matrices

$$\boldsymbol{\beta}_W \mathbf{W} = \begin{bmatrix} 2.048 & 0 & 0 & -0.762 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0.190 & 0 & 0 & -1.048 \end{bmatrix} \begin{bmatrix} 1.303 \\ 0.895 \\ 0.564 \\ 0.445 \end{bmatrix} = \boldsymbol{\beta}_U \mathbf{U} + \boldsymbol{\beta}_\Omega \boldsymbol{\Omega} \tag{Beta6.1}$$
$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3.263 & -2.842 & 0 \\ 0 & 1.895 & -2.263 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0.672 \\ 0.804 \\ 0.487 \\ 0.141 \end{bmatrix} + \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 1.683 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 2.000 \\ -1.268 \\ 1.414 \\ -1.000 \end{bmatrix}.$$

Beta6.1 shows the relative beta calculation of each flexibility element at each point in the investment graph.

The matrix for $\boldsymbol{\beta}_W$ contains the elasticities for the components in $\mathbf{W}$ evaluated at each of the thresholds P_n. When pre–multiplying it on $\mathbf{W}$, the aggregate beta of flex consumed is calculated, thus on the first line (top $P_4 = 4$) $V_{p4} = 1.303$ has a beta itself of 2.048 (at this threshold the put has zero value). However the put has a non zero derivative and contributes to the slope here with a beta of -0.762 expressed as a function of its value at P_1 which is $V_{p1} = 0.445$. Thus the composite beta of flex foregone is $2.048 * 1.303 - 0.762 * 0.445 = 2.329$. This matrix also contains simple one way, put and call betas $(-1, 2)$ at P_3 and P_2 respectively, quantities which are also present (at $P_{4,1}$) in $\boldsymbol{\beta}_U$ which acts on $\mathbf{U}$.

6.2 Special cases

We can also look at the consequences of either matching two thresholds or sending one threshold to an extreme level. Using the formalization of multiple smooth pasting and value matching, the purpose is to be able to model either perfect reversibility (with continuous switching) or irreversibility (by making the attainment of one remote). The former allows cap and floor like flows to be modelled whilst the latter terminates flexibility at one point within a larger system.

The next two cases in Section 6.3 and 6.4 are when an exit threshold becomes arbitrarily low and unobtainable (allowing entry only) or a re-entry threshold becomes unobtainable by being arbitrarily high (to model exit). The case after than in Section 6.5 is when two thresholds coalesce. All are tackled by first using the matrix algebra for finite threshold and differences and then evaluating the solved flexibility values in a limit. Subject to a certain level of numerical precision, the matrix algebra can handle these special situations as limiting cases increasing the modelling scope.

6.3 One time entry

For the hysteresis example in Section 1, consider the limiting case as $P_1 \to 0$. This means that once full, the idle state is not achievable and any entry is "one time" only and final. Although the discount and growth matrices are degenerate when $P_1 = 0$, they still have non zero determinants for finite P_1. Evaluating the two components of end flex within **W**, it is possible to show that in the limit a finite flex value remains for entry V_{i2} but not for exit V_{f1}

$$\lim_{P_1 \to 0} \begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix} = \begin{bmatrix} \frac{1}{a} P_2 \\ 0 \end{bmatrix}. \qquad (\lim \mathbf{W} 6.3)$$

Since exit P_1 is no longer attainable, entry at P_2 yields no further flex ($V_{f2} = 0$) so from value matching the classic relationship between one time entry value and trigger level P_2 can be recovered from the payoff and cost alone

$$V_{i2} = P_2 - X_2 = \frac{1}{a} P_2 \qquad P_2 = \frac{a}{a-1} X_2.$$

Whilst not specifying how the system could have got into a potential launch position without having first been shut (impossible if $P_1 \to 0$), a one time entry option value can be calculated from this payoff for any intermediate $(0 < P_{1.5} < P_2)$ insertion level $P_{1.5}$ via $V_{i1.5} = D_{i1.5,2} V_{i2}$.

Although this last equations show how to recover classic "one shot" real option values, we can use the "solve first and take limits later" feature of the solution in the investment ladder in Table 4. Suppose in Section 4 we wished to model "one time entry" into the power state, with subsequent opportunity to move to the full state and back again, but without the opportunity to suspend back to the idle condition. This would be important when valuing an initial option to invest in such a flexible revenue stream from a clean sheet as opposed to the option to invest having been in a prior idle state. Within the ladder context, such a calculation is evaluated in Section 6.6.

6.4 One time exit

Looking at the classical GBM hysteresis from Sections 1 and 2, we can model permanent exit by allowing $P_2 \to \infty$. This means that once in the idle state, the full threshold P_2 is not achievable, i.e. that any exit is one time and final. Again solving for **W** before taking limits shows

$$\lim_{P_2 \to \infty} \begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix} = \begin{bmatrix} \frac{1}{a}P_2 \\ -\frac{1}{b}P_1 \end{bmatrix} \to \begin{bmatrix} \infty \\ -\frac{1}{b}P_1 \end{bmatrix}. \qquad (\lim\mathbf{W}6.4)$$

Since the payoff when launching at an arbitrarily high level (P_2) is unbounded, so is the option value $\frac{1}{a}P_2$ at that threshold. However due to the effect of discounting, the option flex value at P_1 is bounded (and since $b < 0$, positive) at $-\frac{1}{b}P_1$. Its exercise, which begets $X_1 - P_1$ only (no further flex), is consistent with an optimal put threshold of $P_1 = \frac{b}{b-1}X_1$ which is natural given our understanding of the classical GBM put.

Both one time exit and entry cases can also be represented in a simple sequential model where switching can occur a finite number of times as in Ekern (1993). In Section 4 for sequential hysteresis, to model finite switching ending in closed or open terminal limiting cases can be included with $P_{open} \to \infty, P_{close} \to 0$) whilst incorporating different processes at different stages of the sequence.

6.5 Perfect reversibility

Consider one reversible boundary separating only two regions and a policy of continuous switching at a common point. This form of collapsed hysteresis can be modelled as $P_2 \to P_1$ in Sections 1 and 2 (for GBM). This would be an appropriate policy in a situation where two frictions $K_{2,1}$ incurred at $P_{2,1}$ (contained within **X** in Vm1.5) tend to zero so that hysteresis disappears and action thresholds match.

We would expect the output costs **X** to align as well (since $K_{2,1} \to 0$) but the matrix inversion is not strictly possible if $P_2 = P_1$. This is because as thresholds merge, discounting disappears. Furthermore the key matrix (**S** from Section 2) that requires inversion has a determinant that tends to zero

$$\lim_{P_2 \to P_1} \mathbf{D}, \mathbf{G} \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \det\left[\boldsymbol{\beta}_W \mathbf{G} - \boldsymbol{\beta}_U\right] \to 0.$$

However, analytical progress can be made with the solution before the limit is taken. With the GBM system in Section 1 for example, it is possible to show using L'Hôpital's rule on the quotients[16] that finite values of V_{i2}, V_{f1} result, i.e. for **W**

$$\lim_{P_2 \to P_1} \begin{bmatrix} V_{i2} \\ V_{f1} \end{bmatrix} = \begin{bmatrix} P_1 \frac{1-b}{a^2-ab} \\ P_1 \frac{a-1}{b^2-ab} \end{bmatrix}. \qquad (\lim \mathbf{W}6.5)$$

The common value X of $X_2 = -X_1 = \frac{\delta}{r} P_1$ corresponds to a flow condition $\delta P_1 \gtrless rX$ which says that the perfectly reversible option to switch[17] should be based on the opportunity cost rates of the current flow δP_1 and operational costs rX.

Whilst L'Hôpital's rule works analytically, the same result can be retrieved numerically with arbitrary precision by using a small but finite difference between the two unique thresholds. Safe in the knowledge that a limit is secure, a system with a truly reversible threshold can be modelled numerically to any desired level of accuracy as having an arbitrarily small hysteretic band. Thus within a larger system, reversibility can be built in as a feature at any pair of thresholds that are to act as one (knowing that in Section 4 $X_4 = -X_3$ etc. will be recovered). In the next subsection 10^{-6} (but consistent with the numerical precision available in the computational package) is used for the Investment Ladder threshold differences $P_2 - P_1$ and $P_4 - P_3$.

Along with the one time options, this is another useful modular feature of the solution method of this article all of which are now illustrated.

6.6 Limiting cases for the ladder

The ladder calculations tolerate limiting special cases. The first is as $P_1 \to 0$ when exit from the power state is no longer allowable, and the only entry possible is one time. For the same values of $P_{4,3,2}$ but with $P_1 = 10^{-6}$, the solved flex values and investment costs are (limited only by the precision available within the matrix inversion, 0.000 indicates values less than 10^{-4})

$$\begin{matrix} P_n \\ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 10^{-6} \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathbf{W} \\ \begin{bmatrix} 1.210 \\ 0.772 \\ 0.656 \\ 0.000 \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathbf{U} \\ \begin{bmatrix} 0.579 \\ 0.681 \\ 0.303 \\ 0.000 \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathbf{X} \\ \begin{bmatrix} 1.369 \\ -1.359 \\ 1.061 \\ 0.000 \end{bmatrix} \end{matrix}.$$

[16] This assumes a higher derivative and contact principle (see Dumas 1991 Dumas (1991)).

[17] See Jorgensen (1963) for this user cost of capital argument. Also McDonald and Siegel (1985) and Shackleton and Wojakowski (2007) for reversible flow, cap and floor cases.

Jumping in at a one time entry or insertion point $P_{1.5} = 1.5$ would therefore be worth $\left(\frac{1.5}{2.0}\right)^2 * 0.303 = 0.170$.

Modelling a one time switch from full is also possible. Sending $P_4 \to \infty$ prohibits the transition from power to full but not full to power. Although setting $P_4 = 10^6$ yields two extreme values and investment quantities, at other thresholds these get discounted to zero due to their remoteness

$$\begin{matrix} P_n \\ \begin{bmatrix} 10^6 \\ 3 \\ 2 \\ 1 \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathbf{W} \\ \begin{bmatrix} 5.000 * 10^5 \\ 2.257 \\ 0.261 \\ 0.369 \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathbf{U} \\ \begin{bmatrix} 0.000 \\ 0.123 \\ 0.185 \\ 0.065 \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathbf{X} \\ \begin{bmatrix} 5.000 * 10^5 \\ -3.402 \\ 1.338 \\ -1.304 \end{bmatrix} \end{matrix}.$$

Although it could not have come from P_4, were the policy to be inserted in the full state say at $P_{3.5} = 3.5$, it would be worth $3.5 + \left(\frac{3.5}{3.0}\right)^{-1} * 2.257 = 5.435$. The moderate flex premium to its value of 3.5 reflects the timing option at P_3 to go to power flow with a payoff net of cost savings $(P_3^\gamma - P_3 + X_3)$ of $1.732 - 3 + 3.402 = 2.134$ equal to the flex value use $2.257 - 0.123$.

Finally, to demonstrate perfect reversibility at each of the two thresholds $P_3 = 3$ and $P_2 = 2$ (and two way passage between them), this final example has $P_4 = P_3 + 10^{-6}$ and $P_1 = P_2 - 10^{-6}$

$$\begin{matrix} P_n \\ \begin{bmatrix} 3 + 10^{-6} \\ 3 \\ 2 \\ 2 - 10^{-6} \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathbf{W} \\ \begin{bmatrix} 1.019 \\ 0.803 \\ 0.525 \\ 0.702 \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathbf{U} \\ \begin{bmatrix} 0.803 \\ 1.109 \\ 0.702 \\ 0.525 \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathbf{X} \\ \begin{bmatrix} 1.051 \\ -1.051 \\ 1.591 \\ -1.591 \end{bmatrix} \end{matrix}.$$

Firstly note that as expected the investment divestment quantities match $X_4 = -X_3$ and $X_2 = -X_1$ (zero frictions K). Secondly since payoffs are also matched at $P_{4,3}$ (and also $P_{2,1}$) cross pair elements within $\mathbf{W}, \mathbf{U}$ are also equal. The matrix algebra can only be inverted if all smooth pasting conditions are unique, whereas in the limit both pairs of equations at their common limits have the same WCCM. This system can also be used to search for specific X, say $X_4 = -X_3 = 1$, $X_2 = -X_1 = 1.5$ in which case the solution is near $P_{4,3} = 2.924$ and $P_{2,1} = 1.778$ ($\mathbf{W}, \mathbf{U}$ differ).