Product development with value-enhancing options

Nicos Koussis[1], Spiros H. Martzoukos[2], and Lenos Trigeorgis[3]

This draft, April 2007

JEL classification: G13, G31, O32

Keywords: Real options, economics of R&D, technical risk, path-dependency, sequential (compound) options, early product versions, pilot projects, jump-diffusion.

The authors are grateful to seminar participants at the MIT Sloan Management School, the 2005 FMA Annual Meeting, the 7th Annual International Conference on Real Options and the European Union Workshop Series on Mathematical Optimization for Financial Institutions for helpful comments and discussions; and for partial financial support to the HERMES European Center of Excellence on Computational Finance and Economics at the University of Cyprus. Nicos Koussis also acknowledges financial support from the Cyprus Research Promotion Foundation. The usual disclaimer applies.

---

1. Visiting Scholar, Haas Business School, University of California, Berkeley, Berkeley, CA 94720-1900, Tel: +1-5106427826, Fax: +1-5106431420. Email: kkousis@haas.berkeley.edu
2. Assistant Professor, Department of Public and Business Administration, University of Cyprus, P.O. Box 20537, CY 1678 Nicosia, CYPRUS.Tel.: +357-22892474, Fax: +357-22892460.Email: baspiros@ucy.ac.cy
3. Bank of Cyprus Chair Professor of Finance at University of Cyprus and Visiting Professor, Haas Business School, University of California, Berkeley, Berkeley, CA 94720-1900,
Tel: +1-5106436349,Fax: +1-5106431420. Email: lenos_trigeorgis@haas.berkeley.edu

Product development with value-enhancing options

Abstract

We study costly managerial actions that reveal uncertainty like research, experimentation, and early product versions (pilot projects) or actions that are intended to bring about an increase in value like attribute-enhancing development options and advertising but have an uncertain outcome. Actions are implemented sequentially at an optimal time and involve path-dependent characteristics. We derive two-stage analytic formulas to study product development with optimal timing of product versions and sequencing of value-enhancing actions. We also propose a multi-period solution using a numerical lattice approach. Our analysis reveals that exploration actions are more important when the project is out or at-the-money (near zero NPV), and less important for high project values. In a multi-stage setting, exploration actions are important even for in-the-money projects when follow-on actions exist that can enhance the expected value of the project. With path-dependency, early actions are more valuable since they enhance the impact or reduce the cost of subsequent actions.

JEL classification: G13, G31

Keywords: Real options, economics of R&D, technical risk, path-dependency, sequential (compound) options, early versions (pilot projects) with cash multiplier, jump-diffusion.

1. Introduction

The innovation development process involves exploration and experimentation, research to meet consumer needs or to outperform competitors and attribute enhancing investments. In the area of consumer electronics, for example, Apple's iPod involved experimentation with the use of materials and appearance that make it very attractive even if some materials are more costly to produce (Burrows, Business Week 9/25/2006). Research and development, however, involves considerable risks that affect the profitability and successful launching of the product. McGrecor et al. (Business Week 7/10/2006) discuss several examples of project failures. Apple also faced revenue reductions due to the vulnerability of some of the materials used and for overlooking other features like battery use (Burrows, Business Week 9/25/2006). Samsung's marketing research concerning what consumers considered most important attributes of a flat-screen TV resulted in a more focused development that achieved a higher market penetration (Moon, Business Week 7/3/2006). Even if the immediate cash-flow outcome of an action is negative, there may still be a "learning" effect which may have a positive impact on future decisions. McGrecor et al. (Business Week 7/10/2006) describe how firms have used previous failures to improve future decisions.

We develop a real options model to study costly, interacting managerial exploration actions and actions that are expected to enhance value or reduce the cost of a project, albeit having an uncertain outcome. The information revealed from exploration actions or the resulting uncertainty of development investments, may cause management to deviate from its original plans. Kothari et al. (2002) find that the relation between R&D expenditures and uncertainty of future benefits exhibits a positive correlation. In our model the information revelation of exploration actions and the volatility of direct-value enhancing actions also interact with exogenous demand-driven uncertainty (e.g. capturing changing consumer preferences) which is separately modelled in continuous time using a Brownian motion or a jump-diffusion process.

Pure research or exploration actions include investments in early product versions (pilot projects), experimentation using new processes, or marketing research. These actions resolve uncertainty about the true project value or cost, enabling management to capitalize on new information before irreversible investment is undertaken. Childs et al. (2001) (see also Childs et al., 2002) and Bernardo and Chowdhry (2002) use a filtering approach to study information acquisition in a real options model with noisy assets. Paddock et. al. (1988) study oil reserves risk, while Smith and Thompson (2005) study the choice between interdependent exploration projects. Pindyck (1993) examines sequential multi-stage actions with technical uncertainty that decreases as the project approaches completion. Pindyck assumes continuous reduction of technical uncertainty while we allow for different levels of technical uncertainty resolution between stages. We also allow for interacting actions and derive analytic formulas for the two-stage problem. Childs and Triantis (1999) consider accelerated versus sequential strategies and learning spillovers between projects. They assume that actions affect the Brownian volatility, while we maintain separate demand driven uncertainty and consider path-dependency.

Direct value-enhancing actions include R&D efforts to improve the attributes or quality of a product, enhance customer perceptions through advertising or efforts to reduce cost through adoption of new technologies in production. Similarly to Huchzermeier and Loch (2001) these actions aim at enhancing project value but have an uncertain outcome (see also Weitzman and Roberts, 1981). We assume that decisions are made at discrete points in time and the outcome of such investment actions is realized immediately. Impulse-type actions with uncertain outcome were introduced in control theory by Korn (1997) and in real options by Martzoukos (2000). Childs and Triantis (1999) and Berk et al. (2004) analyze projects that require completion of development stages before commercialization of the product. In our setting, the firm may decide to develop the product immediately, to delay development by exploring further experimentation and development opportunities, or to introduce early product versions. The expected impact, volatility and costs of managerial actions and the cash flows of early product versions may depend on the sequencing of actions (path-dependency). For example, the firm may expect a higher impact of R&D if prior marketing research has been implemented. New information following the results of an experimentation process may also reduce next-stage costs. Childs et al. (1998) focus on potential synergies between actions by comparing sequential versus parallel development.

We derive analytic solutions for a two-stage problem that involves multiple value-enhancing actions. Our analytic solutions nest several known results as special cases, including Geske (1979) and Longstaff (1990) (see Chung and Johnson, 1994 for the multi-stage extendible option). We incorporate path-dependency and optimal timing of managerial exploration and value-enhancing actions. We also allow for optimal timing of early product versions that provide cash flows and information about future product versions, the investment decision in the final version, and abandonment options for partial recovery of invested capital. We extend the model to a multi-stage framework using a numerical lattice approach and provide a numerical application with multiple actions and path-dependency.

Consistent with results in Bernardo and Chowdhry (2002) and Huchzermeier and Loch (2001), we show that managerial exploration actions may be more valuable for projects that are marginal or break-even (close to zero-NPV investments or near at-the-money options). In contrast to these papers, however, we show that in the case of interacting actions, exploration actions may be important even in deep in-the-money projects when follow-on value-enhancing actions are involved. Furthermore, we show that multiple and interchanging decision regions (as a function of project value) between delay, early development, exploration and expected value-enhancing actions are possible. Path-dependency also has a substantial impact on these regions.

The rest of the paper is organized as follows. Section 2 describes the problem and assumptions. Section 3 provides the analytic formulas and discusses the results and main implications. Section 4 provides a generalization to a multistage application in new product development and our proposed numerical solution. The last section concludes.

2. Problem description

Figure 1 illustrates a valuation problem for product development and market introduction which is typical for new products in many industries. With a basic technology already developed, the firm can proceed with the introduction of a basic product immediately (with a set of features the management considers to be an adequate basic version). However, management has the option for further enhancement, adding features (MC_1) and improving quality or investing in advertising (MC_2) to influence customer perceptions. It may also proceed with an early or scaled-down version (L_G) of the product that would provide only a fraction of the cash flows of the complete version but generate valuable information (e.g., customer reaction and product testing) that strengthens the launch of the basic version at a later date. Management may also engage in research or experimentation (L_1) that will enable further value-enhancing development opportunities. Figure 1 illustrates the set of feasible actions and sequencing. For example, research may be followed by an introduction of an early version or follow-on attribute development. The choice of an early action may affect the expected outcome, volatility, and costs of following-on actions due to new information and experience obtained.

[Insert Figure 1 here]

The present value of the basic version S_t (i.e., the present value of project cash flows of the basic version without further enhancements) is assumed to follow a Geometric Brownian motion, adjusted for the impact of optional activation of $i = MC_1, MC_2, ...MC_{N_{MC}}$ value-enhancing managerial actions:[1]

$$\frac{dS_t}{S_t} = adt + \sigma dz + k_i dq_i \tag{1}$$

where a denotes the expected return of the project, σ is the standard deviation of the rate of return, and dz is an increment to a standard Wiener process (describing the exogenous uncertainty). The N_{MC} managerial actions can be optionally activated (at a cost) by the management. Parameter k_i is a random variable that represents the impulse effect $Y_i = 1 + k_i$ on project value of managerial action i, and dq_i is a control variable that takes the value 1 when the action is optimally activated by management and 0 if not. Actions bring about value improvement by increasing the expected value or volatility of the project (thus increasing option value). Exploration actions (also included in the N_{MC} set) help update management's estimate about the true project value.

We assume that the multiplicative impact $Y_i = 1 + k_i$ follows a log-normal distribution of the form:

$$Y = (1 + k_i) \sim log\, N\left(\exp(\gamma_i), \left(\exp(2\gamma_i)\left(\exp\left(\sigma_i^2\right) - 1\right)\right)\right) \tag{2a}$$

where

$$log(1 + k_i) \sim N\left(\gamma_i - 0.5\sigma_i^2, \sigma_i^2\right) \tag{2b}$$

The assumption of log-normality precludes negative asset values. Moreover, conditional on action activation, asset value remains log-normally distributed. The pair (γ_i, σ_i) denotes the expected (exponential) impact (size) and the volatility of action i. We use $\gamma_i > 0$ to describe efforts intended to enhance value with an uncertain outcome. (Alternatively, if S is interpreted as a cost, $\gamma_i < 0$ implies efforts to reduce costs.)

[1] The jump-diffusion case with multiple classes of jumps is discussed in the appendix.

The case $\gamma_i = 0$ (with $\sigma_i^2 > 0$) captures exploration activities with no direct cash-flow value impact. In this case value-enhancement can be achieved through improvement of the information. This situation exists when project values are not observed with certainty and are noisy estimates of true project values, with total uncertainty σ_{TU}^2. This uncertainty is different than demand-driven uncertainty. It might relate to uncertainty in the original cash flow projections, the selection of the optimal development process, or the optimal product features to be included. Each exploration investment reduces uncertainty about the true project value (in log-scale) by an amount equal to σ_i^2. The parameter σ_i^2 determines the expected (ex-ante) amount of information revelation that affects option value. This is consistent with a Bayesian approach where the above parameters of the log-normal distribution are estimated as the parameters of a preposterior distribution (see Kaufman, 1963, and a recent application by Davis and Samis, 2006).

Management can make decisions at N_{dec} discrete (equally-spaced) decision points in time before maturity T: $t_0 = 0,\ t_1 = \frac{T}{N_{dec}},\ t_2 = \frac{2T}{N_{dec}}, \ldots,\ t_{N_{dec}-1} = \frac{(N_{dec}-1)T}{N_{dec}}$. The two-stage ($N_{dec}$ = 2) problem involves decisions at t_0 (= 0) and t_1 (< T). At time T the decision is to either exercise or abandon the project. With an early (pilot) version, the firm may generate additional cash, assumed to a fraction m of final project value (S_T). With an early scaled-down version the firm still has the option to develop the basic version while it can also obtain more information observing the market's reaction. The set of all feasible decisions is:

$$M = \left\{W, A, EE, MC_1, MC_2, \ldots MC_{N_{MC}}, W_1, W_2, \ldots W_{N_{MC}}\right\}$$

This set includes the following: wait (W), abandon (A), exercise early (EE) the investment option, N_{MC} managerial value-enhancing control actions (MC_i), and a set of possible states of inaction (W_i) after an action is activated. We will use the notation L to distinguish actions of pure experimentation or exploration that intend to reduce technical uncertainty from other direct value-enhancing actions. At any decision time t, the set of admissible choices is M_t^+. M_t^+ may not include all decisions and can be a subset of the superset M ($M_t^+ \subseteq M$). The characteristics of the actions (expected impact, volatility, and costs) in the set M_t^+ may be path dependent, affected by the action history M_t^-. The value V^{d_t} of the project under decision d_t is thus a function of the history of actions in M_t^-. (The modes *EE* and *A* are absorbing states since in those states the decision process stops.)

Option value, reflecting the value of the project with the embedded opportunities to enhance value, must satisfy the following set of partial integro-differential equations (PIDEs) between decision points:

$$\underset{d_t \in M_t^+}{\text{Max}} \left\{ \frac{1}{2}\sigma^2 S^2 V_{SS} + (r-\delta) S V_S + V_t + \left[m_i S - X_i + E\left[V(SY_i, t) - V(S,t) \right] \right] dq_i - rV \right\} = 0 \tag{3}$$

Here *r* denotes the riskless rate of interest, and δ represents an opportunity cost of waiting or a shortfall from the equilibrium required rate of return (see McDonald and Siegel, 1984, and Constandinides, 1978). It may also represent exogenous competitive erosion (e.g., Trigeorgis, 1986; Childs and Triantis, 1999). The parameter m_i captures additional cash flows received because of an early (pilot) version when capital expenditure X_i is paid (that also keeps the option to invest in the basic version).[2]

[2] The PIDE in equation (3) is derived as follows. For the continuous part one could follow two alternative approaches. First, one can assume the existence of a "twin security" (or spanning assets) and follow a replication approach (e.g. Merton's (1976)). A second approach is to follow Constantinides (1978) assuming that the intertemporal CAPM holds (Merton, 1973), adjusting required returns to their certainty equivalents. For the discontinuous part, generated by the effect of managerial actions, we follow a similar assumption with Merton (1976) assuming that managerial actions involve firm-specific risks which are uncorrelated with the market portfolio and thus are not priced by investors. We then discount using the risk free rate. Equivalently, one may assume risk-neutral agents. Consistently with these two assumptions, and in contrast with Pindyck (1993) we also allow final investment decisions to be made even when residual uncertainty is left, i.e. when exploration actions do not reveal all uncertainty about true *S* value.

With $Z_{t+\Delta t}$ denoting the accumulated (Brownian) noise between successive decision points from t to $t + \Delta t$, project value must then satisfy:

$$\frac{S_{t+\Delta t}}{S_t} = \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)\Delta t + \sigma Z_{t+\Delta t}\right]\left(1 + k_i dq_i\right) \tag{4}$$

The boundary condition at maturity T of the option is the maximum of the value of the decision to exercise (*EE*) [3], obtaining $S_T - X$ (X being the development cost), or to abandon (*A*) the project for a recovery amount α of total cumulative costs paid *TC*. This resale value may represent resale of innovations, property rights or the value of knowledge capital obtained that may be used for other spin-off products. X represents the necessary costs that will achieve a certain level of performance dictated by current competitive conditions in the market (see also Huchzermeier and Loch, 2001). The firm expects to get S with the option value being affected by competitive erosion and consumer demand uncertainty (Brownian noise). Implementing costly managerial actions may enhance this value but has an uncertain outcome. We assume for simplicity that development costs X are not affected by the selection of the R&D path (although this can be easily relaxed).

In what follows we use the general notation $\gamma\left(M_t^-, d_t\right)$ and $\sigma\left(M_t^-, d_t\right)$ to describe the expected impact (size) and volatility of a managerial action d_t at time t, conditional on the history of decisions M_t^-. For example, implementing two actions in sequence may result in a higher expected impact or lower costs for the second action due to learning-by-doing or new information obtained.

Log-returns between successive decision points t and $t + \Delta t$ follow a normal distribution:

$$ln\left(\frac{S_{t+\Delta t}}{S_t} \mid M_t^-\right) \sim N\left(\left(r - \delta - \frac{1}{2}\sigma^2\right)\Delta t + \gamma\left(M_t^-, d_t\right) - \frac{1}{2}\sigma^2\left(M_t^-, d_t\right), \sigma^2 \Delta t + \sigma^2\left(M_t^-, d_t\right)\right) \tag{5}$$

The distribution of returns conditional on no activation of a managerial action is obtained by removing $\gamma\left(M_t^-, d_t\right)$ and $\sigma\left(M_t^-, d_t\right)$ from equation (5). Actions in general increase the mean and

[3] We use a single mode *EE* to denote exercise whether this decision involves early exercise or at the maturity of the option.

variance of *S*. Increases in mean and volatility increase option value but the firm should weigh these expected benefits against the costs of an action.

3. Two-stage analytic solutions

In this section we provide valuation formulas for investment options with multiple embedded managerial actions in a two-stage framework. Appendix section B provides formulas for the case of jump-diffusion process describing exogenous uncertainty.

Consider a sequential option (call on call) with embedded managerial actions, early product versions providing cash (pilot projects), and early exercise and abandonment at $t_0 = 0$ and $t_1 > 0$. At $t_0 = 0$, the firm may exercise the investment option early, in which case it will obtain $S - X$. Since the project is not yet initiated, abandonment has zero value (abandonment decisions in a later stage may allow recovering part of previously incurred costs). If the firm decides to wait or to invest in a managerial action at $t_0 = 0$, i.e., when ($d_0 \in M \setminus \{EE, A\}$), the payoff $V^{d_1}(.)$ at an intermediary point $t_1 < T$ is:

$$V^{d_1}\left(S_{t_1}, t_1 \mid M, M_1^+, M_1^-\right) = \tag{6}$$
$$I_{d_1 \in \{W, MC_k\}}\left[m(d_0, d_1)S_{t_1} + S_{t_1} e^{-\delta(T-t_1)+\gamma(d_0,d_1)} N\left(a_{d_1,1}\right) - X e^{-r(T-t_1)} N\left(a_{d_1,2}\right) - X(d_0, d_1)\right]$$
$$+ I_{d_1 \in \{A\}} aX(d_0) + I_{d_1 \in \{W, MC_k\}} e^{-r(T-t_1)} a\left(X(d_0) + X(d_0, d_1)\right) N\left(-a_{d_1,2}\right) + I_{d_1 \in \{EE\}}\left(S_{t_1} - X\right)$$

where $a_{d_1,1} = \dfrac{ln\left(S / S_T^*(d_0, d_1, d_2)\right) + (r - \delta)(T - t_1) + \gamma(d_0, d_1) + 0.5\sigma^2(T - t_1) + 0.5\sigma^2(d_0, d_1)}{\left(\sigma^2 T + \sigma^2(d_0, d_1)\right)^{1/2}}$

$a_{d_1,2} = a_{d_1,1} - \left(\sigma^2 T + \sigma^2(d_0, d_1)\right)^{1/2}$, $I_{d_1 \in \{.\}} = 1$, zero otherwise.

Equation (6) above applies when $M_1^- = \{W, MC_k, \ k = 1, 2, ..N_{MC}\}$ for $d_1 \in M_1^+ = \{W, A, EE, MC_i, \ i \neq k = 1, 2, ..N_{MC}\}$. That is, conditional on delaying development or investing in an action at $t_0 = 0$, the firm has the following options at t_1: delay investment (wait) (extension may be costly with $X\left(d_0, W\right) \geq 0$) , early exercise of the development option, engage in costly managerial R&D actions to enhance project value or reveal more information about the true value of the project (with cost $X\left(d_0, MC_i\right)$, $i = 1, 2, \ldots N_{MC}$), or abandon the project (recovering a fraction α% of paid capital). With $d_1 = W$, the second term is a standard call option. With $d_1 = MC_i$, a modified version of the standard call option obtains. It can be easily verified that the payoff is increasing in both the average impact ($\gamma(d_0, d_1)$) and volatility of actions ($\sigma^2(d_0, d_1)$). At t_1 the cash factor, expected impact, volatility and costs may depend on a previous decision.

We define R_{d_1}, $d_1 \in \left\{W, EE, A, MC_1, MC_2, \ldots, MC_{N_{MC}}\right\}$, to be the number of regions where decision d_1 is optimal at t_1. L denotes the lower boundary and H the upper boundary of that region. At maturity we have two decision regions: development or abandonment of the project ($d_2 \in \{EE, A\}$). The value of a sequential two-stage option is then given by:

$$
\begin{aligned}
& CC(. \mid d_o \in M \setminus \{EE, A\} = \{W, MC_1, MC_2, \ldots MC_{N_{MC}}\}) = \\
& Se^{-\delta t_1 + \gamma(d_0)} \left[\sum_{l=1}^{R_{EE}} [N_l(a_{EE,1}^L) - N_l(a_{EE,1}^H)] \right] - Xe^{-rt_1} \left[\sum_{l=1}^{R_{EE}} [N_l(a_{EE,2}^L) - N_l(a_{EE,2}^H)] \right] \qquad (7) \\
& + m(d_0, W) Se^{-\delta t_1 + \gamma(d_0)} \left[\sum_{l=1}^{R_W} [N_l(a_{W,1}^L) - N_l(a_{W,1}^H)] \right] - X(d_0, W) e^{-rt_1} \left[\sum_{l=1}^{R_W} [N_l(a_{W,2}^L) - N_l(a_{W,2}^H)] \right] \\
& + \sum_{\substack{i=1, \\ i \neq d_0}}^{N_{MC}} \left[m(d_0, MC_i) Se^{-\delta t_1 + \gamma(d_0)} \sum_{l=1}^{R_{MC_i}} [N_l(a_{i,1}^L) - N_l(a_{i,1}^H)] - X(d_0, MC_i) e^{-\delta t_1} \sum_{l=1}^{R_{MC_i}} [N_l(a_{i,2}^L) - N_l(a_{i,2}^H)] \right] \\
& + e^{-rt_1} a X(d_0) N(-a_{A,2}) \\
& + Se^{-\delta T + \gamma(d_0)} \left[\sum_{l=1}^{R_W} [N_l(a_{W,1}^L, b_{W,1}, \rho_W) - N_l(a_{W,1}^H, b_{W,1}, \rho_W)] \right] \\
& - Xe^{-rT} \left[\sum_{l=1}^{R_W} [N_l(a_{W,2}^L, b_{W,2}, \rho_W) - N_l(a_{W,2}^H, b_{W,2}, \rho_W)] \right] \\
& + \sum_{\substack{i=1, \\ i \neq d_0}}^{N_{MC}} \left[Se^{-\delta T + \gamma(d_0) + \gamma(d_0, MC_i)} \sum_{l=1}^{R_{MC_i}} [N_l(a_{i,1}^L, b_{i,1}, \rho_i) - N_l(a_{i,1}^H, b_{i,1}, \rho_i)] \right]
\end{aligned}
$$

$$- \sum_{\substack{i=1,\\ i\neq d_0}}^{N_{MC}} \left[Xe^{-rT} \sum_{l=1}^{R_{MC_i}} [N_l(a_{i,2}^{L}, b_{i,2}, \rho_i) - N_l(a_{i,2}^{H}, b_{i,2}, \rho_i)] \right]$$

$$+ e^{-rT} \sum_{l=1}^{R_W} a\left(X(d_0) + X(d_0, W)\right)\left[N_l(a_{W,2}^{L}, -b_{W,2}, -\rho_W)] - N_l(a_{W,2}^{H}, -b_{W,2}, -\rho_W)\right]$$

$$+ e^{-rT} \sum_{\substack{i=1,\\ i\neq d_0}}^{N_{MC}} \sum_{l=1}^{R_{MC_i}} a\left(X(d_0) + X(d_0, MC_i)\right)\left[N_l(a_{i,2}^{L}, -b_{i,2}, -\rho_i)] - N_l(a_{i,2}^{H}, -b_{i,2}, -\rho_i)\right]$$

The following are defined for all decisions d_1 and for each of the R_{d_1} regions:

$$a_{d_1,1}^{(L,H)} = \frac{ln\left(S / S_{t_1}^{*(L,H)}\left(d_0, d_1\right)\right) + \left(r - \delta + 0.5\sigma^2\right)t_1 + \gamma\left(d_0\right) + 0.5\sigma^2\left(d_0\right)}{\left(\sigma^2 t_1 + \sigma^2\left(d_0\right)\right)^{1/2}},$$

$$a_{d_1,2}^{(L,H)} = a_{d_1,1}^{(L,H)} - \left(\sigma^2 t_1 + \sigma^2\left(d_0\right)\right)^{1/2}$$

$$b_{d_1,1} = \frac{\ln\left(S / S_T^*(d_0, d_1, d_2)\right) + (r-\delta)T + \left(\gamma(d_0) + \gamma(d_0, d_1)\right) + 0.5\sigma^2 T + 0.5\left(\sigma^2(d_0) + \sigma^2(d_0, d_1)\right)}{\left(\sigma^2 T + \sigma^2(d_0) + \sigma^2(d_0, d_1)\right)^{1/2}}$$

$$b_{d_1,2} = b_{d_1,1} - \left(\sigma^2 T + \sigma^2\left(d_0\right) + \sigma^2\left(d_0, d_1\right)\right)^{1/2}$$

The notation (L, H) used in the parameters of the cumulative univariate and bivariate normal terms implies that the formula applies separately for the L (low threshold) and for the H (upper threshold) case of each decision region.

For each decision d_1 the correlation coefficient is $\rho_{d_1} = \sqrt{\dfrac{\sigma^2 t_1 + \sigma^2\left(d_0\right)}{\sigma^2 T + \sigma^2\left(d_0\right) + \sigma^2\left(d_0, d_1\right)}}$. Note that the correlation coefficient reduces to the well-known result of compound-sequential options (see Geske, 1979) when none of the managerial actions is activated. For $d_1 = W$ we have $\sigma^2\left(d_0, W\right) = 0$ so that $\rho_W = \sqrt{\dfrac{\sigma^2 t_1 + \sigma^2\left(d_0\right)}{\sigma^2 T + \sigma^2\left(d_0\right)}}$. We use $S_{t_1}^*\left(d_0, d_1\right)$ and $S_T^*\left(d_0, d_1, d_2\right)$ to denote the threshold project value(s) at t_1 and T for actions d_1 and d_2, respectively, conditional on previous actions. These are determined by appropriate value-matching conditions (described next). $N_l\left(.\right), N_l\left(.,.,.\right)$ are the univariate and bivariate cumulative standard normal distribution functions, respectively. Equation (7) is conditional on the decision at t_0 and excludes any additional positive or negative cash flows at t_0.

The term $\sum_{l=1}^{R_{d_1}}\left[N_l\left(a_{d_1,2}^L\right)-N_l\left(a_{d_1,2}^H\right)\right], d_1 \in M_{t_1}$, represents the probability of reaching a particular decision region d_1 at t_1 and $\sum_{l=1}^{R_{d_1}}\left[N_l\left(a_{d_1,2}^L, b_{d_1,2}, \rho_{d_1}\right)-N_l\left(a_{d_1,2}^H, b_{d_1,2}, \rho_{d_1}\right)\right]$, gives the joint probability of reaching decision region d_1 ($d_1 \in M_1^+ \setminus \{EE\} = \{W, MC_1, MC_2, \ldots, MC_{N_{MC}}\}$) at t_1 and exercising the investment option at T. Similarly, the term $\sum_{l=1}^{R_{d_1}}\left[N_l\left(a_{d_1,2}^L, -b_{d_1,2}, -\rho_{d_1}\right)-N_l\left(a_{d_1,2}^H, -b_{d_1,2}, -\rho_{d_1}\right)\right]$ denotes the joint probability of reaching decision region d_1 at t_1 and abandoning the project at T. The term $m\left(d_0, d_1\right) S e^{-\delta t_1+\gamma\left(d_0\right)}\left[\sum_{l=1}^{R_{d_1}}\left[N_l\left(a_{d_1,1}^L\right)-N_l\left(a_{d_1,1}^H\right)\right]\right]$ captures the cash flows that the option holder gets in region d_1 at t_1 (with $m\left(d_0, EE\right)=1$).

$S e^{-\delta T+\gamma\left(d_0\right)+\gamma\left(d_0, d_1\right)} \sum_{l=1}^{R_{d_1}}\left[N_l\left(a_{d_1,1}^L, b_{d_1,1}, \rho_{d_1}\right)-N_l\left(a_{d_1,1}^H, b_{d_1,1}, \rho_{d_1}\right)\right]$ captures the cash flows that the option holder gets at T after a decision $d_1 \in M_1^+ \setminus \{EE\}$ at t_1 (for decision $d_1=W$, $\sigma\left(d_0, d_1\right)=\gamma\left(d_0, d_1\right)=0$).

In equation (7) the number of optimal regions at t_1 for each action and the critical thresholds that separate the regions also need to be determined. At maturity there are two regions, option exercise (*EE*) and abandon (*A*). The critical threshold is determined by applying the value-matching condition: $S_T^*\left(d_0, d_1, d_2\right)-X=a\left(X\left(d_0, d_1\right)+X\left(d_0\right)\right)$. Depending on the action path, the critical trigger point at maturity will differ since $X\left(d_0, d_1\right)$, $X\left(d_0\right)$ depend on the path of actions.

At t_1 exploration actions that reveal volatility (or managerial actions with high volatility and low expected impact) are expected to be more important for near at-the-money options (*S* close to *X*) due to the convexity of the option payoff. Their importance is expected to be less for in-the-money options. Having a fixed cost of activation may make activation suboptimal both at very low values and at high values of *S*. The benefit is defined as the difference between the value conditional on action activation and the next-best choice between abandonment, wait or early development (with the cost of managerial action not yet accounted for). Figure 2 illustrates with the use of numerical example how the benefit of value-enhancing actions varies with different value-enhancing characteristics. If a given fixed cost is surpassed, a positive net gain obtains resulting in an activation of that action.

[Insert Figure 2 here]

We also assume a positive abandonment recovery value. Focusing on exploration or small impact (size) actions, we see that at very low values of *S* these costly actions will be dominated by abandonment or costless wait. For intermediary values of *S* (near at-the-money), they start to exhibit a significant value enhancement that diminishes as *S* gets deeper in the money. This result is consistent with the findings in Bernardo and Chowdhry (2002) and Huchzermeier and Loch (2001). A costly action may remain dominant for high values of *S* if the expected impact is sufficiently positive. At very high values of *S* the payoffs of value-enhancing actions (as with simple delay) do not depend on the univariate cumulative normal terms (which effectively become 1). The maximum of managerial value-enhancement action, $V^{d_1} = m(d_0, d_1)S_{t_1} + S_{t_1} e^{-\delta(T-t_1)+\gamma(d_0,d_1)} - Xe^{-r(T-t_1)}$ (with $d_1 = MC$ or with *L* using $\gamma(d_0, d_1)$ =0), of wait ($d_1 = W$ with $\gamma(d_0, d_1) = m(d_0, d_1) = 0$ in previous equation), and of early exercise ($d_1 = EE$, which gives *S*-*X*) then provides the best decision. An action with zero expected impact, $\gamma(d_0, d_1) = 0$, but positive volatility will have zero net benefit over a costless wait decision at high values of *S* and will be dominated by costless wait for any positive cost action. Similarly, these zero expected impact actions and costless wait will be dominated by early development at high *S* ranges. With a $\gamma > \delta(T - t_1)$, the payoff with a managerial action increases more than the early exercise (one-to-one) increase and will dominate the upper region irrespective of the action cost. When $\gamma = \delta(T - t_1)$, the action may still be preferable over early development in the upper range if the action costs are low. At high S values the slope of the payoff with respect to S will be a key determinant of which action dominates.

With only a single exploration/experimentation action, we observe the following sequence of regions as a function of S (starting from low values of S) in the most general case: abandonment (A), wait (W), exploration/experimentation (L), wait (W), and early development (EE). In the presence of a positive expected impact action (MC), we may get sequences of regions like {A, W, L, W, MC}, {A, W, L, EE, MC}, {A, L, MC, EE}, etc., depending on action characteristics (special cases may look like {A, MC} for a high average-impact action). No general rule applies for the determination of the sequence of regions at the intermediary S values. Here one can use a simple graphical inspection to investigate these regions and apply value-matching conditions to find the critical points where decision regions change. In the numerical multi-stage procedure (described in the appendix section A), optimal decisions are determined at each node of the numerical lattice tree through an optimization algorithm.

The critical point where one would switch from optimal decision f to i is determined by finding the critical (threshold) value of S that solves the appropriate value-matching condition: $V^{i}\left(S_{t_1}^{*}\left(d_0, i\right), t_1 \mid M, M_1^{+}, M_1^{-}\right) = V^{f}\left(S_{t_1}^{*}\left(d_0, f\right)\right), t_1 \mid M, M_1^{+}, M_1^{-}) \quad i, f \in M_1^{+}$.

At the initial stage (t = 0) the formula is evaluated for each possible decision at t = 0 with the optimal decision being the one providing the maximum value net of costs. The application of the general formula (equation 7) with multiple actions in each of the two stages is demonstrated in the next section. We also employ a simplified version to demonstrate the importance of value-enhancing actions and the interactions between actions implemented at different points in time.

3.1. A numerical example, sensitivity analysis and main implications

Consider an application of equation (7) involving a two-stage option with two exploration and two positive-impact value-enhancing actions, with wait, abandonment and early exercise options. We also consider the case of path-dependency in abandonment costs. Here the action subscript denotes the time that the action can be optionally activated. Assume that the set of actions at $t = 0$ are

$M_0 = \{W, EE, L_0, MC_0\}$ and at $t = t_1$ the firm can choose among actions $M_1 = \{W, EE, L_1, MC_1\}$.[4] Action costs are increasing over time and that managerial value-enhancement are more costly than exploration actions so that $X(L_1) = 2X(L_0)$, $X(L_0)$=2.5, $X(MC_1) = 2X(\mathrm{MC}_0)$, $X(MC_0) = 5$. This example may represent a situation were the firm invests in a marketing campaign (L_0) that may be followed by a more costly pilot project (L_1) and attribute-enhancement actions that are increasingly more costly to implement .

To obtain the value of the project, we need to determine the optimal regions at intermediate point t_1 and evaluate equation (7) conditional on W, L_0 or MC_0. The value of the project at $t = 0$ is the maximum of $S - X$, $CC(.\mid d_0 = W) - X(W)$, $CC(.\mid d_0 = MC_0) - X(MC_0)$ and $CC(.\mid d_0 = L_0) - X(L_0)$. For the numerical investigation, we assume the initial value of the project is $S = 100$, the development costs are $X = 100$ and $X(W) = 0$, $r = \delta = 0.05$, $\sigma = 0.2$, $T = 2$ years, and intermediate decision point is at $t_1 = 1$ year. For L_0 we assume $\gamma(L_0) = 0, \sigma(L_0) = 0.5$ and for action MC_0 at $t = 0$ assume $\gamma(MC_0) = 0.1, \sigma(MC_0) = 0.3$. For the second-stage managerial actions we assume the same characteristics for actions (i.e., $\gamma(L_1) = 0, \sigma(L_1) = 0.5$ and $\gamma(MC_1) = 0.1,\ \sigma(MC_1) = 0.3$). To identify the optimal regions at t_1 we compare the payoffs illustrated in Figure 3 (panels A-D).

[Enter Figure 3 here]

At t_1 there are three action regions: $\{W, L_1, MC_1\}$. EE is a dominated strategy and does not appear at t_1 regardless of the decision at $t = 0$. At t_1, at low project values S, W will be the optimal strategy, while for very high values of S the MC_1 action is optimal. The MC_1 payoff grows at a higher rate than other payoffs for high values of S, such that no other payoff can exceed it. This can be seen from

[4] All possible combinations of actions between t_0 and t_1 involve: (EE),(W,W),(W,L_1),(W,MC_1),(W,EE), (L_0,W), (L_0,EE),(L_0,L_1),(L_0,MC_1),(MC_0,W),(MC_0,EE),(MC_0,L_1), (MC_0,MC_1).

the payoffs.[5] For some S values, the payoff of EE can dominate the payoffs of W and L_1 but here it is dominated by MC_1.[6] To determine the critical thresholds we apply the value-matching conditions; for $d_0 = \{W, L_0, MC_0\}$ the lower boundary of W at t_1 is $S_{t_1}^{*L}(d_0, W) = 0$, the threshold where one switches from W to L_1 is $S_{t_1}^{*H}(d_0, W) = S_{t_1}^{*L}(d_0, L_1) = 65.844$ and the high boundary for the decision L_1 is $S_{t_1}^{*H}(d_0, L_1) = S_{t_1}^{*L}(d_0, MC_1) = 131.096$. With the above information, equation (7) gives: $CC(.\,|\, d_0 = W) = 15.888$, $CC(.\,|\, d_0 = L_0) = 24.827$, and $CC(.\,|\, d_0 = MC_0) = 26.159$ (before considering the costs). The value of the complete project at t = 0 net of the costs is 22.327 (with optimal decision being to follow an exploration action). Exploration is of high importance because the option is at the money. In this case, the exploration action at t = 0 increases the probability of development of the option at maturity (we provide further analysis relating this result to the moneyness of the option in this section using a special case of the general formula). In this particular case we also observe an increase in the likelihood of a direct value-enhancement action at t_1, but a decrease in the probability of a second exploration action.

In order to illustrate the effect of path-dependency on optimal decisions, we revisit the previous case assuming a recovery amount (α) of 50% of incurred costs. For example, if the pure research/exploration is activated at t = 0 the firm can recover 1.25 at t_1 while if in addition a managerial enhancement action is exercised at t_1 the firm may recover 1.25 + 5 = 6.25 at T. With abandonment, option values increase, obtaining the following results (net of associated costs): $CC(.\,|\, d_0 = W) - X(W) = 17.264$ (> 15.888), $CC(.\,|\, d_0 = L_0) - X(L_0) = 23.951$ (> 22.327) and $CC(.\,|\, d_0 = MC_0) - X(MC_0) = 23.515$ (> 21.159). The decision regions at t_1 now are {A, W,

---

[5] For example, at S = 250 $\text{Slope}_{MC_1}\big|_{S=250} = e^{-\delta(T - t_{d_1}) + \gamma_i} N(a_{MC_1,1}) = 1.05$ which is higher than the slopes of other payoffs. Even if the slope of learning or wait goes to one for an incremental increase in S it is still not possible for these payoffs to surpass the positive impact managerial enhancement payoff. Note that the costs will not be important since the impact is proportional to S and S values are at a high range.

[6] Changing the base case parameters may result in more complex regions. An interesting case is where the costs of all the managerial actions are doubled. In this case we will have a region where EE is optimal. The following regions at t_1 would then appear: W, L_1, EE, MC_1. The managerial enhancement option appears in the upper region since its slope is higher than the slope of the exercise payoff.

L_1, $MC_1\}$. Again, the optimal decision at $t = 0$ is to undertake an exploration action. However, with abandonment the difference between the exploration and the expected impact enhancement action decreases because some of the high costs of MC_0 can be recovered in the future (the optimal decision may even be reversed depending on the respective costs and the prospects for capital recovery). Further evidence on the importance of path-dependency are provided in the next section focusing on the opportunities of affecting the expected impact of future actions through exploration actions.

In order to draw more insights and provide sensitivity results we focus on an interesting special case of a sequential growth option with two managerial actions activated at $t = 0$ and/or at $t = t_1$.[7] The first action MC_1, with mean impact and volatility ($\gamma(d_0)$, $\sigma(d_0)$), can be activated at $t = 0$ at a cost $X(d_0)$. The second action (MC_2), with distributional characteristics ($\gamma(d_0, d_1)$, $\sigma(d_0, d_1)$), can be activated at $t = t_1$ at a cost $X(d_0, d_1)$. The set of available decisions are $M_0 = \{A, W, MC_1\}$, $M_1 = \{A, MC_2\}$, and $M_T = \{A, EE\}$. In what follows we assume abandonment has zero recovery (positive-value recovery would require two additional terms as in equation (7)). Action MC_2 also generates cash, such that the firm gets a fraction $m(d_0, d_1)$ of S and improves future investment opportunities. In equation (8) below, a *single threshold* exists at $t = t_1$. The value of the sequential-growth option conditional on the activation of action d_0 at $t = 0$ is given by:

$$\begin{aligned} CC(. \mid d_0 \in M_0 = \{W, MC_1\}) &= Se^{-\delta T + \gamma(d_0) + \gamma(d_0, d_1)} N(a_1, b_1, \rho) \\ &- Xe^{-rT} N(a_2, b_2, \rho) + m(d_0, d_1) Se^{-\delta t_1 + \gamma(d_0)} N(a_1) \\ &- X(d_0, d_1) e^{-rt_1} N(a_2) \end{aligned} \tag{8}$$

---

[7] Equation (7) encompasses other cases appearing in the literature as special cases. The extendible option of Longstaff (1990) can be obtained by setting: $\gamma(d_0) = \sigma^2(d_0) = 0$, $N_{MC} = 0$ $R_{EE} = R_W = R_A = 1$, $m(d_0, W) = 0$, $X(d_0, W) > 0$, $\alpha = 0$

The wait mode here is equivalent to an extension option. In this case there are three regions at the intermediary decision point, *A*, *W*, and *EE*, with two thresholds, between *A* and *W* and between *W* and *EE*. Since $S_{t_1}^{*H}(d_0, EE) = \infty$ we then have $N(a_{EE,1}^H) = N(a_{EE,2}^H) = 0$.

where $$a_1 = \frac{ln\left(S / S^*_{t_1}(d_0, d_1)\right) + \left(r - \delta + 0.5\sigma^2\right)t_1 + \gamma(d_0) + 0.5\sigma^2(d_0)}{\left(\sigma^2 t_1 + \sigma^2(d_0)\right)^{1/2}}$$

$$b_1 = \frac{ln(S / X) + (r - \delta)T + \left(\gamma(d_0) + \gamma(d_0, d_1)\right) + 0.5\sigma^2 T + 0.5\left(\sigma^2(d_0) + \sigma^2(d_0, d_1)\right)}{\left(\sigma^2 T + \sigma^2(d_0) + \sigma^2(d_0, d_1)\right)^{1/2}}$$

$$a_2 = a_1 - \left(\sigma^2 t_1 + \sigma^2(d_0)\right)^{1/2}, \quad b_2 = b_1 - \left(\sigma^2 T + \sigma^2(d_0) + \sigma^2(d_0, d_1)\right)^{1/2}$$

$$\rho = \sqrt{\frac{\sigma^2 t_1 + \sigma^2(d_0)}{\sigma^2 T + \sigma^2(d_0) + \sigma^2(d_0, d_1)}}$$

The value of the option, assuming the firm decides to wait at $t = 0$, is obtained by setting $\gamma(d_0) = \sigma^2(d_0) = 0$. The optimal value at $t = 0$ equals $max\left(CC(.\mid MC_1) - X(MC_1), CC(.\mid W) - X(W), 0\right)$. The compound call option of Geske (1979) can be seen as a special case by setting $\gamma(d_0) = \sigma(d_0) = \gamma(d_0, d_1) = \sigma(d_0, d_1) = 0$. The critical value $S^*_{t_1}(d_0, d_1)$ for $d_1 = MC_2$ is found by solving numerically the value-matching condition:

$$m(d_0, d_1)S^*_{t_1}(d_0, d_1) + S^*_{t_1}(d_0, d_1)e^{-\delta(T - t_1) + \gamma(d_0, d_1)}N(\upsilon_1) - X\, e^{-r(T - t_1)}N(\upsilon_2) = X(d_0, d_1)$$

with $$\upsilon_1 = \frac{ln(S / X) + \left(r - \delta + 0.5\sigma^2\right)(T - t_1) + \gamma(d_0, d_1) + 0.5\sigma^2(d_0, d_1)}{\left[\sigma^2 T + \sigma^2(d_0, d_1)\right]^{1/2}}$$

$$\upsilon_2 = d_1 - \left[\sigma^2 T + \sigma^2(d_0, d_1)\right]^{1/2}$$

In Figure 4 we investigate the impact of value-enhancing and exploration actions in the sequential framework using equation (8).

[Insert Figure 4 here]

Panel A shows the effect of changes in the impact $\gamma(d_0)$ and volatility $\sigma(d_0)$ of a managerial action on the value of the option, assuming no further improvement action is available at t_1. The value of a project is increasing in the expected impact and volatility of actions. Similar results apply for $\gamma(d_0, d_1)$ and $\sigma(d_0, d_1)$. Panel B shows that an increase in the volatility of the action in general increases the probability of project development for out-of-the-money (or near-at the-money) options and decreases the probability of project development for in-the-money options. This confirms the intuition developed above on the importance of exploration actions for out-of-money or at-the-money options. Project values are increasing in the volatility (or the information revelation level) as shown in Panel A, but for very high volatility levels the probability of development declines. This concave shape of probability of development as a function of the volatility of the R&D investments is similar with the observations made in Sarkar (2000) about the effect of exogenous volatility on investment. The impulse nature of the endogenously activated actions in our case shows that higher information revelation (increase in volatility) of R&D investments may result in a strictly decreasing relationship with the probability of development when options are in the money. We also observe that the difference in the development probabilities between out- and in-the-money options is decreasing considerably with the volatility of the action, i.e., options that are in-the-money tend to have similar probability of development with out-of-the money options as the volatility increases. In Panel C we investigate the marginal effect of such actions. The figure shows that the incremental value (% benefit) of exploration actions (over passive wait at $t = 0$) is decreasing in the volatility and expected impact of follow-on actions. The intuition behind this result is as follows. A higher information revelation of follow-on actions increases the value of the call-like payoff at t_1. This effectively increases the moneyness of the compound option (the option held at $t = 0$). Since exploration actions are less important for in-the- money options, the marginal value of additional units of information revelation (volatility) is less in the presence of follow-on actions. The same result applies for positive expected-impact follow on actions (the marginal benefit at $t = 0$ of additional volatility revelation is less).

As shown in equation (8), both the volatility and the impact of actions cumulate as more actions are implemented. Thus, despite the lower marginal value of actions, the combined (cumulative) present value increase may exceed the costs. Interestingly, this may result in exercise of exploration actions even at high levels of the underlying project value. This in contrast to a single period case and the results of Bernardo and Chowdhry (2002) and Huchzermeier and Loch (2001) that show that exploration actions are only important for in-between ranges of project values and not for very high values. To see how this may result in our model, assume for simplicity a zero cash factor *m* and no early exercise at t_1. (These features would work further in favour of exploration action at t = 0.) For very high values of *S*, the value of an exploration action (L_1) at $t = 0$ assuming a follow on value-enhancing action MC_2, is at least $Se^{-\delta T+\gamma(L_1,MC_2)} - Xe^{-rT} - X(L_1,MC_2)e^{-rt_1}$ (the univariate and bivariate terms tend to 1 for very high values of *S*). It can then be seen that exploration actions may prevail even at high values of *S*. For example, comparing the exploration payoff with early development at t = 0, shows that for sufficiently high expected impact γ of follow-on actions the benefits would outweigh the lost value due to the erosion δ and the extra cost of action that has to be paid at t_1 Furthermore, the importance of exploration actions increases when they provide enhanced benefits (higher expected impact or lower cost) of follow on actions, e.g., due to learning. In the next section we extend this framework to multiple periods with path-dependent managerial actions focusing on the effect of early exploration actions' on follow on decisions.

4. Multistage product development with path-dependencies

Let us revisit the general problem of Figure 1 with multiple decision points and managerial actions involving optimal timing and path-dependency. Section A of the appendix describes a numerical approach based on a forward-backward algorithm of exhaustive search in a lattice framework for this multistage problem. We focus on the same problem with two exploration actions $\{L_1, L_G\}$ and two positive-impact value-enhancing actions $\{MC_1, MC_2\}$ in a multi-stage framework. The admissible sequences of actions assumed here is that the firm can move from research (L_1), to product development stage I (MC_1), to product development stage II (MC_2). Alternatively, it can move from L_1 to the scaled-down version I (L_G) of the product to product development phase II (MC_2). We assume here that with the pilot project the firm also implement development stage I. Some or all actions can be skipped (see Figure 1) and the characteristics of version I (pilot project L_G) and the managerial actions depend on the sequence (path) followed. Figure 5 provides the base-case parameter values for this problem.

[Enter Figure 5 here]

The total uncertainty that can be resolved by exploration actions is 3 $(0.3)^2$. Direct implementation of the pilot project (at optimal time) resolves 2/3 of this uncertainty. Volatility resolved by each of the other actions is 30% (1/3 of total). The pilot project also provides cash that is a fraction *m* = 10% of the basic product version. The expected impact of the managerial value-enhancing actions is higher if a research action has been performed earlier ($\gamma(L_G, MC_2) = 0.2$, $\gamma(L_1, MC_1) = 0.2$). The associated costs of these actions are $X_{L_1} = X_{MC_1} = X_{MC_2} = 10$, $X_G = 20$, the maturity of the option is $T = 5$ years, and we allow 5 annual decision points before maturity (starting at t_0). We determine the optimal timing of all actions in the problem. For example, the firm may decide to invest in product development of basic attributes in year 2 and in further quality improvements in year 4. We use a 12-step lattice tree per year (dt = 1 month) and an exhaustive search to evaluate all combinations of decisions between nodes of the lattices at each point in time.

Our choices of parameter values are chosen to get closer to average product development situations. The short 5-year horizon of the maturity reflects typical situations in innovation development with firms facing high competitive pressures. The costs of the actions are consistent with empirical observation in Amir et al. (2006) that R&D expenditures are about 8% of the market value of equity for R&D-intensive firms. Grabowski and Vernon (1990) document average returns on pharmaceutical R&D of around 15-30% with a highly skewed distribution with the top decile providing a return around 400-500%. A cost of 10 to 20 provides an increase of 10-20% on an asset value of 100, implying an R&D return of about 50% to 100%. Exogenous uncertainty may in general be high especially for new products. Childs and Triantis (1999) use a standard deviation of 40% to capture volatility attributed to R&D; we use 30%.

[Enter tables 1, 2 here]

Our numerical results show sensitivity with respect to the basic project value *S*, the cash flow factor of the scaled-down version *m*, and the effect of exploration actions on the expected impact of follow-on value-enhancement actions ($\gamma\left(L_1, MC_1\right)$ and $\gamma\left(L_G, MC_2\right)$). Table 1 provides sensitivity with respect to the effectiveness of exploration actions (keeping the cash factor to m = 0.1). The base case reflects the situation where exploration actions provide for a better expected impact of follow-on actions ($\gamma(L_G,MC_2)$ =0.2, and $\gamma(L_1,MC_1)$=0.2) than implementing these actions directly (which provides only a 10% expected increase for each action). Under this specification, the early version dominates in most of the range of S values, with delay being optimal at lower values. The importance of the scaled-down version is enhanced by the early cash it provides (equal to 10% of S), besides the enhancing the impact of follow-on actions. The second panel shows the case when neither the scaled-down version nor the exploration action L_1 provide any additional expected impact on follow-on actions ($\gamma(L_G,MC_2)$ = 0.1, $\gamma(L_1,MC_1)$ = 0.1). The third panel shows the case when the early version does not provide any additional expected value-enhancement, while action L_1 does ($\gamma(L_G,MC_2)$ = 0.1, $\gamma(L_1,MC_1)$ = 0.2). The results show that if neither of the two exploration actions improves the characteristics of future actions, the firm will proceed directly to positive-impact value-enhancing actions. When the early version does not improve the characteristics of future actions while L_1 does, the firm will proceed with L_1. The overall results highlight the importance of optimal sequencing of actions due to path-dependency when actions affect follow-on actions. Contrary to Bernado and Chowdhry (2002) and Huchzermeier and Loch (2001), here even very profitable investments might justifiably be postponed with the firm performing further experimentation when technical uncertainty remains. Path-dependency may reinforce this result since early actions may enhance the impact of follow-on actions. Table 2 provides sensitivity with respect to the level of the early version's cash factor m. As expected, the higher the cash factor (with more revenues provided early on) the more likely management will proceed with the early version and the higher the project (option) value will be at t = 0.

5. Conclusion

We analyzed investment options with embedded explorative research (e.g., experimentation or marketing research) and value-enhancing (attribute or quality improvement actions, or advertisement). These actions improve option value through increases in the expected impact of project value or through information revelation. Our framework can be used for analyzing new product development and deriving optimal decisions. The framework accounts for path-dependency in the mean impact, volatility and cost of actions. We derive analytic solutions for two-stage sequential options, and use a numerical lattice-based model for analyzing the multi-stage problem. Our model also allows for early development, abandonment and early versions of the product that provide cash and resolve uncertainty. Exogenous uncertainty is modeled using a diffusion (or jump-diffusion) process.

We show that there may be an interchanging range of optimal decision regions but in general exploration actions are important when the NPV of the project is close to zero. The marginal value of value-enhancing exploration actions is less important when follow-on actions exist. Exploration actions are shown to be worthwhile even for valuable products when subsequent actions may enhance the expected value of the project. The optimal sequencing of actions is important in the presence of path-dependency.

References

Amir, Eli, Yanling Guan, and Gilad Livne. 2007. The Association of R&D and Capital Expenditures with Subsequent Earnings Variability, *Journal of Business Finance & Accounting*, 34(1) & (2), 222–246.

Anderson, Christopher W., and Luis Garcia-Feijóo. 2006. Empirical Evidence on Capital Investment, Growth Options and Security Returns, *Journal of Finance* LXI, 171-194.

Berk, Jonathan, Richard C. Green, and Vasant Naik. 2004. Valuation and Return Dynamics of New Ventures, *Review of Financial Studies* 17, 1-35.

Bernado, Antonio E., and Bhagwan Chowdhry. 2002. Resources, Real Options, and Corporate Strategy, *Journal of Financial Economics* 63, 211-234.

Black, Fischer, and Myron Scholes. 1973. The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–659.

Burrows, Peter. 2006. The Man Behind Apple's Design Magic, *Business Week* 4002 (9/25/2006, special section), 26-33.

Carr, Peter. 1988. The Valuation of Sequential Exchange Opportunities, *Journal of Finance* 43, 1235-1256.

Childs, Paul D., Steven H. Ott, and Alexander J. Triantis. 1998. Capital Budgeting for Interrelated Projects: A Real Options Approach, *Journal of Financial and Quantitative Analysis* 33, 305-334.

Childs, Paul D., Steven H. Ott, and Timothy J. Riddiough. 2001. Valuation and Information Acquisition Policy for Claims Written on Noisy Assets, *Financial Management* 30, 45-75.

Childs, Paul D., Steven H. Ott, and Timothy J. Riddiough. 2002. Optimal Valuation of Claims on Noisy Assets: Theory and an Application, *Real Estate Economics* 30, 415-444.

Childs, Paul D., and Alexander J. Triantis. 1999. Dynamic R&D Investment Policies, *Management Science* 45, 1359-1377.

Chung, H. Peter, and Herb Johnson. 1994. Pricing Extendible Options: The General Case, Working paper.

Constantinides, George. 1978. Market Risk Adjustment in Project Valuation, *Journal of Finance*

33, 603–616.

Davis, Graham A., and Michael Samis. 2006. Using Real Options to Value and Manage Exploration, Society of Economic Geologists Special Publication 12, ch. 14, 273-294.

Geske, Robert. 1979. The Valuation of Compound Options, *Journal of Financial Economics* 7, 63-81.

Grabowski, Henry, and John Vernon. 1990. A New Look at the Returns and Risks to Pharmaceutical R&D, *Management Science* 36, 804-821.

Grenadier, Steven R., and Allen M. Weiss. 1997. Investment in Technological Innovations: An Option Pricing Approach, *Journal of Financial Economics* 44, 397-416.

Gukhal, Chandrasekhar R. 2004. The Compound Option Approach to American options on Jump-Diffusions, *Journal of Economic Dynamics and Control* 28, 2055-2074.

Huchzermeier, Arnd, and Christoph H. Loch. 2001. Project Management under Risk: Using the Real Options Approach to Evaluate Flexibility in R&D, *Management Science* 47, 85-101.

Hull, John, and Alan White. 1993. Efficient Procedures for Valuing European and American Path-Dependent Options, *Journal of Derivatives* 1, 21-31.

Kauffman, Gordon M. 1963. Statistical Decision and Related Techniques in Oil and Gas Exploration. Englewood Cliff, N.J.: Prentice Hall.

Korn, Ralf. 1997. Optimal Impulse Control when Control Actions Have Random Consequences, *Mathematics of Operations Research* 22, 639-667.

Kothari, S. P., Ted E. Laguerre, and Andrew J. Leone. 2002. Capitalization versus Expensing: Evidence on the Uncertainty of Future Earnings from Capital Expenditures versus R&D Outlays, *Review of Accounting Studies* 7, 355-382.

Longstaff, Francis A. 1990. Pricing Options with Extendible Maturities: Analysis and Applications, *Journal of Finance* 45, 935-957.

Majd, Saman, and Robert Pindyck. 1989. The Learning Curve and Optimal Production under Uncertainty. *RAND Journal of Economics* 20, 331-343.

Martzoukos, Spiros H. 2000. Real Options with Random Controls and the Value of Learning, *Annals of Operations Research* 99, 305-323.

McDonald, Robert and Daniel Siegel. 1984. Option Pricing When the Underlying Asset Earns a Below-Equilibrium Rate of Return: A Note, *Journal of Finance* 39, 261–265.

McDonald, Robert and Daniel Siegel. 1986. The Value of Waiting to Invest, *Quarterly Journal of Economics* 101, 707–727.

McGregor, Jena, William C. Symonds, Dean Foust, Diane Brady, and Moira Herbst. 2006. How Failure Breeds Success, *Business Week* 3992 (10/7/2006), 42-52.

Merton, Robert. 1973. An Intertemporal Capital Asset Pricing Model, *Econometrica* 41, 867–887.

Merton, Robert. 1976. Option Pricing When the Underlying Stock Returns are Discontinuous, *Journal of Financial Economics* 3, 125-144.

Moon, Ihlwan. 2006. Camp Samsung. *Business Week* 3991, 46-48.

Pennings Enrico, and Onno Lint. 1997. The Option Value of Advanced R&D, European *Journal of Operational Research* 103, 83-94.

Pindyck, Robert S. 1993. Investments of Uncertain Cost, *Journal of Financial Economics* 34, 53-76.

Ritchken, Peter, and Bardia Kamrad. 1991. A Binomial Contingent Claims Model for Valuing Risky Ventures, *European Journal of Operational Research* 53, 106-118.

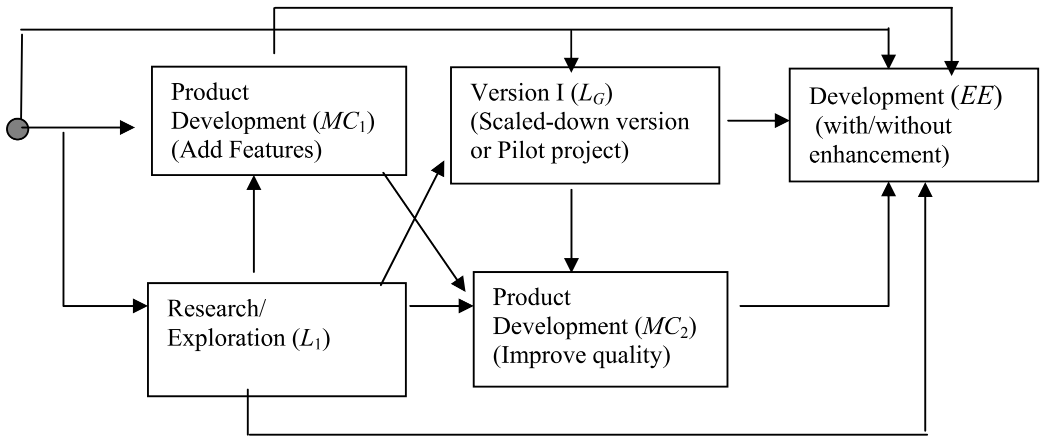
Sarkar, Sudipto. On the investment-uncertainty relationship in a real options model, *Journal of Economic Dynamics and Control*, 24, 219-225.

Smith, James L., and Rex Thompson. 2005. Diversification and the Value of Exploration Portfolios, Working Paper, Southern Methodist University.

Thompson, Andrew C. 1995. Valuation of Path-dependent Contingent Claims with Multiple Exercise Decisions over Time: The Case of Take-or-Pay, *Journal of Financial and Quantitative Analysis* 30, 271-293.

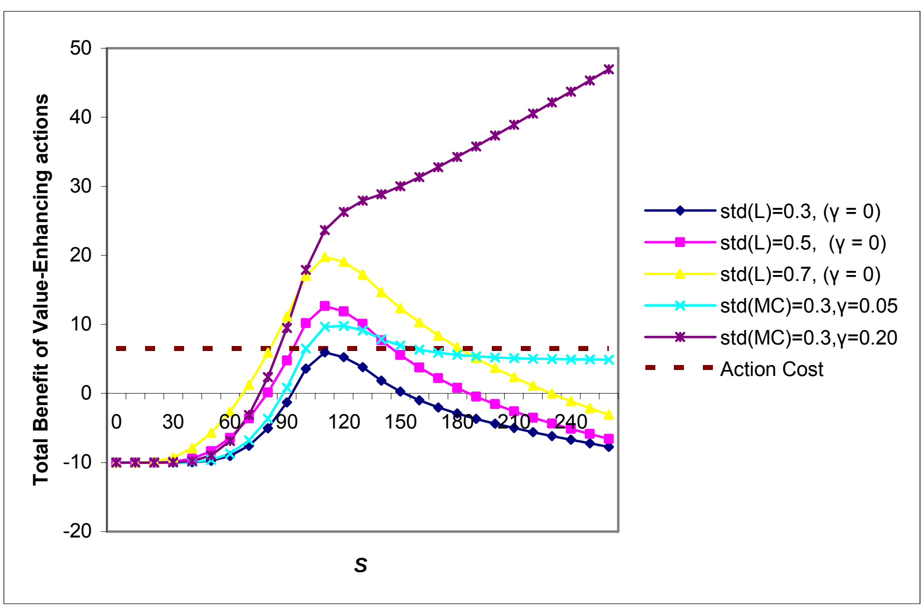
Weitzman, Martin L., and Kevin Roberts. 1981. Funding Criteria for Research, Development, and Exploration Projects, *Econometrica* 49, 1261-1288.

Figure 1. Product development with early product versions and value-enhancing actions



Note: The figure describes a representative case in product development. The firm is considering the development of a product that currently has an expected value S (Development mode (EE) without enhancement). The firm faces exogenous uncertainty about demand and may hold an option to delay development (Wait (W) mode not shown and implied between decisions). The firm can however proceed with alternative development scenarios that involve the choice of research (L_1) (e.g. marketing research or experimentation) or it may proceed to further product development, improving basic features (MC_1), or investing in improving quality (MC_2). These actions are expected to increase value, albeit with uncertain outcome. Another choice involves the launching of an early scaled-down version with limited features (Version I (L_G)) (getting a fraction of cash of the complete version) that may also provide information about the complete basic version. MC_1 and MC_2 can also be alternatively interpreted as advertisement campaigns targeting to enhance the complete version's value with uncertain outcome. All value-enhancing actions can be developed at optimal timing.

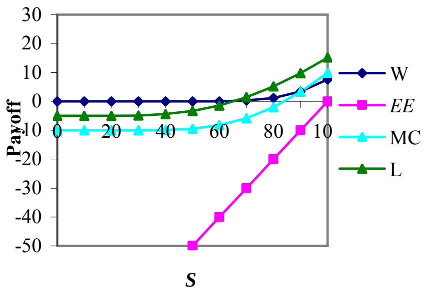
Figure 2. Net benefit of exercising exploration (L) and mean enhancing actions with uncertain outcome (MC)



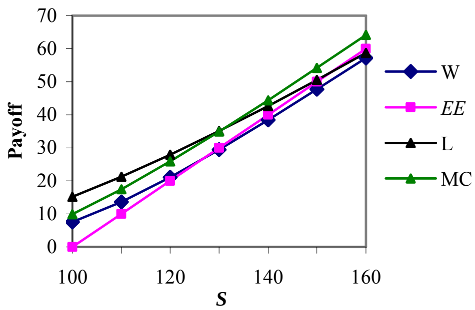
Note: Base parameters values are $m = 0$, development cost $S = 100$, $X = 100$, $r = \delta = 0.05$, $\sigma = 0.1$, $t_1 = 1$, and $T = 2$. We use payoff functions at intermediary decision point t_1 (see equation 6) under alternative parameter values for value-enhancing actions. Total benefit (y-axis) is defined as the difference between the payoff of the value-enhancing action and the next best payoff (the maximum of abandonment, wait or exercise). Abandonment recovery value is assumed to be equal 10, and delay is costless. The net benefits should be compared with the action cost (assumed equal to 6.5) in order to determine the optimal decision.

Figure 3. The payoff functions of the compound-growth option with wait, early exercise and two managerial actions at the intermediate point (t_1)

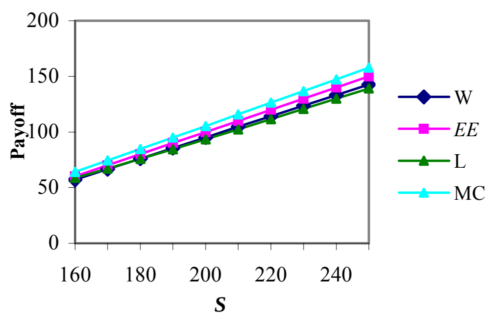
Panel A:



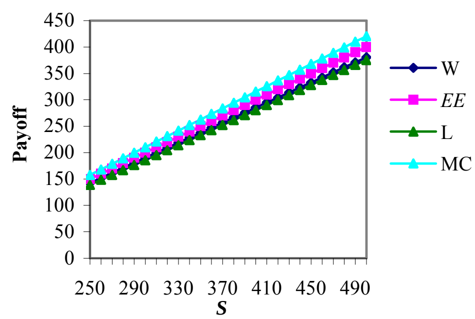
Panel B:



Panel C:



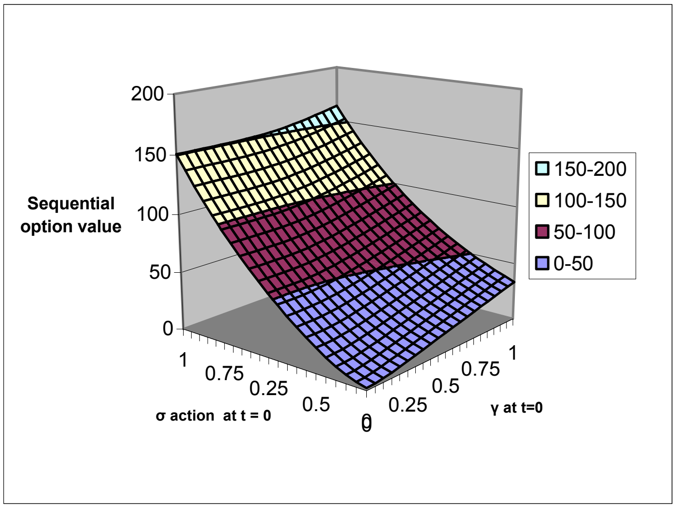
Panel D:



Note: We investigate the payoffs for alternative decisions at t_1 =1 for an investment option with maturity $T = 2$ (see equation 6). The set of possible actions at t_1 is Wait (W), Early Exercise (EE), managerial enhancement action (MC) or exploration/experimentation (L). The general parameters for the problem are: S =100, $r = \delta = 0.05$, $\sigma = 0.2$. For L_1 use $\sigma_L = 0.5$, $X_L = 5$, while the managerial enhancement option use $\sigma_{MC} = 0.3$, $\gamma_{MC} = 0.1$, $X_{MC} = 10$. We also assume $m = 0$.

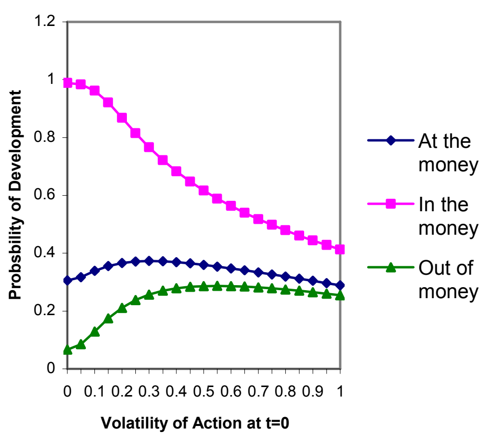
Figure 4. Sensitivity of the sequential investment option, the probability of development and the incremental benefits of value-enhancing actions with respect to actions' expected impact and volatility

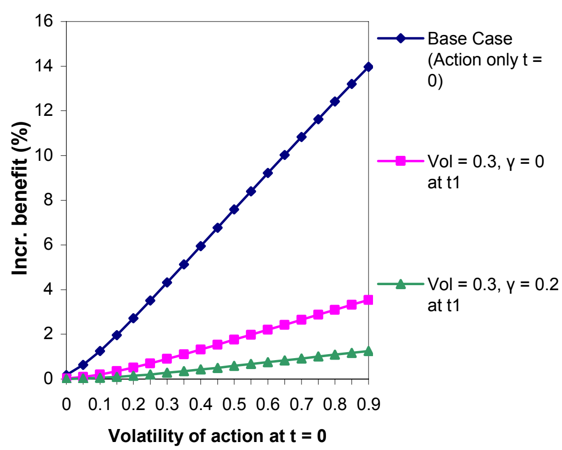
Panel A: Joint effects (at $t = 0$) of expected impact and volatility of value-enhancement actions



Panel B: Probability of development as a function of volatility of actions

Panel C: Net benefit (%) of value enhancing actions as a function of volatility of actions at t_0 with and without action at t_1





Note: Base parameters values (at the money case) are $m = 0$, $S = 100$, $X = 100$, $X(d_0, d_1) = 5$, $r = \delta = 0.05$, $\sigma = 0.1$, $t_1 = 1$ and $T = 2$. For all panels equation (8) is used. Panel A shows joint effect (at $t = 0$) of changes in impact $\gamma(d_0)$ and volatility $\sigma(d_0)$ of a managerial action on the value of the compound-growth option (additionally $\gamma(d_0, d_1)$ =0, $\sigma(d_0, d_1)$ = 0). For panels B $\gamma(d_0)$ = 0, in-the-money S =140 and out-of-the-money case S = 90. For panel C, Incr. benefit (%) is defined as the incremental percentage difference between the value with a managerial value enhancement action and the value with no action at t = 0.

Figure 5. Parameter values for the new product development case

Volatility matrix of actions

From \ To	L_1	L_G	MC_1	MC_2
W	$(0.3)^2$	$2(0.3)^2$	$(0.3)^2$	-
L_1	-	$(0.3)^2$	$(0.3)^2$	-
L_G	-	-	-	$(0.3)^2$
MC_1	-	-	-	$(0.3)^2$

Average impact matrix of actions

From \ To	L_1	L_G	MC_1	MC_2
W	0	0	0.1	-
L_1	-	0	0.2	-
L_G	-	-	-	0.2
MC_1	-	-	-	0.1

Note: For the description of the problem see Figure 1. Base-case parameters are: r = δ = 0.05, σ = 0.1, T = 5, cost for each action is $X_{L_1} = X_{MC_1} = X_{MC_2} = 10$ and $X_G = 20$. Growth factor of pilot project is m = 0.1. The average impact and volatility of managerial actions are given above. We use 5 decision points (N*dec* = 5) and for the numerical lattice N*sub* = 12 per year (*dt* = 1 month).

Table 1. Multi-stage investment program with an exploration L_1, a pilot project (L_G) and value-enhancing actions (MC_1, MC_2). Sensitivity with respect to the effectiveness of exploration actions (L_1, L_G)

	Base case $\gamma(L_G,MC_2) = 0.2$ $\gamma(L_1,MC_1) = 0.2$		$\gamma(L_G,MC_2) = 0.1$ $\gamma(L_1,MC_1) = 0.1$		$\gamma(L_G,MC_2) = 0.1$ $\gamma(L_1,MC_1) = 0.2$	
S	**Value**	**Optimal decision at *t* =0**	**Value**	**Optimal decision at *t* =0**	**Value**	**Optimal decision at *t* =0**
240	170.096	L_G	155.495	MC_1	164.770	L_1
230	158.230	L_G	144.523	MC_1	153.281	L_1
220	146.400	L_G	133.564	MC_1	141.814	L_1
210	134.615	L_G	122.629	MC_1	130.381	L_1
200	122.890	L_G	111.747	MC_1	118.995	L_1
190	111.274	L_G	100.925	MC_1	107.689	L_1
180	99.761	L_G	90.153	MC_1	96.446	L_1
170	88.365	L_G	79.471	MC_1	85.296	L_1
160	77.109	L_G	68.965	MC_1	74.308	L_1
150	66.080	L_G	58.589	MC_1	63.483	L_1
140	55.347	L_G	48.468	MC_1	52.891	L_1
130	44.903	L_G	38.700	MC_1	42.644	L_1
120	34.831	L_G	29.294	MC_1	32.784	L_1
110	25.390	L_G	20.575	MC_1	23.535	L_1
100	16.521	L_G	12.514	MC_1	14.982	L_1
90	9.006	*W*	6.599	*W*	7.840	*W*
80	3.951	*W*	2.720	*W*	3.198	*W*
70	1.180	*W*	0.755	*W*	0.849	*W*
60	0.179	*W*	0.113	*W*	0.119	*W*
50	0.009	*W*	0.006	*W*	0.006	*W*

Note: See problem description and base-case parameters in Figures 1 and 5.

Table 2. Multi-stage investment program with an exploration L_1, a pilot project (L_G) and value-enhancing actions (MC_1, MC_2). Sensitivity with respect to the level m of early version (pilot project) cash flows

	Growth m = 0		Growth m = 0.1		Growth m = 0.2	
S	**Value**	**Optimal decision at t=0**	**Value**	**Optimal decision at t=0**	**Value**	**Optimal decision at t=0**
240	164.770	L_1	170.096	L_G	194.096	L_G
230	153.281	L_1	158.230	L_G	181.230	L_G
220	141.814	L_1	146.400	L_G	168.400	L_G
210	130.381	L_1	134.615	L_G	155.615	L_G
200	118.995	L_1	122.890	L_G	142.890	L_G
190	107.689	L_1	111.274	L_G	130.274	L_G
180	96.446	L_1	99.761	L_G	117.761	L_G
170	85.296	L_1	88.365	L_G	105.365	L_G
160	74.308	L_1	77.109	L_G	93.109	L_G
150	63.483	L_1	66.080	L_G	81.080	L_G
140	52.891	L_1	55.347	L_G	69.347	L_G
130	42.644	L_1	44.903	L_G	57.903	L_G
120	32.784	L_1	34.831	L_G	46.831	L_G
110	23.535	L_1	25.390	L_G	36.390	L_G
100	14.982	L_1	16.521	L_G	26.521	L_G
90	7.840	W	9.006	W	17.430	L_G
80	3.198	W	3.951	W	9.360	W
70	0.849	W	1.180	W	3.920	W
60	0.119	W	0.179	W	1.002	W
50	0.006	W	0.009	W	0.105	W
40	0.000	W	0.000	W	0.002	W

Note: See problem description and base-case parameters in Figures 1 and 5. The middle column represents the base case parameters.

Appendix

Section A: A numerical lattice model for the multi-stage problem

In this section we consider the more general version of the new product development problem that allows for multiple stages, path-dependent actions at optimal timing, growth options, abandonment options and early exercise in a numerical lattice based framework. The multi-stage framework is useful in solving problems like those posed in Figure 1 of the main text .Decisions are again made sequentially at equal periodic intervals. To account for path-dependency we keep track of all previous decisions M_t^-. Remember that $V^{d_t}(.)$ is the payoff conditional on decision d_t. This payoff is a function of the value of cash flows S at that decision point, the characteristics of available actions, the development cost X, the action-specific path-dependent costs $X\left(M_t^-, d_1\right)$, the recovery rate α in case of abandonment, the growth factors $m\left(M_t^-, d_1\right)S$ of pilot projects, etc. t represents the time of a decision point ($t < T$) and Δt the time interval between decision points. Our objective is to maximize the value of the investment option value (V) by making the optimal feasible decisions (d_t) at each t:

$$V^*\left(S_t, t \mid M, M_t^+, M_t^-\right) = \max_{d_t \in M_t}\left\{V^{d_t}\right\}$$

We have the following cases:

$$V^{d_t}\left(S_t, t \mid M, M_t^+, M_t^-\right) = e^{(-r\Delta t)} E_t^{d_t}\left[V^*\left(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-\right)\right] + m\left(M_t^-, d_t\right)S_t - X\left(M_t^-, d_t\right) \text{ for } d_t \in \left\{MC_1, MC_2, \ldots., MC_{N_{MC}}\right\}$$

$V^{d_t}\left(S_t, t \mid M, M_t^+, M_t^-\right) = S_t - X$ for $d_t \in \{EE\}$,

$V^{d_t}\left(S_t, t \mid M, M_t^+, M_t^-\right) = \alpha TC_t\left(M_t^-\right)$ for $d_t \in \{A\}$ where $TC\left(M_t^-\right)$ defines the total investment costs paid until t, and

$$V^{d_t}\left(S_t, t \mid M, M_t^+, M_t^-\right) = e^{(-r\Delta t)} E_t^{d_t}\left[V^*\left(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-\right)\right]$$

for $d_t \in \left\{W, W_1, W_2, \ldots, W_{N_{MC}}\right\}$,

At maturity we have:

$$V^{d_T}\left(S_T, T \mid M, M_T^+, M_T^-\right) = max\left(S_T - X, a\ TC\left(M_T^-\right)\right)$$

Expectation $E_t^{d_t}$ when $d_t \in \left\{MC_1, MC_2, \ldots, MC_{N_{MC}}\right\}$ is taken with respect to the distribution of log-returns that depend both on demand and exploration actions information revelation or expected value-enhancing actions volatility. With delay i.e. $d_t \in \left\{W, W_1, W_2, \ldots, W_{N_{MC}}\right\}$, the expectation is taken only with respect to demand uncertainty. Note that $d_t \in \{EE, A\}$ are terminal/absorbing decision states. We discretize the state-space of S values using a numerical lattice scheme. From equation (5) of the main text, the underlying asset S follows a log-normal distribution between decision points. We approximate the distribution on the time interval $\Delta t \equiv T_{sub} = \dfrac{T}{N_{dec}}$ with a binomial lattice with N_{sub} steps in-between decision points. The per step conditional volatilities $\upsilon^2\left(M_t^-, d_t\right)$ over the interval (t, t +Δt) depend on the current action d_t and all previous actions. They equal $\upsilon^2\left(M_t^-, d_t\right) = \sigma^2 \dfrac{T_{sub}}{N_{sub}} + \dfrac{\sigma^2\left(M_t^-, d_t\right)}{N_{sub}}$ for $d_t \in \left\{MC_1, MC_2, \ldots, MC_{N_{MC}}\right\}$. Managerial actions with uncertain outcome are of impulse type. However, since decisions are at discrete intervals and because of the actions multiplicative impact on project value, the distribution between decision points is not affected if we allocate the total volatility $\sigma^2\left(M_t^-, d_t\right)$ and impact $\gamma\left(M_t^-, d_t\right)$ equally between the N_{sub} steps. We use the following up and down moves for the lattice between stages (decision points) that match the volatility of the continuous process (equation 1 of main text):

$$u\left(M_t^-, d_t\right) = exp\left(\upsilon\left(M_t^-, d_t\right)\right), \qquad d\left(M_t^-, d_t\right) = \frac{1}{u\left(M_t^-, d_t\right)}$$

The probabilities for an up and down move for $d_t \in \left\{MC_1, MC_2, \ldots, MC_{N_{MC}}\right\}$ are obtained by matching the mean of the continuous process:

$$p_u\left(M_t^-, d_t\right) = \frac{exp\left((r-\delta)\frac{T_{sub}}{N_{sub}} + \frac{\gamma\left(M_t^-, d_t\right)}{N_{sub}}\right) - d\left(M_t^-, d_t\right)}{u\left(M_t^-, d_t\right) - d\left(M_t^-, d_t\right)},$$

$$p_d\left(M_t^-, d_t\right) = 1 - p_u\left(M_t^-, d_t\right)$$

For $d_t \in \left\{W, W_1, \ldots, W_{N_c}\right\}$ we set the γ and σ parameters equal to zero. Due to path dependency the optimal value V^* cannot be evaluated in the usual backward dynamic programming fashion. Instead, we take into account all alternative combinations of actions and paths of the state-variable. We thus implement a forward-backward algorithm of exhaustive search (see also Hull and White, 1993, Ritchken and Kamrad, 1991, or Thompson, 1995), and the optimal decision will determine today's option value.

Table A1 shows a comparison between the analytic and lattice based numerical model for the case of a two-stage compound-growth option with different levels of exploration volatility at t_1 and possible positive cash flows (cash factor m) at t_1 At $t = 0$ we assume that only costless wait is possible. We can see that the numerical model provides a very good approximation to the analytic formulas in both cases. Note that the case of zero volatility of action (and zero impact) reflects the case of the compound option of Geske (1979). The results show that the value of exploration options embedded in investment options can be extremely important. In the table we use N_{sub} = 60 steps. We have also implemented the lattice with N_{sub} = 30 steps and the error was again very low (between 0.1% – 0.7%).

Table A1 Comparison of numerical and analytic compound option with learning

Time	Growth option factor m = 0						
	Volatility	S = 80		S = 100		S = 120	
	of action	Analytic	Numerical	Analytic	Numerical	Analytic	Numerical
T = 1	0.000	0.000	0.000	1.103	1.094	14.320	14.315
	0.100	0.001	0.001	1.656	1.662	14.839	14.838
	0.200	0.016	0.015	3.773	3.774	16.864	16.863
	0.300	0.282	0.282	7.079	7.093	19.883	19.892
	0.400	1.743	1.753	10.660	10.678	23.341	23.357
	0.500	4.424	4.438	14.266	14.288	26.991	27.012
T = 2	0.000	0.013	0.013	2.123	2.118	14.100	14.094
	0.100	0.027	0.026	2.648	2.654	14.675	14.675
	0.200	0.126	0.127	4.406	4.410	16.547	16.550
	0.300	0.616	0.618	7.203	7.214	19.310	19.319
	0.400	2.038	2.050	10.447	10.461	22.506	22.519
	0.500	4.400	4.416	13.792	13.811	25.906	25.924
T = 5	0.000	0.302	0.300	3.860	3.859	13.635	13.643
	0.100	0.378	0.380	4.244	4.244	14.091	14.094
	0.200	0.668	0.664	5.427	5.439	15.462	15.464
	0.300	1.338	1.338	7.339	7.348	17.560	17.566
	0.400	2.552	2.563	9.747	9.758	20.091	20.102
	0.500	4.319	4.328	12.402	12.418	22.850	22.866
Time	Growth option factor m = 0.1						
	Volatility	S = 80		S = 100		S = 120	
	of action	Analytic	Numerical	Analytic	Numerical	Analytic	Numerical
T = 1	0.000	2.964	2.963	8.670	8.662	25.992	25.989
	0.100	3.220	3.218	10.239	10.243	26.537	26.536
	0.200	4.511	4.511	13.344	13.349	28.567	28.567
	0.300	6.707	6.712	16.827	16.841	31.587	31.595
	0.400	9.357	9.368	20.413	20.431	35.045	35.061
	0.500	12.223	12.237	24.019	24.041	38.695	38.716
T = 2	0.000	3.133	3.134	9.857	9.846	25.407	25.401
	0.100	3.506	3.505	11.001	11.005	26.045	26.044
	0.200	4.796	4.797	13.576	13.584	27.957	27.961
	0.300	6.838	6.843	16.674	16.685	30.725	30.734
	0.400	9.299	9.310	19.957	19.972	33.921	33.934
	0.500	11.973	11.989	23.304	23.323	37.321	37.339
T = 5	0.000	3.946	3.951	11.345	11.331	23.977	23.986
	0.100	4.320	4.320	12.004	12.009	24.510	24.512
	0.200	5.400	5.403	13.699	13.705	25.991	25.995
	0.300	7.029	7.033	15.970	15.978	28.137	28.144
	0.400	9.006	9.014	18.527	18.540	30.679	30.691
	0.500	11.187	11.199	21.220	21.236	33.440	33.456

Note: Parameters are: $r = \delta = 0.05$, $\sigma = 0.10$, $t_1 = T/2$, $\gamma = 0$, and cost of action $X_1 = 5$. For the analytic formula we use equation 8 of the main text. For the numerical lattice $N_{sub} = 60$ steps.

Section B: Analytic formulas and a numerical lattice implementation for jump-diffusion

In the presence of *i* optional actions $\{MC_1, MC_2, ...MC_{N_{MC}}\}$ and N_j independent classes of jumps the value of the project is defined as:

$$\frac{dS_t}{S_t} = (a - \sum_{j=1}^{N_j} \lambda \bar{k}_j)dt + \sigma dz + k_i dq_i + \sum_{j=1}^{N_j} k_j d\pi_j$$

Jumps have an impact k_j, j = 1, 2,…, N_j, with $d\pi_j$ denoting Poisson processes with frequency of arrival λ_j per year. The partial integro-differential equation (PIDE) that the option should satisfy is given by:

$$rV = \frac{1}{2}\sigma^2 S^2 V_{SS} + \left(r - \delta - \sum_{j=1}^{N_j} \lambda_j \bar{k}_j\right) SV_S + V_t + \sum_{j=1}^{N_j} \lambda_j E\left[V\left(SY_j, t\right) - V\left(S, t\right)\right]$$
$$+ \sum_{i=1}^{N_{MC}} \left[m_i S - X_i + E\left[V\left(SY_i, t\right) - V\left(S, t\right)\right]\right] dq_i$$

Denoting the accumulated (Brownian) noise between successive decision points from *t* to $t + \Delta t$ by $Z_{t+\Delta t}$ we then have that asset values are:

$$\frac{S_{t+\Delta t}}{S_t} = \exp\left[(r - \delta - \frac{\sigma^2}{2})\Delta t + \sigma Z_{t+\Delta t}\right]\left[(1 + k_i dq_i)\prod_j (1 + k_j dq_j)\right]$$

Similarly with the managerial actions, jumps are log-normally distributed impact k_j, and $Y = 1 + k_j$ follows a log-normal distribution:

$$Y = (1 + k_j) \sim \log N\left(\exp(\gamma_j), \exp(2\gamma_j)(\exp(\sigma_j^2) - 1)\right)$$

The risk-neutral distribution of *S* at $t + \Delta t$ conditional on the activation of managerial action *i* and on the realization of $n = \{n_1, n_2, ..., n_{N_j}\}$ jumps is given by:

$$\ln\left(\frac{S_T}{S_t} \mid i, n = \{n_1, n_2, ... n_{N_j}\}\right) \sim N\left((r - \delta - \frac{1}{2}\sigma^2 - \sum_{j=1}^{N_j} \lambda_j \bar{k}_j)(T-t) + \gamma_i + \sum_{j=1}^{N_i} n_j \gamma_j, \sigma^2(T-t) + \sigma_i^2 + \sum_{j=1}^{N_j} n_j \sigma_j^2\right)$$

Analytic formulas

Due to the complexity of the notation we present the formulas for the special case of compound-growth options in equation (8) of the main text. The formulation can be generalized to other more complex cases with multiple actions and regions. Our results are consistent with Gukhal (2004) who prices simple compound options for the jump-diffusion case and our results were derived independently. Furthermore we provide more details on how the correlation coefficient of the compound option formula gets affected by the impact of exogenous jumps and endogenous managerial actions. Project value follows jump-diffusion with N_j sources of jumps and two optional managerial actions at $t = 0$ and $t = t_1$ (and also note that $t_2 = T - t_1$). The compound-growth option conditional on activation of a managerial action at $t = 0$ is given by:

$$CC(.\,|\,d_0 \in M_0 = \{W, MC_1\}) =$$

$$\sum_{n(t_1)_1=0}^{\infty} \cdots \sum_{n(t_1)_{N_j}=0}^{\infty} \sum_{n(t_2)_1=0}^{\infty} \cdots \sum_{n(t_2)_{N_j}=0}^{\infty} \{p\left(n(t_1)_1, ..., n(t_1)_{N_j}\right) p\left(n(t_2)_1, ..., n(t_2)_{N_j}\right)$$

$$[Se^{-\delta T - \sum_{j=1}^{N_j}(\lambda_j \bar{k}_j T) + \sum_{j=1}^{N_j}\left(n(t_1)_j + n(t_2)_j\right)\gamma_j + \gamma(d_0) + \gamma(d_0, d_1)} N(a_{1,n(t_1)}, b_{1,n(t_2)}, \rho_{n(t_1),n(t_2)}) - Xe^{-rT} N(a_{2,n(t_1)}, b_{2,n(t_2)}, \rho_{n(t_1),n(t_2)})]\}$$

$$+ \sum_{n(t_1)_1=0}^{\infty} \cdots \sum_{n(h_1)_{N_j}=0}^{\infty} \{p\left(n(t_1)_1, ..., n(t_1)_{N_j}\right)[m(d_0, d_1)Se^{-\delta T - \sum_{j=1}^{N_j}(\lambda_j \bar{k}_j t_1) + \sum_{j=1}^{N_j} n(t_1)_j \gamma_j + \gamma(d_0)} N(a_{1,n(t_1)}) - X(d_0, d_1)e^{-rt_1} N(a_{2,n(t_1)})]\}$$

Where
$$a_{1,n(t_1)} = \frac{\ln(S / S^*_{t_1}(d_0, d_1)) + (r - \delta - \sum_{i=1}^{N_j}(\lambda_i \bar{k}_i) + 0.5\sigma^2)t_1 + \sum_{j=1}^{N_j} n(t_1)_j \gamma_j + \gamma(d_0) + 0.5\sum_{j=1}^{N_j} n(t_1)_j \sigma_j^2 + 0.5\sigma^2(d_0)}{\left(\sigma^2 t_1 + \sum_{j=1}^{N_j} n(t_1)_j \sigma_j^2 + \sigma^2(d_0)\right)^{1/2}}$$

$$a_{2,n(t_1)} = a_{1,n(t_1)} - \left(\sigma^2 t_1 + \sum_{j=1}^{N_j} n(t_1)_j \sigma_j^2 + \sigma^2(d_0)\right)^{1/2}$$

$$b_{1,n(t_2)} = \frac{\begin{array}{c}\ln(S / X) + (r - \delta - \sum_{j=1}^{N_j}(\lambda_j \bar{k}_j) + 0.5\sigma^2)T + \gamma(d_0) + \gamma(d_0, d_1) \\ + \sum_{j=1}^{N_j}\left(n(t_1)_j + n(t_2)_j\right)\gamma_j + 0.5\sum_{j=1}^{N_j}\left(n(t_1)_j + n(t_2)_j\right)\sigma_j^2 + 0.5\sigma^2(d_0) + \sigma^2(d_0, d_1)\end{array}}{\left(\sigma^2 T + \sigma^2(d_0) + \sigma^2(d_0, d_1) + \sum_{j=1}^{N_j} n(t_2)_j \sigma_j^2\right)^{1/2}}$$

$$b_{2,n(t_2)} = b_{1,n(t_2)} - \left(\sigma^2 T + \sigma^2(d_0) + \sigma^2(d_0, d_1) + \sum_{j=1}^{N_j} n(t_2)_j \sigma_j^2\right)^{0.5}$$

$$\rho_{n(t_1),n(t_2)} = \sqrt{\frac{\left(\sigma^2 t_1 + \sigma_0^2 + \sum_{j=1}^{N_j} n(t_1)_j \sigma_j^2\right)}{\left(\sigma^2 T + \sigma^2(d_0) + \sigma^2(d_0,d_1) + \sum_{j=1}^{N_j}\left(n(t_1)_j + n(t_2)_j\right)\sigma_j^2\right)}}$$

where $p\left(n(t)_1,\ldots,n(t)_{N_j}\right) = \prod_{j=1}^{N_j}\left[e^{-(\lambda_j t)}\left(\lambda_j t\right)^{n(t)_j} / n(t)_j!\right]$ holds for $t = t_1$ and $t = t_2 = T - t_1$. We weight the value of the compound option with the probabilities of occurrence of all combinations of jumps that can be realized until t_1, $\left(n(t_1)_1,..,n(t_1)_{N_j}\right)$, and those realized from t_1 to T, $\left(n(t_2)_1,..,n(t_2)_{N_j}\right)$, with $t_2 = T - t_1$. The critical value $S_{t_1}^*(d_0,d_1)$ is found by solving:

$$m(d_0,d_1)S_{t_1}^*(d_0,d_1) + \sum_{n(t_2)_1}^{\infty}\ldots\sum_{n(t_2)_{N_j}}^{\infty}\{p\left(n(t_2)_1,n(t_2)_2,\ldots,n(t_2)_{N_j}\right)$$

$$[S_{t_1}^*(d_0,d_1)e^{-\delta(T-t_1)-\sum_{j=1}^{N_j}(\lambda_j\bar{k}_j(T-t_1))+\sum_{j=1}^{N_j}(n(t_2)_j\gamma_j)+\gamma(d_0,d_1)} N(d_{1,n(t_2)}) - Xe^{-r(T-t_1)}N(d_{2,n(t_2)})]\} = X(d_0,d_1)$$

$$d_{1,n(t_2)} = \frac{\ln(S/X) + (r - \delta - \sum_{j=1}^{N_j}(\lambda_j\bar{k}_j) - 0.5\sigma^2)(T-t_1) + \sum_{j=1}^{N_j} n(t_2)_j\gamma_j + \gamma(d_0,d_1) + 0.5\sum_{j=1}^{N_j} n(t_2)_j\sigma_j^2 + 0.5\sigma^2(d_0,d_1)}{[\sigma^2(T-t_1) + \sum_{j=1}^{N_j} n(t_2)_j\sigma_j^2 + \sigma^2(d_0,d_1)]^{1/2}}$$

$$d_{2,n(t_2)} = d_{1,n(t_2)} - [\sigma^2(T-t_1) + \sum_{j=1}^{N_j} n(t_2)_j\sigma_j^2 + \sigma^2(d_0,d_1)]^{1/2}$$

Similarly analytic valuation formulas exist for the cases of call on put, put on call and put on put under jump-diffusion assumptions and managerial value-enhancing actions.

Numerical solution for the jump-diffusion case

The value function for early exercise, abandonment as well as the boundary condition at maturity stay the same like the diffusion case. We have the following adjustments to the cases of managerial actions and the decision to delay investment due to the conditioning on the arrival of jumps:

$$V^{d_t}(S_t, t \mid M, M_t^+, M_t^-) =$$

$$e^{(-r\Delta t)}\left[\sum_{n_1=0}^{\infty}...\sum_{n_{N_j}}^{\infty}\left[p(n_1,...,n_{N_j})E_t\left[V^*\left(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-, n=(n_1, n_2,..., n_{N_j})\right)\right]\right]\right]$$
$$+ m(M_t^-, d_t)S_t - X(M_t^-, d_t)$$

for $d_t \in \{MC_1, MC_2,...., MC_{N_{MC}}\}$ and

$$V^{d_t}(S_t, t \mid M, M_t^+, M_t^-) =$$
$$e^{(-r\Delta t)}\left[\sum_{n_1=0}^{\infty}...\sum_{n_{N_j}}^{\infty} p(n_1,...,n_{N_j})\left[E_t\left[V^*\left(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-, n=(n_1, n_2,..., n_{N_j})\right)\right]\right]\right]$$

for $d_t \in \{W, W_1, W_2,..., W_{N_{MC}}\}$

The volatility conditional on the realization of $n = (n_1, n_2,..., n_{N_j})$ jumps is:

$$v^2(M_t^-, d_t \mid n = (n_1, n_2,..., n_{N_j})) = \sigma^2 \frac{T_{sub}}{N_{sub}} + \frac{\sigma^2(M_t^-, d_t)}{N_{sub}} + \frac{1}{N_{sub}}\sum_{j=1}^{N_j} n_j \sigma_j^2$$

The formulas for the up and down steps are like in the diffusion case (but we use the above specification of volatility) and the up and down probabilities are given by:

$$p_u(d_{t-\Delta t}, d_{t+\Delta t}) = \frac{\exp\left((r-\delta)\frac{T_{sub}}{N_{sub}} + \frac{\gamma(M_t^-, d_t)}{N_{sub}} + \frac{1}{N_{sub}}\sum_{j=1}^{N_j} n_j \gamma_j\right) - d(M_t^-, d_{t+\Delta t})}{u(M_t^-, d_t) - d(M_t^-, d_t)},$$

$$p_d(M_t^-, d_t) = 1 - p_u(M_t^-, d_t)$$

The implementation of the numerical framework with both path-dependency of managerial actions and jumps can be computationally intensive and it is only recommended for low intensities of arrival of rare events.