Technology Adoption under Uncertain Innovation Progress

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Abstract

Within uncertain technological progress, a firm has to decide when to update its technology and select a new one among a non-decreasing range over time. Under constant return to scale, the best existing technology is implemented. Replacement is triggered not only when the firm operated technology is sufficiently obsolete but also the wedge between the latest and the state of the arts grades is large enough. This result indicates that the higher the threat a better technology may be released is a cricial determinant in upgrading decision. Effects of the mean and the variance of technological progress on the adoption policy are also examined.

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1 Introduction

In the May 1999 issue of "Communications of the ACM", the computer science magazine tried to explore the following dilemma: how often should a firm buy a new computer and what type of machine should it buy? The article reached the conclusion that a firm should replace its PC at regular intervals using two dominating strategies: either buy high-end machines every 36 months for organizations seeking substantial computer performance or buy intermediate-level computers every 36 months, a cheaper alternative. Changing configurations and declining prices lead to an important characteristic of the PC market: computers must be replaced at regular intervals. In February 2005, IBM unveiled a new computer chip called "Cell" that will run about ten times faster than the chips found in the fastest desktop PCs today. The chip, developed in conjunction with Sony and Toshiba, is being widely hailed as a significant development in the evolution of computing technology and a challenge to Intel, the current market leader.

These observations raise an interesting set of questions. How do looming releases of superior technologies affect upgrading decisions? What is the impact of the speed (drift) and uncertainty (variance) of technological progress on the replacement decision?

In this paper, we propose a tractable continuous time model in which a firm must choose when to scrap its technology and implement a new one when the arrival of innovations on the market is random. Our main contribution lies in the fact we are able to derive the impact of the threat of the arrival of superior technologies (making newly adopted ones obsolete) on the replacement policy.

Adoption of a new technology is by no means a simple issue to study so the literature has tried to disentangle independently the role of several factors. A common feature of all technology adoption models is the trade-off between waiting and upgrading. A change in technology is costly and usually irreversible, so a natural concern for the manager is: how will the market evolve and how fast will technological progress occur? When adoption is decided, the manager may hesitate over the type of new technology to implement: Does the new piece of equipment require specific knowledge to be operated properly? How large will the gains in efficiency be?

A large class of models focuses on the complementarity between technology and skills. There is a trade-off between improving expertise and experience by continuing to operate a given technology (learning by doing) and switching to a more profitable production process that is not fully mastered by the firm right after adoption (Jovanovic and Nyarko (1996), Chari and Hopenhayn (1991)). Parente (1994) proposes a model where learning exhibiting decreasing returns takes time and switching technology induces a loss in know how. These authors emphasize the link between the low pace of diffusion of a technology and the time required to acquire the skills to use it. More recently, Karp and Lee (2002) investigate technology among less advanced and more advanced firms, the latter being more reluctant to scrap a technology they are familiar with. They show that if agents are patient enough, no leapfrogging occurs. Within a learning by doing framework, Mateos-Planas (2004) focuses on the relationship between technology adoption and firm horizon.

Uncertainty is a fundamental factor in adoption of a new technology. Several types of uncertainties have highlighted in the literature. Uncertainty may lie in the quality of the new technology or, more generally, in its profitability. The moment when a technological curiosity becomes a commercial one is hard to define. Mansfield (1968) mentions that in the case of a new piece of equipment, both the supplier and the user often take a considerable risk. Does new necessarily mean more efficient, and if yes for how long? To overcome the first difficulty, Jensen (1982) proposes a model in which the plant manager observes signals from which she can infer the quality of the technology and, therefore, updates her beliefs over time. Similarly, Jensen (1983) presents a firm undertaking trials to evaluate the quality of two competing innovations. Another class of models tries to capture the uncertainty surrounding the arrival of a new technology, in particular the speed of arrival and the size of future innovations. Both Balcer and Lippman (1984) and Farzin, Huisman and Kort (1997) examine the optimal timing of technology adoption in a context of uncertainty regarding the arrival speed and the efficiency of innovations. They show that significant technological improvements and a high rate of innovations delay adoption. As pointed out in Rosenberg (1976), the sunk cost of investing prematurely in a given technology is usually unrecoverable, a manager expecting a major technological breakthrough may choose to delay adoption as she tries to avoid to lock herself in. Grenadier and Weiss (1997) use an option pricing approach to study the adoption of new technologies when the arrival date of the next generation of innovations is random. The model predicts four types of behaviors: i) compulsive adoptions of every innovation, ii) leapfrogging which consists of skipping an early innovation but adopting some subsequent developed technology, iii) sticking to some early purchased technology, and finally iv) a lagging strategy of buying some older technology at some discounted price after waiting the appearance of some new innovation on the market is stochastic.

Indeed, an important issue lies in the description of the range of new technologies appearing on the market and its evolution. Most of the existing models on adoption technology makes for restrictive assumptions on how new technologies become available on the market. Many assume that the firm has no choice but to implement the latest developed technology or that the technological frontier evolves in a deterministic and increasing fashion. Few attempts have been made to relax this assumption. Jovanovic and Rob (1998) construct a deterministic general equilibrium model in which a manager can choose to upgrade among an increasing range of vintages as technological progress continues. Yet since the production function considered exhibits constant returns to scale, the state of the art technology is always purchased. Bar-Ilan and Mainon (1993) introduce a stochastic environment in which the firm must adjust its technological level with respect to the frontier technology. Indeed, in reality, managers pay attention to what type of technology to implement. Why adopt the frontier technology in a recession time?

Finally, our approach focuses on the option value of waiting to adopt a suitable technology as there is uncertainty and the decision taken is irreversible. We lie in the vein of models developed by Abel and Eberly (1996), (1998) and (2004), Abel et al. (1996), Bertola and Caballero (1994), Dixit and Pindyck (1994) or in a context of indivisible durable goods by Grossman and Laroque (1990).

1.1 Results

Adoption of a new technology is governed by economic depreciation due to the arrival of improved technologies as well as the fear that a superior innovation may be released making the newly adopted obsolete. We show that optimally the manager of the firm follows a (s, S) *style* policy and the scrapping decision depends on how far the ratio of the operated technology to best invented technology is from the ratio current state of the technology to best invented. Since we assume constant returns to scale in technology, updating to the cutting edge technology is optimal. We find that the scrapped grade relative to the state of the art technology is a decreasing function of the current state of research relative to the state of the art one. It is never optimal to adopt the state of the art technology when it is released. As in the case of a Russian option (see Shepp and Shiryaev (1993)), the manager of the firm experiences some reduced regret of not having exercised her option at an earlier time as she still has the opportunity of purchasing some previously introduced technologies. Finally, we establish that increasing the average growth of technological progress as well as increasing volatility leads to a more conservative updating strategy as economic depreciation is accelerated.

The paper is organized as follows. Section 2 describes the economic setting. In section 3, we examine the case of a single adoption and investigates the effects of the mean and volatility of the technological progress on the optimal scrapping frontier. Section 4 extents the analysis to multiple adoptions. Section 5 concludes. Proofs of all results are collected in the appendix.

2 The General Economic Setting

Time is continuous. An infinitely lived risk neutral manager has to decide sequentially the quality of the technology her firm (plant) should operate.

2.1 Technology Adoption and Information Structure

Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P)$ on which is defined a *one* dimensional (standard) Brownian motion w. A state of nature ω is an element of Ω. $\mathcal{F}$ denotes the tribe of subsets of Ω that are events over which the probability measure P is assigned.

Technology is embodied in new capital goods. A single variable $a \geq 0$ captures all the relevant attributes of the production process to the operating cash flow. Roughly speaking, a represents the grade of the technology. A_t denotes the latest developed technology and evolves exogenously according to a geometric Brownian motion

$$dA_t = A_t \left(\mu dt + \sigma dw_t \right),$$

where dw_t is the increment of a standard Wiener process under P, μ represents the average growth rate of technological progress and σ is its the volatility. On average, technology becomes better but it can decrease, capturing the fact that some newly released technologies can be worse than some older ones[1]. In general, only superior technologies are released on the market. Alternatively, one can think of variable A as describing the state of current research. If so, A captures the likelihood that an improved technology will appear. In this case, A is both the quality and an index for the state of current research.

At time t, let z_t be the best grade ever invented (frontier technology), starting at $z > 0$ at date 0, i.e.

$$z_t = \max\{z, \sup_{0 \leq s \leq t} A_s\}.$$

Let $\mathcal{F}_t$ be the σ-algebra generated by the observations of the released technologies, $\{A_s; 0 \leq s \leq t)\}$ and augmented. At time t, the investor's information set is $\mathcal{F}_t$. The filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented).

Operating technology grade a is costless and output y is simply equal to a. A risk neutral manager who discounts future at a rate $r > \mu$ has to choose when to upgrade technology and which new technology to implement among the ones available on the market at the time of adoption.

2.2 Timing of Adoption

We follow Jovanovic and Rob (1998). Denoting one particular adoption time by τ, switching technology requires two steps:

- At time τ^-, the firm has to scrap its old technology a_{τ^-}. The underlying idea is that technologies are fully incompatible. We assume thin markets for used machines: the firm activity may be so specific

[1]For instance, the latest version of a software may include some bugs and may not be as good as the previous version. Ultimately, the problems will be fixed and the efficiency of the technology enhanced.

that capital resales only occur at heavy discounts. In our case, the resale price is simply *zero and scrapping is costless*.

- At time τ^+, the firm decides which technology to adopt a_{τ^+} in $[0, z_\tau]$. Obviously, the manager will always select $a_{\tau^+} > a_{\tau^-}$. The price p of one efficiency unit of technology is assumed to be constant, with $0 < p < \frac{1}{r}$. We start by analyzing the simplest case when the firm can only upgrade once. This case carries most of the intuition present in the multiple adoption case.

3 Single Upgrading

3.1 The Firm's Problem

Switching technology implies giving up the cumulative discounted profit at the discount rate r that could have been realized with the technology already in use. Since the forgone profit is strictly positive, the manager is therefore facing an opportunity cost and upgrading cannot be continuous across time. As aresult, technology adoption is lumpy. The firm optimally chooses a stopping time τ [2] and a positive random variable a' that represents the level of the its new technology adopted at τ. At some initial date $t = 0$, given an operated technology a, the state of the art technology is z and the latest technology is A, the firm's problem is

$$F(A, z, a) = \sup_{(\tau \geq 0,\ 0 \leq a'_\tau \leq z_\tau)} E\left[\int_0^\tau a e^{-rs} ds + \int_\tau^\infty a'_\tau e^{-r(s-\tau)} ds - p a'_\tau e^{-r\tau}\right]. \quad (1)$$

Equivalently

$$F(A, z, a) = \frac{a}{r} + \sup_{(\tau \geq 0,\ 0 \leq a'_\tau \leq z_\tau)} E\left[\left((\frac{1}{r} - p) a'_\tau - \frac{a}{r}\right) e^{-r\tau}\right].$$

The first term $\frac{a}{r}$ is the value of operating forever the same technology a whereas the second term is the option of upgrading technology once. It is equal to a perpetual American call option with underlying asset $(\frac{1}{r} - p)a'$ and strike price $\frac{a}{r}$. We now derive some properties of the value function and the optimal grade adopted.

Properties of the Value Function F

Property 1: F is strictly increasing in a, non-decreasing and convex in A and z.

Property 2: F is homogeneous of degree one and adopting the best existing technology is optimal, i.e. $a'_\tau = z_\tau$.

Proof. See appendix 1. ■

The problem can be interpreted in terms of a Russian option as described in Shepp and Shiryaev (1993). The only difference here is the strike price $\frac{a}{r}$, which represents the opportunity cost of giving away the cumulated discounted profit made by operating technology a forever. In the sequel, we explicitly look at the option value of waiting

$$G(A, z, a) = \sup_{\tau \geq 0} E\left[\left((\frac{1}{r} - p) z_\tau - \frac{a}{r}\right) e^{-r\tau}\right].$$

[2] A stopping time τ is a measurable function from the state space $(\mathbb{R}^3_+,\ \mathbb{F})$ to $\mathbb{R}_+$ such that $\left\{(A, z, a) \in \mathbb{R}^3_+,\ \tau(A, z, a) \leq t\right\} \in \mathcal{F}_t$ for all $t \geq 0$. It means that the stopping rule is a non-anticipated strategy or in other terms the decision of switching technology only depends on the information available up to the time of the adoption.

Clearly, the option value of waiting and upgrading once is decreasing in the grade operated by the firm a and increases with the state of the art technology z. We start by examining the case when $a = 0$.

3.1.1 Benchmark Case: $a = 0$

This problem can be seen as a firm that contemplates to enter into a new market. When is the best time to enter? Which technology the firm should then operate? There is an explicit solution given by

$$F(A, z, 0) = \begin{cases} (\frac{1}{r} - p)\frac{\beta_1(\frac{\alpha A}{z})^{\beta_2} - \beta_2(\frac{\alpha A}{z})^{\beta_1}}{\beta_1 - \beta_2} z, & \frac{z}{\alpha} \le A \le z, \\ (\frac{1}{r} - p)z, & 0 \le A \le \frac{z}{\alpha}, \end{cases}$$

where β_1 and β_2 are respectively the positive and negative roots of the quadratic

$$\frac{\sigma^2}{2}\beta^2 + (\mu - \frac{\sigma^2}{2})\beta - r = 0, \tag{2}$$

and

$$\alpha = \left(\frac{1 - \frac{1}{\beta_2}}{1 - \frac{1}{\beta_1}}\right)^{\frac{1}{\beta_1 - \beta_2}} > 1.$$

Proof. See Shepp and Shiryaev (1993). ■

The optimal strategy is to upgrade immediately if the current technology is far away down from the state of the art technology, otherwise wait. This simple case provides a lot of economic intuition regarding the optimal timing of a technological upgrade. As long as the ratio current frontier technology A over the state of the art technology z is large enough, namely above $\frac{1}{\alpha}$, i.e. if the threat that a better technology soon appears on the market is significant, waiting is optimal.

We now study the general case when a is positive for which obsolescence of the technology operated by the firm also matters.

3.2 General case

3.2.1 Inaction Region and Conjecture of the Optimal Policy

Details of the existence of the solution can be found in Øksendal (2000), Chapter 10. The supremum F is the least superharmonic majorant of the reward function $(\frac{1}{r} - p)z$. We define the inaction region IR where no upgrading takes place as

$$IR = \left\{(A, z, a) : A \le z,\ F(A, z, a) > (\frac{1}{r} - p)z\right\}.$$

In appendix 1, we prove that the inaction region is connected and is of the form

$$IR = \{(A, z, a) : a \ge a^*(A, z)\},$$

or equivalently

$$IR = \left\{(A, z, a) : A > A^*(z, a) = zL_0(\frac{a}{z})\right\},$$

for some smooth decreasing function L_0. As mentioned in Grossman and Zhou (1993), z is a continuous increasing process and thus a finite variation process. Moreover, denoting by $[X, Y]$ the quadratic covariation between processes X and Y, we have $d\left[z,w\right]_t = 0$ and $d\left[z,z\right]_t = 0$. For $(A,z,a) \in IR$ and $A < z$, the Hamilton-Jacobi-Bellman (HJB) equation is

$$rF(A_t, z_t, a)dt = adt + E_t\left(dF(A_t, z_t, a)\right).$$

Dropping the time index and applying Ito lemma leads to the following expression for the HJB

$$rF(A,z,a) = a + \mu A F_1(A,z,a) + \frac{\sigma^2}{2}A^2 F_{11}(A,z,a). \tag{3}$$

Since F is homogeneous of degree one, the general solution of the HJB is

$$F(A,z,a) = \frac{a}{r} + a^{1-\beta_1} f(\frac{z}{a})A^{\beta_1} + a^{1-\beta_2} g(\frac{z}{a})A^{\beta_2},$$

where f and g are two smooth positive functions to be determined. In order to do so, it remains to examine what happens at $A = z$. As mentioned in Shepp and Shiryaev (1993) and derived in Grossman and Zhou (1993), in order for F to satisfy the HJB at $A = z$, F must satisfy the additional condition $F_z(z,z,a) = 0$, which implies that for all $x \geq 0$

$$f'(x)x^{\beta_1} + g'(x)x^{\beta_2} = 0. \tag{4}$$

The initial condition is $F(0,z,a) = \max\{(\frac{1}{r} - p)z, \frac{a}{r}\}$ and the value-matching and smooth pasting (free boundary) conditions respectively are

$$\begin{aligned} F(A^*(z,a),z,a) &= (\frac{1}{r} - p)z \\ \nabla F(A^*(z,a),z,a) &= (0, \frac{1}{r} - p, 0), \end{aligned}$$

where $\nabla F = (F_1, F_2, F_3)$ is the gradient of F.

Proposition 1 *The option value is given by*

$$F(A,z,a) = \begin{cases} \frac{a}{r} + \frac{\beta_1 L_0(\frac{a}{z})^{-\beta_2}\left(\frac{A}{z}\right)^{\beta_2} - \beta_2 L_0(\frac{a}{z})^{-\beta_1}\left(\frac{A}{z}\right)^{\beta_1}}{\beta_1 - \beta_2}\left((\frac{1}{r} - p)z - \frac{a}{r}\right), & zL_0(\frac{a}{z}) \leq A \leq z, \\ (\frac{1}{r} - p)z, & 0 \leq A \leq zL_0(\frac{a}{z}), \end{cases}$$

where L_0 is the solution for $u \in [0, 1 - rp]$ of the following ODE

$$uL_0'(u) = L_0(u)\left(1 - \frac{(1-rp)\left(\beta_1 L_0(u)^{\beta_1 - \beta_2} - \beta_2\right)}{\beta_1\beta_2(1 - rp - u)\left(1 - L_0(u)^{\beta_1 - \beta_2}\right)}\right),$$

with $L_0(0) = \frac{1}{\alpha}$ and $L_0(u) = 0$ for $u \in [1 - rp, 1]$. When $a > (1 - rp)z$, no updating takes place; the value of the firm is independent of z and is given by

$$F(A,z,a) = \frac{a}{r} + Da^{1-\beta_1}A^{\beta_1},$$

where $D = \lim\limits_{u \to 1-rp} \frac{-\beta_2}{r(\beta_1 - \beta_2)}(1 - rp)^{\beta_1 - 1}(1 - rp - u)L_0(u)^{-\beta_1}$.

Proof. See appendix 2 ■

When the technology operated by the firm a is close enough to the state of the art technology z, regardless of the threat that a better technology could be released soon on the market $\frac{A}{z}$, no upgrading takes place. Also notice that, in this case, the value of the firm is independent of the state of the art technology z.

We now present some properties of the optimal scrapping frontier.

Proposition 2 *The optimal frontier A^* is homogeneous of degree one in (z, a), $A^*(z, a) = zL_0(\frac{a}{z})$, increasing in z and decreasing in a. It follows that $\frac{a}{z}$ is a decreasing function of $\frac{A}{z}$: upgrading takes place when the gap between the current operated technology a and the state of the art technology z is large enough provided that it is unlikely that a better technology will soon be released. i.e. the wedge between the current frontier technology A and the state of the art technology z is must be significant.*

3.2.2 Uncertainty effects

We have the following proposition.

Proposition 3 *An increase in the project volatility raises the option value and consequently delays adoption.*

Proof. See appendix 3. ■

An increase in the project volatility shifts in the optimal scrapping frontier L_0.

3.3 Numerical Simulations

In this paragraph, we aim at quantifying the impact of the mean and the variance of the technological progress process on the optimal scrapping frontier. We use Mathematica to simulate the ODE defining the optimal frontier L_0 using the initial condition $L_0(0) = \frac{1}{\alpha}$.

3.3.1 Effects of the Average Speed of Technological Progress

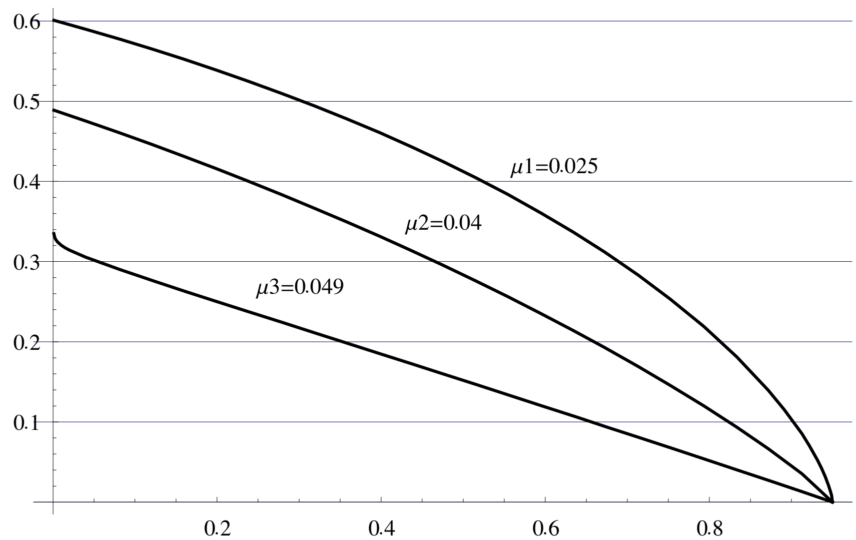

Figure 1 : Effects of the technological progress mean on the optimal scrapping frontier

r=0.05, $\sigma = 0.2$, $p = 1$

The optimal scrapping frontier L_0 is displayed in Figure 1 for several values of the average speed of technological progress μ. As μ increases, the optimal scrapping frontier shifts in: For any given value of $\frac{A^*}{z}$, the relative upgrading trigger point is lower, which indicates that upgrading is delayed.

3.3.2 Effects of the Volatility of Technological Progress

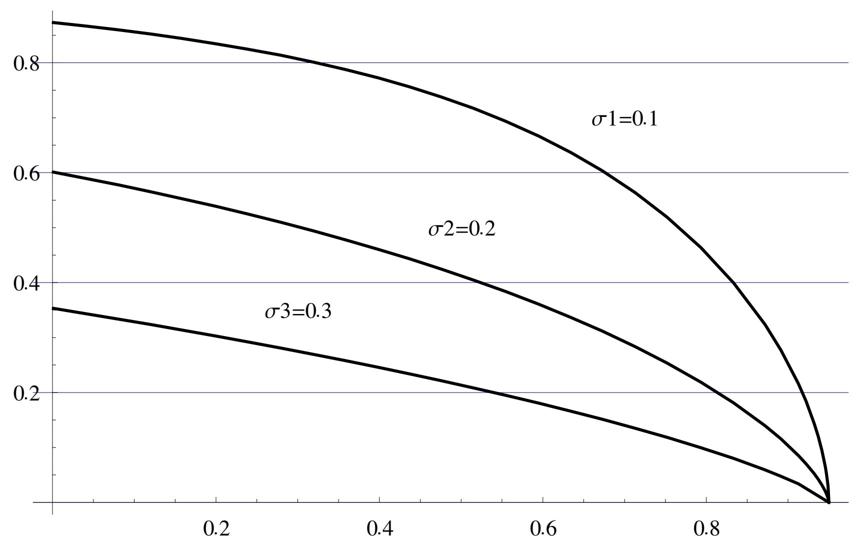

Figure 2 : Effects of the technological progress volatility on the optimal scrapping frontier

r=0.05, $\sigma = 0.2$, $p = 1$

The optimal scrapping frontier L_0 is displayed in Figure 2 for several values of the volatility of technological progress σ. We find similar effects as those described previously when analyzing the impact of parameter μ. As σ increases, the optimal scrapping frontier shifts in: For any given value of $\frac{A^*}{z}$, the relative upgrading trigger point is lower, which indicates that upgrading is delayed.

4 Multiple Upgrading

The firm optimally chooses an increasing sequence of stopping times[3] $\{\tau_k\}_{k=1}^{\infty}$ and a sequence of positive random variables $\{a'_k\}_{k=1}^{\infty} \in [0, z_{\tau_k}]$, where a'_k represents the level of the kth technology adopted at τ_k. This is a typical *impulse control problem* (see Harisson, Sellke and Taylor (1983) and Brekke and Oksendal (1994)). For an initial condition (A_0, z_0, a_0), the value of the firm is

$$F(A_0, z_0, a_0) = \sup_{(\tau_k \geq 0,\ 0 \leq a'_k \leq z_{\tau_k})_{k=1}^{k=\infty}} E\left[\int_0^{\tau_1} a_0 e^{-rs} ds + \sum_{k=1}^{\infty} \left(\int_{\tau_k}^{\tau_{k+1}} a'_k e^{-r(s-\tau_k)} ds - p a'_k e^{-r\tau_k}\right)\right]. \quad (5)$$

Using a recursive approach, the problem can be reformulated as

$$F(A, z, a) = \frac{a}{r} + \sup_{(\tau \geq 0,\ 0 \leq a'_\tau \leq z_\tau)} E\left[\left(F(A_\tau, z_\tau, a'_\tau) - p a'_\tau - \frac{a}{r}\right) e^{-r\tau}\right].$$

[3] A stopping time τ is a measurable function from the state space $(\mathbb{R}_+^3,\ \mathbb{F})$ to $\mathbb{R}_+$ such that $\{(A, z, a) \in \mathbb{R}_+^3,\ \tau(A, z, a) \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. It means that the stopping rule is a non-anticipated strategy or in other terms the decision of switching technology only depends on the information available up to the time of the adoption.

We now derive some properties of the value function.

Property 1: F is increasing in a and z, non-decreasing in A and F is homogeneous of degree one in (A, z, a).

Property 2: F is convex in a so upgrading to the best existing technology is optimal: $a'_\tau = z_\tau$.

Proof. See appendix 4 ■

From property 2, we have

$$F(A, z, a) = \frac{a}{r} + \sup_{\tau \geq 0} E\left[\left(F(A_\tau, z_\tau, z_\tau) - pz_\tau - \frac{a}{r}\right) e^{-r\tau}\right],$$

and the option value of upgrading is

$$G(A, z, a) = \sup_{\tau \geq 0} E\left[\left(F(A_\tau, z_\tau, z_\tau) - pz_\tau - \frac{a}{r}\right) e^{-r\tau}\right].$$

It is follows that G is decreasing in a and increasing in A and z.

Shape of the Inaction Region and Properties of the optimal scrapping frontier

As derived in appendix 4, similar to the single adoption case, the inaction region IR has the following shape

$$IR = \left\{(A, z, a) : A > A^*(z, a) = zL(\frac{a}{z})\right\},$$

where L is a decreasing function to be characterized. In addition, we find that scrapping takes place when, given (A, z) the operated technology a corresponds to a minimum of the value function F.

4.1 Derivation of the Value Function

Inside the inaction region IR, the HJB equation is same as before

$$rF(A, z, a) = a + \mu A F_1(A, z, a) + \frac{\sigma^2}{2} A^2 F_{11}(A, z, a).$$

The initial condition is $F(0, z, a) = \max\ \{(\frac{1}{r} - p)z, \frac{a}{r}\}$ and the value-matching and smooth pasting (free boundary) conditions respectively are

$$\begin{aligned} F(A^*(z,a), z, a) &= F(A^*(z,a), z, z) - pz \\ \nabla F(A^*(z,a), z, a) &= (F_1(A^*(z,a), z, z), F_2(A^*(z,a), z, z) - p), F_3(A^*(z,a), z, z)). \end{aligned}$$

Proposition 4 *The option value is given by*

$$F(A, z, a) = \begin{cases} \frac{a}{r} + \frac{\beta_1 L(\frac{a}{z})^{-\beta_2}\left(\frac{A}{z}\right)^{\beta_2} - \beta_2 L(\frac{a}{z})^{-\beta_1}\left(\frac{A}{z}\right)^{\beta_1}}{\beta_1 - \beta_2}\left((\frac{1}{r} - p)z - \frac{a}{r}\right), & zL(\frac{a}{z}) \leq A \leq z, \\ (\frac{1}{r} - p)z, & 0 \leq A \leq zL(\frac{a}{z}), \end{cases}$$

where L is the solution for $u \in [0, 1 - rp]$ of the following ODE

$$uL'(u) = L(u)\left(1 - \frac{(1 - rp)\left(\beta_1 L(u)^{\beta_1 - \beta_2} - \beta_2 - \left(\left(\beta_2(\beta_1 - 1)L(0)^{-\beta_1} + \beta_1(1 - \beta_2)L(0)^{-\beta_2}\right) L(u)^{\beta_1}\right)\right)}{\beta_1 \beta_2 (1 - rp - u)\left(1 - L(u)^{\beta_1 - \beta_2}\right)}\right),$$

and $L(u) = 0$ *for* $u \in [1 - rp, 1]$. *When* $a > (1 - rp)z$, *no updating takes place; the value of the firm is independent of* z *and is given by*

$$F(A, z, a) = \frac{a}{r} + Da^{1-\beta_1} A^{\beta_1},$$

where $D = \lim_{u \to 1-rp} \frac{\beta_2}{r(\beta_1 - \beta_2)} \frac{(1-rp-u)L(u)^{-\beta_1}}{1-(1-rp)^{1-\beta_1}}$.

Proof. See appendix 5 ■

The determination of the upgrading frontier L (and the value of the firm F) is not complete yet since we still ignore the initial value $L(0)$.

4.2 Complete Characterization of the Scrapping Frontier

As derived in appendix 5, constant D and the initial value $L(0)$ are linked by the following relationship

$$D = \frac{1 - rp}{r(\beta_1 - \beta_2)(\beta_1 - 1)} \left(\beta_2(\beta_1 - 1)L(0)^{-\beta_1} + \beta_1(1 - \beta_2)L(0)^{-\beta_2} \right). \tag{6}$$

It is not possible to determine analytically the initial value $L(0)$ so the ODE defining L cannot be solved numerically in the standard way as we did in the single adoption case using an initial condition. Instead, we need to look for a fixed point.

Double Shooting Method. The ODE defining L can be solved numerically by looking for a fixed point. The method used is called double shooting. We start with some initial guess about $L(0)$ in $(0, 1)$, then we compute numerically the values of L in the range $[0, 1 - rp]$ for instance using Mathematica, and determine D. Finally, we compare the computed value of D with the one given by relationship (8). The operation is repeated until the two values coincide.

4.3 Comparison Between Single and Multiple Adoptions

When multiple adoption are allowed, the option value of waiting is higher since the manager always has the possibility to upgrade only once. As a consequence, we expect the optimal switching frontier for the multiple adoption case to be above the optimal switching frontier for the single adoption case. In fact in appendix 5, we formerly establish that this intuition is correct: for all u in $[0, 1 - rp)$, we have $L(u) > L_0(u)$ and at $u = 1 - rp$, both frontiers coincide and are equal to zero. The firm is less concerned with adopting a technology that may soon be rendered obsolete since it will have the opportunity to upgrade again in the future.

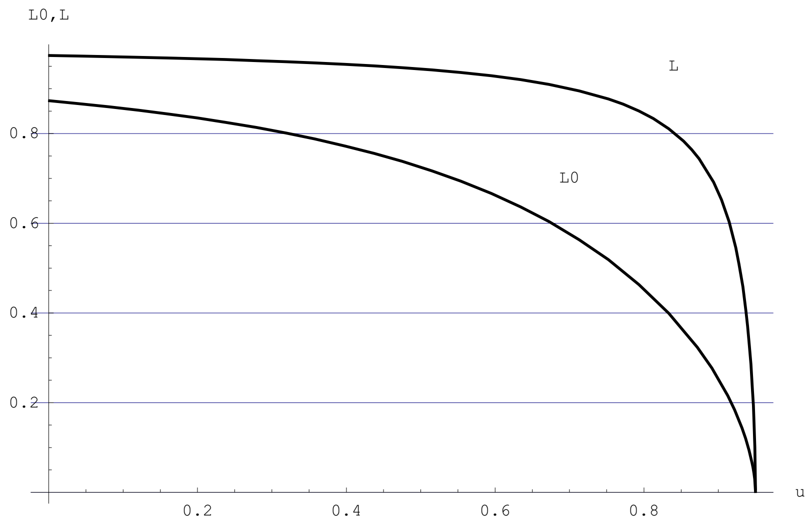

Figure 3 : Optimal scrapping frontiers for single and multiple adoptions

r=0.05, $\sigma = 0.2,\ p = 1$

Figure 3 compares the optimal scrapping frontiers in the case of a single adoption and multiple adoptions. The distance between the two curves first widens as u increases and then shrinks as u gets closer to $1 - rp$. Indeed, having the opportunity to upgrade technology several times leads to a significantly less conservative scrapping policy, in particular for large values of u.

Additional numerical simulations (not displayed here) show that the effects on the mean and volatility of the technological progress on the optimal scrapping frontier are identical to those found in the single adoption case.

5 Conclusion

In this paper, we develop a simple model of innovation adoption allowing for random technological progress. For the sake of simplicity, much of the literature dealing with technology adoption in a dynamic framework chose to examine the special case where the latest developed technology is systematically purchased. We relax this assumption and any technology available within a non-decreasing range across time may be implemented. Our framework shares some common feature with Russian options as presented in Shepp and Shiryaev (1993). Namely, the firm experienced some reduced regret from not adopting a technology as soon as it is released (and would rather wait for the next available innovations) since this opportunity still holds later on. We first examine the case of a single adoption and extend the analysis to the case of multiple adoptions. We find similar results for both frameworks: the firm is all the more reluctant to upgrade the higher the threat that appears on the market a better technology. The single adoption case reinforces this phenomena because the firm has little room for mistake. This result indicates that the introduction of better technologies and the uncertainty surrounded them may be a crucial determinant in upgrading decision. Finally, the impact of the average speed and volatility of the technological progress is to enhance the obsolescence of newly adopted technologies, thus deterring the firm from upgrading. We have considered an extreme case where the new technology implemented is more productive right after adoption. Lag effects such as time to build or time to learn can also have a significant impact on updating decision. In addition, updating decisions are based on expectations about future available technologies. We have taken the arrival of new grades as exogenous. A general equilibrium model would allow us to endogenize it. This is left for future research. .

6 Appendix

6.1 Appendix 1

Proof of property 1. Given relationship (1), the only statement that is not trivial to show the convexity in A. Let λ in $(0,1)$ and two initial values A_0 and A'_0. As shown in the sequel, it is optimal to adopt the best ever invented technology z. Recall that

$$\begin{aligned} z_{\lambda,t} &= \max\{\lambda A_0 + (1-\lambda)A'_0, \sup_{0\le s\le t} \lambda A_s + (1-\lambda)A'_s\} \\ &\le \lambda \max\{A_0, \sup_{0\le s\le t} \lambda A_s\} + (1-\lambda)\max\{A'_0, \sup_{0\le s\le t} A'_s\} \\ &\le \lambda z_t + (1-\lambda) z'_t. \end{aligned}$$

It follows that

$$\begin{aligned} F(A_\lambda, z, a) &= \frac{a}{r} + \sup_{\tau\ge 0} E\left((\frac{1}{r} - p) z_{\lambda,\tau} - \frac{a}{r} \right) e^{-r\tau} \\ &\le \lambda \left(\frac{a}{r} + \sup_{\tau\ge 0} E\left((\frac{1}{r} - p) z_\tau - \frac{a}{r} \right) e^{-r\tau} \right) + (1-\lambda) \left(\frac{a}{r} + \sup_{\tau\ge 0} E\left((\frac{1}{r} - p) z'_\tau - \frac{a}{r} \right) e^{-r\tau} \right) \\ &\le \lambda F(A, z, a) + (1-\lambda) F(A', z, a). \ \blacksquare \end{aligned}$$

Proof of property 2. We first show that F is homogeneous of degree one in (a, A, z). Let $\lambda > 0$ and an initial state $(\lambda a, \lambda A, \lambda z)$, since the law of motion of A is linear at date τ, the frontier level is λz_τ and the current technology level is λA_τ. It follows that

$$\begin{aligned} F(\lambda A, \lambda z, \lambda a) &= \frac{\lambda a}{r} + \sup_{(\tau\ge 0,\ 0\le a'_\tau \le \lambda z_\tau)} E\left((\frac{1}{r} - p) a'_\tau - \frac{\lambda a}{r} \right) e^{-r\tau} \\ &= \lambda \left(\frac{a}{r} + \sup_{(\tau\ge 0,\ 0\le b'_\tau \le z_\tau)} E\left((\frac{1}{r} - p) b'_\tau - \frac{a}{r} \right) e^{-r\tau} \right) \ (b' = \frac{a'}{\lambda}) \\ &= \lambda F(A, z, a). \end{aligned}$$

At the time of adoption, the manager must decide which technology to upgrade and maximize

$$\sup_{0\le a'\le z} (\frac{1}{r} - p) a' - \frac{a}{r}.$$

This leads to $a' = z$. ■

Proof of properties of the optimal frontier A^* and inaction region IR. Let (A, z, a) in IR and $a' > a$. Since F is strictly increasing in a we have

$$\begin{aligned} F(A^*(z,a), z, a') &> F(A^*(z,a), z, a) \\ &> (\frac{1}{r} - p) z, \end{aligned}$$

so (A, z, a') is also in IR and IR must be of the form

$$IR = \{(A, z, a) : a \ge a^*(A, z)\},$$

for some smooth function a^*. Then, if $A^*(z,a') \geq A^*(z,a)$, this implies that

$$F(A^*(z,a'),z,a') \geq F(A^*(z,a),z,a') > (\frac{1}{r}-p)z,$$

which is a contradiction. Hence, A^* is strictly decreasing in a. Finally, as F is homogeneous of degree one, the optimal scrapping frontier is also homogeneous of degree one so we can write

$$A^*(z,a) = zL_0(\frac{a}{z}),$$

for some strictly decreasing function L_0. ■

6.2 Appendix 2

Let

$$M = \{(A,z,a) : 0 \leq A \leq z, 0 \leq a \leq (1-rp)z\}$$

The value matching and smooth pasting conditions lead to

$$\begin{aligned}
\frac{a}{r} + a^{1-\beta_1} f(\frac{z}{a}) A^{*\beta_1} + a^{1-\beta_2} g(\frac{z}{a}) A^{*\beta_2} &= (\frac{1}{r}-p)z \\
\beta_1 a^{1-\beta_1} f(\frac{z}{a}) A^{*\beta_1} + \beta_2 a^{1-\beta_2} g(\frac{z}{a}) A^{*\beta_2} &= 0.
\end{aligned}$$

This yields

$$\begin{aligned}
f(\frac{z}{a}) &= \frac{-\beta_2}{\beta_1-\beta_2}\left((\frac{1}{r}-p)z - \frac{a}{r}\right) A^{*-\beta_1} a^{\beta_1-1} \\
g(\frac{z}{a}) &= \frac{\beta_1}{\beta_1-\beta_2}\left((\frac{1}{r}-p)z - \frac{a}{r}\right) A^{*-\beta_2} a^{\beta_2-1}.
\end{aligned}$$

Differentiating with respect to a, we find that:

$$\begin{aligned}
-\frac{z}{a^2} f'(\frac{z}{a}) &= \frac{-\beta_2}{\beta_1-\beta_2}\left(-\frac{1}{r} A^{*-\beta_1} a^{\beta_1-1} + ((\frac{1}{r}-p)z - \frac{a}{r})\left((\beta_1-1)A^* - \beta_1 a \frac{\partial A^*}{\partial a}) A^{*-(\beta_1+1)} a^{\beta_1-2}\right)\right) \\
-\frac{z}{a^2} g'(\frac{z}{a}) &= \frac{\beta_1}{\beta_1-\beta_2}\left(-\frac{1}{r} A^{*-\beta_2} a^{\beta_2-1} + ((\frac{1}{r}-p)z - \frac{a}{r})\left((\beta_2-1)A^* - \beta_2 a \frac{\partial A^*}{\partial a}) A^{*-(\beta_2+1)} a^{\beta_2-2}\right)\right)
\end{aligned}$$

Using condition (4) we obtain that in the interior of M, A^* must satisfy the following ODE

$$\frac{\partial A^*}{\partial a} = A^* \frac{\beta_1\beta_2((\frac{1}{r}-p)z - \frac{a}{r})\left(\left(\frac{z}{A^*}\right)^{\beta_1} - \left(\frac{z}{A^*}\right)^{\beta_2}\right) + (\frac{1}{r}-p)z\left(\beta_1\left(\frac{z}{A^*}\right)^{\beta_2} - \beta_2\left(\frac{z}{A^*}\right)^{\beta_1}\right)}{\beta_1\beta_2 a((\frac{1}{r}-p)z - \frac{a}{r})\left(\left(\frac{z}{A^*}\right)^{\beta_1} - \left(\frac{z}{A^*}\right)^{\beta_2}\right)}$$

Note that the denominator is strictly negative, so $\frac{\partial A^*}{\partial a}$ is well defined. Writing

$$A^*(z,a) = zL_0(u),$$

for $u = \frac{a}{z} \in U = [0, 1-rp]$, it is easy to check that L_0 must satisfy the following ODE

$$L_0'(u) = L_0(u)\frac{\beta_1\beta_2(1-rp-u)\left(1-L_0(u)^{\beta_1-\beta_2}\right) + (1-rp)\left(\beta_1 L_0(u)^{\beta_1-\beta_2} - \beta_2\right)}{\beta_1\beta_2 u(1-rp-u)\left(1-L_0(u)^{\beta_1-\beta_2}\right)} \tag{7}$$

with $L_0(0) = \frac{1}{\alpha}$, $L_0(1-rp) = 0$. From relationship (9), when u is close to $1-rp$, we have

$$L_0'(u) \underset{1-rp}{\simeq} -\frac{L_0(u)}{\beta_1(1-rp-u)},$$

which implies that

$$L_0(u) \underset{1-rp}{\simeq} B(1-rp-u)^{\frac{1}{\beta_1}},$$

for some $B > 0$. Then define

$$\begin{aligned} x &= \frac{u}{1-rp} \\ y(x) &= L_0((1-rp)x)^{\beta_1-\beta_2}, \end{aligned}$$

it follows that y satisfies the following ODE

$$y'(x) = (\beta_1 - \beta_2)y(x)\frac{\beta_1\beta_2(1-x)\,(1-y(x)) + \beta_1 y(x) - \beta_2}{\beta_1\beta_2 x(1-x)\,(1-y(x))}, \tag{8}$$

for all x in $[0,1]$ with $y(0) = \left(\frac{1}{\alpha}\right)^{\beta_1-\beta_2} = \frac{-\beta_2}{\beta_1}\frac{\beta_1-1}{1-\beta_2}$, $y(1) = 0$. This ODE is an Abel's equation of second kind. Set

$$\varphi(x) = \frac{-\beta_2\,(\beta_1\,(1-x)-1)}{\beta_1\,(1-\beta_2\,(1-x))}.$$

φ is decreasing from $\varphi(0) = \left(\frac{1}{\alpha}\right)^{\beta_1-\beta_2}$ down to $\varphi(1) = \frac{\beta_2}{\beta_1} < 0$. Writing $y(x) = y(0)(1 + mx + o(x))$ and injecting this asymptotic expansion into relationship (10) leads to

$$m = -\frac{1}{\beta_1 - \beta_2(\beta_1 - 1)} < 0.$$

We now show that y is decreasing on $[0,1]$ which is equivalent to show that $y(x) \geq \varphi(x)$ for all x in $[0,1]$. We know that $y(0) = \varphi(0)$ and $y'(0) < 0$. Hence, by continuity of y' there exists a neighborhood $(0,\delta)$, with $\delta > 0$ such that $y'(u) < 0$ for all x in $(0,\delta)$. Now assume that there is a point $x^* > \delta$ such that $y(x^*) = \varphi(x^*)$ and $\eta > 0$ such that $y(x) < \varphi(x)$ for all x in $(x^*, x^*+\eta)$. It follows that y is increasing on $(x^*, x^*+\eta)$. But recall that $y(x^*) = \varphi(x^*)$ and φ is decreasing, which implies that we must have $y(x) > \varphi(x)$ for all x in $(x^*, x^*+\eta)$. This leads to a contradiction and indeed y is decreasing. It follows that L_0 is decreasing and the proof is complete.

Properties of the optimal scrapping frontier. From the firm view point it is optimal to upgrade technology when

$$a^* = zL_0^{-1}(\frac{A}{z}).$$

Note that $\frac{a^*}{z}$ is decreasing in the relative threat $\frac{A}{z}$. It is also easy to see that a^* is decreasing in A and

$$\frac{\partial a^*}{\partial z} = \frac{L_0^{-1}(\frac{A}{z})L_0'(L_0^{-1}(\frac{A}{z})) - L_0(L_0^{-1}(\frac{A}{z}))}{L_0'(L_0^{-1}(\frac{A}{z}))} > 0,$$

since from relationship (9) it is easy to see that $uL_0'(u) - L_0(u) > 0$.

6.3 Appendix 3

Let us consider $\sigma' > \sigma$ and denote $F(A, z, a; \sigma')$ and $F(A, z, a; \sigma)$ the corresponding option values. By definition we have

$$F(A, z, a; \sigma') = \frac{a}{r} + \sup_{(\tau \geq 0,\ 0 \leq a'_\tau \leq z_\tau)} E\left[\left((\frac{1}{r} - p)a'_\tau - \frac{a}{r}\right) e^{-r\tau}\right].$$

Inside the inaction region $IR_{\sigma'}$, we have

$$rF(A, z, a; \sigma') = a + \mu A F_1(A, z, a; \sigma') + \frac{\sigma^2}{2} A^2 F_{11}(A, z, a, \sigma') + \frac{\sigma'^2 - \sigma^2}{2} A^2 F_{11}(A, z, a; \sigma').$$

Since F is homogeneous of degree one, the general solution of the HJB is

$$F(A, z, a) = \frac{a}{r} + a^{1-\beta'_1} m(\frac{z}{a}) A^{\beta'_1} + a^{1-\beta'_2} n(\frac{z}{a}) A^{\beta'_2}.$$

where β'_1 and β'_2 are the roots of the quadratic (2) for parameter σ' and m and n are smooth functions. It is easy to verify that since $\sigma' > \sigma$, $0 < \beta'_1 < \beta_1$ and $\beta_2 < \beta'_2 < 0$. Let $(\varepsilon_1, \varepsilon_2)$ be positive. Alternatively, we can write

$$\begin{aligned} F(A, z, a, \sigma') &= \frac{a}{r} + (\varepsilon_1 + a^{1-\beta_1}) f(\frac{z}{a}) A^{\beta_1} + (\varepsilon_2 + a^{1-\beta_2}) g(\frac{z}{a}) A^{\beta_2} \\ &\quad - \frac{\left(\frac{\sigma'}{\sigma}\right)^2 - 1}{2(\beta_1 - \beta_2)} A^{\beta_1} \int_{A^*(z,a;\sigma)}^{A} x \left(\left(\frac{A}{x}\right)^{\beta_1} - \left(\frac{A}{x}\right)^{\beta_2}\right) F_{11}(x, z, a, \sigma') dx, \end{aligned}$$

where $A^*(z, a; \sigma)$ is the optimal updating frontier for $F(A, z, a, \sigma)$. Note that since $F_{11} > 0$, if $A^*(z, a; \sigma) < A$, then the last term on the RHS of the above equality is negative.

$$\begin{aligned} F(A, z, a, \sigma') &= \frac{a}{r} + a^{1-\beta_1} f(\frac{z}{a}) A^{\beta_1} + a^{1-\beta_2} g(\frac{z}{a}) A^{\beta_2} \\ &\quad - \frac{\left(\frac{\sigma'}{\sigma}\right)^2 - 1}{2(\beta_1 - \beta_2)} A^{\beta_1} \int_{c(z,a)}^{A} \left(\beta'_1(\beta'_1 - 1) a^{1-\beta'_1} m(\frac{z}{a}) x^{\beta'_1 - \beta_1 - 1} + \beta'_2(\beta'_2 - 1) a^{1-\beta'_2} n(\frac{z}{a}) x^{\beta'_2 - \beta_1 - 1}\right) dx \\ &\quad + \frac{\left(\frac{\sigma'}{\sigma}\right)^2 - 1}{2(\beta_1 - \beta_2)} A^{\beta_2} \int_{d(z,a)}^{A} \left(\beta'_1(\beta'_1 - 1) a^{1-\beta'_1} m(\frac{z}{a}) x^{\beta'_1 - \beta_2 - 1} + \beta'_2(\beta'_2 - 1) a^{1-\beta'_2} n(\frac{z}{a}) x^{\beta'_2 - \beta_2 - 1}\right) dx. \end{aligned}$$

where β_1 and β_2 are the roots relative to $F(A, z, a; \sigma)$ defined by relationship (2). Identifying terms, it follows that

$$\begin{aligned} (\varepsilon_1 + a^{1-\beta_1}) f(\frac{z}{a}) &= \frac{\left(\frac{\sigma'}{\sigma}\right)^2 - 1}{2(\beta_1 - \beta_2)} \left(\frac{\beta'_1(\beta'_1 - 1) a^{1-\beta'_1} m(\frac{z}{a})}{\beta_1 - \beta'_1} A^*(z, a; \sigma)^{\beta'_1 - \beta_1} \right. \\ &\quad \left. + \frac{\beta'_2(\beta'_2 - 1) a^{1-\beta'_2} n(\frac{z}{a})}{\beta_1 - \beta'_2} A^*(z, a; \sigma)^{\beta'_2 - \beta_1}\right) \\ (\varepsilon_2 + a^{1-\beta_2}) g(\frac{z}{a}) &= \frac{\left(\frac{\sigma'}{\sigma}\right)^2 - 1}{2(\beta_1 - \beta_2)} \left(\frac{\beta'_1(\beta'_1 - 1) a^{1-\beta'_1} m(\frac{z}{a})}{\beta'_1 - \beta_2} A^*(z, a; \sigma)^{\beta'_1 - \beta_2} \right. \\ &\quad \left. + \frac{\beta'_2(\beta'_2 - 1) a^{1-\beta'_2} n(\frac{z}{a})}{\beta'_2 - \beta_2} A^*(z, a; \sigma)^{\beta'_2 - \beta_2}\right). \end{aligned}$$

Inverting the system, we find that

$$\frac{\beta_1'(\beta_1'-1)(\beta_1-\beta_2)(\beta_1'-\beta_2')a^{1-\beta_1'}m(\frac{z}{a})A^*(z,a;\sigma)^{\beta_1'}}{(\beta_1-\beta_1')(\beta_1'-\beta_2)} = -\beta_2'\left(a^{1-\beta_1}f(\frac{z}{a})A^*(z,a;\sigma)^{\beta_1}+a^{1-\beta_2}g(\frac{z}{a})A^*(z,a;\sigma)^{\beta_2}\right) + (\beta_1-\beta_2')\varepsilon_1A^*(z,a;\sigma)^{\beta_1}-(\beta_2'-\beta_2)\varepsilon_2A^*(z,a;\sigma)^{\beta_2} \quad (9)$$

$$\frac{\beta_2'(\beta_2'-1)(\beta_1-\beta_2)(\beta_1'-\beta_2')a^{1-\beta_2'}n(\frac{z}{a})A^*(z,a;\sigma)^{\beta_2'}}{(\beta_1-\beta_2')(\beta_2'-\beta_2)} = \beta_1'\left(a^{1-\beta_1}f(\frac{z}{a})A^*(z,a;\sigma)^{\beta_1}+a^{1-\beta_2}g(\frac{z}{a})A^*(z,a;\sigma)^{\beta_2}\right) -(\beta_1-\beta_1')\varepsilon_1A^*(z,a;\sigma)^{\beta_1}+(\beta_1'-\beta_2)\varepsilon_2A^*(z,a;\sigma)^{\beta_2} \quad (10)$$

When ε_1 and ε_2 are equal to zero, then m and n are positive functions. We want to impose ε_1 and ε_2 positive and show that it is still the case that m and n are positive functions. To simplify notations, let

$$\begin{aligned}\delta_1 &= A^*(z,a;\sigma)^{\beta_1}\varepsilon_1\\ \delta_2 &= A^*(z,a;\sigma)^{\beta_2}\varepsilon_2.\end{aligned}$$

We would like to choose δ_1 and δ_2 positive in a way such that

$$\begin{aligned}(\beta_1-\beta_2')\delta_1-(\beta_2'-\beta_2)\delta_2 &> 0\\ -(\beta_1-\beta_1')\delta_1+(\beta_1'-\beta_2)\delta_2 &> 0.\end{aligned}$$

This implies that we need to choose $\frac{\delta_1}{\delta_2}$ such that

$$\frac{\beta_2'-\beta_2}{\beta_1-\beta_2'}<\frac{\delta_1}{\delta_2}<\frac{\beta_1'-\beta_2}{\beta_1-\beta_1'}.$$

This is possible if and only if

$$\frac{\beta_2'-\beta_2}{\beta_1-\beta_2'}<\frac{\beta_1'-\beta_2}{\beta_1-\beta_1'},$$

or equivalently

$$(\beta_1'-\beta_2)(\beta_1-\beta_2')-(\beta_2'-\beta_2)(\beta_1-\beta_1')>0.$$

Since

$$(\beta_1'-\beta_2)(\beta_1-\beta_2')-(\beta_2'-\beta_2)(\beta_1-\beta_1')=-\beta_2(\beta_1'-\beta_2')>0,$$

the condition is satisfied. To sum up, given the choice of ε_1 and ε_2 positive and any positive functions f and g, it is possible to choose two positive functions m and n given by relationships (11) and (12). It follows that given the properties of f, g and $A^*(z,a;\sigma)$

$$F(A^*(z,a;\sigma),z,a,\sigma')-\left(\frac{1}{r}-p\right)z=A^*(z,a;\sigma)^{\beta_1}\varepsilon_1+A^*(z,a;\sigma)^{\beta_2}\varepsilon_2>0.$$

Since F is strictly increasing in A it must be the case that for $\sigma<\sigma'$, $A^*(z,a;\sigma')<A^*(z,a;\sigma)$. ■

6.4 Appendix 4

Proof of properties 1 and 2. The first three points of property 1 are obvious from relationship (5). The homogeneity is degree one for F is a direct consequence of the linearity of the law of motion of the technology A, the linearity of adoption constraint $0 \leq a'_\tau \leq z_\tau$ and the expression of F given by relationship (5). To prove property 2, let λ be in $[0,1]$ and a_0 and b_0 in $\mathbb{R}_+$. Denote by $c_0 = \lambda a_0 + (1-\lambda) b_0$ and $c' = \{c'_k\}_{k=1}^{\infty}$ the optimal adoption strategy. We have

$$\begin{aligned}
F(A,z,c_0) &= \sup_{(\tau_k \geq 0,\ 0 \leq c'_k \leq z_{\tau_k})_{k=1}^{k=\infty}} E\left[\int_0^{\tau_1} c_0 e^{-rs} ds + \sum_{k=1}^{\infty}\left(\int_{\tau_k}^{\tau_{k+1}} c'_k e^{-r(s-\tau_k)} ds - p c'_k e^{-r\tau_k}\right)\right] \\
&\leq \lambda \sup_{\tau_1 \geq 0} E\left[\int_0^{\tau_1} a_0 e^{-rs} ds\right] + (1-\lambda) \sup_{\tau_1 \geq 0} E\left[\int_0^{\tau_1} b_0 e^{-rs} ds\right] \\
&\quad + \sup_{(\tau_k \geq 0,\ 0 \leq c'_k \leq z_{\tau_k})_{k=1}^{k=\infty}} E\left[\sum_{k=1}^{\infty} \lambda \left(\int_{\tau_k}^{\tau_{k+1}} c'_k e^{-r(s-\tau_k)} ds - p c'_k e^{-r\tau_k}\right)\right. \\
&\quad \left. + (1-\lambda)\left(\int_{\tau_k}^{\tau_{k+1}} c'_k e^{-r(s-\tau_k)} ds - p c'_k e^{-r\tau_k}\right)\right] \\
&\leq \lambda \sup_{(\tau_k \geq 0,\ 0 \leq c'_k \leq z_{\tau_k})_{k=1}^{k=\infty}} E\left[\int_0^{\tau_1} a_0 e^{-rs} ds + \sum_{k=1}^{\infty}\left(\int_{\tau_k}^{\tau_{k+1}} c'_k e^{-r(s-\tau_k)} ds - p c'_k e^{-r\tau_k}\right)\right] \\
&\quad +(1-\lambda) \sup_{(\tau_k \geq 0,\ 0 \leq c'_k \leq z_{\tau_k})_{k=1}^{k=\infty}} E\left[\int_0^{\tau_1} b_0 e^{-rs} ds + \sum_{k=1}^{\infty}\left(\int_{\tau_k}^{\tau_{k+1}} c'_k e^{-r(s-\tau_k)} ds - p c'_k e^{-r\tau_k}\right)\right] \\
&\leq \lambda F(A,z,a_0) + (1-\lambda) + F(A,z,b_0).
\end{aligned}$$

It follows that $a \mapsto F(A,z,a) - pa$ is also convex and therefore when upgrading, the best technology is adopted. ■

Shape of the inaction region and properties of the optimal scrapping frontier

The inaction region is now defined as

$$IR = \{(A,z,a) : A \leq z,\ F(A,z,a) > F(A,z,z) - pz\}.$$

First of all, notice that if a is in IR, then $a' > a$ is also in IR since

$$F(A,z,a') > F(A,z,a) > F(A,z,z) - pz.$$

Then, using the Envelop condition, we have

$$F_3(A,z,a) = E_0\left[\int_0^{\tau_1^*} e^{-rs} ds\right] \geq 0. \tag{11}$$

Switching exactly means $\tau_1^* = 0$, so $F_3(A^*(z,a),z,a) = 0$. In addition for $a' > a$, $\tau_1^* > 0$, so $F_3(A,z,a') > 0$. Henceforth, for all $a' \geq a$, $F(A^*(z,a),z,a') \geq F(A^*(z,a),z,a)$, which exactly means that a is a minimum for $F(A,z,a)$. From relationship (6), it is then easy to see that given (A,z), there is a unique a^*, such that $F_3(A,z,a^*) = 0$. Then, we claim that given a, $A^*(z,a)$ is unique. Indeed, if $A_1^*(z,a) < A_2^*(z,a)$ are two candidates, then we have $F(A_2^*(z,a),z,a) > F(A_1^*(z,a),z,a)$, which contradicts the fact that $F(A_2^*(z,a),z,a)$ is a minimum. This implies that there is a one to one relationship $A^*(z,a) = zL(\frac{a}{z})$, for some smooth function L. The relationship is invertible so we can write $a = zL^{-1}(\frac{A}{z})$. It follows that L must be monotonic. In appendix 5, we show that

$$L'(0) = \frac{-L(0)(1 - L(0)^{\beta_1 - \beta_2})}{(1 - rp)(\beta_1 - \beta_2 L(0)^{\beta_1 - \beta_2})} < 0,$$

which implies that L is a decreasing function. Clearly, the optimal scrapping frontier has the same properties as in the single adoption case and the inaction region IR can be rewritten

$$IR = \left\{(A,z,a) : A > A^*(z,a) = zL(\frac{a}{z})\right\}.$$

6.5 Appendix 5

Derivation of the optimal scrapping frontier. The value matching and smooth pasting conditions lead to

$$\begin{aligned}
\frac{a}{r} + a^{1-\beta_1} f(\frac{z}{a}) A^{*\beta_1} + a^{1-\beta_2} g(\frac{z}{a}) A^{*\beta_2} &= z^{1-\beta_1} f(1) A^{*\beta_1} + z^{1-\beta_2} g(1) A^{*\beta_2} + (\frac{1}{r} - p)z \\
\beta_1 a^{1-\beta_1} f(\frac{z}{a}) A^{*\beta_1} + \beta_2 a^{1-\beta_2} g(\frac{z}{a}) A^{*\beta_2} &= \beta_1 z^{1-\beta_1} f(1) A^{*\beta_1} + \beta_2 z^{1-\beta_2} g(1) A^{*\beta_2}.
\end{aligned}$$

This yields

$$\begin{aligned}
f(\frac{z}{a}) &= \frac{-\beta_2}{\beta_1 - \beta_2} \left((\frac{1}{r} - p)z - \frac{a}{r} \right) A^{*-\beta_1} a^{\beta_1 - 1} + f(1) z^{1-\beta_1} a^{\beta_1 - 1} \\
g(\frac{z}{a}) &= \frac{\beta_1}{\beta_1 - \beta_2} \left((\frac{1}{r} - p)z - \frac{a}{r} \right) A^{*-\beta_2} a^{\beta_2 - 1} + g(1) z^{1-\beta_2} a^{\beta_2 - 1}.
\end{aligned}$$

Once again, due to the homogeneous nature of the problem, we look for a solution of the form

$$A^*(z,a) = zL(u),$$

with $u = \frac{a}{z}$. It follows that

$$\begin{aligned}
f(\frac{1}{u}) &= \frac{-\beta_2}{\beta_1 - \beta_2} \left((\frac{1}{r} - p) - \frac{u}{r} \right) u^{\beta_1 - 1} L(u)^{-\beta_1} + f(1) u^{\beta_1 - 1} \\
g(\frac{1}{u}) &= \frac{\beta_1}{\beta_1 - \beta_2} \left((\frac{1}{r} - p) - \frac{u}{r} \right) u^{\beta_2 - 1} L(u)^{-\beta_2} + g(1) u^{\beta_2 - 1}.
\end{aligned} \tag{12}$$

We conjecture that $g(1) = 0$ (to be justified later since we need $g(\frac{1}{1-rp}) = 0$) and therefore

$$\begin{aligned}
-\frac{1}{u^2} f'(\frac{1}{u}) &= \frac{-\beta_2}{\beta_1 - \beta_2} \left(-\frac{1}{r} L(u)^{-\beta_1} u^{\beta_1 - 1} + ((\frac{1}{r} - p) - \frac{u}{r}) \left((\beta_1 - 1) L(u) - \beta_1 u L'(u) \right) L(u)^{-(\beta_1 + 1)} u^{\beta_1 - 2} \right) \\
&\quad + (\beta_1 - 1) f(1) u^{\beta_1 - 2} \\
-\frac{1}{u^2} g'(\frac{1}{u}) &= \frac{\beta_1}{\beta_1 - \beta_2} \left(-\frac{1}{r} L(u)^{-\beta_2} u^{\beta_2 - 1} + ((\frac{1}{r} - p) - \frac{u}{r}) \left((\beta_2 - 1) L(u) - \beta_2 u L'(u) \right) L(u)^{-(\beta_2 + 1)} u^{\beta_2 - 2} \right)
\end{aligned}$$

and using the condition $f'(x)x^{\beta_1} + g'(x)x^{\beta_2} = 0$, we find that

$$uL'(u) = L(u) \left(1 - \frac{(1 - rp)\left(\beta_1 L(u)^{\beta_1 - \beta_2} - \beta_2\right) - r(\beta_1 - 1)(\beta_1 - \beta_2) f(1) L(u)^{\beta_1}}{\beta_1 \beta_2 (1 - rp - u)\left(1 - L(u)^{\beta_1 - \beta_2}\right)} \right), \tag{13}$$

with $L(1-rp)=0$. From relationship (15), it is easy to check that

$$L(u) \underset{1-rp}{\sim} B(1-rp-u)^{\frac{1}{\beta_1}}.$$

Using relationship (**??**), by continuity we find that

$$\begin{aligned} f(\frac{1}{1-rp}) &= \frac{-\beta_2}{r(\beta_1-\beta_2)}(1-rp)^{\beta_1-1}B^{-\beta_1} + f(1)(1-rp)^{\beta_1-1} \\ g(\frac{1}{1-rp}) &= g(1)(1-rp)^{\beta_2-1}. \end{aligned}$$

Imposing that f and g are constant on the range $(1, \frac{1}{1-rp}]$ yields

$$\begin{aligned} f(1) &= \frac{-\beta_2}{r(\beta_1-\beta_2)} \frac{B^{-\beta_1}}{(1-rp)^{1-\beta_1}-1} > 0 \\ g(1) &= 0. \end{aligned} \tag{14}$$

Finally, assuming that $L'(0)$ is finite, we must have

$$f(1) = \frac{1-rp}{r(\beta_1-1)(\beta_1-\beta_2)} \left(\beta_2(\beta_1-1)L(0)^{-\beta_1} + \beta_1(1-\beta_2)L(0)^{-\beta_2}\right). \tag{15}$$

Since $f(1)>0$, it must be the case that

$$L(0) > \frac{1}{\alpha}.$$

Hence

$$\frac{uL'(u)}{L(u)} = 1 - \frac{(1-rp)\left(\beta_1 L(u)^{\beta_1-\beta_2} - \beta_2 - \left(\left(\beta_2(\beta_1-1)L(0)^{-\beta_1} + \beta_1(1-\beta_2)L(0)^{-\beta_2}\right) L(u)^{\beta_1}\right)\right)}{\beta_1\beta_2 u(1-rp-u)\left(1-L(u)^{\beta_1-\beta_2}\right)}. \tag{16}$$

Writing

$$L(u) = L(0) + L'(0)u + o(u),$$

and plugging back into relationship (18) we find that

$$\begin{aligned} L'(0)\beta_1\beta_2(1-rp)\left(1-L(0)^{\beta_1-\beta_2}\right) &= \beta_1\beta_2 L(0)(L(0)^{\beta_1-\beta_2}-1) \\ &\quad + \beta_1\beta_2 L'(0)(1-rp)((\beta_2-1)L(0)^{\beta_1-\beta_2} - (\beta_1-1)) + o(u). \end{aligned}$$

Hence, we must have

$$L'(0) = \frac{-L(0)(1-L(0)^{\beta_1-\beta_2})}{(1-rp)(\beta_1-\beta_2 L(0)^{\beta_1-\beta_2})} < 0. \blacksquare$$

Comparison between the single and multiple scrapping frontiers. From the differential equations defining L_0 and L, for u in $[0.1-rp]$, it is possible to write

$$\begin{aligned} L_0'(u) &= -\Gamma(L_0(u)) \\ L'(u) &= -\Gamma(L(u)) - \Delta(L(u)), \end{aligned}$$

for some positive functions Γ and Δ. Set $v = 1-rp-u$ and define two auxiliary functions K and K_0 such that $K_0(v) = L_0(u)$ and $K(v) = L(u)$. We have $K_0(0) = K(0) = 0$ and

$$\begin{aligned} K_0'(v) &= \Gamma(K_0(v)) \\ K'(v) &= \Gamma(K(v)) + \Delta(K(v)). \end{aligned}$$

It follows that

$$\int_0^{K_0(v)} \frac{dx}{\Gamma(x)} = v$$

$$\int_0^{K(v)} \frac{dx}{\Gamma(x)} = v + \int_0^v \frac{\Delta(K(x))}{\Gamma(K(x))} dx.$$

since Γ and Δ are positive functions, it must be the case that the function K is strictly greater that function K_0. ■

7 References

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