GENERALIZED REAL RENOVATION OPTIONS

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Abstract

This paper revisits Williams (1997) on the effect of net rental price, asset quality and variable investment costs on the asset redevelopment decision. This paper develops a quasi-analytical method for solving the generalised real renovation option problem by treating the investment cost as having both a variable and a fixed element and then extends the analysis for the case where both the net rental price and asset quality are stochastic. Analysis on the generalised renovation option shows in common with previous real options analyses on investment opportunities that the renovation decision is triggered when the incremental net rental value exceeds the investment cost by a mark-up factor exceeding unity. The solution to the variable cost investment function is derived from the generalised model as a special case. The quasi-analytical method reveals hitherto latent conditions constraining the scope of values the various parameters can take, and in particular that the net rental price volatility limits the range of the risk-free rate. This result holds also when both net rental price and asset quality are stochastic. The paper finishes by considering the effect of the generalised renovation option on the original option to invest.

JEL Classifications: D81, G31
Keywords: Renovation-refurbishment-replacement options, compound options.

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Introduction

Not all assets improve with age.[1] An important property development segment involves renovation, or upgrading the quality of a deteriorated facility, although most property development real option models assume construction from vacant land, or redevelopment of an existing structure, or sequential demolition and reconstruction[2]. We extend the current literature of real property redevelopment to the generalised case of fixed and variable investment cost, and cover asset renovation and refurbishment. We also demonstrate an easy way to solve models involving several variables, and explore the use of transformations in reducing model dimensionality.

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[1] Excepts being certain vintages of port, some types of art, and possibly some professors of finance.
[2] The extension of Samuelson (1965) to a real land development option in Geltner & Miller (2001), as developed into multi-factor models in Williams (1991), and Patel & Paxson (1998) are examples of the first type; Williams (1997) is an example of the second type; and Paxson (2007) is an example of the third type.

Real property assets possesses the feature that their quality or efficiency to capture rents deteriorates continuously over time owing to usage and consequently, they demand from time to time quality enhancement investments to raise their quality to a more viable level. Real property redevelopment occurs when the necessary investment to increasing its prevailing quality level from a threshold bound is adequately compensated by the incremental net rental income earned from the quality enhancement. By treating the net renal income as the product of an exogenous market rental price net of unit costs and asset quality that deteriorates deterministically over time, Williams (1997) develops a real options model for investigating the optimal threshold and redevelopment quality levels, instantaneously before and after the investment, through adopting a Cobb-Douglas function to represent the investment cost behaviour, which depends on those two quality levels. A comparison is conducted on the effect of multiple redevelopment opportunities versus a single redevelopment opportunity during the asset lifetime that reveals a widening of the threshold and redevelopment quality levels as the number of opportunities drops from multiple to one. In common with other real option analyses in the related field of asset replacement, Ye (1990), Mauer & Ott (1995), and Dobbs (2004), the effect of volatility increases is to lower the threshold level signalling the re-investment. This suggests that property redevelopment is similar to equipment replacement in demanding greater patience on when to optimally re-invest. Further, because Williams (1997) models both the threshold and redevelopment levels, he also finds that increasing uncertainty lowers the quality level following the redevelopment.

From the perspective of model design, the key advantages of the model as formulated by Williams (1997) are the inclusion of a variable investment cost and the development of a variable transformation used for deriving the solution to the two dimensional partial differential equation representing the fundamental valuation relationship. These features can be usefully applied to valuation in alternative contexts where asset quality deterioration rates or quantity depletion rates can be taken to be constant. The two model features of a variable investment cost and the transformation to reduce model dimensionality are intrinsically intertwined. Relaxing the investment cost formulation to include a fixed cost element as well as the existing variable element violates the conditions underpinning the transformation's viability. Generalising the model to include costs that are independent of the threshold and redevelopment quality levels, such as refurbishment fees and levies or the loss of business due to the disruption, means sacrificing his solution methodology and creating an alternative method. The primary aim of this paper is to revisit the Williams (1997) solution methodology and to formulate an alternative way of generating a generalised solution without recourse to numerical solution techniques, which yields his results as a derivative case.

The solution method adopted here is quasi-analytic in the sense that the solution values have to be determined numerically from a small set of simultaneous non-linear relationships since no explicit solution exists. When analytical methods fail, the recourse is to use of a purely numerical solution technique, Brennan & Schwartz (1978), Geske & Shastri (1985), Boyle (1988), and Cortazar (2001), which is applied by Childs, Riddiough, & Triantis (1996) to solve a two dimensional partial differential equation in the context of mix use redevelopment. Although inferior to the explicit solution, the quasi-analytic approach has the advantage of providing a framework from which key derivatives such as "vega" can be derived analytically. Further, since numerical solution techniques have normally to be benchmarked against an analytical solution for a simpler problem to test their efficacy, these techniques rely on the existence of analytical solutions.

By extending the scope of the model to consider fixed and variable investment cost, the quasi-analytical method unearths potentially latent properties of the solution. The real options result that the value of an investment opportunity has to exceed its cost by a mark-up factor, S. Majd & R. S. Pindyck (1987), McDonald & Siegel (1986), Williams (1991), is replicated for asset redevelopment. It is established from the generalised model that the incremental net rental income generated from enhancing asset quality has to exceed the investment cost of effecting that enhancement by a similar mark-up factor. In addition to the usual requirement that the risk-free rate has to exceed the net trend rate of net rental price, it is established that this has to be augmented by extra conditions to ensure solution stability and the avoidance of bizarre results. These extra conditions, which are derived from the requirement imposed by considering asset quality, require that the range of the risk-free rate is limited by the value of the net rental price volatility. Outside of these bounds, the asset value including the redevelopment option becomes unstable. This suggests that the formulation of the investment cost function imposes more limitations on the scope of parametric values than originally envisaged. Finally, since the quasi-analytical method has the versatility for dealing with partial differential equations of order two, the model is extended in its treatment of variables by allowing asset quality as well as net rental price to be stochastic. Even for this enlarged model, it is shown that similar restrictions on the scope of parametric values apply.

The paper is organised as follows. The first section presents the fixed and variable investment renovation model and develops the quasi-analytical approach to generating the solution composed of four simultaneous non-linear equations. Both the cases of a single and multiple renovation options are considered. The equivalence between the current solution for a zero fixed cost element and the Williams (1997) solution is explained in the Appendix. The next section considers the numerical results. The following two sections adapt the basic model first by treating the investment cost as having only a fixed element and secondly by extending the model to include two stochastic variables. The penultimate section examines the effect of the renovation option on the original decision to build. The final section is a conclusion.

Real Renovation Options with Fixed and Variable Investment Costs

A firm owning a real property asset is considering whether or not to redevelop the asset in order to raise its current possibly inferior quality to a superior level. Any quality increase is reflected in incremental future net cash flows generated by the asset but at the sacrifice of the redevelopment investment cost. At each instant of time, the real property asset creates a flow of rental services and generates a cash flow that depends on its quality and the relevant market rental price. A rental service unit is characterised by its current asset quality q measured per unit of time, and its rental price p, defined as net of current operating and maintenance costs, is also measured per unit of time. Both asset quality and rental price change through time but not in the same way. Rental price is taken to evolve stochastically whereas the change in quality is deterministic. Collectively, the asset quality and the rental price determine the asset net rental income $x = pq$ per unit of time.

Market price evolution is taken to follow the geometric Brownian process with drift:

$$dp = \alpha p dt + \sigma_q p dZ_p . \tag{1}$$

In (1), the constant α represents the rental price's mean growth rate per unit of time and the constant σ_p its standard deviation per unit of time, and Z_p denotes the standard Wiener random variable.

Asset quality process is deterministic and follows the process:

$$dq = -\theta q dt . \tag{2}$$

In (2), the constant $\theta > 0$ represents the asset's mean depreciation rate per unit of time and incorporates both physical deterioration and functional obsolescence.

We introduce the valuation function $V(p,q)$, which is defined as the continuance value of the representative incumbent real property asset and its redevelopment option. The value is defined as a function of p and q distinctly instead of the net rental income $x = pq$ since we are interested in seeking the separate net rental price and asset quality trigger levels signifying redevelopment. By applying the general result derived by Shimko (1992), the valuation function has to satisfy the bivariate partial differential equation:

$$\tfrac{1}{2}\sigma_p^2 p^2 \frac{\partial^2 V}{\partial p^2} + \alpha p \frac{\partial V}{\partial p} - \theta q \frac{\partial V}{\partial q} - \mu V + pq = 0 . \tag{3}$$

Paxson & Pinto (2005) demonstrate that for equations of the type (3), their dimensionality can be reduced to univariate by applying the transformation $x = pq$. So defining $F(x) = V(p,q)$ we have:

$$\tfrac{1}{2}\sigma_p^2 x^2 \frac{\partial^2 F}{\partial x^2} + \eta x \frac{\partial F}{\partial x} - \mu F + x = 0, \tag{4}$$

where $\eta = \alpha - \theta$. The solution to (4) is:

$$F = A_{x,1,1} x^{\beta_1} + A_{x,1,2} x^{\beta_2} + \frac{x}{\mu - \eta}, \tag{5}$$

where the $A_{x,1,1}$ and $A_{x,1,2}$ are non-negative constants to be determined and the exponents β_1 and β_2 are evaluated from:

$$\beta_{1,2} = \left(\tfrac{1}{2} - \frac{\eta}{\sigma^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{\eta}{\sigma^2}\right)^2 + \frac{2\mu}{\sigma^2}},$$

with $\beta_1 \geq 1$ provided that the condition ensuring a definite lifetime $\mu - \eta \geq 0$ is not violated, and $\beta_2 < 0$.

Although this transformation reduces the dimensionality of the fundamental valuation relationship to one and facilitates its solution, its disadvantage is the possible ambiguity it can create. Terms in the valuation relationship (5) are normally eliminated by considering the limiting values of the function, that is the asymptotic values of $F(x \to 0)$ and $F(x \to \infty)$. However, the limiting value of say $x = 0$ may arise from either $p = 0$ or $q = 0$. If we presume that $x = 0$ arises from a zero net rental price, then the asset value including the redevelopment option is zero and almost surely no redevelopment would be contemplated. The exponent for x cannot be negative. In contrast, if we presume that $x = 0$ arises from a zero quality level then there is some justification for asset redevelopment. In this case, the asset value including the option would be positive and the exponent for x in the solved valuation relationship cannot be positive. These two conflicting interpretations are probably equally contestable although it can be argued that when the net rental price is stochastic and the asset quality is deterministic, the former is possibly more defensible. To avoid this ambiguity, our approach is to avoid transforming the fundamental valuation relationship and to propose an analytical solution that solves the partial differential equation directly. Although this product transformation is unsuitable for the present formulation, for a different set of model underlying circumstances, it is possible that this transformation is a viable means for reducing dimensionality.

In their solution of a bivariate partial differential equation, Adkins & Paxson (2006) set the homogenous element of the solution to be equal to the product of the variables, each raised to a distinct exponent. In the same way, we can propose that the solution to (3) has the generic form:

$$V = Bp^{\psi}q^{\lambda} + \frac{pq}{\mu - \eta} \tag{6}$$

It is straightforward to show by substituting this generic form in (3), that the valuation relationship is satisfied provided that the following condition on the parameters is fulfilled:

$$Q(\psi, \lambda) = \tfrac{1}{2}\psi(\psi - 1)\sigma_p^2 + \alpha\psi - \theta\lambda - \mu = 0\,. \tag{7}$$

This is the equivalent of the single stochastic variable condition as shown by Dixit & Pindyck (1994), except that (7) includes the additional parameter λ and that it relates to one stochastic and one deterministic variable.

$Q(\psi, \lambda)$ defines a parabola that passes through the axis $\lambda = 0$ when:

$$\psi_{+,-} = \left(\tfrac{1}{2} - \frac{\alpha}{\sigma_p^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{\alpha}{\sigma_p^2}\right)^2 + \frac{2\mu}{\sigma_p^2}}$$

where $\psi_+ \geq 1$ and $\psi_- < 0$, Dixit & Pindyck (1994), and it displays its greatest curvature when λ attains its minimum. When λ is treated as a known, $Q(\psi|\lambda)$ is a quadratic function of ψ and positive change in λ causes the curve to be pulled down vertically by an amount $\theta\lambda$ and to widen the distance between the two roots for ψ:

$$\psi_{1,2} = \left(\tfrac{1}{2} - \frac{\alpha}{\sigma_p^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{\alpha}{\sigma_p^2}\right)^2 + \frac{2(\mu + \theta\lambda)}{\sigma_p^2}} \tag{8}$$

It can be easily verified that $\psi_1 \geq 1 \Leftrightarrow \lambda \geq -\left(\frac{\mu - \alpha}{\theta}\right)$ and $\psi_2 \leq 0 \Leftrightarrow \lambda \geq -\frac{\mu}{\theta}$.

Since for any permissible value of $\lambda \geq -\frac{\sigma_p^2}{2\theta}\left(\frac{1}{2}-\frac{\alpha}{\sigma_p^2}\right)^2-\frac{\mu}{\theta}$ yielding real roots for ψ there exists two possible solutions for ψ, then (6) takes the form:

$$V = B_{1,1}p^{\psi_1}q^{\lambda_1} + B_{1,2}p^{\psi_2}q^{\lambda_2} + \frac{pq}{\mu-\eta} \tag{9}$$

The form of (9) can be simplified through applying the boundary conditions on the limiting behaviour of V. It can be argued that the value of the real property asset including its redevelopment option will tend to zero as the net rental price p approaches zero. This implies that the coefficient $B_{1,2}=0$ to ensure that V does not become unbounded for $p \to 0$. Then (9) simplifies to:

$$V = B_{1,1}p^{\psi_1}q^{\lambda_1} + \frac{pq}{\mu-\eta}. \tag{10}$$

Although no constraints are imposed on the permissible range of values for λ except those stated above, we can surmise that $\lambda_1 < 0$ since for p fixed, the value of the redevelopment option increases as asset quality declines.

From observing both the net rental price commanded by the asset and its quality, management has to deliberate on whether it is economically justified to continue with the asset in its current state or to incur a redevelopment investment cost to raise its quality to a superior level and benefit from the potentially higher net rental incomes. At the point of redevelopment, asset quality has deteriorated to a level denoted by $\underline{q}$. At this asset quality, the value of the asset and its redevelopment option is $V(p,\underline{q})$ for the prevailing price p. The redevelopment investment cost, which is denoted by K, instantaneously raises the asset quality from $\underline{q}$ to a superior level $\overline{q} > \underline{q}$. Following the redevelopment, the value of the asset and the redevelopment option becomes $V(p,\overline{q})$. Management will choose the threshold quality $\underline{q}$, the superior level $\overline{q}$ attained through investment and the net rental price p to maximise the expected gain from the redevelopment. The optimal choices for $\underline{q}$, $\overline{q}$ and p are the solutions to:

$$0=\max_{p,\underline{q},\overline{q}}\left\{V\left(p,\overline{q}\right)-K-V\left(p,\underline{q}\right)\right\}. \tag{11}$$

subject to $\overline{q}>\underline{q}$. The expression being optimized represents the asset value of a newly redeveloped property less the investment cost incurred in raising the quality to a superior level and the asset value at the threshold quality level that is sacrificed. The value matching condition requires that the overall gain is zero. The optimal values determined from (11) are denoted by $\hat{\underline{q}}$, $\hat{\overline{q}}$ and $\hat{p}$ respectively.

The redevelopment investment cost K is assumed to be composed of a fixed element K_f and a variable element K_v, $K=K_f+K_v$. In his real options analysis of property redevelopment, Williams (1997) assumes a variable investment cost that depends on the threshold and superior quality levels, $\underline{q}$ and $\overline{q}$. He adopts the Cobb-Douglas power function with homogeneity greater than one to represent the investment cost behaviour:

$$K_v=c\underline{q}^{\gamma_1}\overline{q}^{\gamma_2}=c\left(\frac{\underline{q}}{\overline{q}}\right)^{\gamma_1}\overline{q}^{\gamma_1+\gamma_2}$$

where $c>0$, $\gamma_1<0$ and $\gamma_2>1-\gamma_1$ are all known constants. Cost behaviour is characterised as a decreasing function of the threshold to superior quality ratio and as an increasing function of the superior quality level. After specifying this variable cost function, Williams (1997) subsequent comments that the redevelopment cost may entail a fixed element, but this is ignored because of the difficulty in forming an analytical solution. An aspect of our analysis is to include both variable and fixed elements in the investment cost function.

The value matching condition is determined from (11) expressed at their optimal values:

$$B_{1,1}\hat{p}^{\psi_1}\hat{\underline{q}}^{\lambda_1}+\frac{\hat{p}\hat{\underline{q}}}{\mu-\eta}=B_{1,1}\hat{p}^{\psi_1}\hat{\overline{q}}^{\lambda_1}+\frac{\hat{p}\hat{\overline{q}}}{\mu-\eta}-K. \tag{12}$$

The asset value including the redevelopment option on the point of redevelopment is equal to the value following the redevelopment to raise the quality to a superior level less the investment cost. The first order conditions for (11) to attain a maximum are the associated smooth pasting conditions, which can be expressed by:

$$\left.\frac{\partial\left\{V\left(p,\bar{q}\right)-K-V\left(p,\underline{q}\right)\right\}}{\partial u}\right|_{u=\hat{u}}=0 \quad \text{for } u = p, \bar{q}, \underline{q}\,.$$

The smooth pasting condition for p is:

$$\psi_1 B_{1,1}\hat{p}^{\psi_1-1}\hat{\underline{q}}^{\lambda_1}+\frac{\hat{\underline{q}}}{\mu-\eta}=\psi_1 B_{1,1}\hat{p}^{\psi_1-1}\hat{\bar{q}}^{\lambda_1}+\frac{\hat{\bar{q}}}{\mu-\eta}\,.$$

It follows that the value of $B_{1,1}$ is:

$$B_{1,1}=\frac{\hat{p}\hat{\bar{q}}-\hat{p}\hat{\underline{q}}}{\psi_1\left(\mu-\eta\right)}\frac{1}{\hat{p}^{\psi_1}\hat{\underline{q}}^{\lambda_1}-\hat{p}^{\psi_1}\hat{\bar{q}}^{\lambda_1}}\,,$$

which is non-negative provided that $\psi_1 > 0$ and $\lambda_1 < 0$. Substituting for $B_{1,1}$ in (12), we obtain:

$$\frac{\hat{p}\hat{\bar{q}}-\hat{p}\hat{\underline{q}}}{\left(\mu-\eta\right)}=\left(\frac{\psi_1}{\psi_1-1}\right)K\,. \tag{13}$$

This states that the difference in the earnings generated by the property asset and evaluated as a perpetuity, just prior and after the redevelopment has to equal the investment cost multiplied by a factor. Since $\psi_1 > 1$ then $\frac{\psi_1}{\psi_1-1} > 1$, which can then be regarded as a mark-up factor. So, redevelopment occurs when the incremental earnings generated by asset redevelopment evaluated as a perpetuity is at least equal to the investment cost adjusted by this markup factor. This is the asset redevelopment version of an equivalent relationship for a risky investment opportunity, McDonald & Siegel (1986) and Dixit & Pindyck (1994). The presence of uncertainty in one of the model variables implies that the value generated by the redevelopment, evaluated as the incremental net earnings expressed as a perpetuity has to exceed its investment cost by a significant amount for redevelopment to be acceptable. This requirement breaches the traditional net present rule that demands that the value of the incremental net earnings has to at least equal the investment cost. Clearly, the behaviour of $\frac{\psi_1}{\psi_1-1} > 1$ is critical in understanding the conditions conducive to redevelopment. It follows from (7) that $\psi_1 > 1$ when

$\mu > \alpha - \lambda\theta$: since λ is expected to be negative this condition is more binding than the requirement that $\mu > \alpha - \theta$.

The smooth pasting condition for $\underline{q}$ is:

$$\lambda_1 B_{1,1} \hat{p}^{\psi_1} \hat{\underline{q}}^{\lambda_1 - 1} + \frac{\hat{p}}{\mu - \eta} = -\gamma_1 c\left(\hat{\underline{q}}\right)^{\gamma_1 - 1}\left(\hat{\overline{q}}\right)^{\gamma_2}.$$

It follows that the value of $B_{1,1}$ is:

$$B_{1,1} = \frac{-1}{\lambda_1}\left(\frac{\hat{p}\hat{\underline{q}}}{\mu - \eta} + \gamma_1 K_v\right),$$

which is non-negative provided $\lambda_1 < 0$ and the sum of net earnings at the threshold level evaluated as a perpetuity and the marginal investment cost due to a change in the threshold quality level is positive. Note that this marginal investment cost is negative due to γ_1. Substituting for $B_{1,1}$ in (12), we obtain:

$$\frac{\hat{p}\hat{\overline{q}}}{\mu - \eta} - \frac{\hat{p}\hat{\underline{q}}}{\mu - \eta}\left(1 - \frac{1}{\lambda_1} + \frac{\overline{Q}}{\lambda_1}\right) = K_f + K_v\left(1 - \frac{\gamma_1}{\lambda_1} + \frac{\gamma_1 \overline{Q}}{\lambda_1}\right), \tag{14}$$

where $\overline{Q} = \left(\dfrac{\hat{\overline{q}}}{\hat{\underline{q}}}\right)^{\lambda_1}$.

The smooth pasting condition for $\overline{q}$ is:

$$0 = \lambda_1 B_{1,1} \hat{p}^{\psi_1} \hat{\overline{q}}^{\lambda_1 - 1} + \frac{\hat{p}}{\mu - \eta} - \gamma_2 c\left(\hat{\underline{q}}\right)^{\gamma_1}\left(\hat{\overline{q}}\right)^{\gamma_2 - 1}.$$

It follows that the value of $B_{1,1}$ is:

$$B_{1,1} = \frac{-1}{\lambda_1}\left(\frac{\hat{p}\hat{\overline{q}}}{\mu - \eta} - \gamma_2 K\right)$$

which is non-negative provided $\lambda_1 < 0$ and that the net earnings at the superior level evaluated as a perpetuity exceeds the marginal investment cost due to a change in the higher quality level. Substituting for $B_{1,1}$ in (12), we obtain:

$$\frac{\hat{p}\hat{\bar{q}}}{\mu-\eta}\left(1-\frac{1}{\lambda_1}+\frac{\underline{Q}}{\lambda_1}\right)-\frac{\hat{p}\underline{\hat{q}}}{\mu-\eta}=K_f+K_v\left(1-\frac{\gamma_2}{\lambda_1}+\frac{\gamma_2\underline{Q}}{\lambda_1}\right), \tag{15}$$

where $\underline{Q}=\left(\frac{\underline{\hat{q}}}{\hat{\bar{q}}}\right)^{\lambda_1}=\frac{1}{\bar{Q}}$.

The condition, which restricts the scope of the exponents ψ_1 and λ_1 in $Q\left(\psi_1,\lambda_1\right)=0$, (7) and the three reduced form relationships that are derived from the value matching and associated smooth pasting conditions, (13), (14) and (15) collectively form a set of simultaneous non-linear equations, from which we can solve the five unknown quantities, ψ_1, λ_1, $\hat{p}$, $\underline{\hat{q}}$ and $\hat{\bar{q}}$. Superficially, this may suggests that there is inadequate information to solve the unknowns uniquely. The model sets out to determine the optimal values for p, $\underline{q}$ and $\bar{q}$ that identify the point of redevelopment by setting the asset values instantaneously before and after the investment to be equal. For a single set of optimal values $\hat{p}$, $\underline{\hat{q}}$ and $\hat{\bar{q}}$, management may wish to consider the effect of a perturbation of p away from $\hat{p}$ and enquire whether this new price level should result in new optimal values for $\underline{q}$ and $\bar{q}$. If the threshold quality level $\underline{q}$ remained at $\underline{\hat{q}}$ but p is observed to increase above $\hat{p}$, this should warrant a greater investment in quality improvement sustained by the increase in p and this should be recognized in an increased value of the superior quality level above $\hat{\bar{q}}$. By perturbing the value of p away from $\hat{p}$, a new optimal superior quality level has been discovered and this generates an alternative set of optimal values $\hat{p}$, $\underline{\hat{q}}$ and $\hat{\bar{q}}$. By considering all possible perturbations, an infinite set of optimal values can be generated that characterise the various trade-offs between the three variables. This infinite set of optimal values can be represented by the function $G\left(\hat{p},\underline{\hat{q}},\hat{\bar{q}}\right)=0$. This function forms the missing equation. In their analysis of stochastic price and cost on operating policy, Dixit & Pindyck (1994) show that the optimal solution is not unique and the function relating the optimal values is represented by a linear ray passing through the origin. The equivalent in our analysis is specified by (13), which reveals that the relationship between the three optimal variables is non-linear rather than linear.

The interpretation concerning the form of the maximand (11) is that redevelopment of the threshold quality level to the superior quality level can be repeated on a countless number of future occasions. Although this representation is a theoretical construct because of the complexity involved in formulating a finite number of repeatable occasions exceeding one, the results it produces does provide an bound to the required threshold and superior levels for a finite number of future redevelopments. At the opposite extreme from a countless number of redevelopment occasions is a single future redevelopment occasion. Similarly, the bounds on the quality levels can be found for a single future redevelopment occasion. Within these two bounds lie the quality levels for any number of future redevelopment occasions.

When the number of future redevelopment occasions is restricted to one, the value maximizing function (11) has to be amended by setting $V(p,\overline{q}) = \frac{p\overline{q}}{\mu - \eta}$. This change incurs no adjustment to the smooth pasting condition for $\underline{q}$ and (14) remains intact. It does involve a change to the smooth pasting condition for p but the reduced form equation (13) remains unaffected. The smooth pasting condition for $\overline{q}$ does require amending and this changes (15) to:

$$\frac{p\overline{q}}{\mu - \eta} = \gamma_2 K_v . \tag{16}$$

Combining this expression with (13) yields:

$$\frac{\hat{p}\hat{\overline{q}}}{(\mu - \eta)}\left(1 - \frac{\psi_1}{\gamma_1(\psi_1 - 1)}\right) - \frac{\hat{p}\hat{\underline{q}}}{(\mu - \eta)} = \left(\frac{\psi_1}{\psi_1 - 1}\right)K_f . \tag{17}$$

It is straightforward to derive the zero fixed investment cost solution from these results by setting $K_f = 0$. In the appendix, we show that this solution is identical to the results produced by Williams (1997).

Numerical Results

In this section, we examine the relationship between the optimal threshold and redevelopment quality levels for various net rental prices, the effect of fixed investment cost on these optimal levels, and the impact of net rental price volatility on the solution. The results are determined from calculations using data mainly supplied by Williams (1997).

The relationship between the optimal threshold and redevelopment quality levels for various net rental prices is exhibited in Figure (1). The domain of the relationships is constrained for low net rental prices owing to the fixed element of the redevelopment investment cost. From there onwards, the relationship is increasing at an increasing rate and higher net rental prices command higher threshold and redevelopment quality levels since these are more affordable when the incremental value from redevelopment is greater. The ratio of the redevelopment to the threshold quality level $\frac{\hat{\bar{q}}}{\hat{\underline{q}}}$ declines as the net rental price increases and tends to the ratio for the case with a zero fixed investment cost element, which is a constant for all net rental prices. Increases in the net rental price imply that the relative value of the fixed investment cost becomes increasingly less important and the relationships for the two cases merge for large p. Before the two relationships do merge, the relationship for the case of a positive fixed cost element always lies below that for the zero fixed cost element, for both the threshold and redevelopment quality levels. This result partly confirms the conjecture of Williams (1997) that “a fixed cost would reduce the quality at which development begins and raise the quality of new construction”, bearing in mind that the introduction of the fixed investment cost element does change the total investment cost. This effect is also revealed in Figure (2) that exhibits the relationship between the threshold and redevelopment quality levels with a fixed investment cost element, which reveals that the relationship is decreasing. The effect of increasing the fixed investment cost element on the overall investment cost is shown in Table (1), together with other interesting parametric values.

Table (1)

Solution values against fixed investment cost

K_f	ψ_1	λ_1	$\underline{\hat{q}}$	$\hat{\hat{q}}$	$\frac{\hat{\hat{q}}}{\underline{\hat{q}}}$	K_v	K	$\frac{K_v}{K}$
0.0	4.2301	-1.5841	1.6760	4.3154	2.575	16.80	16.80	100.0%
0.5	4.2947	-1.4626	1.5932	4.2711	2.681	16.62	17.12	97.1%
1.0	4.3587	-1.3401	1.5096	4.2234	2.798	16.43	17.43	94.3%
1.5	4.4224	-1.2162	1.4251	4.1718	2.927	16.21	17.71	91.5%
2.0	4.4858	-1.0907	1.3396	4.1159	3.073	15.98	17.98	88.9%
2.5	4.5495	-0.9629	1.2529	4.0551	3.236	15.72	18.22	86.3%
3.0	4.6136	-0.8320	1.1649	3.9885	3.424	15.43	18.43	83.7%
3.5	4.6780	-0.6970	1.0751	3.9148	3.641	15.11	18.61	81.2%
4.0	4.7451	-0.5573	0.9836	3.8332	3.897	14.74	18.74	78.7%
4.5	4.8138	-0.4100	0.8894	3.7407	4.206	14.32	18.82	76.1%
5.0	4.8864	-0.2524	0.7920	3.6342	4.589	13.84	18.84	73.5%
5.5	4.9648	-0.0787	0.6899	3.5077	5.085	13.25	18.75	70.7%

Based on the following parametric values:

σ_p	α	θ	μ	c	γ_1	γ_2	p
0.1	0	0.02	0.1	1	-0.2	2.0	1

Table (1) reveals that as the fixed investment cost element increases, the increase in the overall investment cost less. By allowing the variable investment cost element to be freely determined from the model equations and not constraining the overall investment cost, the value maximization function adjusts the variable investment cost element downwards but less than the increase in the fixed investment cost element. Increases in the fixed investment cost element modify the resulting optimal values for ψ_1, λ_1, $\underline{\hat{q}}$ and $\hat{\hat{q}}$ with $\hat{p}$ fixed and in particular, the value of λ_1 declines and may possibly become positive. Now, the conditions underlying the construction of the model presume that λ_1 is negative and since the model does not impose a constraint on its feasible values, a possibility exists that λ_1 will enter its unacceptable range. This occurs when K_f is approximately greater than 6 and over that range, the model fails to produce a viable solution. This aspect does not affect a model having a zero fixed investment cost element since ψ_1 and λ_1 are constants.

The effect of changes in the volatility of net rental price on the optimal threshold and redevelopment quality levels is presented in Figure (3). This reveals that volatility increases produce a decline in the optimal threshold and redevelopment quality levels. This result corroborates the findings of Williams (1997) and other writers on asset replacement, Mauer & Ott (1995) and Dobbs (2004), that greater prudence and patience should be exercised before asset redevelopment for volatility increases in the underlying stochastic variable. Table (2) shows that volatility increases lowers both the threshold and the redevelopment quality levels. Greater uncertainty also implies that the redevelopment quality level declines but that the ratio of the threshold to the redevelopment quality levels increases. The table also reveals the common finding of real options analysis that the value of the asset together with its redevelopment option $V(p,q)$ increases with volatility for fixed values of $p=1$ and $q=3$; q was fixed to ensure consistency across changes in volatility and at a value between $\underline{\hat{q}}$ and $\hat{\hat{q}}$. Table (3) presents the market values for the asset for two distinct quality levels $q=2$ and $q=3$, and the corresponding values of the capitalization rates. The table reveals that both market values and capitalization rates decline with asset deterioration. The effect of increasing volatility is to increase the market values but produce a decline in the capitalization rates. This analysis is replicated for $q=\underline{\hat{q}}$ and $q=\hat{\hat{q}}$ that produces similar results, see Table (4)

As volatility increases, the value of λ_1 declines until it reaches its unacceptable range, at which point the optimal threshold and redevelopment quality levels adopt plausible values but the coefficient $B_{1,1}$ becomes negative instead of positive and $V(p,q)$ dramatically falls. This feature constraining model universality is similarly shared by the model of Williams (1997) in which the fixed investment cost element is zero.

Table 2

Solution values against net rental price volatility

σ_p	ψ_1	λ_1	$\underline{\hat{q}}$	$\hat{\bar{q}}$	$V(p,q)$	$\frac{\hat{\bar{q}}}{\underline{\hat{q}}}$
0.0%	11.6992	-4.9613	1.7096	4.6295	25.13	2.71
5.0%	7.0816	-2.3064	1.4754	4.4191	25.73	3.00
10.0%	4.6134	-0.8320	1.1649	3.9884	28.62	3.42
12.5%	3.9591	-0.4236	1.0223	3.7408	33.58	3.66
14.0%	3.6593	-0.2314	0.9410	3.5873	42.31	3.81
14.5%	3.5711	-0.1740	0.9145	3.5355	48.73	3.87
15.0%	3.4877	-0.1193	0.8884	3.4835	60.64	3.92

Based on the following parametric values:

α	θ	μ	c	γ_1	γ_2	K_f	p
0	0.02	0.1	1	-0.2	2.0	2	1

Table 3

Option value against net rental price volatility

	$q=2$		$q=3$	
σ_p	$V(p,q)$	$\frac{pq}{V(p,q)}$	$V(p,q)$	$\frac{pq}{V(p,q)}$
0.0%	17.6286	11.3%	25.1287	11.9%
5.0%	18.5325	10.8%	25.7324	11.7%
10.0%	21.7430	9.2%	28.6229	10.5%
12.5%	26.8535	7.4%	33.5793	8.9%
14.0%	35.6745	5.6%	42.3056	7.1%
14.5%	42.1316	4.7%	48.7305	6.2%
15.0%	54.0775	3.7%	60.6439	4.9%

Based on the following parametric values:

α	θ	μ	c	γ_1	γ_2	K_f	p
0	0.02	0.1	1	-0.2	2.0	2	1

Table 4

Option values at threshold and redevelopment quality levels

against net rental price volatility

σ_p	$V\left(p,\underline{\hat{q}}\right)$	$\frac{p\underline{\hat{q}}}{V\left(p,\underline{\hat{q}}\right)}$	$V\left(p,\hat{\bar{q}}\right)$	$\frac{p\hat{\bar{q}}}{V\left(p,\hat{\bar{q}}\right)}$
0.0%	16.3416	10.5%	38.5940	12.0%
5.0%	16.0585	9.2%	37.1259	11.9%
10.0%	17.6662	6.6%	36.0955	11.0%
12.5%	22.0552	4.6%	38.9870	9.6%
14.0%	30.4723	3.1%	46.4982	7.7%
14.5%	36.7995	2.5%	52.5249	6.7%
15.0%	48.6177	1.8%	64.0434	5.4%

Based on the following parametric values:

α	θ	μ	c	γ_1	γ_2	K_f	p
0	0.02	0.1	1	-0.2	2.0	2	1

Whether or not the fixed investment cost element is zero, the value of ψ_1 and thereby the value of λ_1 is influenced by the value of σ_p. When the fixed investment cost element is zero, the solution for ψ_1 is found from (38):

$$\tfrac{1}{2}\sigma_p^2\psi_1\left(\psi_1-1\right)+\left(\alpha+\theta\left(\gamma_1+\gamma_2-1\right)\right)\psi_1-\left(\mu+\theta\left(\gamma_1+\gamma_2\right)\right)=0 \qquad (18)$$

Replacing $\psi_1=\phi\left(\lambda_1-1\right)+1$ where $\phi^{-1}=1-\gamma_1-\gamma_2$, (18) can be transformed into a quadratic function of λ_1:

$$\tfrac{1}{2}\sigma_p^2\lambda_1^2-\lambda_1\left(\tfrac{1}{2}\sigma_p^2\frac{\gamma_1+\gamma_2+1}{\gamma_1+\gamma_2-1}+\left(\alpha+\theta\left(\gamma_1+\gamma_2-1\right)\right)\right)-\left(\mu-\left(\tfrac{1}{2}\sigma_p^2+\alpha\right)\frac{\gamma_1+\gamma_2}{\gamma_1+\gamma_2-1}\right)=0 \qquad (19)$$

Since the coefficient of λ_1 in (19) is negative, then for λ_1 to have a negative root:

$$\mu>\left(\tfrac{1}{2}\sigma_p^2+\alpha\right)\frac{\gamma_1+\gamma_2}{\gamma_1+\gamma_2-1}. \qquad (20)$$

Clearly for fixed μ, (20) will be violated as σ_p increases. The model lacks universality for a relatively large σ_p.

Fixed Investment Cost Model

When the redevelopment investment cost is restricted to be constant, $K = K_f$, there exists no expression that constrains the value of the threshold and the superior quality levels and (11) does not yield definite solutions for $\bar{q}$ and $\underline{q}$. To reach a solution, we have to impose a bound on one of the quality levels such that $\bar{q}$ cannot exceed a specified upper limit or $\underline{q}$ is not permitted to fall below a specified lower limit. The model is amended by proposing that management will always raise the asset quality up to a known superior level $\bar{q} = q_0$. In this representation, the asset quality deteriorates with usage and declines to a determined threshold level $\underline{q}$, which is the point of redevelopment. At this threshold level, the asset quality is improved and raised to the superior level $q_0 > \underline{q}$ by an injection of investment K_f. Under this model, the variable $\bar{q}$ is being treated as a constant and consequently it has no smooth pasting condition, which implies that both the numbers of equations and unknowns are reduced by one. The unknowns are λ_1, ψ_1 and $\underline{q}$; the net rental price p is treated as a given quantity for the same reasons as explained above. These three unknowns are determined from the restriction on the parameters for the homogenous solution to the fundamental partial differential equation (8) and the amended reduced form equations (13) and (14) for $\hat{p}$ and $\hat{\underline{q}}$ respectively:

$$\frac{\hat{p}\hat{\bar{q}} - \hat{p}\hat{\underline{q}}}{(\mu - \eta)} = \left(\frac{\psi_1}{\psi_1 - 1}\right) K_f \,, \tag{21}$$

$$\frac{\hat{p} q_0}{\mu - \eta} - \frac{\hat{p}\hat{\underline{q}}}{\mu - \eta}\left(1 - \frac{1}{\lambda_1} + \frac{\bar{Q}_0}{\lambda_1}\right) = K_f \tag{22}$$

where $\bar{Q}_0 = \left(\frac{q_0}{\hat{\underline{q}}}\right)^{\lambda_1}$.

Renovation Option with Two Stochastic Variables

The merit of the Cobb-Douglas power function is its capacity to incorporate both the threshold and redevelopment quality levels in the investment cost function. However, it possesses the disadvantage of introducing a restrictive condition on the value of μ. This section enquires whether this restriction is due to the particular formulation that treats the net rental price p as a stochastic variable and the asset quality q as a deterministic variable. If asset quality is also treated as a stochastic variable, the restriction may no longer arise or its restrictive effect may be relaxed. Representing asset quality by a stochastic evolution is not implausible since wear and tear through usage may be caused by both regular and irregular factors, incidence may occur randomly during the asset lifetime, or the extent of the deterioration may be random. It is now assumed that asset quality is a stochastic variable and that its evolution is represented by a geometric Brownian motion with drift:

$$dq = -\theta q dt + \sigma_q q dZ_q . \tag{23}$$

In (2), the constant σ_q denotes the standard deviation per unit of time, and Z_q the standard Wiener random variable. For completeness, we specify that changes in rental prices and asset quality evolve dependently, which is captured by their covariance $\mathrm{Cov}\left[dp, dq\right] = \rho\sigma_p\sigma_q dt$ with the correlation coefficient $\left|\rho\right| \leq 1$.

We introduce the valuation function $V_J\left(p, q\right)$, which is defined as the continuance value of the representative real property asset and its redevelopment option. The valuation function has to satisfy the bivariate partial differential equation that incorporates stochastic variations in both p and q:

$$\tfrac{1}{2}\sigma_p^2 p^2 \frac{\partial^2 V_J}{\partial p^2} + \tfrac{1}{2}\sigma_q^2 q^2 \frac{\partial^2 V_J}{\partial q^2} + \rho\sigma_p\sigma_q pq \frac{\partial^2 V_J}{\partial p \partial q} + \alpha p \frac{\partial V_J}{\partial p} - \theta q \frac{\partial V_J}{\partial q} - \mu V_J + pq = 0. \tag{24}$$

The solution to (24) is similar to (6):

$$V_J = B_J p^\beta q^\chi + \frac{pq}{\mu - \eta_J}, \tag{25}$$

where $\eta_J = \rho\sigma_p\sigma_q + \alpha - \theta$ and $B_J > 0$, $\beta > 0$ and $\chi < 0$ are unknown parameters whose values are to be determined. By substituting (25) in (24), it can be established that (25) is a generic solution and that the following condition restricting the scope of the parameters has to be fulfilled:

$$Q_J(\beta,\chi) = \tfrac{1}{2}\beta(\beta-1)\sigma_p^2 + \tfrac{1}{2}\chi(\chi-1)\sigma_q^2 + \beta\chi\rho\sigma_p\sigma_q + \alpha\beta - \theta\chi - \mu = 0. \tag{26}$$

The remaining component of the model, which specifies the value maximization condition, is similar to (11):

$$0 = \max_{p,\underline{q},\overline{q}} \left\{ V_J(p,\overline{q}) - K - V_J(p,\underline{q}) \right\}. \tag{27}$$

If investment cost is treated as a power function with a zero fixed cost element, then from the Appendix the following simplifications can be applied:

$$\chi = 1 + (\beta - 1)\phi^{-1} \tag{28}$$

where $\phi = (1-\gamma_1-\gamma_2)^{-1} < 0$ and we set $\varphi = \phi^{-1} < 0$. Substituting (28) in (26) to eliminate χ yields:

$$Q_J(\beta) = \tfrac{1}{2}\beta(\beta-1)\sigma_\beta^2 + \beta\alpha_\beta - \mu_\beta = 0$$

where:

$$\begin{aligned}
\tfrac{1}{2}\sigma_\beta^2 &= \tfrac{1}{2}\sigma_p^2 + \rho\sigma_p\sigma_q\varphi + \tfrac{1}{2}\sigma_q^2\varphi^2, \\
\alpha_\beta &= \alpha + \rho\sigma_p\sigma_q + \tfrac{1}{2}\sigma_q^2\varphi(1-\varphi) - \theta\varphi, \\
\mu_\beta &= \mu + \theta(1-\varphi) + \tfrac{1}{2}\sigma_q^2\varphi(1-\varphi).
\end{aligned}$$

Since $\beta > 1$ then $\mu_\beta > 0$, which requires that $\mu > \theta(\varphi-1) + \tfrac{1}{2}\sigma_q^2\varphi(\varphi-1)$, which should always hold, and $\mu_\beta > \alpha_\beta$, which requires that:

$$\mu > \alpha + \rho\sigma_p\sigma_q - \theta. \tag{29}$$

Substituting (28) in (26) to eliminate β yields:

$$Q_J(\chi) = \tfrac{1}{2}\chi(\chi-1)\sigma_\chi^2 + \chi\alpha_\chi - \mu_\chi = 0$$

where:

$$\tfrac{1}{2}\sigma_\chi^2 = \tfrac{1}{2}\sigma_q^2 + \rho\sigma_p\sigma_q\phi + \tfrac{1}{2}\sigma_p^2\phi^2,$$
$$\alpha_\chi = \alpha\phi + \rho\sigma_p\sigma_q + \tfrac{1}{2}\sigma_p^2\phi(1-\phi) - \theta,$$
$$\mu_\chi = \mu - \alpha(1-\phi) + \tfrac{1}{2}\sigma_p^2\phi(1-\phi).$$

Since $\chi < 0$ then $\mu_\chi > 0$, which requires that:

$$\mu > \alpha(1-\phi) + \tfrac{1}{2}\sigma_p^2\phi(\phi-1). \tag{30}$$

Conditions (29) and (30) must be fulfilled to produce a viable solution. When there is a zero covariance between the changes of net rental price and asset quality $\rho = 0$, the asset quality volatility plays no role in deciding the existence of a viable solution. When disturbance factors cause both a decline in net rental price and asset quality, then $\rho > 0$, which makes condition (29) more binding. Clearly, a negative correlation between the two variables relaxes the condition. However, the more restrictive condition is (30). The effect of the terms $(1-\phi)$ and $\phi(\phi-1)$ is likely to inflate the values of α and σ_p^2 respectively. By adopting the values $\gamma_1 = -0.2$, $\gamma_2 = 2.0$, $\alpha = 0$, $\mu = 0.1$ used by Williams (1997), $(1-\phi) = 2.25$ and $\phi(\phi-1) = 2.8125$, then $\sigma_p < 27\%$. It becomes quite possible that condition (30) is not satisfied for α and σ_p^2 in their acceptable ranges. When this occurs, the parameter χ becomes positive and the asset value becomes unstable.

The simplification analysis cannot be applied to the non-zero fixed investment cost element, but it is surmised that similar kinds of restrictions will result.

Renovation Options with the Option to Construct

Exercising the redevelopment option or the redevelopment option entails owning the asset and ownership can be acquired through either purchase or construction. The decision to construct a real property asset will depend on many factors including its rentable value, the cost of construction and it sale price. This section explores the extent that the decision to construct a real property asset is influenced by the prospective redevelopment option that is created through construction. The decision to construct involves an option opportunity arising from deferability, McDonald & Siegel (1986), M. Majd & R. S. Pindyck (1987), Williams (1991). So, the opportunity to construct carries the option to defer, which carries the redevelopment option and the decision to construct can be interpreted as an option on an option. When this compound option possesses greater value than the single option to construct, the prospective opportunity to redevelop the asset at some subsequent time reduces the hurdle required to exercise the option to construct. It is important therefore to be able to distinguish the conditions that produce a hurdle reduction and to know whether they apply universally.

The value V_b of the option to construct a real property asset is assumed to depend on the net rental earnings x, $V_b = V_b(x)$, where net earning are defined as the product of the net rental price p and the quality of the constructed asset q_b, $x = pq_b$. Management decides on the asset quality for the proposed constructed asset using its knowledge on construction costs and the prevailing net rental price. We treat asset quality as a constant in the sense that its value does not change over time during the construction phase. Following the construction, which is assumed to take place instantaneously, the asset quality begins to deteriorate with usage over its lifetime but before the asset is constructed, its quality is treated as a constant to be determined. Since net rental price evolves according to (1), the behaviour of net earnings described by:

$$\begin{aligned} dx &= \alpha pq_b dt + \sigma_p pq_b dZ_p \\ &= \alpha x dt + \sigma_p x dZ_p . \end{aligned}$$

The value of the option to construct is determined from the fundamental valuation relationship expressed as a partial differential equation:

$$\tfrac{1}{2}\sigma_p^2 x^2 \frac{\partial^2 V_b}{\partial x^2} + \alpha x \frac{\partial V_b}{\partial x} - \mu V_b = 0, \tag{31}$$

since this option generates no cash flow. The solution to (31), which is well documented, is:

$$V_b(x) = A_{b1} x^{\varepsilon_1}, \tag{32}$$

where ε_1 is the positive root of the quadratic equation:

$$Q_\varepsilon(\varepsilon) = \tfrac{1}{2}\sigma_p^2 \varepsilon(\varepsilon - 1) + \alpha\varepsilon - \mu = 0\,, \tag{33}$$

which constrains the parametric values of the homogenous solution, and:

$$\varepsilon_{1,2} = \left(\tfrac{1}{2} - \frac{\alpha}{\sigma_p^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{\alpha}{\sigma_p^2}\right)^2 + \frac{2\mu}{\sigma_p^2}}\,.$$

The value matching condition for asset construction defines the point to build when the value of the option to construct is equal to the value of the asset following construction net of the construction investment cost. The value of the newly constructed asset is the asset value including the redevelopment option at the net rental price p and quality q_b, (10). The construction investment cost denoted by $K_b = K_{bf} + K_{bv}$ is composed of a fixed element K_{bf} and a variable element $K_{bv} = c_b q_b^{\gamma_3}$, where c_b and γ_3 are known positive constants. Since construction is more expensive than redevelopment, $K_b > K$.

At the point of build, the net rental price attains its trigger level $\hat{p}_b$ and the build quality is $\hat{q}_b$. The value matching condition that expresses the equality of value just prior and after construction is:

$$A_1 \hat{p}_b^{\varepsilon_1} \hat{q}_b^{\varepsilon_1} = B_{1,1} \hat{p}_b^{\psi_1} \hat{q}_b^{\lambda_1} + \frac{\hat{p}_b \hat{q}_b}{\mu - \eta} - K_b\,. \tag{34}$$

The associated smooth pasting condition for p is given by:

$$\varepsilon_1 A_1 \hat{p}_b^{\varepsilon_1 - 1} \hat{q}_b^{\varepsilon_1} = \psi_1 B_{1,1} \hat{p}_b^{\psi_1 - 1} \hat{q}_b^{\lambda_1} + \frac{\hat{q}_b}{\mu - \eta}\,.$$

Substituting this expression in (34) yields:

$$\frac{\hat{p}_b \hat{q}_b}{\mu - \eta} = \frac{\varepsilon_1}{\varepsilon_1 - 1} K_b - \left(\frac{\varepsilon_1 - \psi_1}{\varepsilon_1 - 1}\right) B_{1,1} \hat{p}_b^{\psi_1} \hat{q}_b^{\lambda_1}\,. \tag{35}$$

The real options hurdle for accepting an investment is the investment cost K_b adjusted by the mark-up factor $\frac{\varepsilon_1}{\varepsilon_1 - 1}$. (35) reveals that the hurdle for accepting the investment falls when $\varepsilon_1 > \psi_1$ since $\frac{B_{1,1}\hat{p}_b^{\psi_1}\hat{q}_b^{\lambda_1}}{\varepsilon_1 - 1} > 0$. This condition is true based on data set. To identify the conditions supporting the condition, we compare the separate Q functions for ψ_1 assuming a variable investment cost, (38) and for ε_1, (33):

$$Q_\psi\left(\psi_1\right) = \tfrac{1}{2}\sigma_p^2\psi_1\left(\psi_1 - 1\right) + \psi_1\left(\alpha + \theta\left(\gamma_1 + \gamma_2 - 1\right)\right) - \left(\mu + \theta\left(\gamma_1 + \gamma_2\right)\right) = 0,$$

$$Q_\varepsilon\left(\varepsilon_1\right) = \tfrac{1}{2}\sigma_p^2\varepsilon_1\left(\varepsilon_1 - 1\right) + \alpha\varepsilon_1 - \mu = 0.$$

Since both these Q functions are quadratic equations, $\varepsilon_1 > \psi_1$ when $Q_\psi\left(\psi\right) > Q_\varepsilon\left(\psi\right)$. This requires that:

$$\psi_1 = 1 + \frac{1}{\gamma_1 + \gamma_2 - 1}.$$

It is already known that $\psi_1 > 1$, but now we impose a stricter condition to require that the redevelopment option lowers the investment hurdle.

Conclusion

The quasi-analytical method is adopted to develop the solution for the asset renovation option model when the investment cost contains both fixed and variable elements. Although the threshold and redevelopment quality levels may influence the investment cost, other costs of a fixed nature such as consultancy fees, levies and the loss due to disruption may also be important. Although producing a set of simultaneous non-linear equations that are not amenable to yielding explicit solutions, the adopted method has the advantage that these equations are capable of manipulation so in principle important derivatives such as “vega” can be derived analytically. The merit of the solution method also lies in its capacity to cope with either fixed or variable investment cost and from the generalised solution it is possible to derive the special case for a variable investment cost. The derivation applies a transformation of the same form as used by Williams (1997) in his analysis of the case and produces the identical result. Two cases of interest emerge from the adopted solution method analysis. The result common to real options analyses of investment opportunities that the value of the opportunity has to exceed the investment cost by a greater than one mark-up factor is replicated for the renovation option problem. It is established that the incremental net value generated by enhancing the quality level from the threshold to the redevelopment level has to exceed the renovation investment cost by a mark-up factor exceeding one. The second interesting case is that the variable investment cost function introduces additional conditions into the solution that constrain the scope of the parametric values. In particular, the risk-free rate is limited by the values of the trend term for net rental price and its volatility. When asset quality is also allowed to be stochastic, this change does not modify this condition. Using reasonable parametric values, a net rental volatility of 27% was sufficient to upset the solution by making an exponent become positive instead of negative. This caused the asset value and its renovation option to assume a bizarre behaviour.

Although the quasi-analytical method was applied to asset renovation, there exist alternative contexts involving a mix of deterministic and stochastic variables that are amenable to an approach of this kind. Equipment replacements are decided by depreciation charges as well as revenues and costs. In their replacement analysis, Mauer & Ott (1995) treat depreciation as a function of the stochastic variable cost to keep the number of variables in the fundamental valuation relationship to one. Their ploy of maintaining a univariate relationship incurs the unreasonable compromise of treating the depreciation charge as stochastic but, it is possible to reformulate the model comprising a stochastic cost variable and a deterministic depreciation variable in the style of the renovation model and solve it using the described method. Similarly, assets whose volume deplete at a constant rate through extraction or conversion possess valuation relationships involving a stochastic variable that is normally the asset price and a deterministic variable representing depletion, Brennan & Schwartz (1985). Instead of ignoring the depletion effect by treating the asset reservoir as infinite, the quasi-analytical approach is capable of solving the valuation relationship.

A concern with the variable investment cost model is its inadequacy for producing solutions that are universally valid. Stricter restrictions than normal have to be imposed on the parametric values to ensure that the solution is acceptable. Specifically, it was found that not unreasonable parametric values would produce a positive value of a solution variable when it is constrained to be negative. When this jump out of the acceptable range occurs, the value of the asset including the renovation option begins to take on bizarre values and although some solution values are seemingly plausible the overall solution is not acceptable. Part of the explanation may lie in the form of the investment cost function and more research on the solution behaviour is called for. Another explanation may lie in the formulation itself and an alternative similar to Dixit (1989) may provide one avenue of thought.

Appendix: Variable Investment Cost Model

Using the identical assumptions, Williams (1997) examines real property asset redevelopment for variable investment cost and develops solutions for single and multiple redevelopment opportunities. His formulation is a special case of the general model and we now proceed to demonstrate that the solutions are identical. Applying his procedure for reducing the dimensionality of the model, we rewrite (11) as:

$$0 = \max_{p,\underline{q},\overline{q}} \left\{ \left(B_{1,1} p^{\psi_1} \overline{q}^{\lambda_1} + \frac{p\overline{q}}{\mu - \eta} \right) - c\underline{q}^{\gamma_1} \overline{q}^{\gamma_2} - \left(B_{1,1} p^{\psi_1} \underline{q}^{\lambda_1} + \frac{p\underline{q}}{\mu - \eta} \right) \right\}. \tag{36}$$

By setting $\phi = \dfrac{1}{1-\gamma_1-\gamma_2}$, $\lambda_1 = 1 + \dfrac{\psi_1 - 1}{\phi}$, $\overline{r} = p\overline{q}^{\frac{1}{\phi}}$, and $\underline{r} = p\underline{q}^{\frac{1}{\phi}}$, (36) can be expressed as:

$$0 = \max_{p,\underline{q},\overline{q}} \left\{ p^{1-\phi} \left[\overline{r}^{\phi-1} \left(B_{1,1} \overline{r}^{\psi_1} + \frac{\overline{r}}{\mu - \eta} \right) - c\underline{r}^{\phi\gamma_1} \overline{r}^{\phi\gamma_2} - \underline{r}^{\phi-1} \left(B_{1,1} \underline{r}^{\psi_1} + \frac{\underline{r}}{\mu - \eta} \right) \right] \right\}. \tag{37}$$

Clearly, the problem satisfying (37) is identical to the problem satisfying:

$$\tfrac{1}{2}\sigma_p^2 p^2 \frac{\partial^2 V}{\partial p^2} + \alpha p \frac{\partial V}{\partial p} - \theta q \frac{\partial V}{\partial q} - \mu V + pq = 0$$

with the transformation $r = pq^{\frac{1}{\phi}}$ and $F(r) = q^{\frac{1}{\phi}-1} V(p,q)$. Following Williams (1997):

$$\tfrac{1}{2}\sigma_p^2 r^2 \frac{\partial^2 F}{\partial r^2} + \alpha_r r \frac{\partial F}{\partial r} - \mu_r F + r = 0,$$

where $\alpha_r = \alpha + \theta(\gamma_1 + \gamma_2 - 1)$ and $\mu_r = \mu + \theta(\gamma_1 + \gamma_2)$. The solution to this partial differential equation takes the form:

$$F(r) = A_{r1} r^{\psi_1} + A_{r2} r^{\psi_2} + \frac{r}{\mu_r - \alpha_r}.$$

By substituting the solution into the partial differential equation yields the condition constraining the scope of parameters:

$$Q_\psi(\psi) = \tfrac{1}{2}\sigma_p^2 \psi(\psi - 1) + \alpha_r \psi - \mu_r = 0. \tag{38}$$

The solution to this quadratic equation is:

$$\psi_{1,2} = \left(\tfrac{1}{2} - \frac{\alpha_r}{\sigma_p^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{\alpha_r}{\sigma_p^2}\right)^2 + \frac{2\mu_r}{\sigma_p^2}}. \tag{39}$$

Since the redevelopment option has value when the net rental price p is large, the negative exponent would violate this condition and the coefficient A_{r2} attached to the variable r having the negative exponent is set equal to zero. It follows that the valuation relationship for r is:

$$F(r) = A_{r1} r^{\psi_1} + \frac{r}{\mu_r - \alpha_r}. \tag{40}$$

The function $V(p,q)$ is derived from (40).

Evaluating the variable investment cost solution is more straightforward using this method since the value of ψ_1 can be calculated directly from (39) and λ_1 from $\lambda_1 = 1 + \frac{\psi_1 - 1}{\phi}$. The remaining variables $\underline{\hat{q}}$ and $\hat{\hat{q}}$ are solved for a given value of $\hat{p}$ after modifying (14) and (15) for $K_f = 0$:

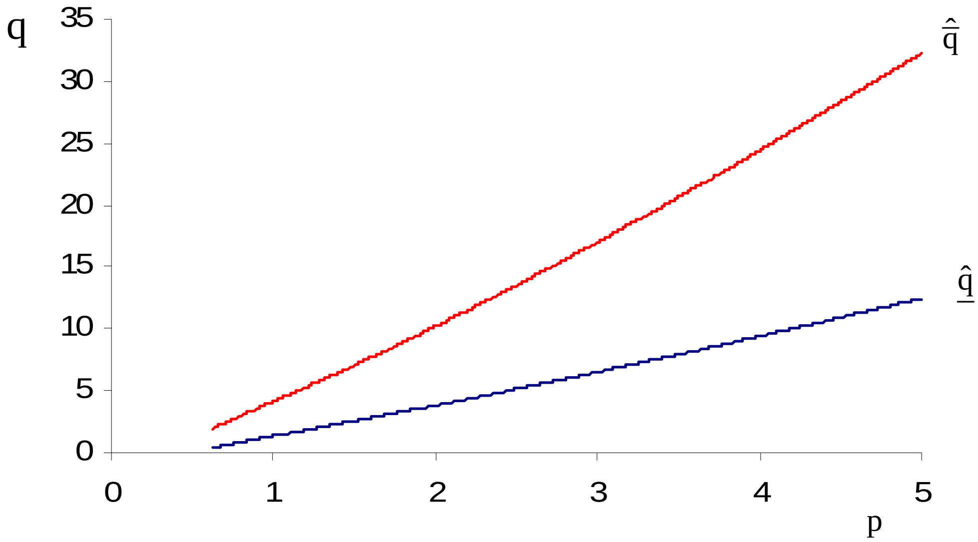
$$\frac{\hat{p}\hat{\hat{q}}}{\mu - \eta} - \frac{\hat{p}\underline{\hat{q}}}{\mu - \eta}\left(1 - \frac{1}{\lambda_1} + \frac{\overline{Q}}{\lambda_1}\right) = K_v\left(1 - \frac{\gamma_1}{\lambda_1} + \frac{\gamma_1 \overline{Q}}{\lambda_1}\right), \tag{41}$$

$$\frac{\hat{p}\hat{\hat{q}}}{\mu - \eta}\left(1 - \frac{1}{\lambda_1} + \frac{\underline{Q}}{\lambda_1}\right) - \frac{\hat{p}\underline{\hat{q}}}{\mu - \eta} = K_v\left(1 - \frac{\gamma_2}{\lambda_1} + \frac{\gamma_2 \underline{Q}}{\lambda_1}\right) \tag{42}$$

Because of (37), (13), (41) and (42) are dependent and it is impossible to derive unique value for $\hat{\underline{q}}$, $\hat{\bar{q}}$ and $\hat{p}$.

It can be established that the ratio of the quality levels before and after redevelopment $\frac{\bar{q}}{\underline{q}}$ is always a constant, or that the superior quality level $\bar{q}$ is strictly proportional to the threshold quality level $\underline{q}$. The proof relies on sequentially eliminating $\hat{p}\hat{\bar{q}}$, $\hat{p}\hat{\underline{q}}$ and K from (13), (41) and (42) to derive a pair of ratios involving $\underline{Q}$ and $\bar{Q}$ as well as the parametric constants. Since $\underline{Q} = \bar{Q}^{-1}$, the ratios can be expressed as a quadratic equation in $\underline{Q}$ (or $\bar{Q}$) from which the positive root is accepted. Since this root depends only on parametric constants then the ratio of $\bar{q}$ to $\underline{q}$ is defined by a constant.

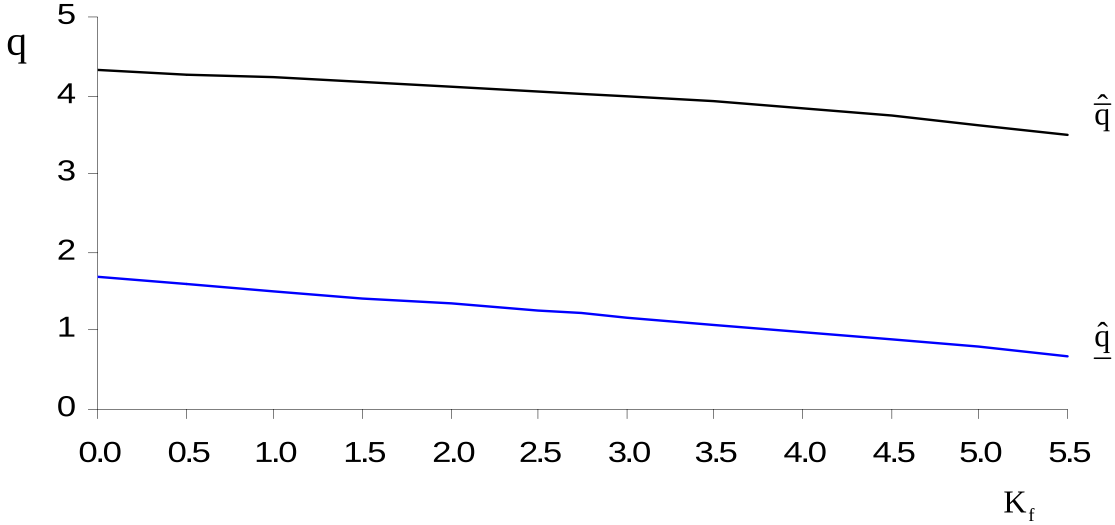
Figure (1)
Relationship between the threshold and redevelopment levels for variations in net rental price



Based on the following parametric values:

σ_p	α	θ	μ	c	γ_1	γ_2	K_f
0.1	0	0.02	0.1	1	-0.2	2.0	2

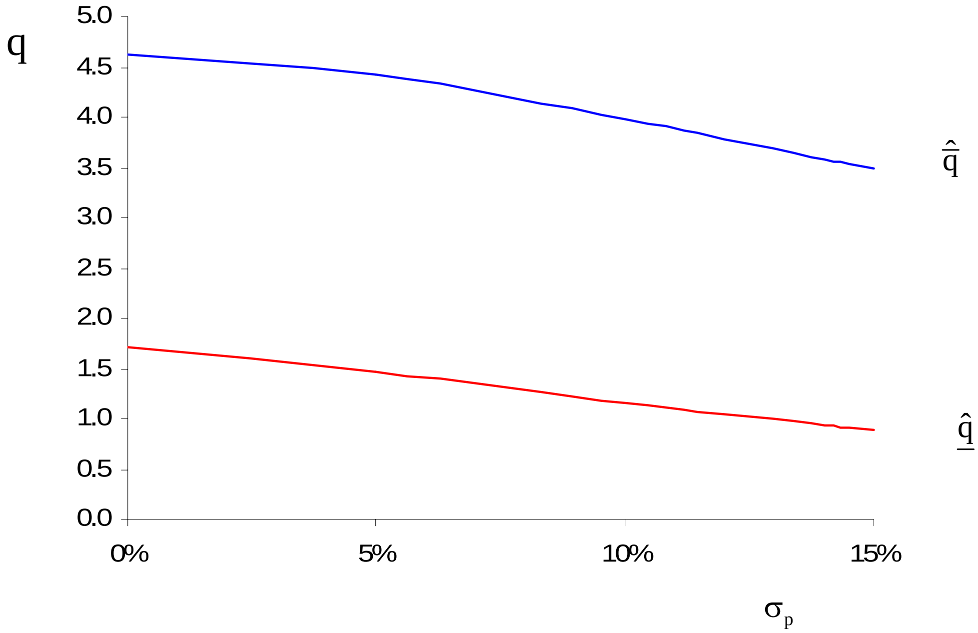
Figure (2)
The relationship between the threshold and redevelopment quality levels and the fixed investment cost



Based on the following parametric values:

σ_p	α	θ	μ	c	γ_1	γ_2	p
0.1	0	0.02	0.1	1	-0.2	2.0	1

Figure (3)
Relationship between the threshold and redevelopment quality levels and the net rental price volatility



Based on the following parametric values:

α	θ	μ	c	γ_1	γ_2	K_f	p
0	0.02	0.1	1	-0.2	2.0	2	1

References

Adkins, R., & Paxson, D. (2006). Optionality in asset renewals, Real Options Conference. New York, USA.

Boyle, P. P. (1988). A lattice framework for option pricing with two state variables. Journal of Financial and Quantitative Analysis, 23(1), 1-12.

Brennan, M. J., & Schwartz, E. S. (1978). Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis. The Journal of Financial and Quantitative Analysis, 13(3), 461-474.

Brennan, M. J., & Schwartz, E. S. (1985). Evaluating natural resource investments. Journal of Business, 58(2), 135-157.

Childs, P. D., Riddiough, T. J., & Triantis, A. J. (1996). Mixed uses and the redevelopment option. Real Estate Economics, 24(3), 317-339.

Cortazar, G. (2001). Simulation and numerical methods in real options valuation. In E. S. Schwartz & L. Trigeorgis (Eds.), Real options and investment under uncertainty: classical readings and recent contributions. Cambridge, Mass.: MIT Press.

Dixit, A. K. (1989). Entry and exit decisions under uncertainty. Journal of Political Economy, 97((June)), 620-638.

Dixit, A. K., & Pindyck, R. S. (1994). Investment under Uncertainty. Princeton (N.J.): Princeton University Press.

Dobbs, I. M. (2004). Replacement investment: optimal economic life under uncertainty. journal of Business Finance & Acounting, 31(5), 729-757.

Geltner, D., & Miller, N. G. (2001). Commercial Real Estate Analysis and Investments. Cinicinnati: South-Western Publishing.

Geske, R., & Shastri, K. (1985). Valuation by approximation: a comparison of alternative option valuation techniques. Journal of Financial and Quantitative Analysis, 20(1), 45-71.

Majd, M., & Pindyck, R. S. (1987). Time to Build, Option Value, and Investment Decision. Journal of Financial Economics, 18(March), 7-27.

Majd, S., & Pindyck, R. S. (1987). Time to build, option value, and investment decisions. Journal of Financial Economics, 18(1), 7-27.

Mauer, D. C., & Ott, S. H. (1995). Investment under uncertainty: the case of replacement investment decisions. Journal of Financial and Quantitative Analysis, 30(4), 581-605.

McDonald, R. L., & Siegel, D. R. (1986). The value of waiting to invest. The Quarterly Journal of Economics, 101(4), 707-728.

Patel, K., & Paxson, D. (1998). Real options based approach to valuation of property development and investment. Property Research Digest, 1-11.

Paxson, D. (2007). Sequential American exchange property options. Journal of Real Estate Finance and Economics, (forthcoming).

Paxson, D., & Pinto, H. (2005). Rivalry under price and quantity uncertainty. Review of Financial Economics, 14(3-4), 209-224.

Samuelson, P. A. (1965). Rational Teory of Warrant Pricing. Industrial Management Review, 6(Spring), 41-50.

Shimko, D. C. (1992). Finance in Continuous Time: A Primer. Miami: Kolb Publishing Company.

Williams, J. T. (1991). Real estate development as an option. Journal of Real Estate Finance and Economics, 4, 191-208.

Williams, J. T. (1997). Redevelopment of real assets. Real Estate Economics, 25(3), 387-407.

Ye, M.-H. (1990). Optimal replacement policy with stochastic maintenance and operations costs. European Journal of Operational Research, 44, 84-94.