Real options using a Continuous-state Markov Process Approximation

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Abstract

Real Options were firstly formulated using traditional financial option models, however in practice an investor can confront with exotic dynamics. Nowadays, approaches based on simulations have been proposed for solving complex options. This paper proposes an alternative appraisal based on a Continuous-state Markov Process Approximation (CMPA) for multivariate Real Option problems. We discuss the viability of the proposal through a study case of a control chart decision (CCD). The proposal is compared with widely used algorithms for CCD problems. The results show the proposal versatility in problems where traditional algorithms can not be used or are inefficient.

Keywords: Real Option, Markov Process, continous state, Control Chart, European Option, American Option, Multivariate Option.

1 Introduction.

Real Options can be traced to Myers (1976), who first defined investments in real assets as mere options. Hence, the real options approach emerges from the idea of applying financial option appraisal theory to capital investment projects. However, the financial options are based on contracts. Conversely, the real options are intrinsic features of strategic within the investments which must be identified and specified (Dixit and Pindyck, 1994). Several methods were developed to value financial options but their direct applications in the real options setting are conditioned to particular characteristics of each problem. In practice, the underlying assumptions of traditional option valuation methods often do not hold when assessing capital investment projects. In this sense, the best-known approaches in optionvaluation, with several possible variants, are: differential equations (Black and Scholes, 1973), lattice methods (Cox et al. 1979) and Monte Carlo simulation (Boyle, 1977).

The Black-Scholes approach has limitations to deal with real option problems due to its basic assumption model is not often fulfilled. Therefore, there are certain limitations to its applicability in the case of real options, for instance it can only be used for not too exotic options. In practice, most real options could have characteristics of American options coupled with several flexibilities and restrictions. The lattice approach is much more robust than the Black-Scholes method. Its simplicity and efficiency make it a proficient appraisal in many real option problems. However, it still has some limitations, such as the the possibility of direct application to any stochastic dynamic. Finally, the Monte Carlo - based method, that could be considered the most versatile approach, but always it has stochastic sampling error and lack of computational efficiency. Besides all those ordinary approaches, a less traditional one is the Markov Chain Approximation (Duan and Simonato, 1999). This approach can be viewed as a generalization of the lattice methods by exploring other potential stochastic dynamics in the variables than just some specific models. This appraisal trades computational efficiency for versatility.

In this paper an alternative approach is analyzed, seeking to apply an intermediate trade-off between the computational effort of the lattice methods and the versatility of the Monte Carlo methods. This alternative is very similar to lattice methods and Markov chain approximations with a significant difference: we avoid the discretization of the stochastic variables by using an interpolating or regressing procedure, thereby maintaining continuous stochastic variables; the time is still discretized using numerical schemes applied to the corresponding equations, which defines the stochastic dynamics of the problem. Therefore, dynamic programming is applied in a typical way but considering infinite points within the stochastic variable interval rather than a finite number of states. The applicability of this approach is not limited by the number of variables or a specific stochastic model, but by the requirement to ensure the existence of a system of stochastic differential equations with a unique solution that characterizes the real option problem and a numerical discretization wich transforms the system in a stochastic recursive equation such that each path has a density function almost everywhere.

This paper presents the formulation of the proposed approach and its implementation in a problem which considers one and two stochastic variables, as well as European and American real options. The results and efficiency of the proposed Continuous-state Markov Process Approximation (CMPA) method is analyzed using a study case about a raising quality control chart problem (Nembhard et al., 2002). We compared the numerical results of the proposed method against the Black-Scholes formula, binomial method and simple Montecarlo simulation results for one variable European options, as well as, binomial method and least square Monte Carlo (LSM) (Schwartz and Longstaff, 2001) results for one variable American options, and finally, the pentanomial method (Boyle, 1988) and simple Monte Carlo simulation results for two variable European options, as well as, pentanomial method and least square Monte Carlo

(LSM) results for two variable American options.

This paper is organized as follows; in section 2 we describes a real option model for control charts; in section 3 we introduce the CMPA, a comparison with the Markov Chain Approximation approach, a description and a simple numerical example; section 4 shows two implementations. Finally, in section 5 we discuss and compare the performance of the proposal with traditional algorithms. The comparisons show that the CMPA method is competitive and versatile with respect to the others methods tested. These results encourage the research in this direction.

2 Quality control charts real option formulation.

In this section we present and formulate the Control charts problem models that will be used as study cases. Control chart is a statistical control that aims to control the quality of a process (Nembhard et al., 2002), helping to find cyclical problems that arise under unpredictable circumstances. Manufacturers use control charts to account for any low quality production in the system, since an over cost of not using control charts exists due to the scrapping by final inspection, returns of defective parts by the consumer or the risk of a reputation of bad quality service or product. Of course using control charts has a cost as well, considering for instance equipments, software and operators (Montgomery,2000). Within this work, the real option problem is based on the hypothesis that there is an option for applying the control charts in order to evaluate it for maximizing the profit that depends on market price and number of sales. In this sense, market price and number of sales are taken into account as uncertain, and therefore, are modeled according to proper stochastic processes. First we consider the simplest model where only the number of sales is variable, and then we go to the model of two variables.

2.1 Financial model with one variable.

Let be $R(t)$ the total sales revenue of the product (the total amount of money received) per time interval beginning at time t, which depends of the stochastic variable number of product sales during a time interval beginning at time t denoted by $S^1(t)$ and the price of the product considered as a constant S^2. Then $R(t)$ can be written as:

(2.1) $R(t) = S^1(t)\, S^2.$

Assuming that the number of sales and the number of units produced per time interval are equal, the total profit per time interval that begins at time t can be defined as a subtraction between revenue and expenses:

(2.2) $P(t) = R(t) - F - S^1(t)\, V,$

where F is the fixed production cost per time interval and V is the variable production cost per unit of product. However, if from now on we consider a more complex model where there may be faults that produce losses and the possibility of control charts, then we must redefine profit. Assuming as constant the fraction of revenue which is lost if control charts are not applied the cost of implementing control charts per time interval, denoted by g and K, respectively; thus the profit without chart per time interval is formulated as:

(2.3) $\widetilde{P}\left(t\right)=\left(1-g\right)S^{1}\left(t\right)S^{2}-F-S^{1}\left(t\right)V,$

and the profit with chart per time interval is represented as:

(2.4) $\widehat{P}\left(t\right)=S^{1}\left(t\right)S^{2}-F-S^{1}\left(t\right)V-K.$

Hence, the profit $D\left(t\right)$ by the mere application of controls charts per time interval is the difference between equations (2.3) and (2.4) which reduces to:

(2.5) $D\left(t\right)=\widehat{P}\left(t\right)-\widetilde{P}\left(t\right)=gS^{1}\left(t\right)S^{2}-K.$

2.2 Financial model with two variables.

The two stochastic variables model consider that the price of the product during the interval time that begins at time t is also stochastic and denoted by $S^{2}\left(t\right)$. Therefore, the total sales revenue $R\left(t\right)$ fulfills the following equation:

(2.6) $R\left(t\right)=S^{1}\left(t\right)S^{2}\left(t\right).$

Therefore, the two variable profits without chart per time interval represented as:

(2.7) $\widetilde{P}\left(t\right)=(1-g)S^{1}\left(t\right)S^{2}\left(t\right)-F-S^{1}\left(t\right)V.$

The two variable profits with chart per time interval are expressed as:

(2.8) $\widehat{P}\left(t\right)=S^{1}\left(t\right)S^{2}\left(t\right)-F-S^{1}\left(t\right)V-K.$

Finally, the two variable profits $D\left(t\right)$ gain per time interval (Nembhard et al., 2002):

(2.9) $D\left(t\right)=gS^{1}\left(t\right)S^{2}\left(t\right)-K.$

2.3 European Option model.

The next valuation arises from the underlying hypothesis wich considers that a decision made in an individual interval does not have any effect on the subsequent intervals, i.e. all the possible decisions are reversible. Consequently, the manufacturer can make a modification of the decision applying control charts at any interval. For instance, if we assume that the manufacturer will elect to use or not the charts every month during a year, then there are twelve decision points and twelve exercise costs for each one. Following this hypothesis, the real option problem can be modeled using a European option for each month because the execution of the option to use the control chart for a period only takes place at its corresponding desicion point, where each option expires at a different time interval (at the desicion points). The final real option value is the total sum of all European option values (Nembhard et al., 2002).

2.4 American Option model.

In this case, we consider that the option will is exercised just once, and thereafter, the choice is going to be remained until the investment horizon. In other words, if we choose to apply the control charts at some time, then the control will be running till the maturity of our investment project. Therefore, throughout the investment maturity, the option can be executed just one time, then we have an American valuation model where a revenue $R(t)$ is collected at each discrete period (months in this case) until the option expiration. Under this assumption, we need to define a total and unique exercise cost for the american option. Then we determine K_A as the total value cost K at the middle point between the beginning and the expiration of the option (Nembhard et al., 2002):

$$(2.10)\ K_A = K\frac{\left(e^{r\Delta t}\right)^{n+1}-1}{(e^{r\Delta t}-1)(e^{r\Delta t})^{n/2}},$$

where there are n intervals.

3 The algorithm approach.

The idea of this approach is strongly inspired by the Markov chain approximation approach, but differs only in that it is considered an uncountable set of states for each stochastic variable. So first we explain the fundamental idea of the latter below. It is well known, that the binomial method attempts to approximate the geometric Brownian motion dynamics considering discrete-time paths that can only evolve in two possible states at each step (Wilmott et al. 1995), assuming such approximation we proceed to calculate the value of the option in each relevant state of each period backwards by the dynamic programming given the appropriate Bellman equation (Bellman, 2003).

The markov chain approximation approach is more general, in that approach we approximate the stochastic continous process S_t in an interval of stochastic states $[c_1, c_2]$ by discrete-time paths X_n (a markov chain) that may evolve into any number of states in the state space $\{A_i\}$ at each next step, to this end, it is necessary to determine the conditional probability of the evolution of a path from one state A_i at the step n to any other A_j in the next step $n+1$. Also it is assigned an unique value a_i to X_n if it belongs to the state A_i such that $i \neq j$ implies that $a_i \neq a_j$. This information is represented in the transition matrix for a fixed step n to $n+1$ (if conditional probabilities keep constant in all steps then there is an unique matrix for all steps, let suppose this is the case):

$$(3.1)\ M_{i,j} = \mathbb{P}\left(\left(X_{n+1} = a_j\right) / \left(X_n = a_i\right)\right).$$

If we consider a column vector V where $V_i = v(a_i)$ for $v : \mathbb{R} \to \mathbb{R}$ (a function estimating the option value) and defining $O := MV$, then the next conditional expectations takes the form:

$$(3.2)\ O_i = \mathbb{E}\left(v\left(X_{n+1}\right)/X_n = a_i\right).$$

The goal of (3.2) is estimating the next conditional expectation:

(3.3) $\mathbb{E}\left(v\left(S_{t_{n+1}}\right)/S_{t_n}\right)(\omega) \approx \mathbb{E}\left(v\left(X_{n+1}\right)/X_n = a_i\right),$

where $S_{t_n}(\omega) = a_i$ for ω in the sample space Ω. CMPA approach takes all points in $[c_1, c_2]$ as states for a Markov proccess $P_n : \Omega \times \mathbb{N} \to [c_1, c_2]$ approximating S_t as much as it is required, a process like that is not a Markov chain because there are uncountable states and instead of using (3.1) we consider $f : [c_1, c_2] \times [c_1, c_2] \to [0, \infty)$ such that $f(x, y)$ is the density function of P_{n+1} if $P_n = y$. Then the left term in (3.3) could be estimated as the expectation of $v(P_{n+1})$ if $P_n = a_i$ and $P_{n+1} \in [c_1, c_2]$:

(3.4) $\mathbb{E}\left(v\left(S_{t_{n+1}}\right)/S_{t_n}\right)(\omega) \approx \int_{c_1}^{c_2} v(x)\, f(x, a_i)\, dx.$

Obviously, the estimation improves when $[c_1, c_2]$ tends to $\mathbb{R}$. This unidimensional example can be extended to multiple dimensions with the same concepts and yet we have not described a real procedure for calculating option value problems, we just compared both approaches.

3.1 Algorithm description.

For the sake of simplicity, within this article, the formulation takes into account only Wiener processes. However, it is important to notice that this approach can be easily extended to more general Levy processes. Let define the next notations:

- The time variable t.
- A vector of n stochastic processes: $X(t) = (X_1(t), X_2(t), \ldots, X_n(t))$.
- A vector of n stochastic differentials: $dX(t) = (dX_1(t), dX_2(t), \ldots, dX_n(t))$.
- A vector of m Wiener processes $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))$, with the differentials: $dW(t) = (dW_1(t), dW_2(t), \ldots, dW_m(t))$ and a correlation matrix ρ such that $\rho_{ij} = corr(W_i, W_j)$.
- A family of n functions $f_i : \mathbb{R}^{n+1} \to \mathbb{R}$ and a family of $n \times m$ functions $g_{ij} : \mathbb{R}^{n+1} \to \mathbb{R}$.

In addition, let define a system of stochastic differential equations with unique solution as follows:

(3.5) $dX_i(t) = f_i(t, X(t))\, dt + \sum_{j=1}^{m} g_{ij}(t, X(t))\, dW_j(t).$

Let be the system of recurrences from (3.5) by using a numerical scheme like Euler-Maruyama or Milstein (Higham, 2001), in an uniform grid for time discretization:

(3.6) $X_i^{k+1} = F_i\left(k, \triangle t, X^k, \triangle W^k\right),$

where k represents the number of steps, $\triangle t$ the time discretization, $X^k = (X_1^k(t), X_2^k(t), ..., X_n^k(t))$ the discretizated vector of stochastic procesess in step k, $\triangle W^k = W((k+1).\triangle t) - W(k.\triangle t)$ and a family of n functions $F_i : \mathbb{N} \times \mathbb{R}^{2m+1} \to \mathbb{R}$.

Suppose that there is a discrete option value V that depends on $\{X_i\}$ and the vector Y containing particular constants parameters from the problem (like the free interest rate), then it is considered a Bellman equation of the form:

$$(3.7)\ V(X^a, t^a) = max_{i \in I(X^a)} \left\{ H_i\left(X^a, Y, \mathbb{E}\left[V_i\left(X^{a+1}, t^{a+1}\right) \diagup X^a\right]\right)\right\},$$

where $H_i : \mathbb{R}^{n+p+2} \to \mathbb{R}$, $t^a = a\triangle t$, p is the dimension of Y, $I(X^a)$ represents all possible decisions on the state X^a, V_i is the option value choosing i and $\mathbb{E}\left[V_i\left(X^{a+1}, t^{a+1}\right) \diagup X^a\right]$ is the conditional expectation of V_i under X^a. Then we define the next function $G : \mathbb{R}^n \times \mathbb{N} \times I(\mathbb{R}^n) \to \mathbb{R}$:

$$(3.8)\ G(P, a, i) = \mathbb{E}\left[V_i\left(X^{a+1}, t^{a+1}\right) \diagup X^a\right](\omega),$$

where $t^a = a\triangle t$, for any ω belonging to a sample space such that $X^a(\omega) = P$ (because X^a is a Markov process) and we have:

$$(3.9)\ G(P, a, i) \approx \int_\Omega J(x_1, ..., x_n)\, V_i\left(x_1, ..., x_n, t^{a+1}\right).dx_1...dx_n,$$

where $J : \mathbb{R}^n \to \mathbb{R}$ is the joint density of X^{a+1} if $X^a = P$ (using (3.6)) and $\Omega_P \in \mathbb{R}^n$ is a $[p_1, q_1] \times ... \times [p_n, q_n]$ region large enough for a good approximation of the integral.

Like other algorithms for option valuing this approach applies dynamic programming taking data from the step closer to time zero to approximate a step immediately previous. Let be $\widetilde{V}$ is our approximation of V and suppose we know a good approximation of the value of the function $V\left(x, t^{a+1}\right)$ (with t^{a+1} fixed) in a region $\Omega_{a+1} \subset \mathbb{R}^n$. Then, we calculate (3.9) (with $\widetilde{V}\left(x, t^{a+1}\right)$) replaced in (3.7) and thus estimate a new $\widetilde{V}(x, t^a)$ in a set of previously selected points $\{P_k\} \subset \Omega_{a+1}$. But to repeat the same procedure and obtain an estimation of the function in the previous time $\widetilde{V}\left(x, t^{a-1}\right)$, we must have a estimation of $V(x, t^a)$ in another region $\Omega_a \subset \mathbb{R}^n$ large enough. For that, we finally make an interpolation or regression of the set of points $\left\{P_k, \widetilde{V}(P_k, t^a)\right\}$ using a predetermined family of functions as basis. A recommended way to do this is through a "local interpolation" where we must make a partition of $\Omega_a = \cup\alpha_i$ and each $V_i : \alpha_i \to \mathbb{R}$ is approximated by an interpolation of $\widetilde{V}_i(x, t^a)$ in $\beta_i \cap \{P_k\}$, where $\alpha_i \subset \beta_i$. The algorithm terminates when the backward procedure reaches $a = 0$. The next pseudo code represents what is described above:

Algorithm 1 CMPA

$N = \frac{T}{\triangle t}$ = number of iterations.

M =cardinality of $\{P_k\}$.

L =cardinality of $I(X)$.

Discretizing the SDE (31) that model the problem.

Determining the joint density function of discrete variables in (32).

FOR $i = 1$ TO N

  FOR $j = 1$ TO M

    FOR $k = 1$ TO L

      $\widetilde{G}(P_j, i, k) = \int_{\Omega_j} J_{P_j}(x_1, ..., x_n) \widetilde{V}_k(x_1, ..., x_n, t^i) .dx_1...dx_n$

    $\widetilde{V}(P_j, t^{i-1}) = max_{k \in I(X)} \left\{ H\left(P_j, Y, k, \widetilde{G}(P_j, i, k)\right)\right\}$

  Interpolating or regressing $\left\{P_j, \widetilde{V}(P_j, t^{i-1})\right\}$.

3.2 A simple numerical example.

Before describing the algorithm we present a simple numerical example for a simple American call option, independent from the main problem of this article. We will choose the following parameters: riskless rate $r = 0.05$, strike price $K = 5$, the asset initial value $S_0 = 10$, the volatility $\sigma = 0.3$, the time $T = 1$ and the time discretization $\Delta t = \frac{1}{3}$.

Assuming that the process followed by the underlying variable S in a risk-neutral world is:

$$(3.10)\ dS = rSdt + \sigma SdW = (0.05)\, Sdt + (0.3)\, SdW,$$

where W is a Wiener process and $\triangle W_n = W_{(n+1)\triangle t} - W_{n\triangle t}$, if we apply the Euler-Maruyama method yields the following discretization:

$$(3.11)\ S_{n+1} = S_n + rS_n \triangle t + \sigma S_n \vartriangle W_n$$
$$= S_n + \left(\frac{1}{60}\right) S_n + (0.3)\, S_n \vartriangle W_n.$$

Observe that for a fixed $S_n = x$, S_{n+1} has a normal distribution with mean $m_x = x + \left(\frac{1}{60}\right) x$ and standard deviation $d_x = \frac{(0.3)x}{\sqrt{3}}$, which implies the following density function:

$$(3.12)\ f_x(y) = \left(\frac{1}{d_x\sqrt{2\pi}}\right) e^{-\left(\frac{(m_x - y)^2}{2d_x^2}\right)}.$$

Here we use the condition in time T: $V(S_3, 1) = max(S_3 - 5, 0)$ to approximate the option value function in the previous instant $t = \frac{2}{3}$ and using the Bellman equation $V(S_n, n\triangle t) = max(S_n - K, e^{-r\triangle t}\mathbb{E}\left(V\left(S_{n+1}, \frac{n+1}{3}\right)/S_n\right))$ backwards. Following the idea put forward earlier an estimation of the function using the next integral is performed in a finite number of values of S_n given by $\{0.1, 12, 24, 36, 48\}$:

$$(3.13)\ \mathbb{E}\left(V\left(S_{n+1},(n+1)\triangle t\right)/S_n\right)(\omega) \approx \int_0^{\infty} V\left(y,(n+1)\triangle t\right) f_x(y)\,dy.$$

Where ω is in a sample space Ω and $S_n(\omega) = x$.

For ω such that $S_2(\omega) = 0.1$, then $m_x = 0.1017$ and $d_x = 0.0173$:

$$(3.14)\ \mathbb{E}\left(V\left(S_3,1\right)/S_2\right)(\omega) \approx \int_0^{76} max(y-5,0).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy$$
$$= 0 \Rightarrow V\left(0.1,\tfrac{2}{3}\right) \approx max\left(0.1-5,0\right) = 0.$$

Remark. As the integrals are calculated using some numerical method, it is considered a finite interval of integration $[0, 76]$ accurate enough for our requirements.

For ω such that $S_2(\omega) = 12$, then $m_x = 12.2$ and $d_x = 2.0785$:

$$(3.15)\ \mathbb{E}\left(V\left(S_3,1\right)/S_2\right)(\omega) \approx \int_0^{76} max(y-5,0).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy$$
$$= 12.1988 \Rightarrow V\left(12,\tfrac{2}{3}\right) \approx max\left(12-5, 11.9972\right) = 11.9972.$$

For ω such that $S_2(\omega) = 24$, then $m_x = 24.4$ and $d_x = 4.1569$:

$$(3.16)\ \mathbb{E}\left(V\left(S_3,1\right)/S_2\right)(\omega) \approx \int_0^{76} max(y-5,0).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy$$
$$= 24.4 \Rightarrow V\left(24,\tfrac{2}{3}\right) \approx max\left(24-5, 23.9967\right) = 23.9967.$$

For ω such that $S_2(\omega) = 36$, then $m_x = 36.6$ and $d_x = 6.2354$:

$$(3.17)\ \mathbb{E}\left(V\left(S_3,1\right)/S_2\right)(\omega) \approx \int_0^{76} max(y-5,0).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy$$
$$= 36.6 \Rightarrow V\left(36,\tfrac{2}{3}\right) \approx max\left(36-5, 35.9951\right) = 35.9951.$$

For ω such that $S_2(\omega) = 48$, then $m_x = 48.8$ and $d_x = 8.3138$:

$$(3.18)\ \mathbb{E}\left(V\left(S_3,1\right)/S_2\right)(\omega) \approx \int_0^{76} max(y-5,0).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy$$
$$= 48.7582 \Rightarrow V\left(48,\tfrac{2}{3}\right) \approx max\left(48-5, 47.9523\right) = 47.9523.$$

Then we approximate the whole function $V\left(S_2,\frac{2}{3}\right)$ in $\mathbb{R}$ by a regression in $\left\{1, x, x^2\right\}$, obtaining $V(x,\frac{2}{3}) \approx -0.093 + 1.0076x - 0.0001x^2$ and repeat the above procedure with the instant $t = \frac{1}{3}$:

For ω such that $S_1(\omega) = 0.1$, then $m_x = 0.1017$ and $d_x = 0.0173$:

$$(3.19)\ \mathbb{E}\left(V\left(S_2,\tfrac{2}{3}\right)/S_1\right)(\omega)$$
$$\approx \int_0^{76}(-0.093+1.0076y-0.0001y^2).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy = 0.0094$$
$$\Rightarrow V\left(0.1,\tfrac{1}{3}\right) \approx max\left(-4.9, 0.0093\right) = 0.0093.$$

For ω such that $S_1(\omega) = 12$, then $m_x = 12.2$ and $d_x = 2.0785$:

(3.20) $\mathbb{E}\left(V\left(S_2, \frac{2}{3}\right)/S_1\right)(\omega)$

$$\approx \int_0^{76}(-0.093+1.0076y-0.0001y^2).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy = 12.1844$$

$\Rightarrow V\left(12, \frac{1}{3}\right) \approx max\,(7, 11.9830) = 11.9830.$

For ω such that $S_1(\omega) = 24$, then $m_x = 24.4$ and $d_x = 4.1569$:

(3.21) $\mathbb{E}\left(V\left(S_2, \frac{2}{3}\right)/S_1\right)(\omega)$

$$\approx \int_0^{76}(-0.093+1.0076y-0.0001y^2).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy = 24.4312$$

$\Rightarrow V\left(24, \frac{1}{3}\right) \approx max\,(19, 24.0274) = 24.0274.$

For ω such that $S_1(\omega) = 36$, then $m_x = 36.6$ and $d_x = 6.2354$:

(3.22) $\mathbb{E}\left(V\left(S_2, \frac{2}{3}\right)/S_1\right)(\omega)$

$$\approx \int_0^{76}(-0.093+1.0076y-0.0001y^2).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy = 36.6473$$

$\Rightarrow V\left(36, \frac{1}{3}\right) \approx max\,(31, 36.0416) = 36.0416.$

For ω such that $S_1(\omega) = 48$, then $m_x = 48.8$ and $d_x = 8.3138$:

(3.23) $\mathbb{E}\left(V\left(S_2, \frac{2}{3}\right)/S_1\right)(\omega)$

$$\approx \int_0^{76}(-0.093+1.0076y-0.0001y^2).\left(\frac{1}{d_x\sqrt{2\pi}}\right)e^{-\left(\frac{(m_x-y)^2}{2d_x^2}\right)}.dy = 48.7911$$

$\Rightarrow V\left(48, \frac{1}{3}\right) \approx max\,(43, 47.9523) = 47.9846.$

Again we approximate the whole function $V\left(S_1, \frac{1}{3}\right)$ in $\mathbb{R}$ by a regression in $\left\{1, x, x^2\right\}$, obtaining $V(x, \frac{1}{3}) \approx -0.0951 + 1.0087x - 0.0001x^2$ and repeat exactly the same procedure with the instant $t = 0$.

4 Implementation

4.1 Implementation for one variable

Using the study case discussed in (Nembhard et al., 2002) for one stochastic variable, we implement an American and European option pricing regarding the following parameters: the number of sales $S_1(0)$ =\$872,640 per month; The price of the product is S_2 =\$5.678; the volatility of the number of sales is 0.930354. When control charts are not implemented for controlling the process, the loss revenue factor is equal to; $g = 0.018$. On the other hand, the cost of implementing control charts is K =\$11,000 per month. Finally, the cost of using control charts for an American option is K_A =\$143,044; the yearly risk-free interest rate is equal to $r = 0.08$ and the maturity is $T = 1$.

We define the stochastic revenue variable $S := g.S_1.S_2$ which follows the same stochastic dynamic than S_1 in a risk-neutral basis:

(4.1) $dS = \mu.S.dt + \sigma.S.dW,$

where W is a Wiener process, $\mu = r = 0.08$ is the expected return in a risk-neutral basis, and $\sigma = 0.930354$ is the volatility. Using the Euler-Maruyama method with the time discretization $\triangle t = \frac{1}{24}$ we have:

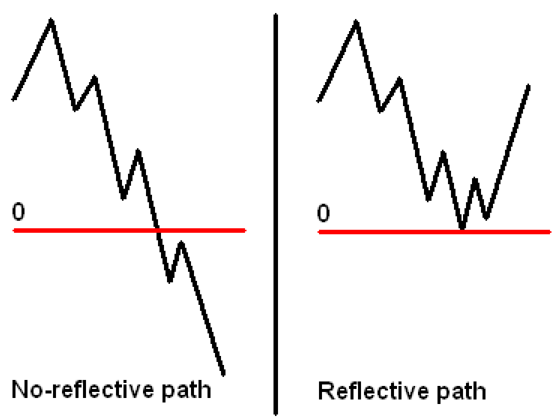
(4.2) $S_{em}^{a+1} = S_{em}^{a} + S_{em}^{a}\left(\frac{0.08}{24}\right) + 0.930354.S_{em}^{a}\triangle W^{a},$

where a is the number of steps and

(4.3) $\triangle W^{a} = W\left(\left(a+1\right)\triangle t\right) - W\left(a\triangle t\right).$

We modificate the original numerical scheme with some reflective assumptions for every hypothetical Euler-Maruyama discretized path, i.e. a discrete path S^a is constructed such that: if $S^a = S_{em}^a$ then we assume that in the next step $S^{a+1} = \left|S_{em}^{a+1}\right|$. This consideration assures us the desired property that S^a never has negative values and as we will see that modifies the density function obtained with the original scheme. Figure 4.1 shows a more intuitive idea.

Fig. 4.1: Graphical example of reflexivity



Therefore, let V_E , V_A, V_e and V_{am} be the European, American and two auxiliar option values respectively (that they will be used to calculate the European and American option values). Let be $\widetilde{V}_E$, $\widetilde{V}_A$, $\widetilde{V}_e$ and $\widetilde{V}_{am}$ the estimations of the last option values. Hence, we have the following Bellman equations :

(4.4) $V_e\left(S^a, t^a\right) = max\left\{0, e^{-r\triangle t}\mathbb{E}\left[V_e\left(S^{a+1}, t^{a+1}\right)/S^a\right]\right\},$

(4.5) $V_{am}\left(S^a, t^a\right) = max\left\{0, e^{-r\triangle t}\mathbb{E}\left[V_{am}\left(S^{a+1}, t^{a+1}\right)/S^a\right]\right\},$

(if a is an odd number)

(4.6) $V_{am}\left(S^a, t^a\right) = max\left\{0, S^a + e^{-r\triangle t}\mathbb{E}\left[V_{am}\left(S^{a+1}, t^{a+1}\right)/S^a\right]\right\}$,

(if a is an even number)

(4.7) $V_A\left(S^a, t^a\right) = max\left\{0, V_{am}\left(S^a, t^a\right) - K_A\right\}$,

(4.8) $V_E\left(S^a, t^a\right) = \sum_{k=0}^{\frac{a}{2}} V_e\left(S^{2k}, t^{2k}\right)$,

(if a is an even number)

where

(4.9) $V_e\left(S^{24}, 1\right) = max\left\{0, S^{24} - 11,000\right\}$,

and

(4.10) $V_{am}\left(S^{24}, 1\right) = max\left\{0, S^{24}\right\} = S^{24}$.

Remark. Assuming that cash flow is monthly, Bellman equation (4.5) corresponds to an intermediate period within the month. Thus, equation (4.5) is used as an extra calculation point in order to improve the estimation of conditional expectations in (4.6) as well as for adapting the option pricing scheme to a discretization grid more dense than $\triangle t = \frac{1}{12}$. If $\triangle t = \frac{1}{12}$, then equation (4.6) can be directly applied.

It can be seen in the Bellman equations that the American real option of this particular problem is not formulated as the traditional American option used in finance. For more details see section (2.4) or (Nembhard et al., 2002).

In this context, we must to calculate the integral (3.9) in enough points to approximate the option values at each point P_i as functions of S_n. Thus, we must choose the points with a sufficient spacing among them with the objective of capturing a range wide enough of possible values, so that it accurately approximates to the option function with an acceptable error in each iteration. Within this work, it has been selected 54 values for P_j such that $P_j = 1.5^j$ and $P_0 = 0$.

Equation (4.2) implies that if $S^a_{em} = P_j$ then S^{a+1}_{em} has a normal distribution with average $P_j + P_j\left(\frac{0.08}{24}\right)$ and standard deviation $0.930354\sqrt{\frac{1}{24}}P_j$, so the modificated discretized variable S^{a+1} with reflexive assumptions follows the next density function:

(4.11) $J_i\left(x\right) = \frac{1_{\left(\mathbb{R}^+\cap\{0\}\right)}(x)\left(A_1(x)+A_2(x)\right)}{\left(0.930354P_i\sqrt{\frac{\pi}{12}}\right)}$,

where:

$$A_1\left(x\right) = e^{-\left(x-P_i-P_i\left(\frac{0.08}{24}\right)\right)^2/\left(2\left(0.930354P_i\sqrt{\frac{1}{24}}\right)^2\right)},$$

$$A_2\left(x\right) = e^{-\left(-x-P_i-P_i\left(\frac{0.08}{24}\right)\right)^2/\left(2\left(0.930354P_i\sqrt{\frac{1}{24}}\right)^2\right)},$$

and $1_D\left(x\right)$ is the indicator function such that $1_D\left(x\right) = 1$ for $x \in D$ and $1_D\left(x\right) = 0$ for $x \notin D$. Then for each P_i and the corresponding region, it satisfies the expression below:

(4.12) $\Omega_i = [L_I, L_S]$

where:

$L_I = P_i + P_i\left(\frac{0.08}{24}\right) - 5\left(0.930354P_i\sqrt{\frac{1}{24}}\right)$

and

$L_S = P_i + P_i\left(\frac{0.08}{24}\right) + 5\left(0.930354P_i\sqrt{\frac{1}{24}}\right)$.

Afterwards we estimate the conditional expectation on the set $\{P_i\}$ using the following integral for the European and American option respectively:

(4.13) $\widetilde{G}_e(P_i, a) = \int_{\Omega_i} J_i(x)\, \widetilde{V}_e\left(x, t^{a+1}\right) .dx$,

(4.14) $\widetilde{G}_{am}(P_i, a) = \int_{\Omega_i} J_i(x)\, \widetilde{V}_{am}\left(x, t^{a+1}\right) .dx$.

Finally, it replaces (4.13) and (4.14) on their respective Bellman equations:

(4.15) $\widetilde{V}_e\left(P_i, t^a\right) = max\left\{0, e^{-r\triangle t}\widetilde{G}_e(P_i, a)\right\}$,

(4.16) $\widetilde{V}_{am}\left(P_i, t^a\right) = max\left\{0, P_i + e^{-r\triangle t}\widetilde{G}_{am}(P_i, a)\right\}$,

(if a is an even number),

(4.17) $\widetilde{V}_{am}\left(P_i, t^a\right) = max\left\{0, e^{-r\triangle t}\widetilde{G}_{am}(P_i, a)\right\}$,

(if a is an odd number) ,

(4.18) $\widetilde{V}_E\left(P_i, t^a\right) = \sum_{k=0}^{\frac{a}{2}} \widetilde{V}_e\left(P_i, t^{2k}\right)$,

(4.19) $\widetilde{V}_A\left(P_i, t^a\right) = max\left\{0, V_{am}\left(P_i, t^a\right) - K_A\right\}$.

At the beginning, integrals (4.13) and (4.14) are calculated using the conditions (4.9) and (4.10), all the integrals are calculated numerically by Simpson's rule, later we work backwards by using the interpolated approximations of $V_e\left(x, t^a\right)$ and $V_{am}\left(x, t^a\right)$ for $n < 24$.

In this sense, let be the family of sets: $C_j = [P_j, P_{j+1})$ for $0 < j < 53$ and $C_{54} = [P_{54}, \infty)$. It is made a "local" approximation of the option value function in each C_j using the resulting function interpolating the next points: $\{\left(P_{j-1}, \widetilde{V}\left(P_{j-1}, t^a\right)\right), \left(P_j, \widetilde{V}\left(P_j, t^a\right)\right), \left(P_{j+1}, \widetilde{V}\left(P_{j+1}, t^a\right)\right), \left(P_{j+2}, \widetilde{V}\left(P_{j+2}, t^a\right)\right)\}$ where $B_j = C_{j-1} \cup C_j \cup C_{j+1}$, by the set of functions $\left\{1_{B_j}, x.1_{B_j}, x^2.1_{B_j}, x^3.1_{B_j}\right\}$. Similarly, for C_0 an approximation of the option value function is obtained interpolating $\{\left(P_0, \widetilde{V}\left(P_0, t^a\right)\right), \left(P_1, \widetilde{V}\left(P_1, t^a\right)\right), \left(P_2, \widetilde{V}\left(P_2, t^a\right)\right), \left(P_3, \widetilde{V}\left(P_3, t^a\right)\right)\}$ by the set of functions $\left\{1_{B_1}, x.1_{B_1}, x^2.1_{B_1}, x^3.1_{B_1}\right\}$, and finally, for C_{53} and C_{54} an approximation of the option value function is estimated by interpolating $\{\left(P_{51}, \widetilde{V}\left(P_{51}, t^a\right)\right), \left(P_{52}, \widetilde{V}\left(P_{52}, t^a\right)\right), \left(P_{53}, \widetilde{V}\left(P_{53}, t^a\right)\right), \left(P_{54}, \widetilde{V}\left(P_{54}, t^a\right)\right)\}$ by the set of functions $\left\{1_{B_{52}}, x.1_{B_{52}}, x^2.1_{B_{52}}, x^3.1_{B_{52}}\right\}$.

4.2 Implementation for two variables

Similarly to the one variable problem, the same study case reported in (Nembhard et al., 2002) is analyzed. For this problem we add the volatility of $S_2(t)$, $\sigma_2 = 0.059634$ and the correlation between $S_1(t)$ and $S_2(t)$: $\rho = 0.111344$.

So let W_1 and W_2 be independent Wiener processes, $r = 0.08$ is the expected return in a risk-neutral basis, then these are the dynamics for $S_1(t)$ and $S_2(t)$ respectively:

$$(4.20)\ dS_1 = 0.08S_1dt + 0.930354S_1dW_1,$$
$$(4.21)\ dS_2 = 0.08S_1dt + 0.059634S_2\left(\rho dW_1 + \sqrt{1-\rho^2}dW_2\right)$$

Using the Euler-Maruyama method with $\triangle t = \frac{1}{24}$, we can state:

$$(4.22)\ S_1^{a+1} = S_1^a + \tfrac{0.08}{24}S_1^a + 0.930354S_1^a\triangle W_1^a,$$
$$(4.23)\ S_2^{a+1} = S_2^a + \tfrac{0.08}{24}S_2^a + 0.059634S_2^a\left(\rho\triangle W_1^a + \sqrt{1-\rho^2}\triangle W_2^a\right)$$

where $\triangle W_i^a = (W_i((a+1)\triangle t) - W_i(a\triangle t))$. Then, the same Bellman equations and conditions at $a = 24$ used in the one variable case are applied for:

$$(4.24)\ S^a = (0.018)\,S_1^aS_2^a.$$

Let be the next sets $\{M_i\}$ such that $M_i = (1.5)^{i-1}$ for $0 < i < 55$, $M_0 = 0$, and $\{N_i\}$ such that $N_i = i$ for $i < 19$. Then let be $\{P_{ij}\} = \{M_i\} \times \{N_j\}$ and defining:

$$(4.25)\ \begin{cases} \mu' = & M_i + \frac{(0.08)}{24}M_i \\ \sigma' = & \frac{(0.930354)M_i}{\sqrt{24}} \\ L_{ij} = & (0.059634)(0.111344)N_j\left(\frac{y-M_i-\frac{(0.08)}{24}M_i}{(0.930354)M_i}\right) \\ \mu" = & N_j + \frac{(0.08)}{24}N_j + L_{ij} \\ \sigma" = & \frac{(0.059634)\sqrt{1-(0.111344)^2}N_j}{\sqrt{24}} \\ H(x,y) = & \frac{1}{2\pi\sigma'\sigma"}\left(e^{-\frac{(x-\mu')^2}{2(\sigma')^2}}e^{-\frac{(y-L_{ij}-\mu")^2}{2(\sigma")^2}}\right) \end{cases}$$

For each $P_{ij} = (M_i, N_j)$ we have:

$$(4.26)\ J_{ij}(x,y) = 1_{(\mathbb{R}^+\cap\{0\})\times(\mathbb{R}^+\cap\{0\})}(x,y)\,H(x,y)\,...$$
$$+1_{(\mathbb{R}^+\cap\{0\})\times(\mathbb{R}^+\cap\{0\})}(x,y)\,H(-x,y)\,...$$
$$+1_{(\mathbb{R}^+\cap\{0\})\times(\mathbb{R}^+\cap\{0\})}(x,y)\,H(x,-y)\,...$$
$$+1_{(\mathbb{R}^+\cap\{0\})\times(\mathbb{R}^+\cap\{0\})}(x,y)\,H(-x,-y)$$

Remark. Note that $H(x,y)$ is the density function obtained directly from Euler-Maruyama while $J_{ij}(x,y)$ is finally the density function where where it is forced the reflective hypothesis to assume probability equal to zero for negative values.

Therefore, we have the following integrals that it will be replaced in the corresponding Bellman equations:

(4.27) $\widetilde{G}_E(P_{ij},a) = \int_{\Omega_{ij}} J_i\,(x,y)\,V_E\left(x,y,t^{a+1}\right).dx.dy,$

(4.28) $\widetilde{G}_{ax}(P_{ij},a) = \int_{\Omega_{ij}} J_{ij}\,(x,y)\,V_{ax}\left(x,y,t^{a+1}\right).dx.dy,$

where:

(4.29) $\Omega_{ij} = \left[M_i\left(1+\frac{0.08}{24}-\frac{(5)(0.930354)}{\sqrt{24}}\right), M_i\left(1+\frac{0.08}{24}+\frac{(5)(0.930354)}{\sqrt{24}}\right)\right]\times$
$\ldots\left[N_j\left(1+\frac{0.08}{24}-\frac{(5)(0.059634)}{\sqrt{24}}\right), N_j\left(1+\frac{0.08}{24}+\frac{(5)(0.059634)}{\sqrt{24}}\right)\right].$

We have implemented an interpolation in each $C_{ij} = [M_i, M_{i+1}) \times [N_j, N_{j+1})$ using the set of functions $\left\{1_{C_{ij}}, x.1_{C_{ij}}, y.1_{C_{ij}}, xy.1_{C_{ij}}\right\}$, but other interestig alternative to the interpolation is implementing a Delaunay triangulation using $\{P_{ij}\}$ as vertices and apply the same triangular or hat functions than finite element method (Pepper and Heinrich, 1992).

5 Numerical Results.

In this section, we discuss the performance of this approach to both one and two variable problems and also we compare these numerical results with those provided by traditional algorithms such as lattice methods and Monte Carlo simulation in the European and American option problems applied in control charts valuation, with one and two stochastic variables.

Using an error analysis of the algorithm (Section 3.1) is observed three types of errors: the first is the error due to discretization of time which can be minimized by reducing the value of $\triangle t$. This error exists in all algorithms except analitic formulations as Black-Scholes. The other two types of error are inherent to our proposal: the first is the error due to the approximation of the value of the option function by using a regression or interpolation, which can be minimized using enough interpolation or regression points $\{P_i\}$ to a better fit of the curve.

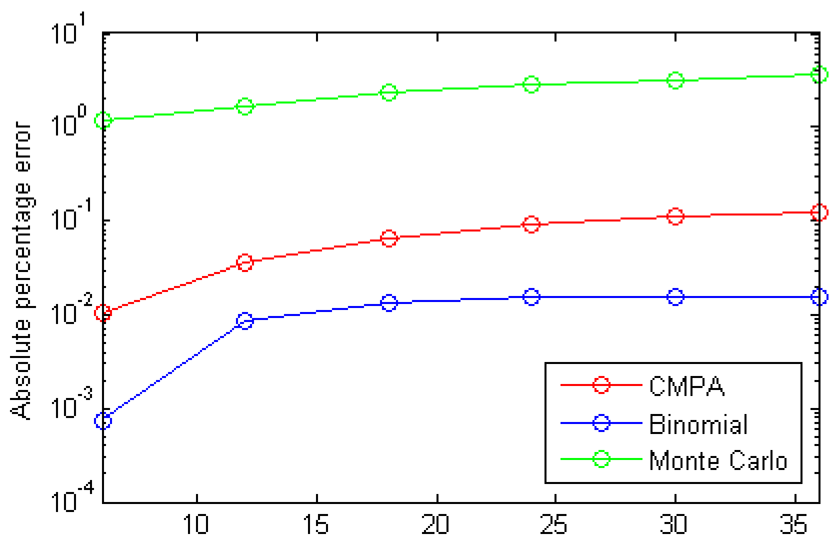
Finally, there is an error in the numerical computation of the integral to estimate the conditional expectation. Since in our implementation the discretized variable follows a normal distribution, then it is reasonable to integrate around the mean. So the distance from the integral limits to the mean, is a variable that affects the precision. In our case, we use the Simpson rule for integration in a ratio of five by the standard deviation from the mean. An analysis of the computational complexity of our own implementation, we consider the following variables:

- $m = \frac{T}{\triangle t}$ (where T is the is the maturity of the European option)
- n =the number of interpolating or regressing points.
- p =the number of points evaluated in the numerical integration.

Then we have the next order of complexity: $O(m.n.p)$

In Figure 5.1 we compare the percentage of absolute error of the European option calculation using the binomial method, Monte Carlo simulation and CMPA by using the absolute value from subtracting the approximated solution and the exact solution given by the Black Scholes formula. Thus, a sensitivity analysis of absolute error depending on the maturity within the sum of European options series problem is done. In this sense, the absolute error of each approach is estimated regarding the maturity parameter within an interval from six months to three years. The Monte Carlo results show the average error of one hundred different realizations each one.

Fig. 5.1: CMPA, Monte Carlo and Binomial percentage absolute errors



In Table 1, a similar sensitivity analysis is done for the case of the one variable American value . In this case the Monte Carlo approach is represented by an implementation of the LSM (Schwartz and Longstaff, 2001) using 1000 paths.

Tab. 1: Comparison between Binomial Method, CMPA and Monte Carlo Simulation for an American option.

	0.5 years	1 years	1.5 years	2 years	2.5 years	3 years
Binomial	481,265	1,016,387	1,551,509	2,086,631	2,621,753	3,156,875
CMPA	481,235	1,016,279	1,551,273	2,086,218	2,621,117	3,155,980
LSM	489,128	986,219	1,618,122	2,186,306	2,698,000	3,281,857

Finally, Tables 2 and 3 compare the performance of all aproaches in multivariable European and American options respectively. Table 2 uses a Monte Carlo simulation with 50000 paths, much more than Table 1 and Table 3 show a multivariable LSM implementation with the relative error of 1% setting 95% of confidence (Fisherman, 2003). In the Fig. 5.1, Table 1, 2 and 3 lattice methods and CMPA have the same time discretization ($\triangle t = \frac{1}{12}, \frac{1}{24}$), but Monte Carlo and LSM simulations have a much thinner discretization ($\triangle t = \frac{1}{1728}$).

Tab. 2: Comparison of the European Option Value between Pentanomial Method, CMPA and Multivariable Monte Carlo Simulation.

	0.5 years	1 years	1.5 years	2 years	2.5 years	3 years
Pentanomial	562,600	1,074,053	1,614,143	2,184,396	2,786,100	3,431,415
CMPA	562,421	1,073,222	1,611,510	2,178,162	2,774,480	3,466,356
Monte Carlo	562,963	1,075,539	1,615,126	2,181,919	2,771,094	3,382,954

Tab. 3: Comparison of the American Option Value between Pentanomial Method, CMPA and Multivariable LSM.

	0.5 years	1 years	1.5 years	2 years	2.5 years	3 years
Pentanomial	495,039	1,068,298	1,667,003	2,292,283	2,945,317	3,627,337
CMPA	494,856	1,067,598	1,665,363	2,289,474	2,941,030	3,607,772
LSM	496,977	1,075,323	1,666,170	2,244,625	2,854,852	3,450,873

6 Conclusion

This paper exposes an alternative method based on an approximation of the stochastic dynamic through infinite states for assessing multivariate real option problems. In order to illustrate the feasibility of the proposed valuation approach, a real option problem related with a control chart implementation was analyzed. The experimental results for Geometric Brownian models of one and two variables suggest to expect a good performance for more exotic dynamics. This approach seems promising for application to problems where some traditional algorithms are no applicable or have not an adequate performance due to its versatility of implementation. In this sense, it offers an interesting trade-off between the precision of the lattice methods and versatility of Monte Carlo methods. In this context, our proposal may be an useful appraisal in some problems where the model versatility is needed but the stochastic error is not desired. Future works could extend this approach to more general dynamic involving Levy processes and multiple option implementations.

References

[1] Bellman, Richard Ernest. Dynamic Programming. Courier Dover Publications. 2003.

[2] Boyle, P. P. Options: A Monte Carlo approach. Journal of Financial Economics 4, no. 3. Journal of Financial Economics: 323-338. 1977.

[3] Boyle, P.P. A Lattice Framework for Option Pricing with Two State Variables. Journal of Financial and Quantitative Analysis, Vol. 23, No. 1, 1988.

[4] Black, Fischer and Scholes M., "The Pricing of Options and Corporate Liabilities", Journal of Political Economy, Vol. 81, No. 3. 1977.

[5] Cox J. C., Ross S. A., and Rubinstein M. "Option Pricing: A Simplified Approach." Journal of Financial Economics 7. 1979.

[6] Dixit, A. K., Pindyck R. S. Investment under uncertainty. Princeton University Press. 1994.

[7] Duan, J. C. and Simonato J. G. American Option Pricing under GARCH by a Markov Chain Approximation. Journal of Economic Dynamics and Control. 1999.

[8] Fisherman G. "Monte Carlo: Concepts, Algorithms, and Applications (Stochastic Modeling)" Kluwer Academic Publishers, Springer .2003

[9] Higham D. J. An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations. SIAM Review 43. 2001. 17

[10] Montgomery D.C. Introduction to Statistical Quality Control, Fourth Edition, John Wiley & Sons, New York, NY, 2000.

[11] Myers, S. C., Determinants of Corporate Borrowing, Journal of Financial Economics, Vol. 5, No. 2. 1977.

[12] Nembhard H. B., Shi L., Aktan M. A real options design for quality control charts. Proceedings of the 32nd conference on Winter simulation. 2002.

[13] Øksendal, Bernt Karsten. Stochastic differential equations: an introduction with applications. Springer.2003.

[14] Pepper, D. W., and Heinrich J. C. The finite element method: basic concepts and applications. Taylor & Francis. 1992.

[15] Wilmott, Paul, Sam Howison, and Jeff Dewynne. The mathematics of financial derivatives: a student introduction. Cambridge University Press. 1995.