On a General Market Portfolio Acting as the Twin Security of an Arbitrary Project

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Abstract

Formation of a riskless portfolio by the derivative and its underlying is the first step of the derivation of Black - Sholes - Merton partial differential equation (BSM), whose solution under certain boundary conditions represents the objective value of options. It is well known that the option value based on BSM is objective as is does not include a trend term, which represent the risk preference inherent to individuals, in its solution. In general real options, even for a simple option such as to defer investment, application of BSM is difficult unless the objective value of the underlying, which is the project itself, is clear. The value of a twin asset, traded in a market, will be a good substitute for objective project value, which is usually difficult to determine as the project itself is not traded in the market. On the other hand, it used to be a common sense that finding a perfect twin asset is virtually impossible.

Necessary conditions of a twin asset to derive the BSM has been discussed and determined. Consequently, perfect correlativity and volatility matching have been determined as the necessary conditions for a twin asset whose value will represent the objective value of a project that could be used for a parameter in the BSM solution.

Efficient searching procedure for a portfolio, which satisfies the necessary conditions as a twin asset, has been proposed and the existence of a twin asset with appropriate congeniality for an arbitrary project has been demonstrated.

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1. Introduction

Formation of a riskless portfolio by the derivative and its underlying is the first step of the derivation of Black - Sholes - Merton partial differential equation (BSM), whose solution under certain boundary conditions represents the objective value of options. It is well known that the option value based on BSM is objective as is does not include the trend term, which represent the risk preference inherent to individuals, in its solution. In general real options, even for a simple option such as to defer investment, application of BSM is difficult unless the objective value of the underlying, which is the project itself, is clear and its twin asset, having enough liquidity to continuously form a riskless portfolio in combination with the real options, exists.

Many aspects of a twin asset have been discussed so far. Hubalek and Schachermayer (2001) pointed out that the no-arbitrage assumption does not work even if a twin asset can be found. On the premise that BSM solution is available but a twin asset is virtually impossible to specify, Copeland and Antikarov (2001) proposed a “market asset disclaimer”, in which the NPV of a project, assuming certain discount rate as well as eliminating managerial flexibility, is regarded as a substitute of the value of an underlying asset.

Formation of a riskless portfolio by a long position of the project with managerial flexibilities and an appropriate amount of short position of the underlying, which represent the project value without any managerial flexibility, is necessary to apply BSM type solution to identify the theoretical value of the real options. Twin asset with enough liquidity, which represents the NPV of a project without any managerial flexibility, can be considered as such underlying. But it used to be a common sense that finding a perfect twin asset is virtually impossible.

In this study, necessary conditions of a twin asset to derive the BSM has been discussed and determined. Also an efficient searching procedure for a portfolio, which satisfies the necessary conditions as a twin asset, has been proposed and its existence with appropriate congeniality for an arbitrary project has been demonstrated.

2. Determination of a twin asset

2.1. Necessary conditions of a twin asset

According to Trigeorgis (1993), “The existence of a traded "twin asset" (or dynamic portfolio) that has the same risk characteristics (i.e., is perfectly correlated) with the non-traded real asset in complete markets is sufficient for real option valuation". Finding a twin asset is the first step to apply the BSM to real option valuation. In general, following conditions must be satisfied so that BSM can be applied to real option valuation.

- A twin asset, having the same risk characteristics as the non-traded underlying asset, exists in the financial market.
- The parameters used in BSM are equal between the non-traded underlying asset and its twin asset.

Having the same risk characteristics means having perfect correlation each other.

Table 1. Corresponding parameters of a financial asset.

Financial asset	Project
Price	Project value (NPV)
Exercise price	Amount of investment
Volatility of the price [%]	Volatility of project value [%]
Term to maturity	Term to decision making
Risk free rate [%]	Risk free rate [%]

Table 1 shows the corresponding parameters of a financial asset for respective parameters of a real asset. Concerning the second condition, not all these parameters are necessary to be matched to be identified as a twin asset. For example, Price and project value could be different as the difference can be adjusted by delta to form a risk less portfolio. Exercise price, term to maturity are not included in the underlying asset. Risk free rate is common to both financial asset as well as project. Consequently, the necessary condition to be a twin asset, in terms of parameters, is volatility matching.

The necessary characteristics of a twin asset to apply BSM to real option valuation can be summarized as followings:

1) The twin asset has perfect correlation with the intended project.
2) The volatility of the twin asset is equal to that of the intended project.

When the price of the financial asset i at time t, $S_i(t)$, and the value of the intended project $X(t)$ follow

$$dS_i(t) = \alpha_i S_i(t)dt + \sigma_i S_i(t)dW_t \tag{1}$$

$$dX(t) = \mu_X X(t)dt + \sigma_X X(t)dW_t \tag{2}$$

where

α_i : expected rate of return for asset i

σ_i : volatility of asset i

μ_X : expected rate of return of the intended project

σ_X : volatility of the intended project

The value of portfolio $P(t)$, which is a combination of n assets, is represented as

$$P(t) = {}^t\mathbf{w} \cdot \mathbf{S} \tag{3}$$

when

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ \vdots \\ w_n \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} S_1(t) \\ S_2(t) \\ S_3(t) \\ \vdots \\ \vdots \\ S_n(t) \end{pmatrix} \tag{4}$$

where

$\mathbf{w}$: weight of each asset

Consequently, the two necessary conditions can be expressed as followings;

1) Perfect correlation

$$\rho_{X,P} = 1 \tag{5}$$

where $\rho_{X,P}$: coefficient of correlation between the project value and the price of twin asset

2) Volatility matching

$$\sigma_P{}^2 = {}^t\mathbf{w} \cdot \mathbf{V} \cdot \mathbf{w} = \sigma_X{}^2 \tag{6}$$

$$\mathbf{V} = \begin{pmatrix} \sigma_1{}^2 & Cov_{1,2} & Cov_{1,3} & \cdots & \cdots & Cov_{1,n} \\ Cov_{2,1} & \sigma_2{}^2 & & \ddots & & Cov_{2,n} \\ Cov_{3,1} & & \sigma_3{}^2 & & \ddots & \vdots \\ \vdots & \ddots & & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & Cov_{n-1,n} \\ Cov_{n,1} & Cov_{n,2} & \cdots & \cdots & Cov_{n,n-1} & \sigma_n{}^2 \end{pmatrix} \tag{4}$$

where $\mathbf{V}$: variance-covariance matrix

$Cov_{i,j}$: covariance between asset i and asset j

σ_P : volatility of the portfolio

The condition for perfect correlation can be rewritten as followings:

$$\begin{aligned} \rho_{X,P} &= \frac{Cov(X(t), P(t))}{\sqrt{Var(X(t))}\sqrt{Var(P(t))}} \\ &= \frac{Cov(X(t), P(t))}{Var(X(t))} \cdot \frac{\sqrt{Var(X(t))}}{\sqrt{Var(P(t))}} \\ &= \beta_{X,P} \cdot \frac{\sigma_X}{\sigma_P} \end{aligned} \tag{5}$$

From eq.(5) and (5),

$$\beta_{X,P} \cdot \frac{\sigma_X}{\sigma_P} = 1 \tag{6}$$

From, eq.(6),

$$\frac{\sigma_X}{\sigma_P} = 1 \tag{7}$$

So that,

$$\beta_{X,P} = 1 \tag{8}$$

2.2. Basic methodology to form a twin asset

The basic procedure to form a twin asset is as followings:

Step1

Calculate both σ_i and $\beta_{i,X}$ for all available assets.

Step2

Randomly select n assets to form a portfolio.

Step3

Under the condition of $\beta_{X,P}=1$, determine the weight $\mathbf{w}$ that gives a value of σ_P that is the closest to σ_X.

The fulfillment of the condition $\beta_{X,P}=1$ is prioritized ahead of that of $\sigma_P=\sigma_X$, to

reduce the amount of required calculations. Assuming a population of N assets, the total number of components to calculate σ is $N^2/2 - N/2$. In this study, 40,335 calculations are required to derive β, but 813,435,945 for σ.

If the condition in eq. (8) is satisfied, the relationship between σ_X and σ_P is always,

$$\sigma_X \leq \sigma_P \tag{9}$$

Because, from equations (6) and (8),

$$\rho_{X,P} = \frac{\sigma_X}{\sigma_P} \tag{10}$$

Since $\rho_{X,P}$ is the coefficient of correlation it fulfills

$$-1 \leq \rho_{X,P} \leq 1 \tag{11}$$

As it is obvious that both σ_P as well as σ_X are greater than 0, eq. (9) holds if $\beta_{X,P}=1$.

The volatility of the portfolio, whose $\beta_{X,P}=1$, is always larger than that of the real asset.

Therefore, for step 3 mentioned above, it is sufficient to find a set of weight $\mathbf{w}$ that minimizes the value of σ_P.

3. Data description

3.1. Considered market data

Commercial database for stocks and indexes from all over the world, for a period from April 2001 to December 2010 has been used as the population for the twin asset. The number of covered assets is 40,335, including 17,869 from the US, 10,754 from Asia, 7,299 from the EU, 1,998 from Canada, and 2,425 from other markets.

3.2. Case project

A business extension project in a material industry is provided as the case project in this study. Free cash flows on every quarter from April 2001 to December 2010 are used. Figure 1 shows the change in the NPV of the 3 years' project discounted at WACC of the company (in this case 20%). Figure 2 shows the rate of return of the real asset. The trend is -2.8% and the volatility is 14.3%.

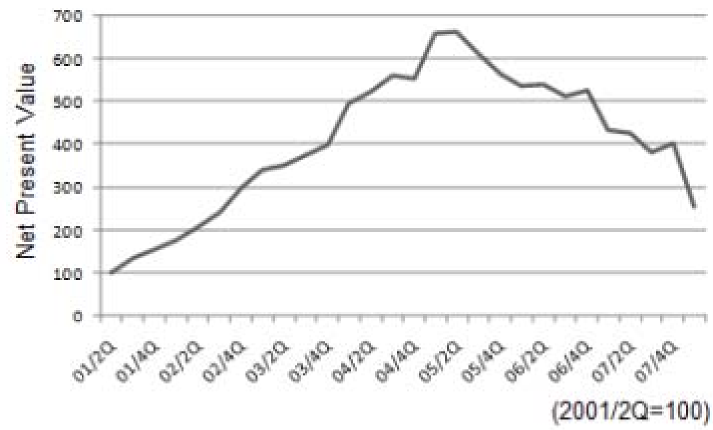

Figure 1. Change in NPV of the case project normalized by the 1st quarter of 2001 (=100)

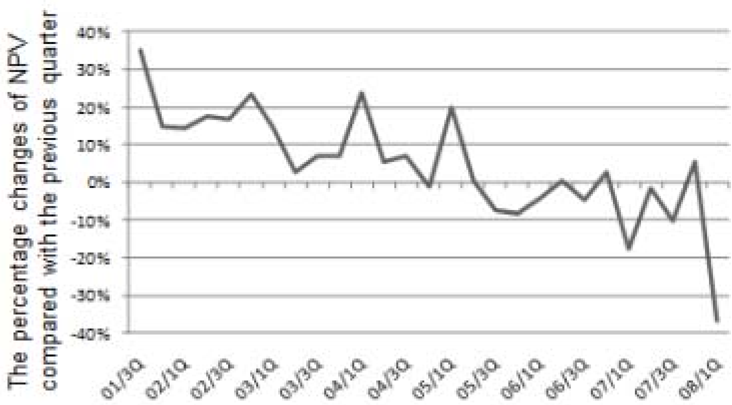

Figure 2. Change in rate of return of project NPV

4. Development of efficient procedure to form a twin asset

Efficient procedure to identify a twin asset for an arbitrary project has been developed. The probability to find a twin asset increases according to the number of assets incorporated in the portfolio. But the number of random combination of assets as well as calculation needed to derive σ_P increase drastically according to the number of assets in the portfolio.

4.1. Effect of increasing number of trial and increasing number of assets incorporated in the portfolio

Two assets are randomly selected from the population of 40,335 available assets to examine the effect of increasing number of trial (random selection) to the probability a twin asset being identified. The number of trial has been changed from 25,000 to 100,000. The weights of each asset were determined so as to satisfy the $\beta_{X,P}=1$ constraint with minimal σ_P.

Figure 3 shows the change of σ_P distribution according to the change in number of trial. The mode of σ_P distribution virtually does not change even though the number of trial has been increased. As $\sigma_X \leq \sigma_P$, it is obvious that the increase in the number of trial does not increase the probability of twin asset identification.

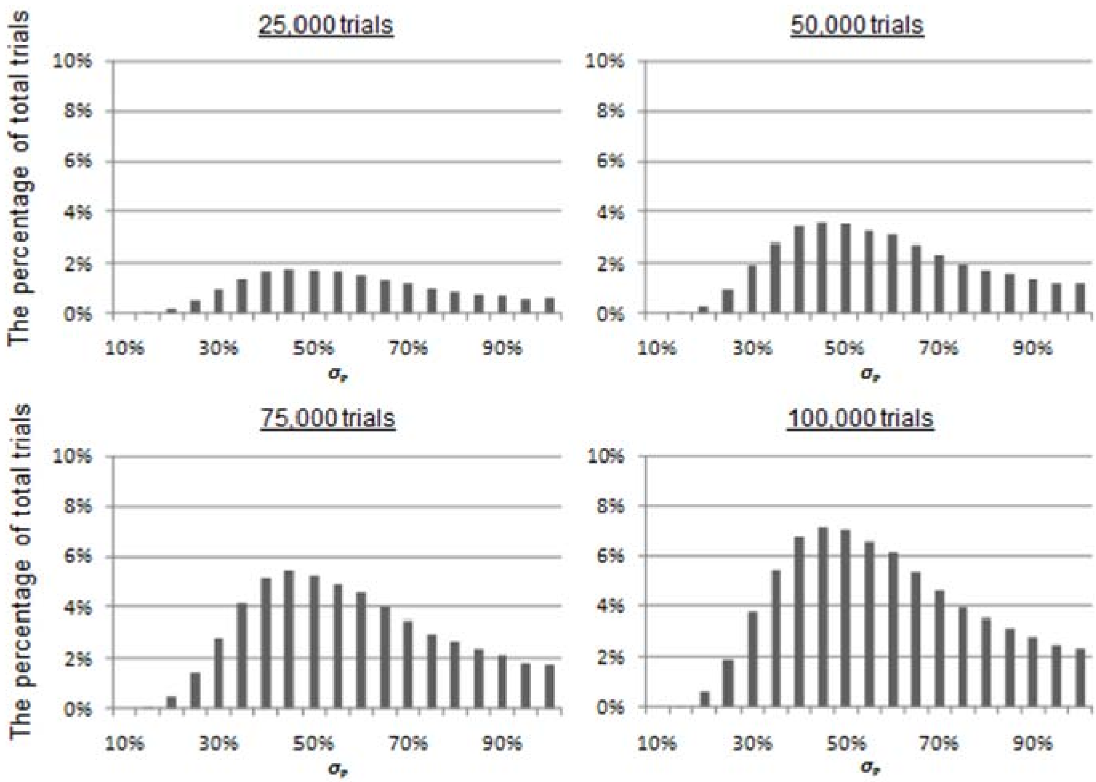

Figure 3. Distribution of volatilities as a percentage of total trials

Number of randomly selected assets has been increased from 3 to 6 and 10,000 trials have been done respectively for each number of randomly selected assets. The weights of each asset were again determined so as to satisfy the $\beta_{X,P}=1$ constraint with minimal σ_P.

Figure 4 shows the change in distribution of σ_P according to the number of assets incorporated in the portfolio. The mode is moving to the lower σ_P direction according to the increase in the number of assets. It is clear that the probability finding a twin asset is increasing according to the increase in the number of incorporated assets.

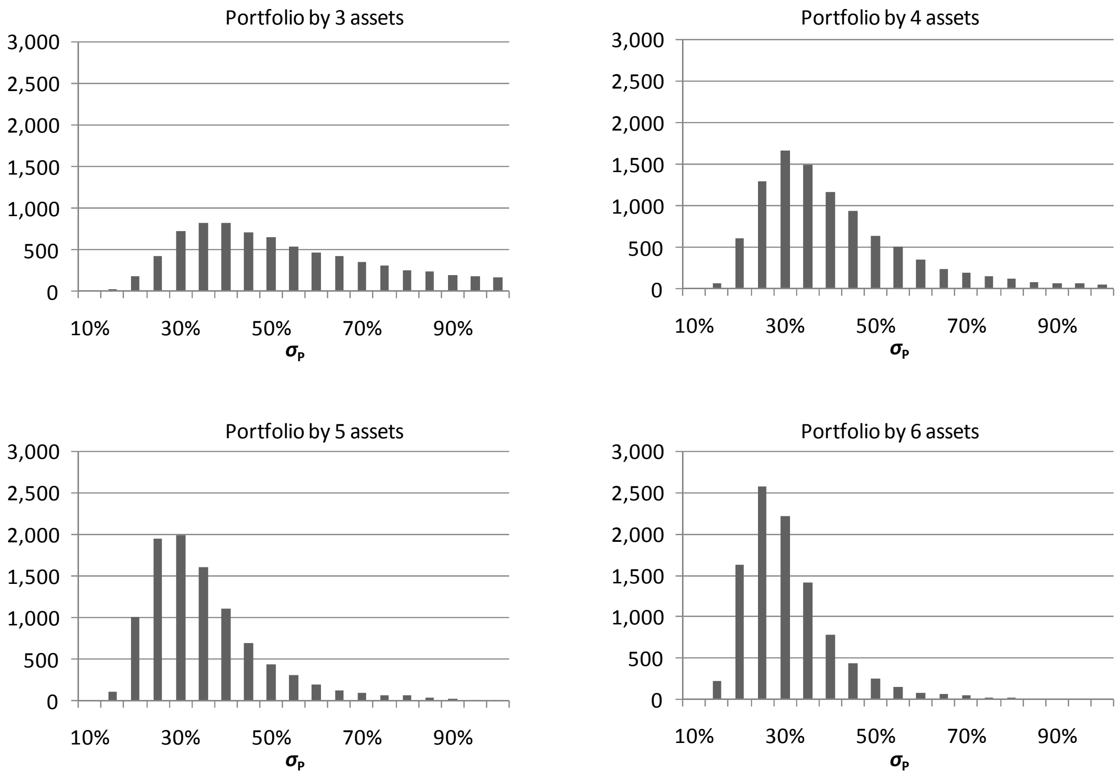

Figure 4. Distribution of volatilities resulting from 10,000 trials.

4.2. Multiple-stage twin asset forming procedure

Yet a perfect twin asset of the case project could not be formed in the former section. Considering the constraints of PC capability it is unrealistic to further increase the number of trials or number of incorporating assets.

A multiple-stage procedure as followings instead of random selection has been proposed as an efficient process to identify the appropriate twin asset.

Step 1: Two assets are randomly sampled from 40,335 assets.
Step 2: Under the $\beta_{X,P} = 1$ constraint, weights w to minimize σ_P is determined.
Step 3: Step 1 and 2 are repeated for 100,000 times
Step 4: The combination of two assets which gives the lowest value of σ_P out of the 100,000 trials will be identified.

Step 5: An asset chosen from the rest of the population will be added to the portfolio.

Step 6: Under the $\beta_{X,P} = 1$ constraint, weights w to minimize σ_P is determined.

Step 7: Repeat Step 5 and Step 6 to chose an asset which gives the minimum σ_P.

Step 8: Step 5 to Step 7 will be repeated to stepwise increasing the number of asset incorporated in the portfolio one by one, until finding an appropriate twin asset.

The distributions of σ_P of the portfolios of 3 to 6 assets, based on the optimal portfolio of 2 to 5 assets respectively, as a result of Step 5 and 6, are shown in Figure 5.

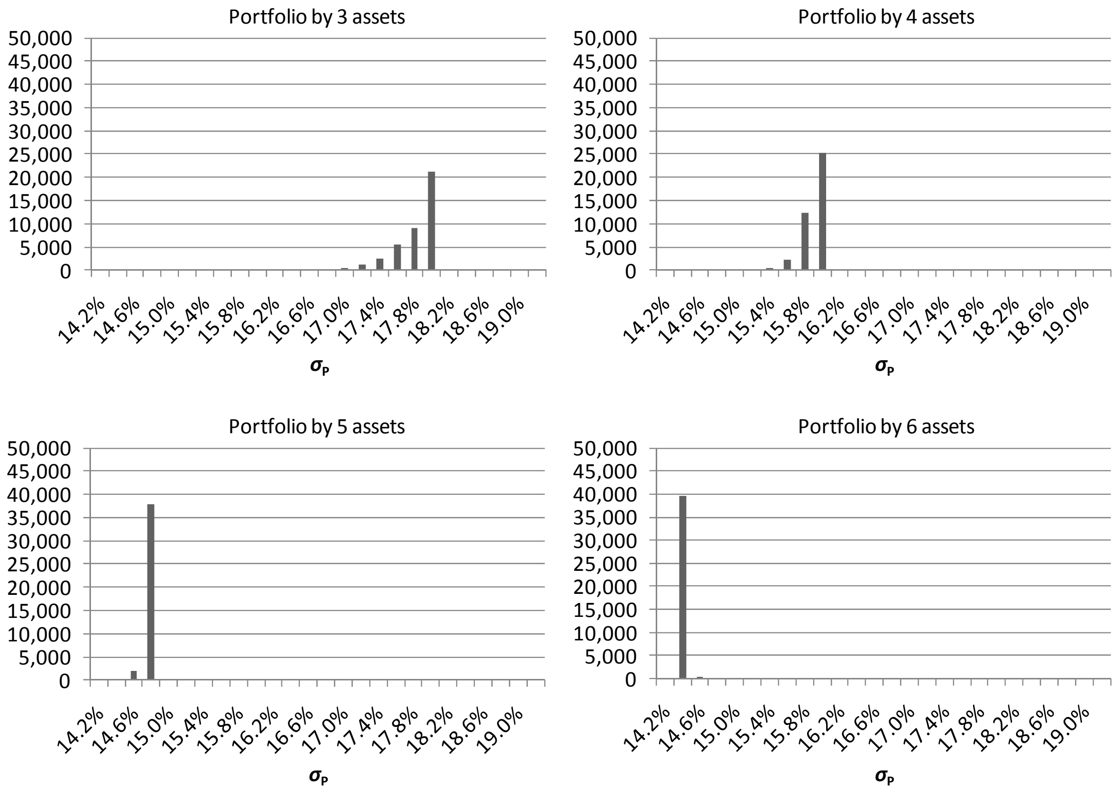

Figure 5. Distribution of volatilities for 100,000 trials.

Table 2. Average and distribution of the volatilities of the portfolios.

Number of assets	Average of σ_P [%]	Distribution of σ_P [%]
3	0.180	3.39×10^{-2}
4	0.160	1.92×10^{-2}
5	0.149	0.68×10^{-2}
6	0.145	1.59×10^{-3}

Table 2 shows the average and distribution of σ_P, corresponding to the number of assets in Figure 5. As the number of incorporated assets increases, the distribution of the volatilities σ_P of the newly created portfolios becomes tighter and the average closer to the volatility of the case project, 14.3%. This can be explained by two reasons. First, as the number of assets increases, the impact of the volatility of the newly added asset on the volatility σ_P of the base portfolio becomes relatively smaller. Second, an asset added to the base portfolio is constrained strongly by the assets already included in the portfolio, so that the distribution of revised σ_P becomes limited according to the increase in the number of incorporated assets.

As shown in Table 3, the integrity of the best candidate for the twin asset increases according to the increase in the number of incorporated assets. The difference in volatility between the best candidate for the twin asset and the project rate of return under $\beta_{X,P} = 1$ constraint is only 0.7% in case of 6 assets.

Table 3. Property of the best candidate of twin asset

Number of assets	Minimum σ_P	σ_X / σ_P [%]
3	16.1	88.9
4	14.9	95.8
5	14.5	98.3
6	14.4	99.3

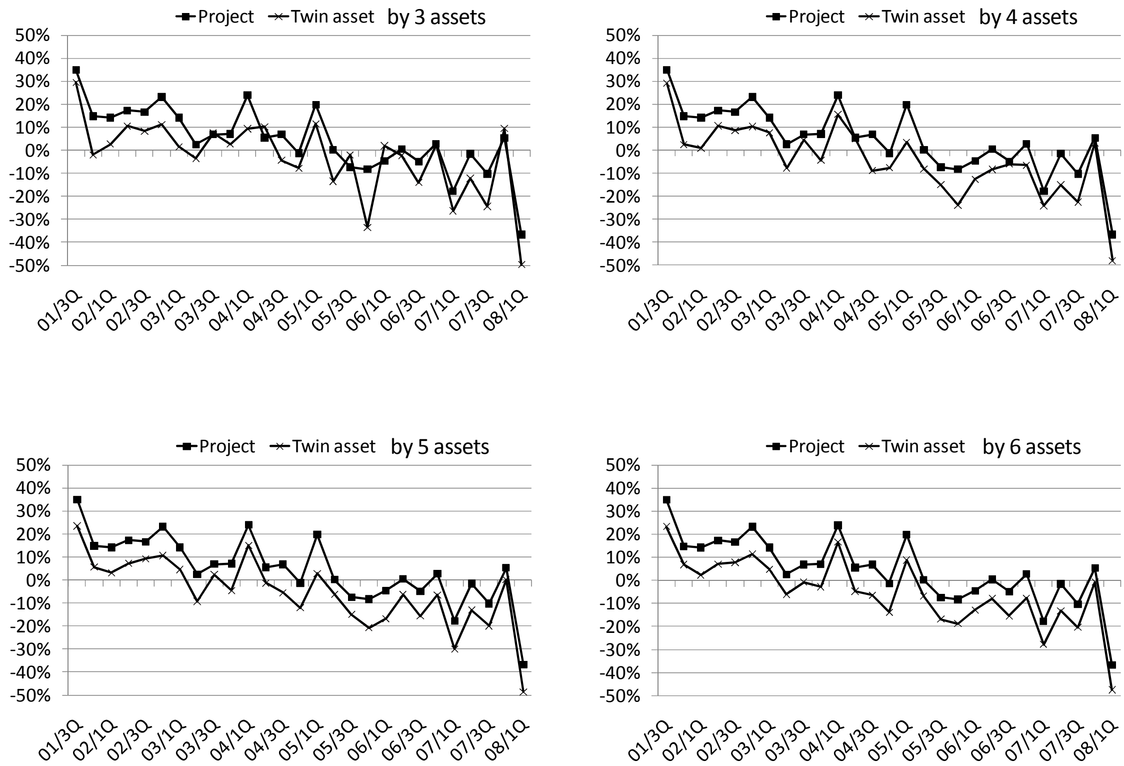

Figure 6. Improvement in the fitting according to the increase in number of incorporated assets

Figure 6 shows the comparison between return of the best candidate for the twin asset and the project rate of return during the covered period for each number of incorporated assets. It is clear that the congeniality is considerably increasing according to the increase in the number of incorporated assets.

4.3. Evaluation of error caused by uncongeniality of the twin asset

Despite a high congeniality level of identified candidate for a twin asset in the former section, it is still an approximation so that accuracy compared to the theoretical option value must be evaluated. Suspension option as shown in Table 4 has been assumed to calculate the theoretical option value by BSM.

Table 4. The conditions of the case project.

Variables in BSM	Values for the case project
Present value [10^3\$]	44,777
Investment [10^3\$]	59,150
Volatility per annum [%]	29
Terms for decision [year]	3
Risk free rate [%]	1.716

The theoretical option value derived by BSM is 5,165. A series of option values have been calculated changing the volatility. $100 \times \left(\sigma_X / \sigma_P\right)$ is defined as a fitting ratio. The result is shown in Table 5. The volatilities of the candidates for twin asset are always larger than the volatility of the return of the project NPV.

Table 5. Relation between the fitting ratio and the error of option values.

Fitting ratio [%]	Volatility [%]	Option value [10^3\$]	Error in option value [%]	Comment
100.0	28.6	5,165	0.0	Theoretical value
99.3	28.8	5,224	1.2	Best candidate
99.0	28.9	5,252	1.7	-
98.0	29.2	5,342	3.4	-
97.0	29.5	5,433	5.2	-
96.0	29.8	5,526	7.0	-
95.0	30.1	5,622	8.9	-

When the fitting ratio decreases by 5%, the error of the option value increases to 8.9%. For example, if the decision maker wants to keep the error for the option value below 5%, a 98.0% fitting ratio will be sufficient for the twin asset.

5. Conclusion

Perfect correlativity and volatility matching have been determined as the necessary conditions for a twin asset whose value will represent the objective value of a project that could be used for a parameter in the BSM solution.

Random selection of assets to form a twin asset could not identify its reasonable candidate in realistic time. An efficient searching procedure for a portfolio, which satisfies the necessary conditions as a twin asset, has been proposed and the accuracy of the option values calculated by the candidates for a twin asset with different congenialities has been evaluated.

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