Pricing Hydrogen Infrastructure Investment with Barrier

This version: 8th June 2011

Ye Li [a], Peter-Jan Engelen [a,b], Clemens Kool [a]

[a] Utrecht University, School of Economics
[b] RODEO Research Centre

Abstract

First passage model specifies a credit default, when the underlying drops below a certain barrier. An investment failure often occurs unexpectedly and involves significant losses to the project value, which makes a great similarity to a default event preventing the investor paying back its debt. In this paper we aim to link the two theories, where an investment failure is determined through the evolution of firm's underlying value. Once the asset value hits the lower barrier, it will result partial or complete failure. This paper will investigate whether a real option with barrier model can be used to count for investment opportunity with choice of failure, where the barrier of which may act as a lower bound for the underlying variation. We will apply on a case of hydrogen infrastructure investment in Netherlands and further determine the barrier through a pessimistic scenario. Sensitivity analysis shows where we set the barrier level have a strong impact on the option value, in addition to the aggregated volatility. Options valuation model are theory, and like all models, are more limited than the real world they attempt to represent. Together with scenario analysis that often used as analyzing alternative possible outcomes, it provides the additional down-side barrier and thus acts as an important tool for facilitating decision making in innovation projects.

Key words: real option with barrier, uncertainty and failure, hydrogen infrastructure investment

1. Introduction

Growing concerns of declining fossil energy resources, environmental pollution, along with climate change, has led to a pressing need for a sustainable energy supply. Unlike fossil fuel, hydrogen is free of carbon; therefore, no carbon-dioxide during combustion or use in a fuel cell (Gasafi et al., 2008). Sandy Thomas (2009) compared the societal benefits of deploying various alternative transportation options (including hybrid electric vehicles and plug-in hybrid fueled by gasoline, diesel fuel, natural gas, and ethanol, and all-electric vehicles powered by either batteries or fuel cells), by which they suggest hydrogen powered fuel cell vehicle is the best option to reduce greenhouse gases. Fuel-cell vehicle is power directly from hydrogen-oxygen reaction; it can achieve high system efficiency in an extremely quiet operation process with zero tailpipe emissions (Smit et al, 2007). Optimistic learning rates suggest a decade or longer time before fuel-cell automotive component costs fall to acceptable levels (Ekdunge and Råberg, 1998). While, as an energy carrier, hydrogen can not be directly extracted like natural gas or oil; it must be produced from a primary source and transmitted to the consumption place. As a result, the upfront construction expenses will be massive and could persist for a decade or more, delaying profitability until sufficient number of vehicles can be produced and moved into consumer markets. A number of studies have previously analyzed and compared the performance of different hydrogen pathways; many of these papers include scenario planning. Scenario analysis often used in energy studies, it starts from the current position and further explores the complexities and dynamics of the possible future states of the world. It can be particular useful, when there is no clear picture of a complete series of infrastructure for a hydrogen-based transport system will look like in detail, in addition to the data uncertainty. Thomas et al (1998) established different market penetration scenarios to estimate the likely number of fuel cell vehicles might be sold in the United Sates over the next decades. Mulder et al (2007) developed a top-down penetration scenario, by which he focus on assessing different technology configuration in terms of chain efficiency and CO_2 emissions. According to their research, no chain (production, storage and transportation) can be selected as an obvious winner according to primary energy demand, emission and cost. Wietschel et al (2006) construct a similar study; he argues that the purpose is not to speculate on a particular pathway, but using various scenarios to broaden the perspective of decision-makers and stakeholders.

Investing on hydrogen infrastructure is also highly uncertain; the significant initial costs will possibly results insufficient project cash inflows to justify on any traditional risk aversion model (e.g. NPV). The promising for the hydrogen fuel cell vehicle must be weighted against the added complexity and cost of developing a hydrogen refueling infrastructure (Ogden, 1999); a radically different model is needed. Option pricing emphasizes potential value, not just net present value. The study by Van Bethem et al (2006) first applied real option theory on Hydrogen infrastructure, by which they argue that an immediate investment is unprofitable, while evaluating it as an option to delay will address extra value by allowing flexibility. Their result indicates the fact that initial additional costs for hydrogen infrastructure and vehicles will turn into savings. Intrinsically, real option transforms uncertainties into flexibilities that confer a large value to the investors under its valuation structure. It suggests to the investors that, instead of calculating what the acquisition would be worth if they started developing them today, they should value the opportunity as an option to develop if they started developing them today (Leslie and Michaels, 1997). By taking the opportunity as an option to develop, the management has the privilege to fulfill but not necessary to do so. He may decide to exercise its right under the favorable conditions, or forgo it in that of an adverse condition, which will create extra value by leaving room for flexible response to the outcome.

However, some innovative investments may have barrier feature that traditional real option fail to capture. For instance, the transition to hydrogen-powered transportation will need to overcome many barriers; it includes creating a market for new and unfamiliar vehicles, and achieving economies of scale in vehicle production while providing an attractive selection of vehicle makes and models for car-buyers. Moreover, technological uncertainty can not be ignored on evaluate such innovative projects. It relates to uncertainty of the technology and how it will develop, of crucial technological challenges and possibilities to solve them, on maturity of the technology, and on competing alternatives. In order to commercialize a FCV (fuel-cell vehicle) this must obtain an equal performance as a regular internal combustion engine (ICE) vehicle today, and in addition, perform better in terms of harmful emissions. And a new product such as hydrogen fuel will only have a chance to be successful if it is not perceived to be a risk and at the same time is able to fulfill the customer's expectations (Schulte et al, 2004). Last but not least, institutional factors, such as, regulation, standards and taxation might affect the utilization of the product into markets; it includes political support for establishing infrastructure, public procurement of vehicles and energy services, and regulation.

It is plausible that a real option to delay investment in a hydrogen infrastructure investment equivalent to a down-and-out type of barrier call option; as a path-dependent process, once the underlying reaching the predetermined level (barrier), which means the minimal requirement of the investment return can not be met. The evolution of such innovative technology will mostly like act as a complex social process involving technological, economic, social and institutional factors. Thus, investment project gets cancelled when expected future return falling below a critical level. The barrier attached will be jointed determined in a mesh of these interactions, which can be adjusted according to the anticipation of the investors. When the new products come together with technological innovations, there is also considerable uncertainty with respect to the actions of a competitor or changes in environment before or soon after technological improvements. Intuitively, technological uncertainty is related to that of the market and not separable. Failure is an inevitable part of the innovation process, that are often potentially good ideas but have been rejected or postponed due to budgetary constraints, lack of fiscal support or poor fit with current goals. In actual investments, investors might relinquish their plans (the options) when the chance of a very low expected present value of the underlying investment appears that might cause by an accidental event. The value below this threshold should not be counted, even though it might shift back before the maturity of the option. Real options with barrier are still options, but have payoffs calculated path dependently with trigger prices. If the trigger price is touched at any time before maturity, it causes an option with pre-determined characteristics to cease to exist. Beyond the typical real option, we can have a specific view about the price path that underlying will evolve over the lifetime of the structure. With the extra constraint, barrier options are always cheaper than a similar option without barrier. And how much cheaper depends on the location of the trigger.

Option theory considers the value of uncertainty, and the main task for option pricing is to determine the present value of the project with uncertainty, namely the price of an option. The valuation based on the hypothesis that the underlying changes over time in a highly volatile way namely follow a stochastic process where only the present value of a variable is relevant for predicting the future. The term uncertainty describes the possibility of a deviation from an expected condition with the concern of different environmental conditions may vary. The barrier imposed works as a restriction to the process, which can cause the stochastic process to stop earlier. Investing in green energy is subject to resource constraints (limited capital, limited ability to borrow) and there might be alternative investments that compete for funding. In addition, governments and regulators may intervene to cancel or take over a project in some circumstances. During the waiting period to when the decision to proceed or not must be made, it is possible that the policy makers will cancel or cut the funding if technological progress stays below their expectation. Barrier options contain provisions which allow them to be effectively cancelled if the price of some underlying asset drops below a threshold barrier level, which may represent some threshold for profitability of an enterprise. However, most real options are created within organizations rather than purchased in the financial market. Hence, there is no direct observable barrier to guide managements' valuations. This will make it difficult to value in practice, which is where we see as a point of convergence between the real option theory and scenario analysis.

Investing on hydrogen fuel-cell technology will face a correlated market and technological uncertainty. This article will transform it into a two-dimensional Brownian motion. A similar model has been proposed by Cortazar et al (2001) to evaluate natural resource exploration investment. By structuring a new state variable, both price and geological-technical uncertainties have been rolling together for an increased volatility. However, beyond their approach, we also consider the factor of an investment failure: market uncertainty reflecting the fluctuation in the value stream of the underlying; if it falls below a certain barrier threshold, i.e., the option is a standard call option with the additional feature that the contract is only relevant once the underlying value pass the level. This paper introduces an alternative analysis and planning methodology for estimating innovative projects. By comparing with a barrier level, an investment will be determined whether it will be terminated. One of the main contributions is that we provide a fresh angel by relating a risky investment opportunity with chance of failure to a down-and-out barrier option. The decision-making depends on whether the underlying as a call option is more valuable than the exercise price and no failure occurs before maturity.

The rest of the paper is organized as follows. Section 2 introduces the investment problem and decision making process. Section 3 sets up the valuation framework. With the model in place, Section 4 discusses the results and conduct sensitivity analysis. Finally, section 5 summarizes and concludes.

2. The decision-making diagram

A hydrogen energy chain starts with hydrogen production, then hydrogen transport and distribution and finally hydrogen conversion and end use. Relatively large-scale hydrogen production plants produce hydrogen at some distance from the end use centers. The hydrogen produced will have to be transported and distributed to refilling station for end users. These infrastructures will most likely be built by the energy companies with substantial governments support. Based on HyWays [1], hydrogen-based vehicle rollout in Netherlands will expect to happen in three phases; a pre-commercial phase from 2010 to approximately 2015 comprised of technology refinement and market preparation. It will be 30 H_2 stations set up to serve around 1000 cars; the early-commercialization phase II (2015-2025) is expected to start with a continuous ramp-up to 100 H_2 stations and further to lead a mass market up to 5000 fuel-cell vehicles; finally, full-commercialization will start from 2025. Based the present study, the fuelling station capacity is assumed to be 500kg/day, each of which is estimated to serve up to 180 cars (Murthy Konda, 2011). By the end of Phase III approximately 20,000 hydrogen vehicles will be on roads around 80% of the population will have local access to hydrogen fuelling stations. It makes

[1] HyWays is a research project conducted by the European Commission with the aim of developing a validated and well-accepted roadmap for the introduction of hydrogen in the energy system in Europe

approximately 350 hydrogen refilling stations and 7% of the total number of fueling station national wide.

As indicated in Figure, the first step of the evaluation is to determine the revenues and relevant costs. Investors will receive revenues from hydrogen fuel retail, bearing the cost of hydrogen production and transportation. To support such business, they need to build refueling stations and distribution systems. And the cost to this transition is too high for industry to bear on its own, and given the public benefits, it is entirely appropriate and essential for government to support this transition (Shayegan et al, 2006). What follows, we can examine the investment problem by the real option model, by which the undertaken of an investment opportunity is equivalent to exercising one option. Investors will then make their choice of whether to exercise their right or wait and see based on estimation of the level of uncertainty and chance of investment failure. The value this staging investments are not primarily determined by the cash flows coming from the initial investment but also by the future investment opportunities provided by the original investment. Each stage can be viewed as an option on the value of subsequent stages and valued as a compound option. After the completion of each phase, investors are automatically enter the following phase. It is important to note that Phase III (full-commercialization) cannot proceed without the completion and execution of Phase II, which itself will only take place upon the successful transition from Phase I. In addition, we impose a down-side barrier though scenario planning, which will explores the complexities and dynamics of the business landscape from the current position and speculate a pessimistic possible future state. If we consider each phase valuation as real option with barrier, the whole pricing process involves multiple barriers which must be hit in a pre-specified sequence. That is, the second barrier is only activated after the first barrier is hit, while the third barrier only activated after the second barrier is hit, and so on. As a result, an investor decides that he will not invest even the underlying value overcome the initial infrastructure investment unless its variation over the option continuously stays over a certain pessimistic scenario they consider. With the first and second placed respectively above the underlying, the investor makes sure they receive full benefit from the payoff if the price does indeed never go down below their bottom line.

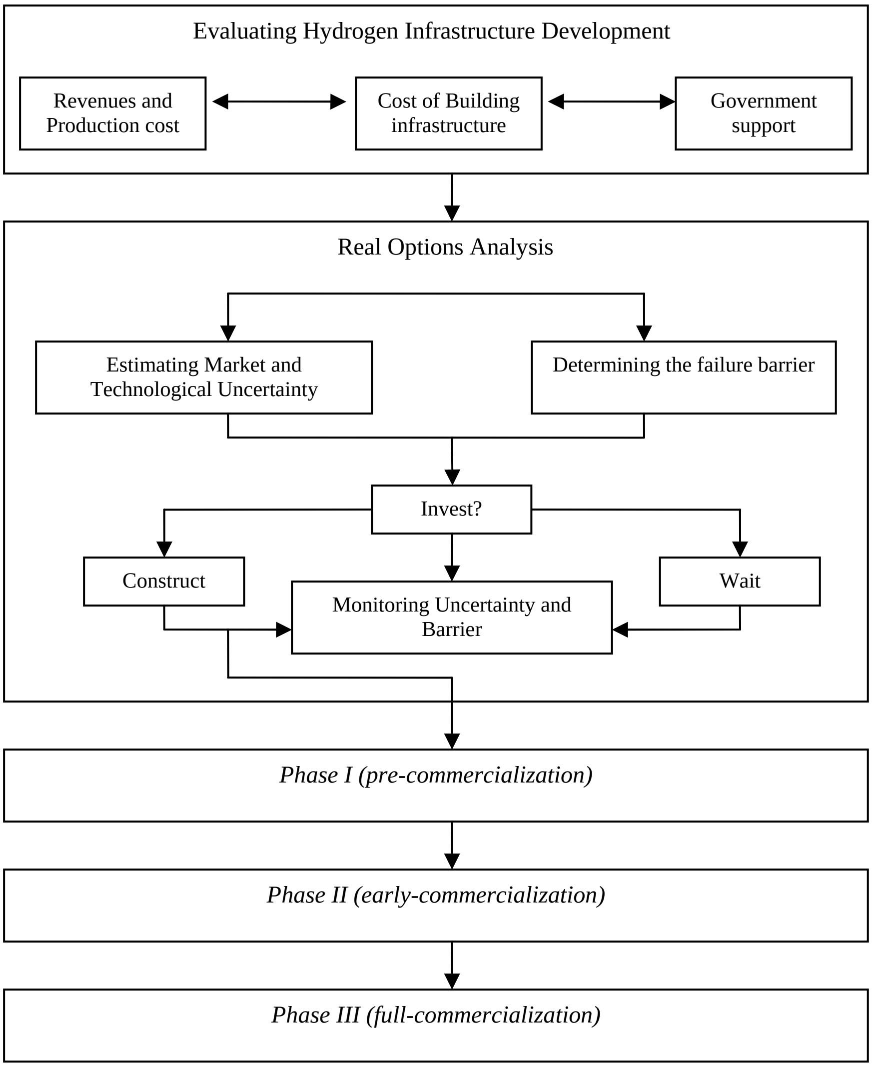

Figure 1: Summary of the deployment phases in the European Hydrogen Roadmap

3. The Model

3.1. Assumptions

The costs of hydrogen infrastructure vary with different types of hydrogen production technologies, forms of storage and methods of transportation and dispensing; we will not address any technical aspect in detail and the calculation below will only act an approximation. Assuming that infrastructure cost includes building a large coverage of refueling stations, which will decrease with the cumulative of experiences. Hydrogen will be transported by using tanker trucks and the cost of which is considered in the production cost. The retail price for hydrogen is assumed constant and the demand will estimated through expected fuel-cell vehicles on the road as the transitional plan.

3.2. Theoretical framework

Assuming a risky investment project will generate a stream of stochastic cash flows, denoted by V_t as its market value at time t; the market is complete with no transactions costs hold. The dynamics of dV_t is driven by a Brownian motion defined on a probability space (Ω, F, P):

$$dV_t = \mu V_t dt + \sigma_M V_t dW_M + \sigma_T V_t dW_T + dq_t, \tag{1}$$

Where

μ	The expected rate of return on the project
σ_M	Market uncertainty
σ_T	Technological uncertainty
dW	Stochastic variable dW follows a Wiener Process in which $dW_t \sim N(0, \sqrt{dt})$
$dW_M dW_T = \gamma_{MT} dt$ [2]	Instantaneous covariance matrix between
dq	Political incentive $q_t = \exp[\alpha - \lambda t(e^{\alpha} - 1)]$ with degree of incentives $\phi = e^{\alpha} - 1$

Therefore, dV_t will follow a two-dimensional Brownian motion $\tilde{\sigma} V_t dW_{MT}$:

$$dV_t = \mu V_t dt + \tilde{\sigma} V_t dW_{MT} + dq_t \tag{2}$$

Some of the reasons for our choice are:

1. The term $\tilde{\sigma} dW$ precludes the possibility of negative values and imposed stochastic variation to capture highly uncertain phenomenon.
2. The complexity of the model can be reduced by rolling both market and technological uncertainty into one factor, that is $\tilde{\sigma}$. Their joint effect on project value can be representing by a modified volatility.
3. To describe this correlation more precisely, we define the correlation coefficient ε as relation between the two Wiener processes. We define $\tilde{\sigma} = \sqrt{\sigma_M{}^2 + 2\varepsilon\sigma_M\sigma_T + \sigma_T{}^2}$ as the total effective volatility of the option.

Correlation is used as a measure of the extent to which the underlying stochastic process for multi-dimensional Brownian motion moves together. It can take values between -100% (perfectly anti-correlated), through 0 (uncorrelated), up to 100% (perfectly correlated). The total uncertainty $\tilde{\sigma}$ is higher when market uncertainty σ_M and technological uncertainty σ_T are positive correlated (positive shocks in market demand are reinforced by technological breakthrough and vice versa), and is lower when they are negatively correlated (increasing difficulty in technology progress need be compensate by a stronger market push). We assume that the fiscal support for

[2] γ_{MT} is the identity matrix.

hydrogen fuel-cell technology q_t will be given, which will add extra value to the project and lead a proportional ϕV increase. In our setting, consumers' acceptance, technological progress and political incentives are the three critical issues that will determine the success of this hydrogen fuel-cell transition.

The following step is to determine an initial estimation of the project value V, that is V_0. Assuming that a significant proportion of consumers will regard fuel-cell vehicles as a small but real improvement compared to gasoline cars and will be willing to pay a slightly higher price. Costs remain constant over any period considered are not affect by technological development or feedstock price change. The initial estimated present value of revenues R_t and relevant costs P_t are represented by:

$$R = \sum_{l=1}^{L} F \cdot \partial \cdot X_l \cdot H \cdot e^{-r(T_l - T_0)} \ ; \qquad P = \sum_{l=1}^{L} (F \cdot \partial \cdot X_l \cdot CU + I \cdot M + CL_l) e^{-r(T_l - T_0)}$$

Where

F = Number of hydrogen vehicles
X_l = Yearly average of distance travel (km)
∂ = Fuel efficiency (kg/ km)
H = Hydrogen retail price (€/kg)
CU = Production costs (€/kg)
r = Risk free interest rate
L = Estimated useful life of the infrastructure
I = Investment for the plant
M = Operation and maintenance coefficient
CL_l = Average labor cost per year

The initial estimation of the project value V_0 is computed from the following:

$$V_0 \approx R - P = \sum_{l=1}^{L} F[\partial \cdot X_l (H - CU) + I \cdot M + CL] \cdot e^{-r(T_l - T_0)}$$

The underlying project value dynamic becomes

$$V_t = V_0 (1 + \phi) \exp[(r - \lambda\phi - \frac{1}{2}\sigma^2)t + \widetilde{\sigma} dW_{MT}]$$

3.2. Option valuation

According the risk neutral valuation principle, the value of a down-and-out call option at time zero with maturity time T, strike price I and barrier B is

$$C(t,V) = e^{-r(T-t)} E^Q[(V_T - I)^+ 1_{\{\max_{0 \le t \le T} V_t \ge B\}}] \qquad (3)$$

Here, E^Q denotes the expectation under the risk neutral measure Q. Then $C(t,V)$ is the price at time t of the option with price $C=(V_T-I)^+1_{\{\max_{0\le t\le T}V_t\ge B\}}$, at maturity time T.

Define $\beta=\frac{B}{V}$ and assume that $C\in[(0,T)\times(B,\infty)]$ (i.e., C is continuously differentiable in the first variable and twice continuously differentiable in the second variable) satisfies partial differential equation

$$\frac{\partial C(t,V)}{\partial t}=rC(t,V)-rV\frac{\partial C(t,V)}{\partial V}-\frac{1}{2}\tilde{\sigma}^2V^2\frac{\partial^2C(t,V)}{\partial V^2} \tag{4}$$

Extra boundary conditions will be determined by the nature of the barrier. Failure occurs at the first time t , where $t\in[0,T]$ at which the firm's value V_t falls below the level B, or the default even does not occur at all. The partial differential equation formulation implies that knock out occurs when the barrier is breached at any time during the life of the option. As soon as the value of firm's assts crosses this lower threshold, the investment project fails. With the time range of $\{(t,V):0\le t<\mathrm{T},\ \mathrm{B}\le V,\}$ and the boundary conditions

$$C(t,B)=0, \qquad 0\le t<T, \tag{5}$$

$$C(T,V)=(V-I)^+, \qquad B\le V,\beta\le 1 \tag{6}$$

To assume no-arbitrage opportunity, $B\le I$, this condition must hold to ensure that the payoff to the investor at the default time τ never exceeds the up-front investment expenditures. As V becomes large the likelihood of the barrier being activated becomes negligible. Here, the first boundary condition is applied at $V=B$ rather than at $V=0$. When the failure occur, there are two possibilities; either the option holder will receive zero payoff or he might has chance to get some recovery by the scrap value R from the infrastructure. We first consider the scenario when V ever reaches B, the option will expire worthless; this financial condition translates into the mathematical condition that of the option payoff is zero, $R=0$.

Apply the solution of the heat equation with the initial condition $A(x,0)=F(e^x)$

$$A(x,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(e^{x+\sigma\sqrt{\tau}\xi})e^{-\frac{1}{2}\xi^2}d\xi \tag{7}$$

The option pricing model can be obtained via solving and further transforming the heat equation:

$$C(V,T)=e^{-\lambda\phi T}V(1+\phi)[N(a)-(\frac{\beta}{1+\phi})^{\frac{2r}{\tilde{\sigma}^2}+1}N(c)]-Ie^{-\lambda T}[N(b)-(\frac{\beta}{1+\phi})^{\frac{2r}{\tilde{\sigma}^2}-1}N(d)] \tag{8}$$

Where

$$a=\frac{\ln[\frac{V(1+\phi)}{I}]+(r+\frac{1}{2}\tilde{\sigma}^2-\lambda\phi)T}{\tilde{\sigma}\sqrt{T}}, \quad b=a-\tilde{\sigma}\sqrt{T},$$

$$c=a+\frac{2}{\tilde{\sigma}\sqrt{T}}\ln[\frac{\beta}{1+\phi}], \quad d=b+\frac{2}{\tilde{\sigma}\sqrt{T}}\ln[\frac{\beta}{1+\phi}],$$

Proof. In Appendix.

When the underlying cross the barrier level, it might accompanied by a payoff to the option holder. In actual investments, it is equivalent to a certain scrap value comes from the initial put in. In other words, in the case of $R \neq 0$, pricing formulas can be derived by applying static hedging. Boundary condition equation (5) becomes

$$C(t,B)=R, \qquad 0 \le t < T,$$

The price can be found through a transformed barrier B^{a+} with zero scrap value $R=0$ written on a converted process V^{a+}:

$$R\cdot E^Q[e^{-r\tau}1_{\{\tau<T\}}]=\frac{R}{B^{a+}}\{V_0^{a+}-e^{-rT}E^Q[V_T^{\ a+}1_{\{\inf u\le T \cup V_u^{a+}>B^{a+}\}}]\}$$

Therefore, a scrap value-R barrier-B call option will be identical to a long position in the zero-scrap call, R/B^{a+} units of $a+$ security, and R/B^{a+} units short in the strik-0, rebate-0 barrier-B^{a+} call on the $a+$ security:

$$\sigma \to a_+\sigma, \; V \to V^{a+}, \text{ and } B \to B^{a+}.$$

Detail. Refer to Carr & Picron (1999)

4. Analysis and discussion

4.1. Implications of the results

The strategic value of phase I can be calculated as compound option C_3 with three orders, in which its time to maturity and exercise price given by T_1 and I_1; with value of phase II as underlying asset a compound call of order 2, which underlies on a European call (phase III) with exercise date and price given by T_3 and I_3. The barrier provision requires the breaching of the three barrier levels at a pre-determined sequential order. Given the asset price V, asset price V_t at time t conditional on non-breaching of the sequential barrier provision (first barrier B_1 then B_2 and finally B_3) is given by

$$C_1(V,T_1)=\begin{cases} C_2(V,T_1)-I_1 & \text{if} \quad V \geq V_1^* \wedge V \geq B_1 \\ 0 & \quad\quad V < V_1^* \vee V < \mathrm{B}_1 \end{cases}$$

The idea is to start at the end and then work backwards, using the solution for each stage in the boundary conditions for the previous stage:

$$C_3 = Ve^{-\lambda\phi T_1}V(1+\phi)[N(a_1,a_2,a_3;\Xi_1{}^3)-(\frac{B_3}{V(1+\phi)})^{\frac{2r}{\tilde{\sigma}^2}+1}N(c_1,c_2,c_3;\Xi_1{}^3)]-I_3e^{-rT_1}[N(b_1,b_2,b_3;\Xi_1{}^2)+$$

$$+(\frac{B_3}{V(1+\phi)})^{\frac{2r}{\tilde{\sigma}^2}+1}N(d_1,d_2,d_3;\Xi_1{}^3)]-I_2e^{-rT_2}[N(b_1,b_2;\Xi_1{}^2)-(\frac{B_2}{V(1+\phi)})^{\frac{2r}{\tilde{\sigma}^2}+1}N(d_1,d_2;\Xi_1{}^2)-$$

$$-I_1e^{-rT_3}[N(b_1)-(\frac{B_1}{V(1+\phi)})^{\frac{2r}{\tilde{\sigma}^2}+1}N(d_1)]$$

$$a_k=\frac{\ln[\frac{V(1+\phi)}{V_k^*}]+(r+\frac{1}{2}\tilde{\sigma}_k{}^2-\lambda\phi)T_k}{\widetilde{\sigma_k}\sqrt{T_k}},\quad b_k=a_k-\widetilde{\sigma_k}\sqrt{T_k},$$

$$c_k=a_k+\frac{2}{\widetilde{\sigma_k}\sqrt{T_k}}\ln[\frac{B_k}{V(1+\phi)}],\quad d_k=b_k+\frac{2}{\widetilde{\sigma_k}\sqrt{T_k}}\ln[\frac{B_k}{V(1+\phi)}],\quad k=1,2,3$$

Where $N(a_1,a_2,a_3;\Xi_1{}^3)$ is the standard trivariate normal distribution function with correlation coefficient $\Xi_1{}^3=\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{12}\rho_{13}+\sqrt{(1-\rho^2{}_{12})(1-\rho^2{}_{13})} \\ \rho_{31} & \rho^2{}_{12}+\sqrt{(1-\rho^2{}_{12})^2} & 1 \end{pmatrix}$.

To calculate hydrogen demand for fuel cell passenger vehicles, one can assume that each vehicle will use approximately 0.7 kg of hydrogen each day[3]. For an average fuel cell vehicle with a fuel economy of 50 to 60 miles per kg, this would accommodate about 35 to 40 miles of driving on an average day. By the time of commercialization, approximately 20,000 hydrogen vehicles will be on roads, which request hydrogen fuel demand $V_3=166.74$ Million Euros. Other relevant parameter refers to Table 1. We assume that hydrogen retail price is constant. The production cost given includes all the relevant expenses, for instance, CCS[4] and transportation to the refilling station.

Exercise price is equivalent to the total capital expenses on build manufacturing faculties for fuel-cell vehicles. The cost to build and operate a hydrogen refilling station depends upon many factors, including the type of station, location, equipment manufacturing volume and continuing technology advancements. For calculation, we

[3] one kilogram of hydrogen is approximately equal to one gallon of gasoline on an energy basis.
[4] Carbon Capture and Storage

take CGH2 model with input data summarized in Table 1. A rough calculation of the expected infrastructure costs are based on the required number of infrastructure units and their investment cost shown in Table 2. As more stations are deployed, costs will likely decrease as a result of economies of scale and learning by the following formula:

$$\varpi = \alpha \cdot N^{-b}$$

where

ϖ = Investment of the N^{th} unit
α = Investment of the 1^{st} unit
N = Number of units
b = Learning parameter

Early station costs can vary greatly depending upon the specific technology used, site conditions and experience, the 1st unit capital cost α is approximately €0.49 million[5]

$$I = a \cdot \int_1^N N^{-b} dN$$

Table 1.

Hydrogen retail price H :	€10/kg
Production costs CU [6]:	€7/kg at phase I and II; €5/kg at phase III
Risk free interest rate r	0.04
Political incentive: ϕ	50%
Chance of political support λ	0.35

Table 2
CGH2/Filling Station/in 2.0 Mpa; out 88.0 MPa (120 t/yr)

Concept	value
Investment 1st Unit (EUR/unit)	496000
Average investment 50 units	305000
Average investment 100 units	231000
Average investment 500 units	211000
O&M coefficient	2.7% of investment/yr
Useful lifetime	20yr
Average labor cost per year	3200000 Eur/yr
Annual full load hours	8760h/yr

Our investment rule will take the form of a critical value V_k^* such that it is optimal to invest once $V_k \geq V_k^*$ at each phase $k = 1, 2, 3$. Uncertainty abstracted as volatility σ is one of the key factors in real option, where a higher value of σ_k will result in a higher V_k^* , that is, a greater value to delay the actual investment. Hydrogen

[5] Vision for Rollout of Fuel Cell Vehicle and hydrogen Fuel Stations California Fuel Cell Partnership
[6] Production cost for the reference scenario.

infrastructure investment is a typical multi-stage project with high-risk, in particular the estimation of its volatility can be troublesome especially when there is hardly historical source of uncertainty data. Given the investors have different goal and task at different stages, the risk characteristic consequently differs. Technological uncertainty relates to uncertainty of the fuel-cell technology itself and how it will develop, of crucial technological barriers and possibilities to solve them, on maturity of the technology, and on competing alternatives. The level and the quality of technological knowledge inside the corporation have been related to its ability to achieve product and process innovations and then to is future economic performance (McGrath et al., 1996). In early phases, technological uncertainty is greater and more difficult to control and much more correlated with levels of market acceptance. As visualized by the tendency chart of Figure 2, innovation diffusion often believed can be broken down into five different segments, based on their propensity to adopt a specific innovation. The adoption process begins with a tiny number of visionary, imaginative innovators. The future of the technology will be then decided in the market. If market fails, it can disappear for a long time or forever. Otherwise, once the benefits start to become apparent, early adopters leap in. It might still remain isolated or become economically significant. Early majorities, who are influenced by mainstream fashion and wary of fads, will step in during the commercialization phase. It comes with frequent increases in technical efficiency; productivity and precision in processes, the regular changes in products to achieve better quality reduce costs or widen their range of uses. Taking only the first three segments, they are about to achieve market shares 5%, 27% and 68% respectively.

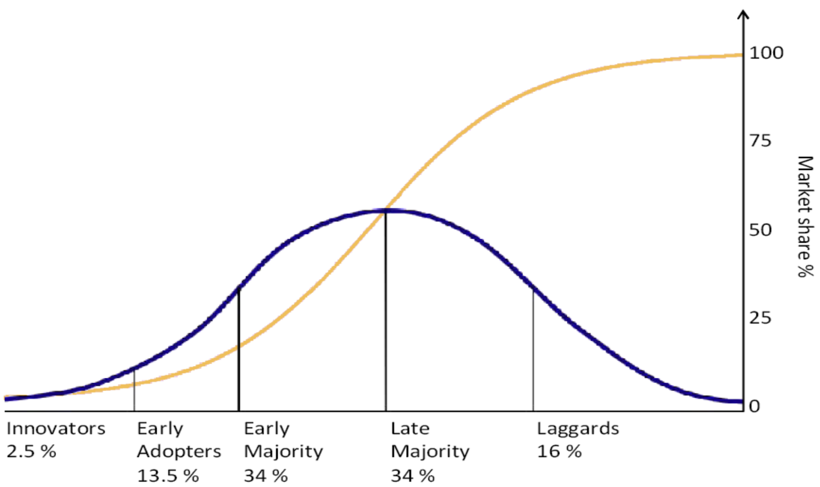

Figure 2: innovation diffusion Source: Rogers (2003)

Since hydrogen passenger vehicles have only recently been introduced to the public to be driven and refueled, there is a relatively short research history focused on the observed consumer response to hydrogen as a transportation fuel. Few studies to date have explored the direct interaction of consumers with a fleet of hydrogen personal vehicles over an extended time period. O'Garra et al (2005) explored determinants of awareness and acceptability of hydrogen vehicles through a 400-person socioeconomic survey in London. This study found that awareness was a function of gender, age, and environmental knowledge, whereas acceptability was primarily determined by previous knowledge of hydrogen technologies. Schulte et al (2004), the degree to which the early adopters felt safer than later adopters was statistically significant at the 10% level during the first phase. Market and technological uncertainty estimation is approximated based on the results from questionnaire and fixed constant per phase; $\sigma_M = 0.4, 0.3, 0.2$ and $\sigma_T = 0.4, 0.2, 0.1$. We select an estimate of 0.61 to be a realistic volatility of the project return at the first stage of the project. With the successful transition, we believe a volatility estimation of 0.37 for the second phase accurately reflects the lower uncertainty. These values are shown in Table 3. As technology diffuses and market shares increases, the volatility will decrease to 0.23 during the final commercialization phase.

Table 3
Phased uncertainty data

Phase	V	B	I	$\tilde{\sigma}$	ε
I	-	5.2	16.57	0.61	30%
II	-	21.05	32.83	0.37	20%
III	166.75	90	101.66	0.23	10%

Given the trends and uncertainties identified, participants should strive to identify base case and the worst scenario. Reference base case scenarios were established in HyWay. Optimistic and pessimistic scenarios with a certain probability have been building up by the degree of policy support and level of technical learning. Once consensus is reached, these extremes serve as bounds within which a variety of possible. We determine the knock-out barrier through one of the future scenarios that hydrogen fuel-cell will has very low technical learning, hence no extra policy support given. Under this pessimistic scenario, hydrogen production cost are substituted by €8/kg at phase I and II; €6/kg at phase III. Starting from phase III, we set government incentive $q = 0$, together with $B = 80\%V$, which would be the case that investors will automatically knock-out the chance of entering such investment. According to the model proposed, $B = \beta \cdot V = 90$, it simplifies the analysis by precluding very low level of expected future return, which would require considering the decision to abandon infrastructure investment plan. The first two phases need to overcome barrier B_1 and B_2 , which assumed to be the minimal government subsidies investors request. Table 3 present the results of the hydrogen infrastructure development projects. It is optimal to start investing in phase I of the project when the value of the underlying asset is large than €91.49 million. In the event that phase I is successful, the critical asset value in phase II decrease to€77.23 million. Finally, in phase III, when the most substantial part of the investment is about to begin, it decreases again to €62.28 million. With these values, the decision to install the infrastructure was taken; the expected option value would amount to €39.97 M. The results give considerable support for the proposition that failure is best indexed by include a barrier threshold.

4.2. Sensitivity Analysis

This section will analyze the performance of some sensitivity results; the way that real option with barrier will behave as the underlying varies by classifying the strike and barrier levels with regard to volatility, and then combining their effects. Specifically, we focus on testing the effect of the barrier acting as a failure threshold. In figure 3 and 4, option delta is the number of shares that underlying has the same instantaneous exposure as the option. We begin by analyzing how the delta ratio[7] on the underlying and option value converges for models with and without barrier and rebate respectively. Below the barrier, the down-and-out call is worthless and has zero delta. Above the barrier, delta is always positive. As the underlying price moves up from the barrier, the call value inflates rapidly, with a delta just above the barrier that can be larger than that of the corresponding standard call. (Based on V=118.466 M; I=100 M)

Notice the sharp curve in the gradient of the underlying value at $V = B$. This is when the underlying hit the barrier. A higher barrier raises the probability that the underlying will fall to zero. As the failure approaching, the relationship is reversed and the delta of the barrier option will eventually exceed the non-barrier one. In fact, the option with barrier reformed significantly leveraged near the trigger barrier ($B = 90$), much more dominated than an option without barrier. While for the case of real option value with barrier and rebate, there is only a slight variation when underlying come cross the barrier. This is due the fact that payoff is no longer zero even the underlying lays below 90. Hence the higher sensitivity of option prices with chance of failure near the threshold will most likely carrier it over to the value. When underlying evolves far above barrier level, the delta value for real option with or without barrier does not make much difference. For barrier options with non-zero rebates,

For our second illustration, we will demonstrate the effect of implied volatility as a function of the option price by several levels of the barrier B. As plotted in Figure 5, it shows how the volatility is influence by a barrier, in which option variation becomes more pronounced by a lower barrier. For underlying near the barrier, an increase in volatility would actually make knock-out for this particular option more likely, and so decrease overall option value. Higher implied volatilities suggest a greater probability of triggering the barrier and knocking out the option. It will still be cheaper than the plain vanilla option but not by very much. In that case, volatility increases with decreases in the asset value due to a higher leverage. Furthermore this leverage effect is amplified by a higher default barrier. But the point is the barrier's influence dominates the others. To measure the risk of an underlying, the volatility of the price movement is needed in order to determine the volatility of the rate of return.

Figure 6 indicates that uncertainty correlation influences the price of real option without barrier much more significantly than that of with barrier. Given market uncertainty and technological uncertainty $\sigma_M = 0.3$ and $\sigma_T = 0.4$. When there is zero correlation ($\beta = 0$), the total volatility is $\tilde{\sigma} = 0.5$. Take correlation $\beta \in [-1, +1]$, which makes a total volatility $\tilde{\sigma} \in [0.36, 0.61]$. Option value with barrier stays stable with the change of uncertainty correlation and total volatility. In contrast, option value without barrier is much more volatile with the variation of total volatility.

[7] The delta of an option is the rate of change in its value with respect to changes in the price of the underlying.

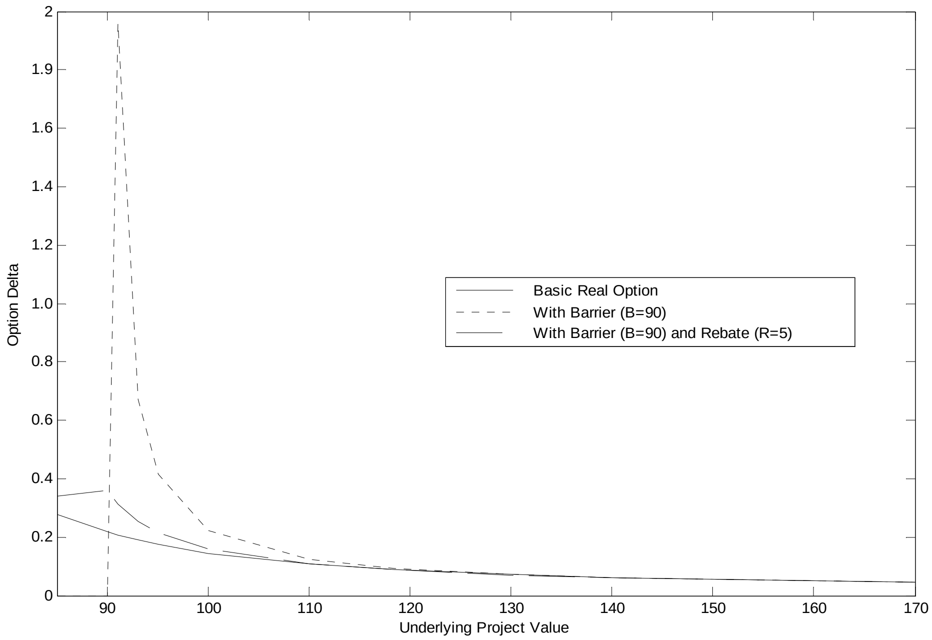

Figure 3: Underlying value to change of delta.

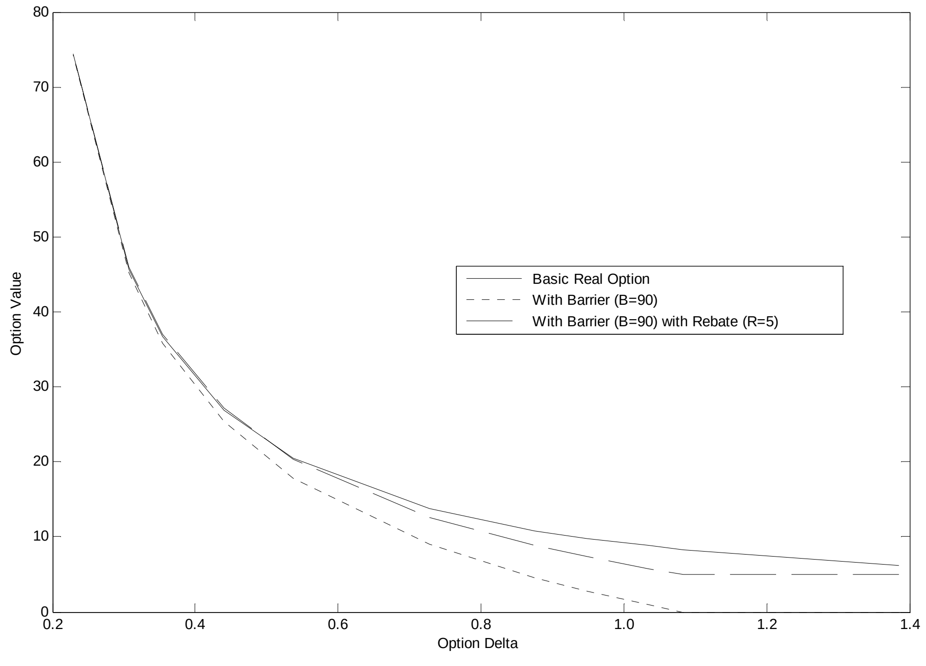

Figure 4: Option value to changes in delta. Calculated with $T = 5$

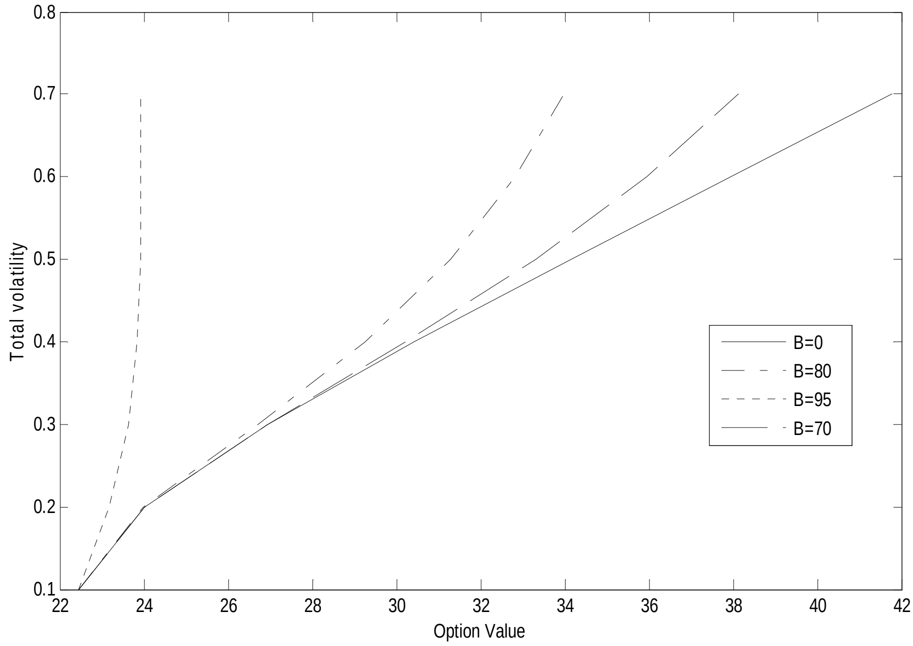

Figure 5: Effect barrier level on option value and total volatility

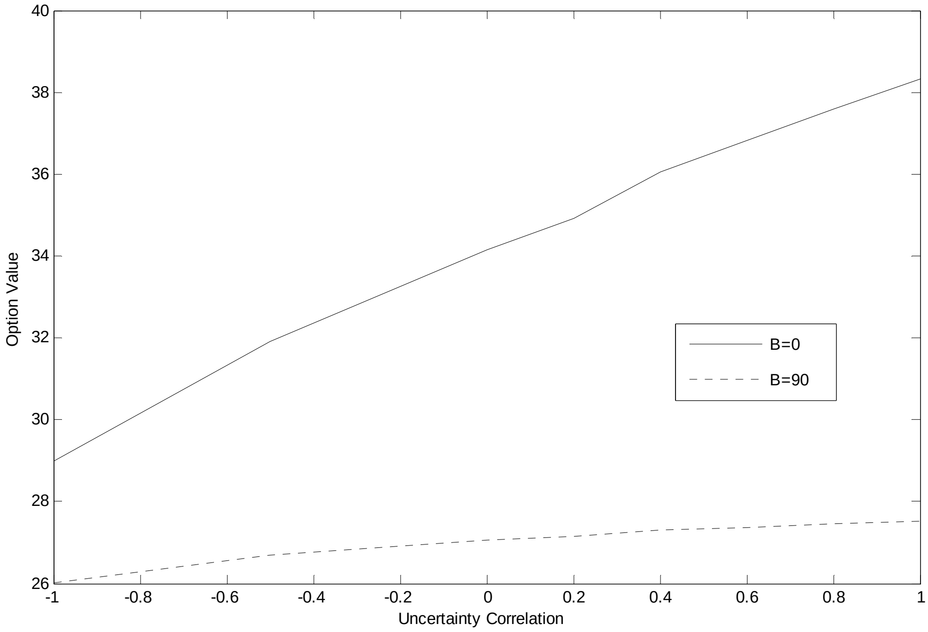

Figure 6: Correlation (market and technological uncertainty)

5. Summary and conclusion

The premise of coming hydrogen transition relies on strategic planning and necessary investments; Energy, economic and environmental analyses must be undertaken in concert with research on improved production, storage, and distribution technologies; to assist the transition, an adequate valuation approach is vital. When commercializing a new technology requires the resolution of both technological and market uncertainty, one might not anticipate the best path forward from the very beginning. Even the amount of planning and research can not help much on revealing the full facts; instead, models that count multiple uncertainties should be highly recommended. A desirable way of handling uncertainties is better off with an initial estimation on the most possible market feedbacks and speed of technological progress. Then further appraise the uncertainty and adapt investment plans in response to the market as they go along, and make adjustments as more information becomes available.

Scenario planning is a process for structured thinking in which stories are created that bring together factual data and human insight to create scenario 'plots' exploring possible futures; by imposing an endogenous barrier through which, this paper proposed an alternative approach to view the chance of an investment failure. Scenario analyses as a process of analyzing possible future events generate a combination of an optimistic, a pessimistic, and a most likely scenario. In our case study, this pessimistic scenario is used to estimate the down-and-out barrier, under which hydrogen infrastructure investments will be delayed. As a path-dependent process, once the underlying reaching the predetermined level (barrier), which means the minimal requirement of the investment return can not be met. We demonstrate a real example from HyWays; numerical results show that a significant fraction of total project value is due to the flexible options available to investors. Taking into consideration of failure chance will reduce the option value, the degree of largely depends on the barrier setting. Without setting the barrier, a direct application of real option model on innovative technology projects are much likely to over-price the problem. In addition, compared to the standard real options, there are some different features in using Real option with barrier model Most of these phenomena can be explained by analyzing the influences from sensitivity analysis.

References

HyWays: Hydrogen Energy in Europe. www. hyways.de

Adamson, K., & Pearson, P. (2000), "Hydrogen and methanol: A comparison of safety, economics, efficiencies and emissions," *Journal of Power Sources, 86*(1-2), 548-555. doi:DOI: 10.1016/S0378-7753(99)00404-8

Baumet, K.A., T. Herzog, and J. Pershing. (2005), " Navigating the numbers: Greenhouse Gas Data and International Climate Policy. *World Resources Institute.*

Carr, P. & Picron, J. (1999), " Static Hedging of Timing Risk," *Journal of Derivatives 6,3,57-70.*

Cortazar, G., Schwartz, E. S., & casassus, J. (2001), "Optimal exploration investments under price and geological-technical uncertainty: A real option model," *R&D Management, 31, 181-190*

Ekdunge, P., & Råberg, M. (1998), "The fuel cell vehicle analysis of energy use, emissions and cost," *International Journal of Hydrogen Energy, 23*(5), 381-385.

Gasafi E., Reinecke My, Kruse A, Schebek L. (2008), "Economic analysis of sewage sludge gasification in supercritical water for hydrogen production. *Biomass Bioenerg. 32(12): 1085- 1096.*

Leslie, K.J., Michaels, M.P. (1997), *The Real Power of Real Options,* The McKinsey Quarterly. n.3, p.4-22.

McDonald, R. and D. Siegel (1986), "The Value of Waiting to Invest", *The Quarterly Journal of Economics, 18*(1), 7-27.

Murthy Konda, N. V. S. N., Shah, N., & Brandon, N. P. (2011), "Optimal transition towards a large-scale hydrogen infrastructure for the transport sector: The case for the Netherlands", *International Journal of Hydrogen Energy, 36*(8), 4619-4635.

Mulder, G., Hetland, J., & Lenaers, G. (2007), "Towards a sustainable hydrogen economy: Hydrogen pathways and infrastructure", *International Journal of Hydrogen Energy, 32*(10-11), 1324-1331.

Ogden, J. M. (1999), "Developing an infrastructure for hydrogen vehicles: A southern California case study", *International Journal of Hydrogen Energy, 24*(8), 709-730.

O'Garra, T., S. Mourato, and P. Pearson (2005), "Analyzing Awareness and Acceptability of Hydrogen Vehicles: A London Case Study", *International Journal of Hydrogen Energy,* Vol.30, 2005, pp.649-659.

Rogers, E.M. (2003), *Diffusion of Innovations,* Fifth Edition, *Free* Press, New York.

Sandy Thomas, C. E. (2009), “Transportation options in a carbon-constrained world: Hybrids, plug-in hybrids, biofuels, fuel cell electric vehicles, and battery electric vehicles”, *International Journal of Hydrogen Energy, 34*(23), 9279-9296.

Schulte, I., Hart, D., & van der Vorst, R. (2004), “Issues affecting the acceptance of hydrogen fuel,” *International Journal of Hydrogen Energy, 29*(7), 677-685.

Schwartz, E.S. and Moon, M. (2000), “Evaluating R&D investments,” In Brennan, M. and Trigeorgis, L. (eds), *Project Flexibility, Agency and Competition.* Oxford: Oxford University Press, pp. 85-106.

Shayegan, S., Hart, D., Pearson, P., & Joffe, D. (2006), “Analysis of the cost of hydrogen infrastructure for buses in London,” *Journal of Power Sources, 157*(2), 862-874.

Smit, R., Weeda, M., & de Groot, A. (2007), “Hydrogen Infrastructure Development in the Netherlands”, *International Journal of Hydrogen Energy, 32*(10-11), 1387-1395.

Thomas, C. E., James, B. D., & Lomax, F. D. (1998), “Market penetration scenarios for fuel cell vehicles”, *International Journal of Hydrogen Energy, 23*(10), 949-966.

van Benthem, A. A., Kramer, G. J., & Ramer, R. (2006), “An options approach to investment in a hydrogen infrastructure”, *Energy Policy, 34*(17), 2949-2963.

Wietschel, M., Hasenauer, U., & de Groot, A. (2006), “Development of European Hydrogen Infrastructure Scenarios—CO_2 Reduction Potential and Infrastructure Investment,” *Energy Policy, 34*(11), 1284-1298.

Appendix

Equation (7)

Let W_τ be a standard Brownian motion. If we consider a function $f(x+W_\tau)$, then from Ito's lemma:

$$df(x+W_\tau) = \frac{\partial f(x+W_\tau)}{\partial W_\tau} dW_\tau + \frac{1}{2}\frac{\partial^2 f(x+W_\tau)}{\partial W_\tau{}^2} d\tau$$

Note that we want to treat x as a parameter rather than a variable, and hence have ignored it in deriving the stochastic differential equation. If we integrate this equation with respect to τ then we obtain:

$$f(x+W_\tau) = f(x) + \int_0^\tau \frac{\partial f(x+W_s)}{\partial W_s} dW_s + \frac{1}{2}\int_0^\tau \frac{\partial^2 f(x+W_s)}{\partial W_s{}^2} ds$$

Where we have used the fact that $W_0 = 0$. We then notice that differentiating $f(x+W_\tau)$ with respect to W_τ is the same as differentiating it with respect to x, that is:

$$\frac{\partial f(x+W_\tau)}{\partial W_\tau} = \frac{\partial f(x+W_\tau)}{\partial x} \quad \text{and} \quad \frac{\partial^2 f(x+W_\tau)}{\partial W_\tau{}^2} = \frac{\partial^2 f(x+W_\tau)}{\partial x^2}$$

Substituting the above into $f(x+W_\tau)$ and take an expectation on each side of the equation. Stochastic integral vanishes due to martingale property, and then we obtain:

$$E[f(x+W_\tau)] = f(x) + \frac{1}{2}\int_0^\tau \frac{\partial^2 E[f(x+W_s)]}{\partial x^2} ds \tag{11}$$

If we define the function

$$A(x,\tau) = E[f(x+W_\tau)]$$

Then equation (11) becomes

$$A(x,\tau) = f(x) + \frac{1}{2}\int_0^\tau \frac{\partial^2 A(x,s)}{\partial x^2} ds$$

Differentiating with respect to τ, we see that $A(x,\tau)$ satisfies the heat equation

$$\frac{\partial A(x,\tau)}{\partial \tau} = \frac{1}{2}\frac{\partial^2 A(x,\tau)}{\partial x^2} \tag{12}$$

Furthermore, if we evaluate $A(x,\tau)$ at $\tau = 0$ we see that

$$\begin{aligned} A(x,0) &= E[f(x+W_0)] \\ &= E[f(x)] \\ &= f(x) \end{aligned}$$

That is, $A(x,\tau)$ satisfies the initial condition $A(x,0) = f(x)$. Thus, we now have a recipe for solving the heat equation subject to a given initial condition. Specifically, if $A(x,\tau)$ satisfies the heat equation (12) and is subject to the initial condition $A(x,0) = f(x)$, then

$$\begin{aligned} A(x,\tau) &= E[f(x+W_\tau)] \\ &= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(x+\xi) e^{-\frac{\xi^2}{2\tau}} d\xi \end{aligned}$$

Somewhat more generally, but by an identical argument, we find that

$$A(x,\tau) = E[f(x+W_\tau)] \tag{13}$$

Satisfies the equation

$$\frac{\partial}{\partial \tau} A(x,\tau) = \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} A(x,\tau) \tag{14}$$

Subject to the initial condition $A(x,0) = f(x)$, for fixed τ, the random variable becomes $W_\tau \sim N(0, \sigma\sqrt{\tau})$. We can therefore rewrite the solution (13) as $A(x,\tau) = E[f(x+\sigma\sqrt{\tau}Z)]$

Where Z is a standard $N(0,1)$ random variable. Explicitly writing out the expectation we have $A(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+\sigma\sqrt{\tau}\xi) e^{-\frac{1}{2}\xi^2} d\xi$ (15)

Equation (8):

The multi-dimensional Ito rule is a straight-forward generalization of the one-dimensional case. If $C(V,t)$ is the value of a derivative at time t which expires at time T, equation (2) must also satisfy the partial differential equation

$$\frac{\partial C(V,t)}{\partial t} = rC(V,t) - rV \frac{\partial C(V,t)}{\partial V} - \frac{1}{2}\tilde{\sigma}^2 V^2 \frac{\partial^2 C(V,t)}{\partial V^2} \tag{16}$$

Now in order to reduce the above PDE to the heat equation, we will make a series of crafty transformation. Set $C(V,t) = \alpha(V,\tau)$, where $\tau = T - t$ is a new time coordinate which still runs over the same interval $[0,T]$ as t, but in the opposite direction. We need to reverse the direction of time, so that the terminal payout of the option becomes the initial condition for the heat equation. The time derivatives of $C(V,t)$ and $\alpha(V,\tau)$ are related by

$$\frac{\partial C}{\partial t} = -\frac{\partial \alpha}{\partial \tau}$$

While all the other derivatives remain the same. Hence the

$$\frac{\partial \alpha}{\partial \tau} = \frac{1}{2}\tilde{\sigma}^2 V^2 \frac{\partial^2 \alpha}{\partial V^2} - r\alpha + rV\frac{\partial \alpha}{\partial V} \qquad (17)$$

This equation now has the "right" sign for the time derivative, and has the initial condition

$$\begin{aligned}\alpha(V_T,0) &= C(V_T,T)\\ &= F(V_T,T)\end{aligned}$$

We now want to eliminate the $r\alpha$ term. We can do this by introducing a "discount factor" $e^{-r\tau}$ explicitly into the equation. Set $\alpha(V,\tau) = \beta(V,\tau)e^{-r\tau}$. The time derivative is then

$$\frac{\partial \alpha}{\partial \tau} = (\frac{\partial \beta}{\partial \tau} - r\beta)e^{-r\tau}$$

And hence equation (17) can be written as

$$\frac{\partial \beta}{\partial \tau} = \frac{1}{2}\tilde{\sigma}^2 V^2 \frac{\partial^2 \beta}{\partial V^2} + rV\frac{\partial \beta}{\partial V}$$

To proceed further, we want to write the equation in terms of the operator $V\partial/\partial V$. This can be easily accomplished by rearranging the second order term,

$$\frac{\partial \beta}{\partial \tau} = \frac{1}{2}\tilde{\sigma}^2 V\frac{\partial}{\partial V}(V\frac{\partial \beta}{\partial V}) + (r - \frac{1}{2}\tilde{\sigma}^2)V\frac{\partial \beta}{\partial V} \qquad (18)$$

We can simplify the operator $V\partial/\partial V$ by defining the new variable $Y = \ln V$, and noting that

$$V\frac{\partial}{\partial V} = \frac{\partial}{\partial Y}$$

If we then introduce the new function $\gamma(Y,\tau) = \beta(V,\tau)$, we see that the differential equation (18) becomes

$$\frac{\partial\gamma}{\partial\tau} = \frac{1}{2}\tilde{\sigma}^2 \frac{\partial^2\gamma}{\partial Y^2} + (r - \frac{1}{2}\tilde{\sigma}^2)\frac{\partial\gamma}{\partial Y} \tag{19}$$

Define $X = Y + (r - \frac{1}{2}\tilde{\sigma}^2)\tau$, and set $A(X,\tau) = \gamma(Y,\tau)$. The partial derivative of γ with respect to τ is then given by

$$\begin{aligned}\frac{\partial\gamma}{\partial\tau} &= \frac{\partial A}{\partial\tau} + \frac{\partial A}{\partial X}\frac{\partial X}{\partial\tau} \\ &= \frac{\partial A}{\partial\tau} + \frac{\partial A}{\partial X}(r - \frac{1}{2}\tilde{\sigma}^2)\end{aligned} \tag{20}$$

However, since $\frac{\partial\gamma}{\partial Y} = \frac{\partial A}{\partial X}$ (21)

It follows that if we substitute (20) and (21) into equation (19) then the first order derivatives with respect to X cancel and we obtain the heat equation.

$$\frac{\partial A}{\partial\tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 A}{\partial X^2}$$

That is identical to equation (14) above. Now we would like to solve the option price $C(V_t, t)$ subject to the terminal condition

$$C(V_T, T) = F(V_T)$$

Where $F(V_T)$ is a prescribed function, that is, the payoff function of the derivative. As noted earlier, $t = T$ corresponds to $\tau = 0$, which is why the terminal payoff function of the derivative is actually an initial condition for $A(x,\tau)$. If we follow through the various transformations made above, then we see that the relation between $C(V_t, t)$ and $A(x,\tau)$ is

$$\begin{aligned}C(V_T, T) &= \alpha(V_t, T - t) \\ &= \beta(V_t, T - t)e^{-r(T-t)} \\ &= \gamma(\log V_t, T - t)e^{-r(T-t)} \\ &= A(\log V_t + [r - \frac{\sigma^2}{2}][T - t], T - t)e^{-r(T-t)}\end{aligned}$$

In particular the derivative payoff function can be written as

$$\begin{aligned}F(V_T) &= C(V_T, T) \\ &= A(\log V_T, 0)\end{aligned}$$

Hence the initial condition on $A(x,\tau)$ at $\tau = 0$ is

$A(x,0)=F(e^x)$

Without the barrier, a call option has

$A(x,0)=\max(e^x-I,0)$

Applied equation (15) for the solution of the heat equation with the initial condition $A(x,0)=F(e^x)$,

$$A(x,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(e^{x+\sigma\sqrt{\tau}\xi})e^{-\frac{1}{2}\xi^2}d\xi$$

Using this value of $A(x,\tau)$ and the transformation (15) we can then write the derivative price as

$$\begin{aligned}C(V,t)&=A(\log V_t+[r-\frac{\sigma^2}{2}][T-t],T-t)e^{-r(T-t)}\\&=\frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(V_te^{r(T-t)+\sigma\sqrt{\tau}\xi-\frac{1}{2}\sigma^2\tau})e^{-\frac{1}{2}\xi^2}d\xi\end{aligned}$$

In particular, if we set $t=0$, then we obtain the initial price of the derivative with no barrier:

$$C_0=\frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(V_0e^{rT+\sigma\sqrt{T}\xi-\frac{1}{2}\sigma^2T})e^{-\frac{1}{2}\xi^2}d\xi.$$

We see that the present value of the derivative depends on the expiry date T, the initial asset price V_0, the volatility σ, the risk-free interest rate r and the specification of the payoff function $F(V_T)$.

Taking into the down-and-out barrier B, the payoff $C(V,t)$ is zero for all V below the strike I; this translates into for $\mathrm{V}<\log(\frac{\mathrm{I}}{\mathrm{B}})$. We set the barrier below the strike to ensure that $\log(\frac{I}{B})>0$.

Let $V=Be^x$, $t=T-\tau/\frac{1}{2}\sigma^2$, $C=Be^{\alpha x+\beta\tau}u(x,\tau)$,

With $\alpha=\frac{1}{2}(1-k)$, $\beta=-\frac{1}{4}(k-1)^2-k$ and $k=r/\frac{1}{2}\sigma^2$.

$C(V,t)=C(Be^x,t(\tau))=Be^{\alpha x+\beta\tau}U(x,\tau)$

Thus $U(x,\tau)=e^{-\alpha x-\beta\tau}C(Be^x,t(\tau))/B$

$$U(-x,\tau) = e^{\alpha x - \beta\tau} C(Be^{-x}, t(\tau)) / B$$

We can now put the pieces together to show that the barrier option value is

$$\begin{aligned}
C(V,t;I) &= Be^{\alpha x + \beta\tau} u(x,\tau) \\
&= Be^{\alpha x + \beta\tau} (U(x,\tau) - U(-x,\tau)) \\
&= C(Be^{x}, t; I) - e^{2\alpha x} C(Be^{-x}, t; I) \\
&= C(V,t;I) - (\frac{V}{B})^{2\alpha} C(\frac{B^2}{V}, t; I)
\end{aligned}$$