Real Options, Variational Approach and Stefan Problem*

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Abstract: The paper deals with optimal stopping problems which arise in real options theory. We describe a variational approach to the solution of optimal stopping problems for diffusion processes, as an alternate to the traditional approach based on the solution of the Stefan (free-boundary) problem. The connection of this variational approach to smooth pasting conditions is established. We present some examples where the solution to the Stefan problem is not the solution to an optimal stopping problem. On the base of the proposed approach, we obtain the solution to an optimal stopping problem for a two-dimensional geometric Brownian motion with a non-linear payoff function. As an application we consider an optimal investment timing model taking into account tax exemptions.

1 Introduction. Real options

Let us consider an investment project of creating a new firm in the real sector of economy. This project is characterized by a pair $(V_t, t \geq 0,\ I)$, where V_t is a Present Value of the firm created at time t, and I is a cost of investment required to create the firm. V_t is assumed to be a stochastic process, defined at a probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbf{P})$, where $\mathcal{F}_t$ specifies the information accessible for investor up to the time t. This model starting from the well-known McDonald-Siegel model (see [11], [12]) supposes that:

*The work was supported by Russian Foundation for Basic Researches (projects 07-01-00541, 08-06-00154)

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- a decision on investing is based on the observed information about market (stochastic) prices on input and output production;
- at any moment, a decision-maker (investor) can either *accept* the project and proceed with the investment or *delay* the decision until he obtains new information;
- investment are considered to be instantaneous and irreversible so that they cannot be withdrawn from the project any more and used for other purposes.

The investor's problem is to evaluate the project and to select an appropriate time for the investment. In real options theory there are two different approaches to solving this problem (see [12]).

The value of project under the first approach is the maximum of net present value (NPV) from the future firm over all stopping times (regarding to the flow of σ-algebras $\mathcal{F}_t$:

$$F = \max_{\tau} \mathbf{E}(V_\tau - I)e^{-\rho\tau} = \mathbf{E}(V_{\tau^*} - I)e^{-\rho\tau^*}, \tag{1}$$

where $\mathbf{E}$ stands for an expectation at initial (Real World) measure $\mathbf{P}$), and ρ is the given discount rate. An optimal stopping time τ^* in (1) is viewed as optimal investment time (investment rule).

Within the second approach an opportunity to invest is considered as an option – the right but not obligation to buy the asset on predetermined price. At that an exercise time is viewed as investment time, and value of option is accepted as a value of investment project. In these framework a project is spanned with some traded asset S, which price S_t is completely correlated with present value of the project V_t.

In order to evaluate a (rational) value of this real option it is usually used two different (but interlinked) ways.

The first one is based on classical Financial Options Pricing theory. Namely, we have BS market (see, e.g., [2]) with risk-free interest rate r and risky asset S, which dynamics $S_t = S_t(\mu)$ is described by geometric Brownian motion with drift μ, volatility σ, and flow of dividends at rate δ. On this market one can consider an American option with payoff $f_t = g(S_t) = (S_t - I)^+$ if exercised at time t. Then the value of this option equals to $\sup\limits_{\tau} \mathbf{E}_Q e^{-r\tau} f_\tau$, where sup is taken over all stopping times, and $\mathbf{E}_Q$ stands for an expectation at risk-neutral (martingale) measure Q, such that $\{S_t e^{-(r-\delta)t},\ t \geq 0\}$ is Q-martingale. After the change of measure (see, e.g. [2], [5]) the value of option can be written as

$$\sup_{\tau} \mathbf{E} e^{-r\tau} g(S_\tau(r - \delta)), \tag{2}$$

where expectation is taken relative to initial (Real world) measure, and risky asset S is evolved as geometric Brownian motion with drift $\rho-\delta$ and volatility σ. The above mentioned formula for the value of American option holds in more general setting with any payoff function $g(S)$.

The second way (see classical textbook [12]) is based on Contingent Claims Analysis and considers a replicated portfolio on the above BS market. The value of real option is defined in this way as a solution to the following free-boundary problem (Stefan problem):

$$\begin{aligned}
&0.5\sigma^2 S^2 F^{''}(S) + (r-\delta)SF^{'}(S) - rF(S) = 0, \quad 0 < S < S^*;\\
&F(S^*) = g(S^*);\\
&F'(S^*) = g'(S^*).
\end{aligned} \tag{3}$$

It is commonly accepted that a solution to Stefan problem (3) gives also the value of the corresponding optimal stopping problem (2). This is really so for a classical American call option with the payoff $g(S) = (S-I)^+$, but for the general payoff function a relation between solutions to problems (2) and (3) remains open. Commonly speaking, what is a connection between optimal stopping problem for diffusion processes and appropriate Stefan problem (with free boundary).

Similar problems arise also for options defined on multiple underlying assets. For a multidimensional case an optimal stopping problem is very hard for studying whereas Stefan problem can be solved numerically (including computer-based methods).

The paper proposes the new (variational) approach to the solution of optimal stopping problems for diffusion processes, as an alternate to the traditional approach based on the solution of the Stefan (free-boundary) problem. The main idea of this approach is a variation of certain functional defined on the sets from the given "continuation regions". In the framework of this approach a smooth-pasting condition can be viewed as first-order optimality condition, and we give some "non-exotic" examples when solution to Stefan problem doesn't give a solution to optimal stopping problem.

On the base of the proposed approach, we obtain the solution to an optimal stopping problem for two-dimensional geometric Brownian motion with a non-linear homogeneous payoff function. As an application we then consider an optimal investment timing model under uncertainty taking into account tax exemptions and derive a closed-form formula for an optimal investment rule.

The paper is organized as follows. In Section 3 we describe a variational approach for solving an optimal stopping problem which is an alternate to the traditional approach based on solving the Stefan (or free-boundary) problem – see Section 2.

For the case, when the class of continuation regions is one-parametric family of sets in $\mathbb{R}^n$, an optimal stopping problem can be reduced to the maximization of some (enough simple) function. We set that smooth pasting conditions are equivalent (in the case considered) to stationarity, or first-order, condition for some function. We give also few examples when a solution to Stefan problem is not a solution to optimal stopping problem.

In Section 4 it is demonstrated how a variational approach works. We show how this approach can be used in known problem of "Russian option" pricing. An optimal stopping problem for two-dimensional Brownian motion and homogeneous (of any non-negative order) payoff function is studied also. A linear version of this problem, arising in investment timing models, is studied, at heuristic level, in [11] and was a base for real option theory (see, for example, [12]). The rigorous proof of an optimality, and the conditions at which the formula for optimal investment time (stated at [11]) is hold, appeared later in [13].

A reduction of optimal investment timing problem (taking into account a taxation of enterprises) to optimal stopping problem for two-dimensional Brownian motion and linear payoff function is conducted in Section 5.1. For solving this problem we use a variational approach. A case with non-linear homogeneous (of the first order) payoff function, that arises in real option theory under tax exemptions is also considered in Section 5.2.

2 Optimal stopping problem

Let $X_t,\ t \geq 0$ be a diffusion process with values in $\mathbb{R}^n$ defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbf{P})$.

Let us consider an optimal stopping problem for this process:

$$U(x) = \sup_{\tau} \mathbf{E}^x g(X_\tau) e^{-\rho\tau}, \tag{4}$$

where $g:\ \mathbb{R}^n \to \mathbb{R}^1$ is payoff function, $\rho \geq 0$ is discount rate, and $\mathbf{E}^x$ means the expectation for the process X_t starting from the initial state x. The maximum in (4) takes over some class of stopping times (s.t.) τ, usually over the class $\mathcal{M}$ of all s.t. with respect to the natural filtration $\mathcal{F}_t^X = \sigma\{X_s,\ 0 \leq s \leq t\},\ t \geq 0)$.

Traditional solving of problem (4) is to find s.t. τ^*, at which sup in (4) is attained, as well as the value function $U(x)$, for all initial states x (see, e.g., [1]). In other words, (4) is considered as the family of problems depending on the parameter x ("mass" setting).

Optimal stopping time for the problem (4) can be represented as the first exit time of the process X_t out of the continuation set $C = \{U(x) > g(x)\}$. Usually it is proposed to find unknown function $U(x)$ and set C as a solution to free-boundary problem (Stefan problem):

$$\mathbb{L}_X U(x) = \rho U(x), \quad x \in C; \tag{5}$$
$$U(x) = g(x), \quad x \in \partial C; \tag{6}$$
$$\operatorname{grad} U(x) = \operatorname{grad} g(x), \quad x \in \partial C \tag{7}$$

([1, Chapter III]; where $\mathbb{L}_X$ is the infinitesimal generator of X_t, ∂C is the boundary of the set C). The condition (6) is called "continuous fit", and (7) – "smooth fit" condition. A proof of necessity of the condition (7) for one-dimensional processes one can find in [1]. The similar result for multi-dimensional case was obtained (under some additional assumptions) in [3].

3 Description of a variational approach

In the paper we develop another approach to solving an optimal stopping problem which we shall refer as a variational. In this framework it is a priori defined a class of "continuation regions", and we find the optimal region over this given class. Unlike the mass setting of an optimal stopping problem, we study the individual problem (4) for the given (fixed) initial state of the process $X_0 = x$.

Let $\mathcal{G} = \{G\}$ be a given class of regions in $\mathbb{R}^n$, $\tau_G = \tau_G(x) = \inf\{t \geq 0 : X_t \notin G\}$ be a first exit time of process X_t out of the region G (obviously, $\tau_G = 0$ whenever $x \notin G$), and $\mathcal{M}(\mathcal{G}) = \{\tau_G, \ G \in \mathcal{G}\}$ be a set of first exit times for all regions from the class $\mathcal{G}$. In the sequel we will suppose that $\tau_G < \infty$ (a.s.) for any $G \in \mathcal{G}$.

Under fixed initial value x for any continuation region $G \in \mathcal{G}$ we define the following function (of sets)

$$V_G(x) = \mathbf{E}^x g(X_{\tau_G}) e^{-\rho \tau_G}. \tag{8}$$

Outside the region G this function equals payoff function g (i.e. $V_G(x) = g(x), \ x \notin G$), and inside G the function $V_G(x)$ can be derived (under some assumptions) as a solution to boundary Dirichlet problem:

$$\begin{array}{ll} \mathbb{L}_X u(x) = \rho u(x), & x \in G; \\ u(y) \to g(x), & y \in G, \ y \to x \in \partial G. \end{array} \tag{9}$$

In order to calculate functions of the type (8) one can use martingale methods also (see, for example, [1], [4]).

Thus, a solving an optimal stopping problem over a class s.t. $\mathcal{M}(\mathcal{G})$ can be reduced to a solving the following variational problem:

$$V_G(x) \to \max_{G \in \mathcal{G}}. \tag{10}$$

If G^* is an optimal region in (10), the optimal stopping time in the class $\mathcal{M}(\mathcal{G})$ coincides with the first exit time from this region: τ_{G^*}.

If the class of regions $\mathcal{G}$ is chosen "well", it is possible to prove that s.t. τ_{G^*} will be also an optimal stopping time for problem (4) over all s.t. $\mathcal{M}$. In Section 3 below such approach will be realized for two-dimension geometric Brownian motion X_t and homogeneous payoff function g.

In one-dimensional case a variational approach (namely, maximization over exit levels for geometric Brownian motion) was exploited in [7] as a method for finding an optimal investment time for the creation of a new enterprise.

A close approach is developed in в [8], where an optimal stopping problem for one-dimensional diffusion is solved by mathematical programming technique. However those method uses a few properties of one-dimensional diffusion, which are not valid in multi-dimensional case.

As for multi-dimensional processes, the first-order conditions as a heuristic method for finding boundaries of optimal "continuation sector" in optimal stopping problem for bivariate geometric Brownian motion and homogeneous (of first order) payoff function was used in [4]. A general idea of a variational approach with application to optimal stopping problems arisen in investment timing models was stated in [9],[10].

Let us note, that the calculation of the optimal stopping time over a given class of regions represents, to our opinion, a practical interest. Indeed, the Stefan problem has no (as a rule) explicit solution for multi-dimensional diffusion processes. Therefore, it has a sense to restrict our consideration to more simple regions, for which it is possible to derive the function of sets $V_G(x)$. Also, numerical methods can be applied for solving the problem (10) with fix initial state $X_0 = x$.

An idea of variational approach is general enough and can be applied not only for a diffusion processes and payoff functions of the type (4).

3.1 A variational approach for one-parametric class of "continuation regions"

Under some additional assumptions a general variational problem (10) can be simplified and be made more convenient for study.

Let D be a set of possible initial states of the process X_t (it can contain, for example, one point only). Let $\mathcal{G} = \{G_p,\ p \in P \subset \mathbb{R}^1\}$ be one-parametric class of regions in $\mathbb{R}^n$, $\tau_p = \inf\{t \geq 0 : X_t \notin G_p\}$, $V_p(x) = V_{G_p}(x)$.

We will call function $F(p, x)$, defined on $P \times D$, a terminal-initial function if $V_p(x) = F(p, x)$ for $p \in P, x \in G_p$.

In the sequel we will use the following representation for the function $V_p(x)$:

$$V_p(x) = \begin{cases} F(p, x), & x \in G_p \\ g(x), & x \notin G_p \end{cases}, \tag{11}$$

where $F(p, x)$ is a terminal-initial function.

It is assumed that "continuous pasting" condition holds at the boundary of set G_p:

$$F(p, x) = g(x),\ x \in \partial G_p. \tag{12}$$

Further, we assume that a family of regions G_p satisfies the following conditions:

1) $G_{p_1} \subset G_{p_2}$ whenever $p_1 < p_2$ – monotonicity;

2) every point $x \in D$ belongs to the boundary of the unique set from the class $\mathcal{G}$ (a parameter of those set will be refered as $q(x)$, so $x \in \partial G_{q(x)}$) – "thickness".

Under a thickness property for continuation regions the continuous pasting condition (12) can be written as follows:

$$F(q(x), x) = g(x) \quad \forall x \in D. \tag{13}$$

Under the stated above assumptions a solution to maximization of $V_p(x)$ in p is given by the following

Theorem 1. Let for $x \in D$ a terminal-initial function $F(p, x)$ have a unique maximum (in $p \in P$) at the point $p^*(x)$, and $F(p, x)$ decreases in p whenever $p > p^*(x)$. Then $\max_{p \in P} V_p(x) = V_{p^*(x)}(x)$, i.e. $\tau_{p^*(x)}$ is an optimal stopping time in the problem (4) over the class of s.t. $\mathcal{M}(\mathcal{G})$.

Thus, under the above assumptions a maximization (in p) of the function $V_p(x)$, which has the composite structure (11), can be reduced to maximization of "more simple" terminal-initial function $F(p, x)$. In subsequent sections we demonstrate how this theorem can be used for finding a solution to a number of optimal stopping problems.

3.2 A variational problem and smooth pasting principle

In the framework of a variational approach we give a new look to a smooth pasting principle, which is one of the base component in solving Stefan problem with a free boundary.

Let the set D is opened (at $\mathbb{R}^n$), the general assumptions from Section 3 hold, and furthermore, functions $F(p,x)$, $g(x)$, as well as $q(x)$ (a parameter of the region whose boundary passes through the point x), are differentiable (in all arguments). Note, that the function $q(x)$ will be smooth, for example, in the case when regions' boundaries ∂G_p are specified by surfaces of the type $\{\Psi(p,y)=0,\ y \in \mathbb{R}^n\}$, where $\Psi(p,y)$ is continuously differentiable in (p,y) and $\Psi'_p(p,y)$ is non-zero.

Let $\bar{p}(x)$ be a stationary point of the function $F(p,x)$ in p, i.e. $F'_p(\bar{p}(x),x)=0$ $(x \in D)$. The condition (13) implies

$$F'_p(q(x),x)\ \operatorname{grad} q(x) + F'_x(q(x),x) = \operatorname{grad} g(x), \quad x \in D. \tag{14}$$

Thus, if x belongs to the boundary of the region $G_{\bar{p}(x)}$, then $q(x)=\bar{p}(x)$ and, therefore,

$$F'_x(\bar{p}(x),x) = \operatorname{grad} g(x), \quad x \in \partial G_{\bar{p}(x)}. \tag{15}$$

This equality can be viewed as a variant of smooth pasting condition at the boundary of the set $G_{\bar{p}(x)}$, whose parameter is a stationary point of a function $F(p,x)$.

Note, the set of such x that (15) holds can be empty. Consider the case, when stationarity points $\bar{p}(x)=\bar{p}$ do not depend on x. In this case the set of such x that relation (15) valid, is not empty. Such a situation emerges, in particular, for the case of geometric Brownian motion and homogeneous payoff function (the function $F(p,x)$ in this case can be represented as a product of a function of x and a function of p). Defining the function $\bar{F}(x)=F(\bar{p},x)$, the relation (15) can be written as follows:

$$\operatorname{grad} V_{\bar{p}}(x) = \operatorname{grad} \bar{F}(x) = \operatorname{grad} g(x), \quad x \in \partial G_{\bar{p}}. \tag{16}$$

Taking into account that $\bar{F}(x)$ for $x \in G_{\bar{p}}$ is a solution to Dirichlet problem (9), the equality (16) is a traditional smooth pasting condition, and, therefore, the pair $(\bar{F}(x), G_{\bar{p}})$ is a solution to Stefan problem (5)-(7). The region $G_{\bar{p}}$ will be a candidate for an optimal continuation set. If maximum of the function $F(p,x)$ in p is attained at the point $p^* \in \operatorname{int} P$, then smooth pasting condition (16) with $\bar{p}=p^*$ holds.

Let us note, if for the family of regions $\{G_p\}$ it is hold $\operatorname{grad} q(x) \neq 0$ for $x \in D$, then (as one can see from (14), the smooth pasting condition (16) is equivalent to stationarity of $F(p, x)$ (in p) at the point $\bar{p}$.

Stationarity points of the function F can not be the points of extremum. Hence, for such a case a solution to Stefan problem (which there is exists), can not give a solution to optimal stopping problem.

3.3 A solution to Stefan problem can not give a solution to Optimal Stopping Problem: Examples

Let us consider (one-dimensional) geometric Brownian motion $X_t = x\exp\{w_t\}$ (where w_t be standard Wiener process), payoff function $g(x)$=$(x-1)^3 + x^\delta$ ($\delta > 0$) for $x \geq 0$, and discount rate $\rho = \delta^2/2$. The function g is smooth and increasing for all $\delta > 0$.

Let p^* be a boundary point for the optimal continuation set. Due to monotonicity of the payoff function we consider the interval $(0, p^*)$ as an unknown continuation set.

A Stefan problem for finding unknown function $U(x)$ and boundary p^* is the following one:

$$\begin{cases} \frac{1}{2}x^2U''(x) + \frac{1}{2}xU'(x) = \rho U(x), \qquad 0 < x < p^* \\ U(p^*) = g(p^*) \\ U'(p^*) = g'(p^*) \end{cases} \tag{17}$$

To study this problem we begin with solving an auxiliary Dirichlet problem for finding the function $V_p(x) = \mathbf{E}^x g(X_{\tau_p})e^{-\rho\tau_p}$, where $\tau_p = \min\{t \geq 0 : X_t \geq p\}$:

$$\begin{cases} \frac{1}{2}x^2V''(x) + \frac{1}{2}xV'(x) = \rho V(x), \qquad 0 < x < p \\ V(p) = g(p). \end{cases} \tag{18}$$

The solution to this problem is $V_p(x) = \mathbf{E}^x g(X_{\tau_p})e^{-\rho\tau_p} = h(p)x^\delta \quad (0 < x < p)$, where $h(p) = g(p)p^{-\delta} = (p-1)^3/p^\delta + 1$.

As one can see, the smooth pasting condition $V'_{p^*}(p^*) = g'(p^*)$ is equivalent to $h'(p^*) = 0$.

For $\delta \leq 3$ the Stefan problem (17) has the unique solution: $U(x) = V_1(x) = x^\delta, \quad p^* = 1$. However, the stopping time τ_1 (a first exit over the level 1) is not optimal s.t. since $V_p(x) \to \infty$ when $p \to \infty$ for any $x > 0$ (when $\delta < 3$), and $V_p(x) \uparrow V(x) = 2x^3$ (when $\delta = 3$).

Thus, in this case there exists a solution of Stefan problem, whereas the optimal stopping problem has no solution. Moreover, in the case $\delta = 3$ there are exist both a solution of Stefan problem (function $U(x)$) and optimal value function $V(x)$ at optimal stopping problem, but its are not the same: $U(x) \neq V(x)$ for all $x > 0$.

For $\delta > 3$ the Stefan problem (17) has two solutions:

$$U(x){=}V_1(x){=}x^\delta,\ p^* = 1 \quad \text{and} \quad U(x){=}V_{p_\delta}(x){=}h(p_\delta)x^\delta,\ p^*{=}p_\delta{=}\delta/(\delta{-}3).$$

Note that $V_{p_\delta}(x) > V_1(x)$. It can be shown (using, for example, [8, Theorem 3]), that s.t. p_δ is optimal over all s.t. $\mathcal{M}$.

Thus, one of the solutions to Stefan problem (corresponding to the boundary $p^* = 1$) does not give a solution to the optimal stopping problem (which there exists, in contrast to the previous case).

Now let us demonstrate, how a variational approach works for the above example. As a class of continuation sets we take intervals $(0, p)$, $p > 0$, so the corresponding s.t. are $\tau_p = \min\{t \geq 0 : X_t \geq p\}$. From the solution to Dirichlet problem (18) one can see, that $F(p, x) = [(p-1)^3/p^\delta + 1]x^\delta$ can be viewed as the terminal-initial function. Let us note, that for $\delta \leq 3$ the function $F(p, x)$ increases (in p) infinitely for any $x > 0$, and, therefore, an optimal s.t. does not exist. When $\delta > 3$, the function $F(p, x)$ attains maximum (in p) for every $x > 0$ at the point $p^*{=}p_\delta$, and all conditions of Theorem 1 hold. Therefore, τ_{p_δ} is optimal s.t. over the class $\{\tau_p,\ p > 0\}$.

4 How the variational approach works

In this Section we demonstrate how a variational approach described above can be applied for solving a concrete optimal stopping problems. Below we consider two such problems: the first one connects with well-known "Russian Option", and the second one concerns two-dimensional geometric Brownian motion and homogeneous payoff function.

4.1 "Russian option"

Russian options were introduced in [5] as a special case of the perpetual (American) options with a path-dependent payoff. As shown in [6], the pricing in *"Russian Option"* can be reduced to an optimal stopping problem (with payoff function $g(x){=}x$) for the diffusion process $(\psi_t, t \geq 0)$ with reflection:

$$d\psi_t = -\psi_t(r dt + \sigma dw_t) + d\varphi_t,$$

where non-negative process $(\varphi_t,\ t{\geq}0)$ grows whenever $(\psi_t,\ t{\geq}0)$ attains boundary $\{1\}$.

Consider the class of stopping times $\tau_p = \min\{t \geq 0 : \psi_t \geq p\},\ p > 1$ that was proposed in [6] for the "simplified version" of the appropriate optimal stopping problem.

Following the explicit formula for $V_p(x) = \mathbf{E}^x \psi_{\tau_p} e^{-\rho\tau_p}$ (obtained from the Dirichlet problem), we can view

$$F(p,x) = p \cdot \frac{\beta_2 x^{\beta_1} - \beta_1 x^{\beta_2}}{\beta_2 p^{\beta_1} - \beta_1 p^{\beta_2}}, \quad p \geq 1,\ x \geq 1,$$

where $\beta_1,\ \beta_2$ are roots of the equation $\sigma^2\beta^2 - (\sigma^2 + 2r)\beta - 2\rho = 0 \quad (\beta_1 < 0,\ \beta_2 > 1)$, as the terminal-initial function.

$F(p,x)$ attains the unique maximum (in $p{\geq}1$) for all $x{>}1$ at the point $p^*{=}\left[\dfrac{\beta_2(1{-}\beta_1)}{\beta_1(1{-}\beta_2)}\right]^{1/(\beta_2-\beta_1)}$, and decreases for $p{>}p^*$. Thus, Theorem 1 implies that τ_{p^*} is optimal stopping time over the class $\{\tau_p,\ p{>}1\}$.

4.2 Two-dimensional geometric Brownian motion and non-linear payoff function

As we see at the next Section, an optimal stopping problems for two-dimensional geometric Brownian motion and homogeneous payoff function arise in investment timing models.

Let consider two-dimensional geometric Brownian motion $X_t{=}(X_t^1, X_t^2),\ t{\geq}\ 0$

$$\begin{array}{ll} dX_t^1 = X_t^1(\alpha_1 dt + \sigma_{11} dw_t^1 + \sigma_{12} dw_t^2), & X_0^1 = x_1, \\ dX_t^2 = X_t^2(\alpha_2 dt + \sigma_{21} dw_t^1 + \sigma_{22} dw_t^2), & X_0^2 = x_2, \end{array} \tag{19}$$

where (w_t^1, w_t^2) is standard two-dimensional Wiener process (with independent components).

Let payoff function $g(x_1, x_2)$ (see (4)) be continuous and positive homogeneous of order $m \geq 0$, i.e. $g(\lambda x) = \lambda^m g(x)$ for all $\lambda > 0$, $x_1, x_2 \geq 0$.

Initial states of the process X_t will be take from the region $D = \{(x_1, x_2) : x_1, x_2 > 0\}$, and we will consider $G_p = \{(x_1, x_2) \in D :\ x_1 < p x_2\},\ p > 0$ as a class of continuation sets. The first exit time of the process (19) from the region G_p: $\tau_p(x) = \min\{t \geq 0 :\ X_t^1 \geq p X_t^2\}$ coincides with the first exit time of $\xi_t = X_t^1 / X_t^2$ over the level p. The explicit formulas for geometric Brownian motion imply that

$$\xi_t = \frac{x_1}{x_2} \exp\left\{\left(\alpha_1{-}\alpha_2 + \frac{\sigma_2^2{-}\sigma_1^2}{2}\right) t + \sigma_1 \tilde{w}_t^1 - \sigma_2 \tilde{w}_t^2\right\}, \tag{20}$$

where $\sigma_i^2 = \sigma_{i1}^2 + \sigma_{i2}^2$, and $\tilde{w}_t^i = (\sigma_{i1} w_t^1 + \sigma_{i2} w_t^2)/\sigma_i$ are standard Wiener processes $(i = 1, 2)$.

It is easy to see that $\tau_p(x)$ is homogeneous (of zero order) function of $x = (x_1, x_2)$, i.e. $\tau_p(\lambda x) = \tau_p(x)$ for all $\lambda > 0$, moreover, under the following condition

$$\alpha_1 - \tfrac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) \geq \alpha_2 - \tfrac{1}{2}(\sigma_{21}^2 + \sigma_{22}^2) \tag{21}$$

$\tau_p(x)$ is finite a.s. for all $x \in D$ and $p > 0$, since Wiener process attains every level (with probability 1).

The homogeneities of the process X_t (in initial state) and the function g imply:

$$\begin{aligned} V_p(\lambda x) &= \mathbf{E}^{\lambda x} e^{-\rho \tau_p(\lambda x)} g(X_{\tau_p(\lambda x)}) = \mathbf{E}^{\lambda x} e^{-\rho \tau_p(x)} g(X_{\tau_p(x)}) \\ &= \mathbf{E}^{x} e^{-\rho \tau_p(x)} g(\lambda X_{\tau_p(x)}) = \lambda^m V_p(x), \end{aligned}$$

i.e. $V_p(x)$ is a homogeneous (of order m) function.

In the region G_p we will represent $V_p(x)$ as follows:

$$V_p(x_1, x_2) = x_2^m v_p(y), \quad \text{где } y = x_1/x_2, \; v_p(y) = V_p(y, 1), \; 0 < y < p.$$

Since the infinitesimal generator of the process (19) has the following form

$$\mathbb{L}u(x) = \alpha_1 x_1 u'_{x_1} + \alpha_2 x_2 u'_{x_2} + \tfrac{1}{2}\sigma_1^2 x_1^2 u''_{x_1 x_1} + (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) x_1 x_2 u''_{x_1 x_2} + \tfrac{1}{2}\sigma_2^2 x_2^2 u''_{x_2 x_2},$$

where σ_1, σ_2 are defined in (20), the elliptic operator in Dirichlet problem (9) for $V_p(x)$ transforms PDE to the ordinary differential equation for the function $v_p(y)$ (or, briefly $v(y)$):

$$\frac{1}{2} y^2 v''(y) \tilde{\sigma}^2 + y v'(y) \left(\bar{\alpha}_1 - \bar{\alpha}_2 - \frac{m-1}{2} \tilde{\sigma}^2 \right) - v(y)(\rho - \bar{\alpha}_2 m) = 0, \tag{22}$$

where $\bar{\alpha}_i = \alpha_i + \frac{1}{2}(m-1)\sigma_i^2 \quad (i = 1, 2), \quad \tilde{\sigma}^2 = (\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2$.

Any solution of equation (22) for $0 < y < p$ has the form $v(y) = C_1 y^{\beta_1} + C_2 y^{\beta_2}$, where $\beta_1, \; \beta_2$ are roots of the quadratic equation

$$\frac{1}{2} \tilde{\sigma}^2 \beta(\beta - 1) + \left(\bar{\alpha}_1 - \bar{\alpha}_2 - \frac{m-1}{2} \tilde{\sigma}^2 \right) \beta - (\rho - \bar{\alpha}_2 m) = 0. \tag{23}$$

Assume that

$$\rho > \max(\bar{\alpha}_1, \bar{\alpha}_2) m. \tag{24}$$

Then one can see that one of the roots in (23) is positive, and another – negative. Let, for certainty, $\beta_1 > 0, \; \beta_2 < 0$.

Returning to initial function we have

$$V_p(x_1,x_2)=C_1 x_1^{\beta_1} x_2^{m-\beta_1}+C_2 x_1^{\beta_2} x_2^{m-\beta_2}, \quad \text{for } 0<x_1<px_2,\ x_2>0. \tag{25}$$

It is easy to see that for all $x=(x_1,x_2)\in\mathbb{R}_+^2$ $|V_p(x_1,x_2)|$ is bounded by the function $C(x_1^m+x_2^m)$, where $C=2^m \max\limits_{0\le y\le 1}|g(y,1-y)|$.

This fact implies that $C_2=0$ in representation (25). The constant C_1 in (25) is found from the boundary condition in Dirichlet problem (9): $V_p(px_2,x_2)=C_1 x_2^m p^{\beta_1}=g(px_2,x_2)=x_2^m g(p,1)$, i.e. $C_1=g(p,1)p^{-\beta_1}$.

Therefore, as a terminal-initial function $F(p,x)$ in representation (11) for the considered optimal problem we can take the following function

$$F(p,x)=h(p)x_1^{\beta_1}x_2^{m-\beta_1}, \quad h(p)=g(p,1)p^{-\beta_1}. \tag{26}$$

Let us note, maximum of the function $F(p,x)$ in p is attained at the same point p^* as maximum of the function $h(p)$, i.e. this point does not depend on x.

The class of regions $\{G_p,\ p>0\}$ satisfies the requirements of monotonicity and thickness for continuation regions (see Section 1), and τ_p are stopping times. The class of these s.t. will be denoted $\mathcal{M}^0=\{\tau_p, p>0\}$. Thus, applying Theorem 1 to the optimal stopping problem for two-dimensional process (19) and homogeneous (of order m) payoff function we obtain the following result.

Theorem 2. Let (21), (24) hold, $\tilde{\sigma}>0$, and p^* be the unique maximum point of the function $h(p)$, defined in (26), moreover $h(p)$ decreases for $p>p^*$. Then $\tau^*=\min\{t\ge 0:\ X_t^1\ge p^* X_t^2\}$ is optimal stopping time over the class of s.t. $\mathcal{M}^0$.

It is turned out, the class of continuation regions for the considered problem is chosen "well" and s.t. τ^* (specified in Theorem 2) will be also optimal (under some additional assumptions) over the class of all s.t.

Theorem 3. Let all conditions of Theorem 2 hold, $g\in C^2(\mathbb{R}_+^2)$, $p^*>0$ be the unique maximum point of the function $h(p)$ and for $f(p)=g(p,1)$

$$f'(p)p^{-\beta_1+1} \quad \text{decreases for } p>p^*. \tag{27}$$

Then $\tau^*=\min\{t\ge 0:\ X_t^1\ge p^* X_t^2\}$ is optimal stopping time over the class of all s.t. $\mathcal{M}$.

In order to prove the optimality of stopping time τ^* over the class $\mathcal{M}$ we use the following "verification theorem", based on variational inequalities method. Below we formulated it for our case.

Let us denote $\mathbf{P}^x$ – the distribution of the process X_t (in sample space) starting from the point $X_0 = x$.

Verification Theorem ([13],[15]). Suppose, there exists a function $\Phi : \mathbb{R}^n_+ \to \mathbb{R}^1$, satisfying the following conditions:
1) $\Phi \in C^1(\mathbb{R}^n_+)$, $\Phi \in C^2(\mathbb{R}^n_+ \setminus \partial\Gamma)$; where $\Gamma = \{x \in \mathbb{R}^n_+ : \Phi(x) > g(x)\}$,
2) $\partial\Gamma$ is locally the graph of Lipschitz function and $\mathbf{E}^x \int_0^\infty \chi_{\partial\Gamma}(X_t)\, dt = 0$ for all $x \in \mathbb{R}^n_+$;
3) $\Phi(x) \geq g(x)$ for all $x \in \mathbb{R}^n_+$;
4) $\mathbb{L}\Phi = \rho\Phi$ for $x \in \Gamma$;
5) $\mathbb{L}\Phi \leq \rho\Phi$ for $x \in \mathbb{R}^n_+ \setminus \overline{\Gamma}$ ($\overline{\Gamma}$ is a closure of the set Γ);
6) $\bar{\tau} = \inf\{t \geq 0 : X_t \notin \Gamma\} < \infty$ a.s. (with respect to P^x) for all $x \in \mathbb{R}^n_+$;
7) the family $\{g(X_\tau)e^{-\rho\tau}, \ \tau \leq \bar{\tau}\}$ is uniformly integrable (with respect to $\mathbf{P}^x$) for all $x \in \Gamma$.

Then $\bar{\tau}$ is an optimal stopping time for the problem (4) over all s.t., and $\Phi(x)$ is the correspondent value function.

Let us give a corollary of Theorem 3 for the linear payoff function $g(x_1, x_2)$ ([13]). Just this case arises in a classical investment timing problem (see [11],[12]).

Corollary. Let $g(x_1, x_2) = c_1x_1 - c_2x_2$ ($c_1, c_2 > 0$), $\tilde{\sigma} > 0$, condition (21) hold, and $\rho > \max(\alpha_1, \alpha_2)$. Then the optimal stopping time for the problem (4) over all s.t. $\mathcal{M}$ is $\tau^* = \min\{t \geq 0 : X^1_t \geq p^* X^2_t\}$, where $p^* = c_2c_1^{-1}\beta(\beta - 1)^{-1}$, and β is a positive root of the quadratic equation

$$\frac{1}{2}\tilde{\sigma}^2\beta(\beta - 1) + (\alpha_1 - \alpha_2)\beta - (\rho - \alpha_2) = 0.$$

(As one can see, in this case $f(p) = g(p, 1) = c_1p - c_2$, the maximum of the function $h(p) = (c_1p - c_2)p^{-\beta}$ is attained at $p^* = c_2c_1^{-1}\beta(\beta - 1)^{-1}$ and all requirements of Theorem 3 hold.)

Remark. As a terminal-initial function for the problem with linear payoff function $g(x_1, x_2) = c_1x_1 - c_2x_2$ ($c_1, c_2 > 0$) we can take $F(p, x) = (c_1p - c_2)p^{-\beta}x_1^\beta x_2^{m-\beta}$ that satisfies all conditions of Theorem 1.

5 Optimal investment timing problem under taxation and uncertainty

Optimal stopping problem with homogeneous payoff function arises in a natural way for an investment timing model, where an agent wants to invest into a creation of new firm under the presence of tax exemptions. We show below that taking into account the tax exemptions which usually accompany a creation of a new firm in real sector of economy leads certainty to optimal stopping problem with non-linear (but homogeneous) payoff function.

5.1 Taxation without exemptions

Let us suppose that investment into creating a firm is made at time $\tau \geq 0$, I_τ be cost of investment required to create firm at time τ, $\pi^{\tau}_{\tau+t}$ be the flow of profit from the firm, $D^{\tau}_{t+\tau}$ denotes the flow of depreciation charges, that diminish the tax base (at time $t+\tau,\ t \geq 0$). So, the after-tax cash flow of the firm at time $t+\tau$ is equal to $\pi^{\tau}_{\tau+t}-\gamma(\pi^{\tau}_{\tau+t}-D^{\tau}_{t+\tau})=(1-\gamma)\pi^{\tau}_{\tau+t}+\gamma D^{\tau}_{t+\tau}$, where γ is the corporate profit tax rate.

The present value of the firm (discounted to the investment time τ) can be expressed as the following formula:

$$V_\tau = \mathbf{E}\left(\int_0^\infty [(1-\gamma)\pi^{\tau}_{\tau+t} + \gamma D^{\tau}_{t+\tau}] e^{-\rho t}\, dt \middle| \mathcal{F}_\tau \right), \tag{28}$$

where ρ is discount rate.

The investor's decision problem is to find such a stopping time τ (investment rule), that maximizes the expected net present value (NPV) from the future firm

$$\mathbf{E}\left(V_\tau - I_\tau\right) e^{-\rho\tau} \to \max_{\tau}, \tag{29}$$

where the maximum is considered over all stopping times $\tau \in \mathcal{M}$.

Assume that the process of *profits* $(\pi^{\tau}_{\tau+t},\ t,\tau \geq 0)$ is represented as: $\pi^{\tau}_{\tau+t} = \pi_{\tau+t}\xi^{\tau}_{\tau+t},\ t,\tau \geq 0$, where $(\pi_t,\ t \geq 0)$ is geometric Brownian motion, specified by the stochastic equation

$$d\pi_t = \pi_t(\alpha_1 dt + \sigma_{11} dw^1_t) \qquad (\pi_0 > 0,\ \sigma_{11} > 0), \quad t \geq 0,$$

w^1_t is standard Wiener process, and $(\xi^{\tau}_{\tau+t},\ t \geq 0)$ is a family of non-negative diffusion processes, homogeneous in $\tau \geq 0$, defined by the stochastic equations:

$$\xi^{\tau}_{\tau+t} = 1 + \int_\tau^{\tau+t} a(s-\tau, \xi^{\tau}_s)\, ds + \int_\tau^{\tau+t} b_1(s-\tau, \xi^{\tau}_s)\, dw^1_s + \int_\tau^{\tau+t} b_2(s-\tau, \xi^{\tau}_s)\, dw^2_s, \quad t,\tau \geq 0, \tag{30}$$

where w^2_t is standard Wiener process independent on w^1_t, and given functions $a(t,x)$, $b_1(t,x)$, $b_2(t,x)$ satisfy the standard conditions for the existence of the unique strong solution in (30) (see, e.g. [15, Chapter 5]).

The process π_t can be related to the (external) prices of produced goods and consumed resources (external uncertainty), whereas fluctuations $\xi^\tau_{\tau+t}$ can be generated by the firm created at time τ (firm's uncertainty).

In order all values in the model were well-defined we will suppose that $\mathbf{E}\pi_{\tau+t} < \infty$ for all $t, \tau \geq 0$.

The cost of the required investment I_t is also modelled by the geometric Brownian motion as follows:

$$dI_t = I_t(\alpha_2 dt + \sigma_{21} dw^1_t + \sigma_{22} dw^3_t), \quad (I_0 > 0) \quad t \geq 0,$$

where standard Wiener process w^3_t is independent on w^1_t , w^2_t, and $\sigma_{21} \geq 0,\ \sigma_{22} > 0$.

The flow of *depreciation charges* at a time $t + \tau$ will be represented as $D^\tau_{\tau+t} = I_\tau a_t, \quad t \geq 0$, where $(a_t,\ t \geq 0)$ is the "depreciation density" (per unit of investment), characterizing a depreciation policy, i.e. non-negative function $a : \mathbb{R}^1_+ \to \mathbb{R}^1_+$, such that $\displaystyle\int_0^\infty a_t\, dt = 1$.

In [14] we derived the following formula: for all $t \geq 0$ and stopping time τ

$$\mathbf{E}(\pi^\tau_{\tau+t}|\mathcal{F}_\tau) = \pi_\tau B_t, \qquad \text{where} \ \ B_t = \mathbf{E}(\pi_t \xi^0_t)/\pi_0. \tag{31}$$

Using this formula one can obtain that

$$V_\tau = \gamma I_\tau A + (1-\gamma)\pi_\tau B, \quad \text{where } B = \int_0^\infty B_t e^{-\rho t}\, dt, \quad A = \int_0^\infty a_t e^{-\rho t} d. \tag{32}$$

Therefore, the investment timing problem (29) can be rewritten as the following optimal stopping problem for two-dimensional geometric Brownian motion (π_t, I_t) and linear payoff function:

$$\mathbf{E}\left[(1-\gamma)\pi_\tau B - I_\tau(1-\gamma A)\right] e^{-\rho\tau} \to \max_\tau, \tag{33}$$

Application of Corollary (from the previous Section) gives immediately the following formula for the optimal investment threshold: $p^* = \dfrac{1-\gamma A}{(1-\gamma)B} \cdot \dfrac{\beta}{\beta - 1}$.

5.2 Tax holidays on payback period

A creation of a new firm in real sector of economy is usually accompanied by certain tax benefits. One of the popular incentives tools is tax holidays, when the new firm is exempted from profit tax during the payback period ν_τ:

$$\nu_\tau = \inf\{\nu \geq 0 : \ \mathbf{E}\left(\int_0^\nu \pi^\tau_{\tau+t} e^{-\rho t} dt \,\middle|\, \mathcal{F}_\tau\right) \geq I_\tau\} \tag{34}$$

(if infimum is not attained, then we put $\nu_\tau = \infty$).

For this case the present value of the firm (discounted to the investment time τ) can be expressed as the following formula:

$$V_\tau = \mathbf{E}\left(\int_0^{\nu_\tau} \pi^\tau_{\tau+t} e^{-\rho t} dt + \chi_\tau \int_{\nu_\tau}^\infty [(1-\gamma)\pi^\tau_{\tau+t} + \gamma D^\tau_{t+\tau}] e^{-\rho t}\, dt \,\middle|\, \mathcal{F}_\tau\right), \tag{35}$$

where χ_τ is an indicator function of the set $\Omega_\tau = \{\omega : \nu_\tau < \infty\}$.

If $\omega \in \Omega_\tau$, then (31) and (34) imply:

$$\begin{aligned} V_\tau &= I_\tau + (1-\gamma)\mathbf{E}\left(\int_0^\infty \pi^\tau_{\tau+t} e^{-\rho t} dt - \int_0^{\nu_\tau} \pi^\tau_{\tau+t} e^{-\rho t} dt \,\middle|\, \mathcal{F}_\tau\right) + \gamma I_\tau A(\nu_\tau) \\ &= I_\tau[1 + \gamma A(\nu_\tau)] - (1-\gamma)\left(I_\tau - \pi_\tau \int_0^\infty B_t e^{-\rho t} dt\right) \\ &= \gamma I_\tau[1 + A(\nu_\tau)] + (1-\gamma)\pi_\tau B, \end{aligned} \tag{36}$$

where B is specified in (32), and $A(\nu) = \displaystyle\int_\nu^\infty a_t e^{-\rho t} dt \ \ (\nu \geq 0)$.

If $\omega \notin \Omega_\tau$ (i.e. $\nu_\tau = \infty$), then

$$V_\tau = \mathbf{E}\left(\int_0^\infty \pi^\tau_{\tau+t} e^{-\rho t} dt \,\middle|\, \mathcal{F}_\tau\right) = \pi_\tau B. \tag{37}$$

If $\omega \in \Omega_\tau$, then from a definition of payback period (34) and formula (31) it is follows that

$$I_\tau = \mathbf{E}\left(\int_0^{\nu_\tau} \pi^\tau_{\tau+t} e^{-\rho t} dt \,\middle|\, \mathcal{F}_\tau\right) = \pi_\tau \int_0^{\nu_\tau} B_t e^{-\rho t} dt. \tag{38}$$

Let us define the following function $\nu(\cdot)$ as

$$\nu(p) = \inf\{\nu > 0 : \ \int_0^\nu B_t e^{-\rho t} dt \geq p^{-1}\}, \quad p > 0 \tag{39}$$

(we put $\nu(p) = \infty$ if infimum is not attained).

Then (38) implies that $\nu_\tau = \nu(\pi_\tau / I_\tau)$ for $\omega \in \Omega_\tau$. It is easy to see that $\Omega_\tau = \{\nu_\tau < \infty\} = \{\nu(\pi_\tau / I_\tau) < \infty\}$.

Combining (36) and (37) we can write the following formula for the present value of the created firm (35):

$$V_\tau = \begin{cases} \gamma I_\tau(1 + A(\nu(\pi_\tau/I_\tau))) + (1-\gamma)\pi_\tau B, & \text{if } \nu(\pi_\tau/I_\tau) < \infty \\ \pi_\tau B, & \text{if } \nu(\pi_\tau/I_\tau) = \infty. \end{cases}$$

So, the investment timing problem (29) is reduced to optimal stopping problem for bivariate geometric Brownian motion (π_t, I_t):

$$\mathbf{E} g(\pi_\tau, I_\tau) e^{-\rho\tau} \to \max_{\tau \in \mathcal{M}}, \tag{40}$$

with non-linear homogeneous (of first order) payoff function

$$g(\pi, I) = (1-\gamma)(\pi B - I) + \gamma I A(\nu(\pi/I)),$$

where $B,\ \nu(\cdot),\ A(\cdot)$ are defined above (we put formally $A(\infty) = 0$).

As one can see if τ^* is an optimal s.t. for the problem (40) and $\nu(\pi_{\tau^*}/I_{\tau^*}) < \infty$ (a.s.), then τ^* is the optimal investment time for the problem (29).

Let β be a positive root of the quadratic equation

$$\tfrac{1}{2}\tilde{\sigma}^2\beta(\beta-1)+(\alpha_1-\alpha_2)\beta-(\rho-\alpha_2)=0, \qquad \tilde{\sigma}^2=(\sigma_{11}-\sigma_{21})^2+\sigma_{22}^2.$$

Then Theorem 3 implies

Theorem 4. *Let $a_t, B_t \in C^1(\mathbb{R}_+)$ and all conditions of Theorem 3 hold. Then the optimal investment time for the problem (29) is $\tau^* = \min\{t \geq 0 : \pi_t \geq p^* I_t\}$, where p^* is a root of the equation*

$$\beta(1-\gamma) + \frac{\gamma a_{\nu(p)}}{p B_{\nu(p)}} = (1-\gamma)(\beta-1)pB + \beta\gamma A(\nu(p)).$$

More simple version of the above model as well as a proof of optimality of s.t. τ^*, which doesn't use Theorem 3, one can find in [14].

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