A Multi-Cycle Two-Factor Model of Asset Replacement

February 2009

João Z. Oliveira
Department of Management and Economics
University of Madeira
Campus da Penteada
9050-590 Funchal, Portugal
Tel: (351) 917 267 681
Fax: (351) 291 705 040
Email: jzo@uma.pt

João C. Duque
Institute of Economics and Management (ISEG)
Technical University of Lisbon
Rua Miguel Lupi, 20,
1249-078 Lisboa, Portugal
Tel: (351) 213 925 800
Fax: (351) 213 922 808
Email: jduque@iseg.utl.pt

A Multi-Cycle Two-Factor Model of Asset Replacement

Abstract: The aim of this article is to analyse the asset replacement problem in the perspective of optimal replacement time given a certain tax environment and depreciation policy. Using a real options approach, our model minimises current operation and maintenance costs and permits a new valuation of the replacement flexibility under a multi-cycle environment. The innovation on the valuation process comes from adding an autonomous salvage value factor. Results from partial differential equations reveal relevant differences from those observed in one-factor models, specifically in optimal replacement levels and in the non-monotonous effects of salvage value variation.

This paper provides enhancements to existing literature on equivalent annual cost by formulating a cost-minimisation problem conditioned by autonomous salvage value dynamics, and it contributes to Real Options literature by introducing a salvage value factor in the valuation model.

JEL classifications: D81, D92, H25

Keywords: Replacement, Real Options, Uncertainty, Equivalent Annual Cost, First Passage Time.

1 Introduction

One of the traditional approaches of determining asset optimal replacement level consists of the use of the minimum Equivalent Annual Cost (EAC) of assets in competition. This methodology assumes cost structure consistency, deterministic cost flows and known salvage value. The major problem of the deterministic EAC results from considering both implicit and explicit uncertainty. Rust (1985) tries to overcome some of these problems by adopting the presumption that higher cost values indicate bigger asset deterioration and by modelling cost accumulation as an arithmetic Brownian motion with constant drift and variance. Ye (1990) follows in this way, assuming that asset deterioration increases stochastically and considering a replacement process return to its initial state, each time occurs an asset swap. Mauer & Ott (1995) enhance Ye's model, modelling cost dynamics using a Geometric Brownian Motion (GBM).

2 Replacement model formulation

We could begin our analysis by using a single-asset model that sets salvage value as a function of operation and maintenance costs (OMC), where the critical cost level triggers the replacement process. Instead, our model sets a salvage value function to trigger asset replacement, which results from the following outcome:

$$\varsigma\left(C_t, S_t\right) = C_t S_t \tag{1.1}$$

This function It represents a product of costs C_t and salvage value S_t. Function $\varsigma(C_t, S_t)$ reflects a market functioning principle that costs and salvage value are inversely proportional (when operational and maintenance costs increase, salvage value decreases). Instead of assuming a constant proportion, our model assumes a mean-reversion process. Therefore, significant changes in $\varsigma(.)$ indicate relevant variations in C_t or in S_t. As a result, asset replacement should be observed when $\varsigma(.)$ exceeds a certain level $\varsigma^*(.)$, by an increase in either C_t or S_t. Therefore, each time $\varsigma(.)$ reaches a trigger level, it triggers the replacement of the current asset by another stochastically equivalent. For a risk-free interest rate r_f, the valuation expression is the following:

$$V(C_t, S_t, t) = \min_{\varsigma_t} E\left[\int_0^\infty \left(C_t(1-\tau) - \tau\delta^a\vartheta(C_t, S_t, t)\right)e^{-r_f t}dt\right] \tag{1.2}$$

The asset valuation is a function of cost flow that results from the difference between the after-tax costs $C_t(1-\tau)$ and the tax shield $\tau\delta^a\vartheta(C_t, S_t, t)$. In expression (1.2), C_t corresponds to OMC, τ is the tax rate, δ^a represents the depreciation rate, and $\vartheta(.)$ indicates the book value, which is given by:

$$\vartheta(t) = P(1-\varphi)e^{-\delta^a t}, \tag{1.3}$$

with P as the acquisition price and φ as the investment tax credit rate.

$$\hat{\vartheta}(C) = P(1-\varphi)\left(\frac{C}{C_N}\right)^{-\frac{\delta^a}{Z}} \tag{1.4}$$

The adoption of an infinite time horizon permits the relaxation of the functional dependence between t and the function $V(.)$. As a result, we consider the following diffusion processes:

$$dC = \alpha_C C dt + \sigma_C C dz_C \tag{1.5}$$

$$d\varsigma = \mu\left(\bar{\varsigma} - \varsigma\right)\varsigma dt + \sigma_\varsigma \varsigma dz_\varsigma \tag{1.6}$$

The expression (1.6) stems from the one described by Gibson & Schwartz (1990), which represents convenience yield properties as part of oil price evolution, and from another expression proposed by Dixit & Pindyck (1994). There is also strong evidence of a mean reversion presence in the futures market and agricultural products as well as some modest evidence of mean reversion in the financial markets (Bessembinderetal *et al.*, 1995).

Assuming the distribution of the risk using financial assets and using the contingent claims approach, it is possible to obtain the differential equation of $V(.)$:

$$\frac{1}{2}\sigma_C^2 C^2 V_{CC} + \alpha_C^* C V_C + \rho_{C\varsigma}\sigma_C\sigma_\varsigma C V_{C\varsigma} + \frac{1}{2}\sigma_\varsigma^2 V_{\varsigma\varsigma} + \left(\mu\left(\bar{\varsigma} - \varsigma\right)\right)V_\varsigma + C\left(1-\tau\right) - \tau\delta^a P\left(1-\varphi\right)\left(\frac{C}{C_N}\right)^{-\frac{\delta^a}{Z}} = r_f V \cdot \tag{1.7}$$

This contains the partial derivatives of $V(.)$ with respect to the variables C and ς, the risk-adjusted drift rate of costs $\alpha_C^* = r_f - \delta_C$, the risk-adjusted drift rate of salvage value $\alpha_S^* = r_f - \delta_S$, the risk free interest rate r_f and convenience yields δ_C, δ_S. The general solution of equation (1.7) is: (see Appendix 8.1)

$$V\left(C,\varsigma\right) = V_H\left(C,\varsigma\right) + k_A C + k_B C^\xi \tag{1.8}$$

with $V_H\left(C,\varsigma\right)$ described by:

$$V_H\left(C,\varsigma\right) = C^{\frac{\sigma_\varsigma}{\sigma_C}\left(1-\beta_1+k_C^2\right)}\varsigma^{\beta_1+k_C^2}\left[k_D H\left(h_1,h_2,h_3\right) + k_E L\left(l_1,l_2,l_3\right)\right], \tag{1.9}$$

where $k_{C,D,E}$, are constants, $H\left(h_1,h_2,h_3\right)$ is a hypergeometric function (see Appendix 8.2), $L\left(l_1,l_2,l_3\right)$ is a *Laguerre* polynomial (see Appendix 8.3), $\sigma_\varsigma,\sigma_C$ correspond to the standard deviation of ς, C, β_1 and the terms $k_{A,B}$ defined as:

$$k_A = \frac{1-\tau}{r_f - \alpha_C^*} \tag{1.10}$$

$$k_B = -\frac{\left(\frac{1}{C_N}\right)^{\xi} \tau\delta^a P(1-\varphi)}{r_f - \alpha_C^*\xi + \frac{1}{2}\xi\sigma_C^2 - \frac{1}{2}\xi^2\sigma_C^2} \tag{1.11}$$

In order to achieve the general solution of $V(C,\varsigma)$, we should determine constant values *k* using boundary equations described in the next section.

3 Boundary Conditions

After establishing a general solution, it is possible to determine a replacement critical level by applying appropriate boundary conditions set specifically to the problem. The first boundary condition, commonly called the value matching condition, sets equal the options to abandon and invest . This condition ensures the function's continuity at a critical level, by disregarding the difference in making an investment immediately before or just after the value passes the critical level:

$$V\left(C^*,\varsigma^*\right) = V\left(C_N,\varsigma_N\right) + P(1-\varphi) - \left[\frac{\varsigma^*}{C^*} - \tau\left\{\frac{\varsigma^*}{C^*} - \hat{\vartheta}\left(C^*\right)\right\}\right]. \tag{1.12}$$

Apart from ensuring a value-matching at critical level, we need to grant the same slope between the payoff and function $V(.)$as demonstrated in the following equations:

$$V_C\left(C^*,\varsigma^*\right) = V_C\left(C_N,\varsigma_N\right) - \tau\,\hat{\vartheta}_C\left(C^*\right), \tag{1.13}$$

$$V_\varsigma\left(C^*,\varsigma^*\right) = V_\varsigma\left(C_N,\varsigma_N\right) - 1 + \tau\ , \tag{1.14}$$

with the book value derivative $\hat{\vartheta}_C(C)$:

$$\widehat{\vartheta}_C(C) = -\frac{\delta^a}{Z}P(1-\varphi)\left(\frac{C}{C_N}\right)^{-\frac{\delta^a}{Z}-1} \tag{1.15}$$

When costs reach very high values, assets can be replaced before being completely written off. Therefore, the cost unitary increase leads to an increase in $V(.)$. To determine the value of that growth, we consider an expression representing the present value of expected costs:

$$V(C,\varsigma) = (1-\tau)\int_0^\infty e^{-r_f t}Ce^{\alpha_C^* t} - \tau\delta^a P(1-\varphi)\left(\frac{1}{C_N}\right)^\xi \int_0^\infty e^{r_f t}C\exp\left\{\left(\xi\alpha_C^* + \frac{1}{2}\xi(\xi-1)\sigma_C^2\right)t\right\}, \tag{1.16}$$

$$V(C,\varsigma) = \frac{(1-\tau)}{\delta_C}C - \frac{\tau\delta^a P(1-\varphi)\left(\frac{1}{C_N}\right)^\xi}{r_f - \xi\alpha_C^* - \frac{1}{2}\xi(\xi-1)\sigma_C^2}C^\xi, \tag{1.17}$$

being $\xi = -\frac{\delta^a}{Z}$ and δ_C representing the convenience yield. Partial derivative of $V(C,\varsigma)$ in respect of C is given by:

$$V_C(C,\varsigma) = \frac{1-\tau}{\delta_C} + \frac{\left(\frac{1}{C_N}\right)^\xi \tau\delta^a P(\varphi-1)}{r_f - \alpha_C^*\xi + \frac{1}{2}\xi\sigma_C^2 - \frac{1}{2}\xi^2\sigma_C^2}\xi C^{\xi-1}, \tag{1.18}$$

As $\xi < 0$:

$$\lim_{C\to\infty}\left(\frac{\left(\frac{1}{C_N}\right)^\xi \tau\delta^a P(\varphi-1)}{r_f - \alpha_C^*\xi + \frac{1}{2}\xi\sigma_C^2 - \frac{1}{2}\xi^2\sigma_C^2}\xi C^{\xi-1}\right) = 0 \tag{1.19}$$

which results in the following boundary equation:

$$\lim_{C\to\infty} V_C(C,\varsigma) = \frac{1-\tau}{\delta_C} \tag{1.20}$$

When S reaches zero and ς takes equal value, then the cost function is no longer affected by salvage value. In this situation, function $V(.)$ exclusively depends on C:

$$V(C,0)=V(C) \tag{1.21}$$

Defining $V(.)$ in R^2, Dineen (2000) says that there exists a local minimum (C_N,ς_N) where:

$$V_C(.)=0, V_\varsigma(.)=0 \tag{1.22}$$

$$V_{CC}(.)>0, V_{\varsigma\varsigma}(.)>0, V_{C\varsigma}(.)>0 \tag{1.23}$$

and where following conditions are satisfied:

$$V_C(C_N,\varsigma_N)=0, V_\varsigma(C_N,\varsigma_N)=0 \tag{1.24}$$

These last conditions play the rule of not allowing $V(C,\varsigma)$ to take values less than $V(C_N,\varsigma_N)$.

4 Numerical case

To test this new model, we consider a parameter set for which we calculate the optimal replacement time. We've assumed a two-factor cost function where one factor (salvage factor) follows a mean-reverting process and the other factor behaves as a geometric Brownian motion. The factor ς has a mean-reversion rate μ, a standard salvage factor ς_S and a standard deviation σ_ς whose values are contained in Table 4-1.

Table 4-1: Mean Reverting Parameters

Parameter	Symbol	Value
Instantaneous standard deviation	σ_ς	0,2
Initial value of salvage factor	ς_N	8
Standard level of salvage factor	ς_S	8
Mean-reversion rate	μ	0,5

Regarding parameter values, we consider a salvage factor standard deviation that is higher σ_ς than the cost standard deviation σ_C. For ς_S, we adopt a value of 8, which represents 80% of purchase value. As we need a fast rate of mean reversion, it has been adopted a value of $\mu = 0,5$.

5 Characteristics of solution

Given the assumption of a constant tax regime and a two factor cost function, Table 5-1 presents the outcomes of the numerical case described in Appendix 8.4:

Table 5-1: Numerical solution of the two factor cost function

#	C*	S*	E[T*]	V(C*)	V(CN)
1	1,137	5,999	1,085	82,184	78,325
2	1,148	6,746	1,222	93,054	88,630

The values inside Table 5-1 confirm a slightly positive change in critical level value, resulting from considering a new two-factor cost function on multi-cycle environment. When we compare critical cost values obtained from various models, it is possible to find substantial differences in cost critical level, supporting the notion that flexibility assessments produced by previous asset replacement models are incorrect. In Table 5-1, the main difference between cost function #1 and cost function #2 is in the way salvage value has been considered (directly or implicitly). Another relevant difference is in the replacement cycles' number, which was assumed.

6 Sensibility Analysis

In this section, we refine our analysis, changing some parameter values:

1. The value of mean reverting rate μ will vary between $\mu = 0.40$ and $\mu = 0.60$, at intervals of 0.10;
2. The standard value ς_S will vary between $\varsigma_S = 6$ and $\varsigma_S = 10$ at intervals of 2;
3. The standard deviation value σ_ς will vary between $\sigma_\varsigma = 0.10$ and $\sigma_\varsigma = 0.30$, at intervals of 0.10.

In order to follow the mean reverting rate ς in the direction of ς_S, we study the effect on replacement critical level by parameters that constitute mean reverting rate ς. From Paxson (2005), it is possible to know that incentive for increasing investment is related to the increase in the speed of mean reversion, given the lack of the variance of the long-term technology factor.

Table 6-1: Effect of changing the mean reverting rate

[redacted]

In Table 6-1, we can observe some changes in C* as a result of changes in the mean reverting rate. It also possible to see an increase in the replacement level C* for higher values of mean reverting rate μ. When μ takes higher values, there is a faster return to the average value of the salvage factor, and the period between asset replacement $E\left[T^*\right]$ increases. Given a standard salvage factor ς_S, we test the model's response to this parameter's changing ($\varsigma_S \neq 8$). From literature, when we give ς_S higher values, we will expect an enlargement between *C* and *S* values because these two parameters are complementary. For similar reasons, low values of ς_S will tend to limit the ranges of *C* and *S*.

Table 6-2: Effect of changes in standard salvage factor

ς_S	C*	S*	E[T*]	V(C*)	V(CN)
6	1,153	6,458	1,278	93,626	88,970
8	1,148	6,746	1,222	93,054	88,630
10	1,145	6,943	1,184	92,652	88,387

We have filled the table above with changes in the main parameters of the replacement model resulting from changes in the standard factor. Its observation allows us to collect evidence that lower values of replacement level C^* and thinner replacement periods $E\left[T^*\right]$ result from higher values of ς_S. The reason associated with this behaviour is that higher levels of salvage value joined with lower costs make investing in assets more attractive. We were expecting a similar effect from introducing volatility σ_ς but, in this case, variation will depend on the magnitude of σ_C and σ_ς relationship. This yields the assumption that $\sigma_\varsigma > \sigma_C$ determines the direction of critical cost variation C^*. In Dobbs (2002), periodic asset exchange accelerates from including σ_ς and reducing replacement critical level.

Table 6-3: Effect of changes in the salvage factor standard deviation

σ_ς	C*	S*	E[T*]	V(C*)	V(CN)
0,10	1,154	6,393	1,291	102,524	97,816
0,20	1,148	6,746	1,222	93,054	88,630
0,30	1,126	8,077	0,953	87,307	83,980

It can be seen in Table 6-3 that variation in volatility has significant effects on the replacement critical level C^*. As a result, it's possible to observe that lower critical values C^* result from higher volatility σ_ς values. A possible explanation for this behaviour seems to lie in the fact that higher volatility levels of salvage factor could create more opportunities for reaching earlier optimal replacement levels, either by increasing cost value or by increasing salvage value.

7 Conclusions

Our work demonstrates a new asset replacement policy based on the ability to measure salvage value's hidden flexibility. Previous literature determines replacement level using a one-factor function. This article enhances the cost function formulation, introducing an innovative dynamic salvage factor. It adapts a constant factor to a salvage value mean reversion process, where the standard salvage factor equals the previous constant salvage factor. This way of addressing the problem of optimal asset replacement assumes a multi-cycle and a constant tax regime. Comparing these results with the previous one found in Oliveira & Duque (2007), we found no significant differences, in spite of using a unique-cycle framework and a geometric Brownian motion to directly emulate salvage value. When we decided not to admit first degree homogeneity, we assumed the risk of not obtaining analytical solutions. As a result, main results have been obtained from numerical solutions of the numerical case illustrated in Appendix 8.4. Thus, our model provides the ability to predict replacement timing, demonstrating the salvage value's relevance to the asset replacement process. Our next step will be to extend this study to a variable tax regime environment.

8 Appendices

8.1 General solution of the differential equation

In order to determine the general solution of the homogeneous equation (1.7), we use the method of characteristics (Polyanin, 2001) to swap it to its canonical form with a new coordinate system. This swapping will allow us to determine a partial differential equation on which it is possible to separate variables (Weinberger, 1995). Following Polyanin (2001), we consider a general form of a second-order partial differential equation:

$$aV_{CC} + 2bV_{C\varsigma} + cV_{\varsigma\varsigma} + dV_C + eV_\varsigma + fV = g\,, \tag{1.25}$$

where a, b, c, d, e, f, g are equation coefficients. Including the new coordinate system (θ, η), we obtain a function $V(\theta, \eta)$ whose canonical form is:

$$V_{\theta\theta} = \phi\left(\theta, \eta, V, V_\theta, V_\eta\right).$$

Adjusting to equation , we obtain the following coefficients:

$$a = \frac{1}{2}\sigma_C^2 C^2\,,\ b = \frac{1}{2}\sigma_C\sigma_\tau C\rho_{C\tau}\,,\ \text{e}\ \ c = \frac{1}{2}\sigma_\varsigma^2\,,$$

From which results the determinant expression:

$$b^2 - ac = \frac{1}{4}\sigma_C^2\sigma_\varsigma^2 C^2\left(\rho_{C\varsigma}^2 - 1\right).$$

If we assume Equation (1.7) as a parabolic equation, we need to switch between coordinates systems $(C,\varsigma)->(\theta,\eta)$ and solve the characteristic equation:

$$\eta_C + \frac{\sigma_\varsigma}{\sigma_C}\frac{1}{C}\eta_\varsigma = 0,$$

with $\eta_C = \frac{\partial\eta}{\partial C}$ and $\eta_\varsigma = \frac{\partial\eta}{\partial\varsigma}$, from which we obtain the following solution:

$$\frac{d\varsigma}{dC} = \frac{b}{a} = \frac{\sigma_\varsigma}{\sigma_C}\frac{\varsigma}{C},$$

where:

$$\frac{d\varsigma}{\varsigma} = \frac{\sigma_\varsigma}{\sigma_C}\frac{dC}{C},$$

$$ln(\varsigma) = \frac{\sigma_\varsigma}{\sigma_C} ln(C) + \varsigma_0,$$

with constant ς_0. Applying the exponential function to both sides of the previous equation, we determine the value of ς

$$\varsigma = C^{\frac{\sigma_\varsigma}{\sigma_C}} e^{\varsigma_0}.$$

As $\varsigma_1 = e^{\varsigma_0}$,

$$\varsigma_1 = \frac{\varsigma}{C^{\frac{\sigma_\varsigma}{\sigma_C}}}.$$

Equalising $\eta(C,\varsigma) = \varsigma_1$,

$$\eta(C,\varsigma) = \frac{\varsigma}{C^{\frac{\sigma_\varsigma}{\sigma_C}}}.$$

For function θ, we can choose any function that intercepts other characteristic curve, as:

$$\theta(C,\varsigma)=C\ .$$

Determining partial derivatives of $\eta(.)$ and $\theta(.)$ with respect to C and ς, we obtain:

$$\eta_C(C,\varsigma)=-\frac{\sigma_\varsigma}{\sigma_C}\frac{\varsigma}{C^{\frac{\sigma_\varsigma}{\sigma_C}+1}},$$

$$\eta_\varsigma(C,\varsigma)=\frac{1}{C^{\frac{\sigma_\varsigma}{\sigma_C}}},$$

$$\theta_C(C,\varsigma)=1,$$

$$\theta_\varsigma(C,\varsigma)=0,$$

with $\theta_C=\frac{\partial\theta}{\partial C}$ and $\theta_\varsigma=\frac{\partial\theta}{\partial\varsigma}$.

Given $v(\theta,\eta)=V(C,\varsigma)$

$$V_C=\theta_C v_\theta+\eta_C v_\eta,$$

$$V_\varsigma=\theta_\varsigma v_\theta+\eta_\varsigma v_\eta,$$

from which we obtain the following expressions:

$$V_C=v_\theta-\frac{\sigma_\varsigma}{\sigma_C}\frac{\varsigma}{C^{\frac{\sigma_\varsigma}{\sigma_C}+1}}v_\eta,$$

$$V_\varsigma=\frac{1}{C^{\frac{\sigma_\varsigma}{\sigma_C}}}v_\eta,$$

$$V_{CC}=v_{\theta\theta}-\frac{2\sigma_\varsigma}{\sigma_C}\frac{\varsigma}{C^{\frac{\sigma_\varsigma}{\sigma_C}+1}}v_{\theta\eta}+\left(\frac{\sigma_\varsigma}{\sigma_C}\frac{\varsigma}{C^{\frac{\sigma_\varsigma}{\sigma_C}+1}}\right)^2 v_{\eta\eta},$$

$$V_{\varsigma\varsigma} = \left(\frac{1}{C^{\frac{\sigma_\varsigma}{\sigma_C}}}\right)^2 v_{\eta\eta},$$

$$V_{C\varsigma} = \frac{1}{C^{\frac{\sigma_\varsigma}{\sigma_C}}} v_{\theta\eta} - \frac{\sigma_\varsigma}{\sigma_C}\frac{\varsigma}{C^{\frac{2\sigma_\varsigma}{\sigma_C}+1}} v_{\eta\eta},$$

Substituting previous expressions in the following equation:

$$\frac{1}{2}\left(V_C\sigma_C C + V_\varsigma\sigma_\varsigma\right)^2 = \frac{1}{2}V_{CC}\sigma_C^2 C^2 + V_{C\varsigma}\sigma_C\sigma_\varsigma C + \frac{1}{2}V_{\varsigma\varsigma}\sigma_\varsigma^2 \tag{1.26}$$

we obtain its canonical form:

$$\frac{1}{2}\sigma_c^2\theta^2 v_{\theta\theta} + \alpha_C^*\theta v_\theta + \left(\left(\bar{\varsigma} - \left(\theta^{\frac{\sigma_\varsigma}{\sigma_C}}\eta\right)\right)\mu - \frac{\alpha_C^*\sigma_\varsigma}{\sigma_C}\right)\eta v_\eta = r_f v \tag{1.27}$$

In order to solve equation (1.27) using variable separation (Weinberger, 1995), we introduce some new variables with the following values:

$$m = \theta^{\frac{\sigma_\varsigma}{\sigma_C}}\eta \text{ e } n = \eta \tag{1.28}$$

Setting the equivalence $v(\theta,\eta) = v(m,n)$ yields the following expressions:

$$v_\theta = \frac{\sigma_\varsigma}{\sigma_C}\frac{1}{\theta}v_m, \tag{1.29}$$

$$v_\eta = \frac{1}{n}v_m + v_n \tag{1.30}$$

After substituting them in equation (1.27), it is possible to confirm this expression:

$$\frac{1}{2}\sigma_\varsigma^2 v_{mm} = -(\bar{\varsigma} - m)\mu v_m - \left((\bar{\varsigma} - m)\mu - \alpha_C^*\frac{\sigma_\varsigma}{\sigma_C}\right)n v_n + r_f v. \tag{1.31}$$

After doing a variable switching in order to obtain $v(m,n)=v(q,r)$ and having m and n given by expressions (1.28), function $v(q,r)$ takes the following form:

$$v(q,r)=Q(q)R(r), \tag{1.32}$$

which represents the product of two functions and permits a solution based on two ordinary differential equations. Adopting equal notation:

$$Q'=\frac{dQ}{dq} \text{ and } R'=\frac{dR}{dr},$$

from which we can take following equalities:

$$v_m=v_q=Q'R,$$

$$v_n=v_r=QR',$$

$$v_{mm}=v_{qq}=Q''R.$$

Applying these expressions to equation (1.31) permits the assembly of an expression like this one:

$$\frac{1}{2}\sigma_\varsigma^2 Q''R+\left(\left(\bar{\varsigma}-q\right)\mu\right)Q'R+\left(\left(\bar{\varsigma}-q\right)\mu-\alpha_C^*\frac{\sigma_\varsigma}{\sigma_C}\right)rQR'-r_fQR=0, \tag{1.33}$$

with the free-risk interest rate r_f. We manipulate equation (1.33) in order to split components that are functions of Q or of R:

$$\frac{\frac{1}{2}\sigma_\varsigma^2 Q''+\left(\left(\bar{\varsigma}-q\right)\mu\right)Q'-r_fQ}{\left(\left(\bar{\varsigma}-q\right)\mu-\alpha_C^*\frac{\sigma_\varsigma}{\sigma_C}\right)Q}=-r\frac{R'}{R}=-k^2 \tag{1.34}$$

with a constant ratio $-k^2$ (Abell and Braselton, 1997). From previous expression we can create the following ordinary differential equations:

$$\frac{1}{2}\sigma_{\varsigma}^{2}Q''+\left(\left(\bar{\varsigma}-q\right)\mu\right)Q'+\left[k^{2}\left(\left(\bar{\varsigma}-q\right)\mu-\alpha_{C}^{*}\frac{\sigma_{\varsigma}}{\sigma_{C}}\right)-r_{f}\right]Q=0, \qquad (1.35)$$

and

$$rR'-k^{2}R=0. \qquad (1.36)$$

For equation (1.35) there exists a general solution of the form (Abramowitz & Stegun, 1965):

$$Q(q)=q^{\beta_1}\left[K_1 H\left(h_1,h_2,h_3\right)+K_2 L\left(l_1,l_2,l_3\right)\right], \qquad (1.37)$$

where $K_{1,2}$ are constants and $H(.)$ is confluent hypergeometric function pleaded with the following parameters:

$$\beta_1=-\frac{2\bar{\varsigma}\mu-\sigma_{\varsigma}^{2}-\dfrac{\sqrt{4\bar{\varsigma}^{2}\mu^{2}\sigma_{\varsigma}-4\left(1+2k\right)\bar{\varsigma}\mu\sigma_{C}\sigma_{\varsigma}^{2}+\sigma_{\varsigma}^{2}\left(8r_{f}\sigma_{C}+8k\alpha_{C}^{*}\sigma_{C}\sigma_{\varsigma}^{2}\right)}}{\sqrt{\sigma_{C}}}}{2\sigma_{\varsigma}^{2}},$$

$$h_1=-\frac{1}{2\sigma_{\varsigma}^{2}}\left(2\bar{\varsigma}\mu-\sigma_{\varsigma}^{2}-2k\sigma_{\varsigma}^{2}-\sigma_{\varsigma}^{2}\sqrt{\frac{1}{\sigma_{C}\sigma_{\varsigma}^{4}}\left(4\left(\bar{\varsigma}\mu\right)^{2}\sigma_{C}+8r_{f}\sigma_{C}\sigma_{\varsigma}^{2}-4\bar{\varsigma}\mu\sigma_{C}\sigma_{\varsigma}^{2}-8k\bar{\varsigma}\mu\sigma_{C}\sigma_{\varsigma}^{2}+8k\alpha_{C}^{*}\sigma_{\varsigma}^{3}+\sigma_{C}\sigma_{\varsigma}^{4}\right)}\right)$$

$$h_2=1+\frac{1}{\sqrt{\sigma_{C}}\sigma_{\varsigma}^{2}}\left(\sqrt{\left(4\left(\bar{\varsigma}\mu\right)^{2}\sigma_{C}+8r_{f}\sigma_{C}\sigma_{\varsigma}^{2}-4\bar{\varsigma}\mu\sigma_{C}\sigma_{\varsigma}^{2}-8k\bar{\varsigma}\mu\sigma_{C}\sigma_{\varsigma}^{2}+8k\alpha_{C}^{*}\sigma_{\varsigma}^{3}+\sigma_{C}\sigma_{\varsigma}^{4}\right)}\right),$$

$$h_3=\frac{2q\mu}{\sigma_{\varsigma}^{2}},$$

and $L(.)$ represents a *Laguerre* generalised polynomial with these parameters:

$$l_1=-\frac{1}{2\sigma_{\varsigma}^{2}}\left(2\bar{\varsigma}\mu-\sigma_{\varsigma}^{2}-2k\sigma_{\varsigma}^{2}-\sigma_{\varsigma}^{2}\sqrt{\frac{1}{\sigma_{C}\sigma_{\varsigma}^{4}}\left(4\sigma_{C}\left(\bar{\varsigma}\mu\right)^{2}+8r_{f}\sigma_{C}\sigma_{\varsigma}^{2}-4\bar{\varsigma}\mu\sigma_{C}\sigma_{\varsigma}^{2}\left(1+2k\right)+8k\alpha_{C}^{*}\sigma_{\varsigma}^{3}+\sigma_{C}\sigma^{4}\right)}\right)$$

$$l_2=\frac{1}{\sqrt{\sigma_{C}}\sigma_{\varsigma}^{2}}\left(\sqrt{4\bar{\varsigma}^{2}\mu^{2}\sigma_{C}+8r_{f}\sigma_{C}\sigma_{\varsigma}^{2}-4k\mu\sigma_{C}\sigma_{\varsigma}^{2}-8k\bar{\varsigma}\mu\sigma_{C}\sigma_{\varsigma}^{2}+8k\alpha_{C}^{*}\sigma_{\varsigma}^{3}+\sigma_{C}\sigma_{\varsigma}^{4}}\right),$$

$$l_3=h_3=\frac{2q\mu}{\sigma_{\varsigma}^{2}}.$$

Developing equation (1.36) to seek a function $R(.)$:

$$\frac{dR}{R} = k^2 \frac{dr}{r},$$

$$\ln(R) = \ln\left(r^{k^2}\right) + r_0, \tag{1.38}$$

with constant r_0. Applying the exponential function to both sides of equation (1.38), we obtain:

$$R(r) = K_r r^{k^2}$$

for $K_r = e^{r_0}$. Thus, function $v(.)$ (1.32) takes the following form:

$$v(q,r) = q^{\beta_1} r^{k^2} \left[K_1 K_r H(h_1,h_2,h_3) + K_2 K_r (l_1,l_2,l_3)\right] \tag{1.39}$$

Setting the equivalences $K_A = K_1 K_r$ and $K_B = K_2 K_r$, we then have:

$$v(q,r) = q^{\beta_1} r^{k^2} \left[K_A H(h_1,h_2,h_3) + K_B L(l_1,l_2,l_3)\right]. \tag{1.40}$$

Unfolding the above expression for the original coordinate system, we get:

$$V_H(C,\varsigma) = C^{\frac{\sigma_\varsigma}{\sigma_C}\left(1-\beta_1+k^2\right)} \varsigma^{\beta_1+k^2} \left[K_A H(h_1,h_2,h_3) + K_B L(l_1,l_2,l_3)\right] \tag{1.41}$$

Henceforth, $V_H(C,\varsigma)$ will represent a homogenous part of the two-factor cost function $V(C,\varsigma)$, whose parameters are already set out above, with the exception of l_3 and h_3, which take the following form:

$$l_3 = h_3 = \frac{2\varsigma\mu}{\sigma_\varsigma^2}.$$

8.2 Hypergeometric function

$H(.)$ is a hypergeometric confluent function, also known as *Kummer's* function, whose value can be taken from the following series expansion:

$$
{}_1H_1\left(a,b,z\right)=\sum_{n=0}^{\infty}\frac{\left(a\right)_n}{\left(b\right)_n}\frac{z^n}{n!},
$$

where a and b are integers and z is the variable. The function representation in integral form is:

$$
{}_1H_1\left(a,b,z\right)=\frac{\Gamma\left(b\right)}{\Gamma\left(b-a\right)\Gamma\left(a\right)}\int_0^{\infty}e^{-zt}t^{a-1}\left(1+t\right)^{b-a-1}dt, \tag{1.42}
$$

with $\Gamma\left(a\right)$ equal to the following integral sum:

$$
\Gamma\left(a\right)=\int_0^{\infty}t^{a-1}e^{-t}dt, \tag{1.43}
$$

where a and b represent polynomial degrees.

8.3 Laplace Function

Generalised *n degree Laguerre polynomial*, $L_{n,a}\left(.\right)$ is given by:

$$
L_{n,a}\left(z\right)=\frac{\left(a+1\right)_n}{n!}\,{}_1H_1\left(-n,a+1,\varsigma\right), \tag{1.44}
$$

for $a>-1$. Substituting t for n , one defines *Laguerre* function as:

$$
L_{n,a}\left(z\right)=\frac{1}{\Gamma\left(t+1\right)}\,{}_1H_1\left(-t,a+1,z\right), \tag{1.45}
$$

whose value corresponds to a multiple of an *n*-degree *Laguerre* polynomial.

8.4 Numerical Case

Parameter	**Symbol**	**Value**
Risk free interest rate	r_f	0.07
Cost Drift Rate	α_C	0.06
Cost Volatility	σ_C	0.10
Market Risk Price	η	0.4
Initial operation and maintenance	C_N	1

Asset purchasing price	P	10
Tax credit rate	φ	0
Tax rate	τ	0.30
Depreciation rate	δ^a	0.50

9 References

Abell, M. e Braselton, J. (1997). *Differential Equations with Mathematica*. London: Academic Press.

Adkins, R. (2005). *Real options analysis of capital equipment replacement under revenue uncertainty*. Working paper, 106. University of Stanford.

Bernstein, P. e Damodaran, A. (1998). *Investment Management*. New York: John Wiley & Sons.

Bessembinder, H., Coughenour, J., Seguin, P. e Smoller, M. (1995). Mean Reversion in Equilibrium Asset Prices: Evidence from the Futures Term Structure. *Journal of Finance*, **50** (1), 361-375.

Brach, M. (2002). *Real Options in Practice*. New Jersey: Wiley, John & Sons, Incorporated.

Dineen, S. (2000). *Functions of Two Variables.* Chapman Hall/CRC Press.

Dixit, A. (1989). Entry and exit decisions under uncertainty. *Journal of Political Economy*, **97**, 620-38.

Dixit, A. e Pindyck, R. (1994). Investment under uncertainty. *Princeton University Press*.

Dobbs, I. (2002). *Replacement investment: Optimal economic life under uncertainty.* Working paper. University of Newcastle.

Keles, P., & Hartman, J. (2004). Case study: Bus fleet replacement. *The Engineering Economist,* **49** (3), 253–278.

Gibson, R. & Schwartz, E. (1990). Stochastic Convenience Yield and the Pricing of Oil Contingent Claims. *Journal of Finance*, **45**, 959-976.

Hull, J. (1993). *Options, Futures and Other Derivatives.* New Jersey: Prentice Hall.

Margrabe, William (1978). The value of an option to exchange one asset for another. *Journal of Finance*, **33** (1), 177-186.

Mauer, D. & Ott, S. (1995). Investment under Uncertainty: The Case of Replacement Investment Decisions. *Journal of Financial and Quantitative Analysis*, **30**, 581-605.

Merton, R. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science*, **4**, 141-183.

Mun, J. (2003). *Options Analysis Course: Business Cases and Software Applications*. New Jersey: Wiley, John & Sons, Incorporated.

Oliveira, J. & Duque, J. (2007). Asset Management: A replacement model under uncertainty. *Proceedings of the Global Management 2007, International Association for the Scientific Knowledge*.

Polyanin, A. (2001). *Handbook of Linear Partial Differential Equations for Engineers and Scientists.* London: CRC Press.

Rust, J. (1985). Stationary Equilibrium in a Market for Durable Assets. *Econometrica*, **53** (4), 783-805.

Ye, M.H. (1990). Optimal Replacement Policy with Stochastic Maintenance and Operation Costs. *European Journal of Operational Research*, **44**, 84-94.

Weinberger, H. (1995). *A First Course in Partial Differential Equations With Complex Variables and Transform Methods*. New York: Courier Dover Publications.