# Entry and Exit Decision Problem with Implementation Delay: The Probabilistic Approach*

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**Abstract**

Using the probabilistic approach we analyze the investment and disinvestment decisions under the Parisian implementation delay. We prove the independence between the Parisian stopping times and solve the constrained maximization problem obtaining an implicit solution for the optimal investment level and an explicit one for the optimal disinvestment level. A sufficient condition for solving the unconstrained maximization problem and obtaining the Parisian optimal-levels correctly ordered is derived. Also, an interesting contact with Wald's identity is discussed. Finally, comparing the results with the instantaneous entry and exit case, we show that an increase in the uncertainty of the underlying process hastens the decision to invest (disinvest), extending the results of the current literature.

**Keywords**: Brownian excursion, Implementation Delay, Parisian Option, Optimal Stopping, Wald's Identity.

**JEL Classifications**: C60, C61, C65, G13.

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# 1 Introduction

The first and simplest real option model was pioneered by Mossin (1968), but a formal discussion on the investment decision problem, also called entry decision problem, was initially proposed by McDonald and Siegel (1986). The authors claim that a firm with the opportunity to invest into a project possesses the option to wait for better conditions before starting to implement the investment. This option is in fact very much like a financial derivative where the underlying consists of the economic variables that will condition the future value of the project (for instance, the value of the product bought or sold by the firm).

The combined investment and disinvestment decisions problem, also called entry and exit decision problem, was initially discussed by Brennan and Schwartz (1985). Applying the option pricing theory developed by Black, Merton and Scholes in the 1973 , they evaluated active and inactive firms, and defined the concepts of option to enter and option to abandon as part of the firms value. A formal and complete discussion of the combined problem using differential equations was presented by Dixit (1989). In particular, he focused on entry and exit trigger prices as fundamental indicators for firms decision policies. A mathematical rigorous treatment of the investment and disinvestment decisions problem was proposed by Brekke and Oksendal (1994). The authors analyzed the entry-exit decisions problem by applying both the option pricing theory and the dynamic programming theory, gave a formal proof of the existence of a solution, and extended the classical approach considering the case of a finite resource.

Although these papers have made a great step toward a better understanding of the investment and disinvestment decisions, they assume that the project is brought on-line immediately after the decision to invest is made, and similarly for the disinvestment. As such, the complexity of these decisions, or the constraints under which they are taken, is not properly modeled. A critical factor is the presence of a delay between the decision to invest (or disinvest) and the implementation process. Numerous investment (disinvestment) decisions, from a practical point of view, are characterized by a significant implementation delay.[1]

For instance, one of the major characteristics of the capital budgeting process is the delay existing between the investment decision and its implementation. This implementation delay is generally associated with the decision process within the firm or the gathering of the financing funds necessary to undertake the investment or disinvestment spending. Harris and Raviv (1996) assert that projects are generally initiated from the bottom up, suggesting a centralization of the capital allocation process. Depending on the nature and the size of the investment, projects that have been approved at the division level may have to be submitted to headquarters. This

[1] Our analysis differs significantly from the "construction-lag" or "time-to-build" literature, where the lag refers to the time between the decision to invest and the receipt of the project's first revenues (see Majd and Pindyck (1987) and Pindyck (1991, 1993)). In our case the lag measures a systematic delay that comes before the investment (disinvestment) project takes effectively place, i.e. before the consequence of hitting the trigger price gets activated.

requires time. Thought the capital budgeting process offers a typical example of investment in the presence of implementation delay (disinvestment with implementation delay exists too), numerous industrial-production processes are characterized by an implementation delay in the decision to invest and disinvest.[2] An interesting example, among others, is observable in the energy-industry. In particular, we refer to fuel-burning utilities that have co-production options, i.e. they generate electricity burning either gas or coal. Each facility conveniently adapts its fuel-inputs according to the price evolution of these factors on international exchanges. However, a utility needs some time to implement the fuel-switching.[3] Ideally, the decision to fuel-switch takes place when the underlying variable, i.e. the fuel price, hits a pre-specified barrier level. In fact, the existence of physical and technical constraints permit the implementation of the new production-process only after a given time-interval (see Burtraw (1996), and Tauchmann (2006)).

Although the existence of such a delay is very common in practice, the problem has not received much attention in the literature whereas it has relevant consequences. In fact, depending on the evolution of the decision variable during this implementation lag, the investment (disinvestment) opportunity may have lost part of its attractiveness. Gauthier and Morellec (2000) have discussed the implementation delay which affects the investment decision in the capital budgeting process. The paper addresses the issue of the investment decision under the Parisian implementation delay using the probabilistic approach. The Parisian-delay reflects the will of any firm to verify that the market conditions remain favorable during the implementation-lag of the investment decision.

The Parisian criterion originates from a relatively new type of financial option-contracts introduced by Chesney et al. (1997) and termed Parisian options. Such a contract corresponds to a generalization of a barrier-type option. More precisely, a Parisian option gets activated (resp. disactivated) if the underlying process has spent a sufficient amount of time above (resp. below) the barrier level. Literature addressing mathematical and computational aspects related to this new option-contract is extensive (see Schröder (2003), Avellaneda and Wu (1999) and Haber et al. (1999), and references therein). However, up to our knowledge, only Gauthier and Morellec (2000) use the Parisian criterion for the appraisal of investments in a real-option context.

In common with the last-mentioned paper, we apply the probabilistic approach but we discuss the combined investment and disinvestment decisions under the Parisian implementation delay. This approach leads to more tractable valuation results compared to the PDE approach. Relying on standard mathematical results, we prove the independence between the Parisian stopping times under both historical and a new probability measure ($\mathbb{P}^*$). This makes our study significantly different from the study in Gauthier and Morellec (2000). Also, an interesting contact with Wald's

[2] In the industrial production context, invest indicates the decision to undertake or start a production process, whereas disinvest indicates the decision to undo or change a specific production process.

[3] Fuel-switching is the extend to which a producer can reduce the use of one type of energy - coal, for instance - and uptake of another source of energy - gas - in its place.

identity is discussed. Furthermore, we derive a sufficient condition for solving the unconstrained maximization problem and we obtain the Parisian optimal-levels which are correctly ordered (the investment level is higher than the disinvestment level). Finally, a numerical exercise is performed to compare the investment and disinvestment decisions problem with implementation delay and the instantaneous investment and disinvestment decisions problem. Our results confirm that an increase in uncertainty delays (instantaneous) investments (see Pindyck (1991)) and that the value of the investment and disinvestment decision problems is lower under the Parisian criterion than in the instantaneous case (see Gauthier and Morellec (2000) for the investment case). Moreover, extending the results of Bar-Ilan and Strange (1996) to the disinvestment case, we show that an increase in the uncertainty of the underlying process hastens the decision to invest (disinvest).

The paper is organized as follows. In Section 2 we introduce the model for the investment and disinvestment decisions problem in the context of the Parisian stopping times. In Section 3 we solve the constrained maximization problem and derive the sufficient condition to obtain the optimal triggering levels correctly ordered. Section 4 concludes with a numerical comparison. In the Appendix we show under which conditions the Parisian time and the drifted Brownian motion stopped at that time are independent. Furthermore, we make a connection with Wald's identity in the context of Parisian stopping time. Finally, we compute the Laplace transform of the Parisian investment (resp. disinvestment) time under a new measure; we calculate the moment generating function for the underlying process stopped at the Parisian investment (resp. disinvestment) time; and we evaluate the first hitting time of the underlying process which starts from the Parisian investment time.

## 2 Model

One of the major characteristic of several decision processes is the delay existing between the investment (resp. disinvestment) decision and its implementation. As mentioned in the introduction, in the capital budgeting process the implementation delay that affects investment decisions can be due to the research of an investment opportunity on an emerging market or to the time spent gathering the financing funds. Different reasons, instead, cause implementation delays which typically affect both the investment and disinvestment decisions in several industrial-production processes. In order to keep the following presentation simple, we consider the simplest environment possible in the case of a capital budgeting process.

As in Harris and Raviv (1996), our firm is composed of headquarters and a single division. The investment decision is initiated by the division manager which must obtain capital from the headquarters. The disinvestment decision is initiated by the headquarters and implemented by the division. In both situations an implementation lag due to complex economical and legal situations is present. The decentralization of the investment and disinvestment decisions is attributable to the specific human capital of the division manager. We assume that the division manager has no incentive to misrepresent his information, however the process of information-transfer within the firm-divisions and the decisions concerning the capital allocation take time. Furthermore, we focus on discretionary investments (disinvestments) generally characterized by such bottom-up (bottom-down) process.

In our setting agents are risk-neutral and the firm has an investment opportunity in a non-traded asset yielding stochastic returns. Markets are incomplete in the sense that it is impossible to buy an asset or a dynamic portfolio of assets spanning the stochastic changes in the value of the project. There is no futures market either for the decision variable or the size of the investment project prevents the firm from taking a position on such a market. We also assume the perpetual setting for our model.

In our set-up, the Parisian criterion reflects the will of the firm (the headquarters and the operational division) to check that the market conditions remain favorable (resp. unfavorable) during the implementation lag of the investment (resp. disinvestment) decision. In this section we give the mathematical formulation of Parisian criterion and in the next section we determine the value of the investment and disinvestment decisions under the Parisian policy.

At any time $t$ the firm can invest in a project yielding an operating profit that depends on the instantaneous cash flow $(S_t, t \geq 0)$. We assume that the dynamics of $S_t$ follow a geometric Brownian motion,

$$\frac{dS_t}{S_t} = \mu dt + \sigma Z_t, \qquad S_0 = x, \tag{1}$$

where $\mu$ and $\sigma$ are constants and $(Z_t, t \geq 0)$ is a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

We denote by $V_t$ the expected sum of the discounted cash flows from $t$ to infinity,

$$V_t = \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)} S_u du\Big], \tag{2}$$

where the discount rate $\rho$ is constant and $\mathbb{E}_t[\cdot]$ stands for the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$. The dynamics of $V_t$ are given in the next lemma. According to standard real-option literature, we must assume that the drift term of the geometric Brownian motion is smaller than the discount rate, i.e. $\mu < \rho$. This is necessary to obtain the integral in equation (3) finite.

**Lemma 2.1** *Assume that $S_t$ is the geometric Brownian motion and that $\rho > \mu$. Then $V_t$ is a geometric Brownian motion and moreover*

$$V_t = \frac{S_t}{\rho - \mu}.$$

**Proof.** We have that

$$V_t = \mathbb{E}_t\Big[\int_t^{\infty} e^{-\rho(u-t)} S_t e^{(\mu-\frac{\sigma^2}{2})(u-t)+\sigma(Z_u-Z_t)} du\Big]$$

Next we search for a martingale, use Fubini's theorem and get:

$$V_t = S_t \int_t^{\infty} e^{-\rho(u-t)} e^{\mu(u-t)} du = \frac{S_t}{\rho-\mu}. \blacksquare$$

Based on the above Lemma, $V_t$ satisfies the following SDE

$$\frac{dV_t}{V_t} = \mu dt + \sigma Z_t, \qquad V_0 = \frac{S_0}{\rho-\mu}. \tag{3}$$

Since the agents are risk-neutral, the value of the investment and disinvestment decisions problem can be written as a discounted expectation

$$\mathbb{E}\Big[e^{-\rho\tau_I}(V_{\tau_I} - C_I)^+ + e^{-\rho\tau_D}(C_D - V_{\tau_D})^+\Big],$$

where $C_I$ (resp. $C_D$) represents the direct investment (resp. disinvestment) costs, which we make constant, and $\tau_I$ (resp. $\tau_D$) represents the first instant when the process $V$ spends consecutively $d$ units of time above (resp. below) a specific thresholds. This satisfy the Parisian criterion, i.e. the manager invests (resp. disinvests) at $\tau_I$ (resp. at $\tau_D$) only if the decision variable $V_t$ had reached a pre-specified level and had remained constantly above (resp. below) this level for a time interval longer than a fixed amount of time (so-called time-window). The time-window corresponds to the implementation delay whereas the pre-specified level is set at an optimal value: the best investment (resp. disinvestment) threshold $h_I^*$ (resp. $h_D^*$). We assume that the time-window associated to the investment (resp. disinvestment) is a fixed amount of time $d_I$ (resp. $d_D$). The decision triggering criterion is then the so-called Parisian stopping time which depends on the size of the excursions of the state variable over (resp. below) the optimal thresholds (see Figure 1).

The firm (the headquarters and the operational division) maximizes the present value of its opportunities, namely it solves

$$VF(V_0) = \max_{\tau_I<\tau_D} \mathbb{E}\Big[e^{-\rho\tau_I}(V_{\tau_I} - C_I)^+ \ 1_{\{\tau_I<\infty\}} + e^{-\rho\tau_D}(C_D - V_{\tau_D})^+ \ 1_{\{\tau_D<\infty\}}\Big].$$

In the maximization problem one stochastic process is involved, and the geometric Brownian motion is continuous. Furthermore, in the perpetual case, the exercised boundaries are constant. Consequently, the investment (resp. disinvestment) decision will occur at the first instant when $V_t$ hits some constant optimal threshold $h_I^*$ (resp. $h_D^*$).

Letting $\tau_I$ and $\tau_D$ be the stopping times corresponding to the Parisian criterion with time-

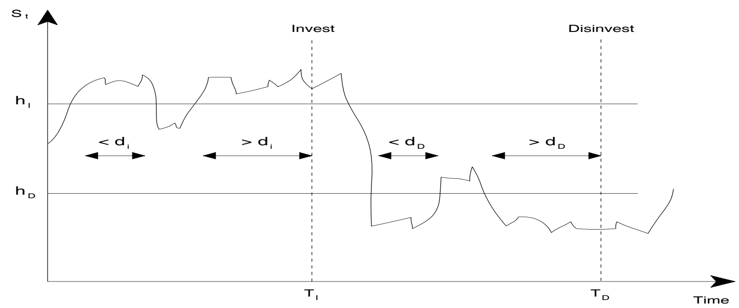


Figure 1: Evolution of the underlying process and the investment $h_I^*$ (resp. disinvestment $h_D^*$) optimal-levels.

windows $d_I$, $d_D$ and levels $h_I$, $h_D$ respectively, the present value of the investment and disinvestment decisions problem becomes

$$VF(V_0) = \max_{h_D \le h_I,\ V_0 \le h_I} \mathbb{E}_0 \Big[ e^{-\rho \tau_I} (V_{\tau_I} - C_I)\ 1_{\{\tau_I < \infty\}} \ +\ e^{-\rho \tau_D} (C_D - V_{\tau_D})\ 1_{\{\tau_D < \infty\}} \Big]. \qquad (4)$$

# 3 Solution

In this section we solve the maximization problem and obtain an explicit solution for the optimal disinvestment threshold, and an implicit solution for the optimal investment threshold. In order to proceed, as usually done in the literature on Parisian options, we translate the problem in terms of the drifted Brownian motion. We write

$$V_t = V_0 e^{\sigma X_t}, \quad \text{where } X_t = bt + Z_t, \quad \text{and} \quad b = \frac{\mu - \frac{\sigma^2}{2}}{\sigma}. \qquad (5)$$

and construct the probability measure $\mathbb{P}^*$ under which $X_t$ becomes a $\mathbb{P}^*$-Brownian motion,

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{\frac{b^2}{2}t - bX_t}. \qquad (6)$$

Then, using Girsanov theorem, we perform a change of measure. It is worth noticing that under the new measure both $\tau_I < \infty$ and $\tau_D < \infty$ hold almost sure. Following the notations in (5) and recalling the independence result from Theorem 5.10, we obtain[4]

$$\mathbb{E}_{\mathbb{P}^*}\Big[ e^{-(\rho + \frac{b^2}{2})\tau_I} \Big] \cdot \mathbb{E}_{\mathbb{P}^*}\Big[ e^{bX_{\tau_I}} (V_0 e^{\sigma X_{\tau_I}} - C_I) \Big] \qquad (7)$$

[4] All terms are calculated in the appendix.

for the first term in the maximization problem. Similarly, the second term results

$$\mathbb{E}_{\mathbb{P}^*}\Big[e^{-(\rho+\frac{b^2}{2})\tau_D}\Big]\cdot\mathbb{E}_{\mathbb{P}^*}\Big[e^{bX_{\tau_D}}(C_D - V_0 e^{\sigma X_{\tau_D}})\Big]. \tag{8}$$

Then, we compute the Laplace transform of the Parisian investment (resp. disinvestment) time under the new measure $\mathbb{P}^*$ defined in (6). We calculate the moment generating function for the process $X_t$ defined in (5), stopped at the Parisian investment (resp. disinvestment) time. Finally, we evaluate the first hitting time of $X$ which starts from the Parisian investment time. Factoring out the common terms, we can re-write the maximization problem (4) as

$$VF(V_0) = \max_{h_D\le h_I,\ V_0\le h_I}\Big(\frac{V_0}{h_I}\Big)^{\theta_1}\frac{\phi(b\sqrt{d_I})}{\phi(\sqrt{(2\rho+b^2)d_I})}\Big\{h_I\frac{\phi(\sqrt{d_I}(\sigma+b))}{\phi(b\sqrt{d_I})} - C_I +$$

$$+\Big(\frac{h_I}{h_D}\Big)^{\theta_2}\frac{\phi(-\sqrt{(2\rho+b^2)d_I})}{\phi(\sqrt{(2\rho+b^2)d_D})}\frac{\phi(-b\sqrt{d_D})}{\phi(b\sqrt{d_I})}\Big(C_D - h_D\frac{\phi(-(b+\sigma)\sqrt{d_D})}{\phi(-b\sqrt{d_D})}\Big)\Big\} \tag{9}$$

where, to simplify the already complicated formulas, we made the following notations:

$$\theta_1 = \frac{-b+\sqrt{2\rho+b^2}}{\sigma} \quad \text{and} \quad \theta_2 = \frac{-b-\sqrt{2\rho+b^2}}{\sigma}, \tag{10}$$

and $\phi$ is defined in (18).

We now solve the unconstrained maximization problem (9) when $d_I = d_D = 0$, i.e. recovering the well-known case of the instantaneous investment and disinvestment problem. We obtain that the optimal instantaneous-disinvestment threshold (labeled $h^*_{ND}$) is

$$h^*_{ND} = \frac{\theta_2 C_D}{\theta_2 - 1}$$

whereas the optimal instantaneous-investment threshold is $h^*_{NI} = \max\{V_0, x^*\}$, where $x^*$ is the largest between the two solutions of the implicit equation

$$x = \frac{\theta_1 C_I}{\theta_1 - 1} + \Big(\frac{x}{h^*_{ND}}\Big)^{\theta_2}\frac{\theta_2-\theta_1}{\theta_1-1}\frac{C_D}{1-\theta_2}. \tag{11}$$

However, the solutions of the unconstrained maximization problem, $h^*_{ND}$ and $h^*_{NI}$, are not necessarily ordered; imposing $C_D < C_I$, one obtains a solution which satisfies the constrains of the constrained maximization problem (9), i.e. $h^*_{ND} \le x^*$ and, consequently, $h^*_{ND} \le h^*_{NI}$. Therefore, for time-windows $d_I = d_D = 0$ and imposing $C_D < C_I$, the (instantaneous) unconstrained problem has a naturally-ordered solution.

We now look for a solution of the investment and disinvestment problem with Parisian delay for more general time-windows, i.e. $\{d_I, d_D\} > 0$. Let us first solve the unconstrained problem corresponding to (9).[5] Taking the partial derivative with respect to $h_D$ and solving for the critical value, we obtain an explicit solution for the optimal disinvestment threshold $h_D^*$.

$$h_D^* = \frac{\phi(-b\sqrt{d_D})}{\phi(-(b+\sigma)\sqrt{d_D})}\frac{\theta_2 C_D}{\theta_2 - 1}. \tag{12}$$

It is immediately observable that $h_{ND}^* = h_D^*$ when $d_D = 0$. Intuitively, $h_D^*$ increases when the disinvestment fixed-costs $C_D$ increase. This means that, similarly to the instantaneous investment and disinvestment problem, the higher are the disinvestment costs the sooner the firm wants to exit. Furthermore, since $\phi$ is an increasing function, we obtain that $h_{ND}^* \leq h_D^*$, or that the firm decides to disinvest earlier in the presence of a disinvestment-delay.

Taking the partial derivative with respect to $h_I$ and solving for the critical value, we obtain an implicit solution for $h_I^*$, as in the case of instantaneous investment and disinvestment problem. In particular, we claim that $h_I^* = \max\{V_0, x^*\}$, where $x^*$ is the largest between the two solutions of the implicit equation

$$x = \frac{\theta_1 C_I}{\theta_1 - 1}\frac{\phi(b\sqrt{d_I})}{\phi((b+\sigma)\sqrt{d_I})} + \left(\frac{x}{h_D^*}\right)^{\theta_2}\frac{\theta_2 - \theta_1}{\theta_1 - 1}\frac{\phi(-\sqrt{(2\rho + b^2)d_I})}{\phi(\sqrt{(2\rho + b^2)d_D})}\frac{\phi(-b\sqrt{d_D})}{\phi((b+\sigma)\sqrt{d_I})}\frac{C_D}{1-\theta_2}. \tag{13}$$

Denoting the right hand side of the implicit equation (13) by $f(x)$, we now prove that it has two solutions, one of which is larger than $h_D^*$, if one imposes the condition (14).

**Lemma 3.1** *Let $f(x)$ be the right hand side of (13) and $h_D^*$ defined in (12). Then the following relations hold.*

*(a) The function $f$ is increasing in $(0, \infty)$, and*

$$\lim_{x \searrow 0} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \frac{\theta_1 C_I}{\theta_1 - 1}\frac{\phi(b\sqrt{d_I})}{\phi((b+\sigma)\sqrt{d_I})},$$

*(b) If the following inequality holds*

$$C_D \frac{\phi(-b\sqrt{d_D})}{\phi(-(b+\sigma)\sqrt{d_D})} < C_I \frac{\phi(b\sqrt{d_I})}{\phi((b+\sigma)\sqrt{d_I})}, \tag{14}$$

*then $f(h_D^*) > h_D^*$.*

**Proof.** Since $\theta_1 > 1$, part (a) follows easily. We now prove part (b). The following relations hold

[5] Here the constraint $V_0 \leq h_I$ is implicitly assumed. The investigation of a possible condition similar to (14) which controls for $V_0 \leq h_I$ can be subject of future research.

since $\phi$ is increasing.

$$\frac{\theta_1}{\theta_1-1}=\frac{1}{1-\theta_2}\Big(-\theta_2+\frac{\theta_1-\theta_2}{\theta_1-1}\Big)\geq\frac{1}{1-\theta_2}\Bigg(-\theta_2+\frac{\theta_1-\theta_2}{\theta_1-1}\times$$

$$\times\frac{\phi(-\sqrt{(2\rho+b^2)d_I})}{\phi(\sqrt{(2\rho+b^2)d_D})}\frac{\phi(-(b+\sigma)\sqrt{d_D})}{\phi((b+\sigma)\sqrt{d_I})}\Bigg).$$

We now multiply the left and right hand side terms of the above inequality with the terms in (14) to obtain

$$\frac{\theta_1 C_I}{\theta_1-1}\frac{\phi(b\sqrt{d_I})}{\phi((b+\sigma)\sqrt{d_I})}>\frac{C_D}{1-\theta_2}\frac{\phi(-b\sqrt{d_D})}{\phi(-(b+\sigma)\sqrt{d_D})}\Bigg(-\theta_2+\frac{\theta_1-\theta_2}{\theta_1-1}\times$$

$$\times\frac{\phi(-\sqrt{(2\rho+b^2)d_I})}{\phi(\sqrt{(2\rho+b^2)d_D})}\frac{\phi(-(b+\sigma)\sqrt{d_D})}{\phi((b+\sigma)\sqrt{d_I})}\Bigg).$$

Regroupping the terms we obtain $f(h_D^*)>h_D^*$. ■

Relying on the previous lemma, we deduce that the implicit equation $x=f(x)$ has two solutions, termed $x_1^*$ and $x_2^*$, where $0<x_1^*<h_D^*<x_2^*$. Hence, the unconstrained maximization version of (9) has two critical points, $(x_1^*,h_D^*)$ and $(x_2^*,h_D^*)$. Out of these two points, only $(x_2^*,h_D^*)$ satisfies the constrain $h_D^*<x^*$. Thus $h_I^*=\max\{V_0,x^*\}$ is well defined, and the conditions $h_D^*\leq h_I^*$, $V_0\leq h_I^*$ are satisfied. It can easily be deduced that the critical point $(x_2^*,h_D^*)$ is a local maximum point for the unconstrained maximization version of (9), and thus $(h_I^*,h_D^*)$ is the unique solution for maximization problem (9). We summarize our results in the following theorem.

**Theorem 3.2** *Consider the investment and disinvestment decisions of a firm under the Parisian criterion with time windows $d_I,d_D$. If (14) holds, then the optimal investment and disinvestment thresholds satisfy the following equations,*

$$h_D^*=\frac{\phi(-b\sqrt{d_D})}{\phi(-(b+\sigma)\sqrt{d_D})}\frac{\theta_2 C_D}{\theta_2-1}$$

*and $h_I^*=\max\{V_0,x^*\}$, where $x^*$ solves the implicit equation*

$$x=\frac{\theta_1 C_I}{\theta_1-1}\frac{\phi(b\sqrt{d_I})}{\phi((b+\sigma)\sqrt{d_I})}+\Big(\frac{x}{h_D^*}\Big)^{\theta_2}\frac{\theta_2-\theta_1}{\theta_1-1}\frac{\phi(-\sqrt{(2\rho+b^2)d_I})}{\phi(\sqrt{(2\rho+b^2)d_D})}\frac{\phi(-b\sqrt{d_D})}{\phi((b+\sigma)\sqrt{d_I})}\frac{C_D}{1-\theta_2}.$$

# 4 Model Results

After we obtain an explicit solution for the optimal exit-level and an implicit solution for the optimal entry-level under the Parisian criterion, we present a brief discussion of the optimal investment and disinvestment thresholds $h_I^*$ and $h_D^*$ in terms of the time windows $d_I$ and $d_D$.

**(a)** If $d_I = d_D = 0$ we recover the well-known case of instantaneous investment and disinvestment problem, and so $h_I^* = h_{NI}^*$ and $h_D^* = h_{ND}^*$,

**(b)** If $d_D = 0$ and $d_I \geq 0$, then $h_D^* = h_{ND}^*$,

**(c)** If $d_D \to \infty$, then $h_I^*$ converges to

$$h_{OI}^* = \frac{\theta_1 C_I}{\theta_1 - 1} \frac{\phi(b\sqrt{d_I})}{\phi((b+\sigma)\sqrt{d_I})},$$

where $h_{OI}^*$ represents the optimal investment threshold for time-window $d_I$ while disinvestment is not possible. This value was first discovered in Gauthier and Morellec (2000).

**(d)** If $d_D \geq 0$ and $d_I \geq 0$, then $h_D^* \geq h_{ND}^*$.

For illustrative purposes we perform a numerical evaluation to compare the investment (disinvestment) decision problem with implementation delay and the instantaneous investment (disinvestment) decision problem. The results are summarized below. In the first table we report the ratio of the value of the Parisian investment and disinvestment decisions problem with respect to the instantaneous investment and disinvestment decisions problem, at their respective optima. Since the first column and row correspond to the values of $d_I, d_D$ respectively, we expect this ratio to be equal to 1 when the delay $d_I = d_D = 0$. This is the case, as we observe in the upper-left corner of Table 1.

| d | 0 | 0.5 | 1 | 1.5 | 2 |
|---|---|---|---|---|---|
| 0 | 1.0000 | 0.9999 | 0.9998 | 0.9997 | 0.9996 |
| 0.5 | 0.9872 | 0.9871 | 0.9870 | 0.9869 | 0.9868 |
| 1 | 0.9738 | 0.9737 | 0.9736 | 0.9735 | 0.9734 |
| 1.5 | 0.9600 | 0.9599 | 0.9598 | 0.9598 | 0.9597 |
| 2 | 0.9461 | 0.9460 | 0.9459 | 0.9458 | 0.9457 |

Table 1: Ratio of Parisian value problem and instantaneous value problem. The parameters we used are $\rho = 0.13; \mu = 0.05; \sigma = 0.40; C_D = 0.5; C_I = 1.7; V_0 = 1$.

We observe that the value of the investment and disinvestment decision problems is lower under the Parisian criterion than in the instantaneous case, for reasonable parameter values. This is to be expected, because the time-lag under the Parisian criterion measures a systematic (and unavoidable) delay that forces the firm to "postpone" the investment (disinvestment) procedure.

Extremely interesting is the presence of an asymmetry effect: the larger $d_I$ is the stronger the impact on the investment-value compared to the impact of an equal value for $d_D$. This is a result of the disinvestment opportunity. Because the firm can exit costly the firm's profits are a convex function of the stochastic underlying and the expected profits increase in an uncertain environment. Therefore, a firm will invest at a lower level when the implementation delay forces the firm to decide in advance whether to enter or not a few periods ahead. Since instantaneous and Parisian investment (disinvestment) optima are linear functions of both $C_D$ and $C_I$, only the impact of the process-volatility on the optimal thresholds requires further investigation.

| $\sigma$ | $\sigma$=.05 | | | $\sigma$=.20 | | | $\sigma$=.40 | | |
|---|---|---|---|---|---|---|---|---|---|
| d | 0 | 0.5 | 1 | 0 | 0.5 | 1 | 0 | 0.5 | 1 |
| 0 | 2.8291 | 2.8291 | 2.8291 | 3.5735 | 3.5735 | 3.5734 | 5.3185 | 5.3113 | 5.3080 |
| 3 | 2.3221 | 2.3221 | 2.3221 | 2.1571 | 2.1571 | 2.1570 | 2.0858 | 2.0823 | 2.0807 |
| 5 | 2.0973 | 2.0973 | 2.0973 | 1.8080 | 1.8079 | 1.8079 | 1.5447 | 1.5417 | 1.5403 |

Table 2: Parisian optimal investment value. The parameters we used are $\rho = 0.13; \mu = 0.05; C_D = 0.5; C_I = 1.7; V_0 = 1$.

Our findings confirm the results of numerous papers that report that an increase in uncertainty delays (instantaneous) investments, see Pindyck (1991) for survey. The first raw of Table 2 show the effect of an increase of uncertainty without investment delay. As $\sigma$ goes from 0.05 to 0.40, $h_I^*$ rises from 2.8291 to 5.3185, while $h_D^*$ falls from 0.4882 to 0.2620 (not reported in the Table). The higher (instantaneous) investment threshold and the lower (instantaneous) disinvestment threshold imply that more uncertainty delays both entry and exit, and thus generates more so-called inertia. The intuition behind such conventional result is that a firm delays in order to avoid learning of bad-news after it has made decision to enter (exit). Since the likelihood of observing a bad-news rises with uncertainty, so does the benefit of waiting. However, waiting has an opportunity cost due to the foregone income during the period of inaction and this is more evident in the presence of delays. As a result, conventional finding on the effect of the uncertainty of the underlying process on the investment (disinvestment) are reversed when there are time-lags. Similarly to Bar-Ilan and Strange (1996), in Table 2 we show that an increase in the uncertainty hastens the decision to invest (disinvest). For instance when $d_I = 5$, $h_I^*$ falls from 2.0973 to 1.5447, while $h_D^*$ raises from 0.5201 to 0.5884 (not reported in the Table). Since the firm can exit costly, the downside risk of the project is bounded. This makes profit a convex function of the stochastic underlying the expected return of the project therefore raises with uncertainty. Therefore, a higher volatility hastens investment (disinvestment) when delays force a firm to decide in advance to undertake a decision or not in the future.

An interesting direction for future research is the analytical study of the behavior of the optimal thresholds in the Parisian decision problem as functions of the delays $d_I, d_D$. Moreover, one can look for explicit conditions when the inequality (14) holds. Though possible, such an analysis requires a careful study of the properties of the function $\phi$.

# 5 Appendix

## 5a Definitions and Results

The Brownian meander and the Parisian criterion are closely related. Following we define the Brownian meander and list some of its properties. Then, we present the connection existing between the Brownian meander and the Parisian criterion.

Let $(Z_t, t \geq 0)$ be a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. For each $t > 0$, we define the random variables

$$g_t = \sup\{s : s \leq t, Z_s = 0\}, \tag{15}$$

$$d_t = \inf\{s : s \geq t, Z_s = 0\}. \tag{16}$$

The interval $(g_t, d_t)$ is called the interval of the Brownian excursion which straddles time $t$. For $u$ in this interval, $sgn(Z_t)$ remains constant. In particular, $g_t$ represents the last time the Brownian motion crossed the level 0. It is known that $g_t$ is not a stopping time for the Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$, but for the slow Brownian filtration $(\mathcal{G}_t)_{t\geq 0}$, which is defined by $\mathcal{G}_t = \mathcal{F}_{g_t} \vee \sigma(sgn(Z_t))$. The slow Brownian filtration represents the information on the Brownian motion until its last zero plus the knowledge of its sign after this.

The Brownian meander process ending at $t$ is defined as

$$m_u^{(t)} = \frac{1}{\sqrt{t - g_t}} |Z_{g_t + u(t-g_t)}|, \quad 0 \leq u \leq 1. \tag{17}$$

The process $m_u^{(t)}$ is the non-negative and normalized Brownian excursion which straddles time $t$ and it is independent of the $\sigma$-field $(\mathcal{G}_t)_{t\geq 0}$. When $u = 1$ and $t = 1$, we conveniently denote $m_1 = m_1^{(1)}$. The random variable $m_1$ will play a central role in the calculation of many other variables that will be introduced later on. The distribution of $m_1$ is known,

$$\mathbb{P}(m_1 \in dx) = x \exp(-\frac{1}{2}x^2) 1_{x>0} dx,$$

and the moment generating function $\phi(z)$ is given by

$$\phi(z) = \mathbb{E}(\exp(zm_1)) = \int_0^\infty x \exp(zx - \frac{1}{2}x^2) dx. \tag{18}$$

We now look at the first instant when the Brownian motion spends $d$ units of time consecutively above (resp. below) the level 0. For $d \geq 0$, we define the random variables

$$H_d^+ = \inf\{t \geq 0 : t - g_t \geq d, \quad Z_t \geq 0\} \tag{19}$$

$$H_d^- = \inf\{t \geq 0 : t - g_t \geq d, \quad Z_t \leq 0\} \tag{20}$$

The variables $H_d^+$ and $H_d^-$ are $\mathcal{G}_t$-stopping times, and hence $\mathcal{F}_t$-stopping times. See Revuz and Yor (1991) for more details. From equation (17) we can easily deduce that the process

$$\left(\frac{1}{\sqrt{d}}|Z_{g_{H_d^+}+ud}|\right)_{u\leq 1} = \left(m_u^{(H_d^+)}\right)_{u\leq 1}$$

is a Brownian meander, independent of $\mathcal{G}_{g_{H_d^+}}$. In particular, $(1/\sqrt{d})Z_{H_d^+}$ is distributed as $m_1$,

$$\mathbb{P}(Z_{H_d^+} \in dx) = \frac{x}{d}\exp(-\frac{x^2}{2d})1_{x>0}dx. \tag{21}$$

and the random variables $H_d^+$ and $Z_{H_d^+}$ are independent.

Similarly, $(1/\sqrt{d})Z_{H_d^-}$ is distributed as $-m_1$,

$$\mathbb{P}(Z_{H_d^-} \in dx) = \frac{-x}{d}\exp(-\frac{x^2}{2d})1_{x<0}dx. \tag{22}$$

and the random variables $H_d^-$ and $Z_{H_d^-}$ are independent.

The Laplace transform of $H_d^+$ was first calculated in Chesney et al. (1997). We present the result in the next theorem.

**Theorem 5.1** *Let $H_d^+$ be the stopping time defined in (19) and $\phi$ the moment generating function defined in equation (18). For any $\lambda > 0$ ,*

$$\mathbb{E}[\exp(-\lambda H_d^+)] = \frac{1}{\phi(\sqrt{2\lambda d})}. \tag{23}$$

The proof of the theorem relies on a very clever use of the Arzela martingale, $\mu_t = sgn(Z_t)\sqrt{t-g_t}$ - a remarkable $(\mathcal{G}_t)$ martingale. The same results holds also when $H_d^+$ is replaced with $H_d^-$.

So far we only looked at the Brownian motion excursions above or below level 0. More generally, we can define for any $a \in \mathbb{R}$ and any continuous stochastic process $X$,

$$g_t^{X_0,a}(X) = \sup\{s : s \leq t, X_0 = X_0, X_t = a\}, \tag{24}$$

$$H^+_{(X_0,a),d}(X) = \inf\{t \geq 0 : t - g_t^{X_0,a} \geq d, X_0 = X_0, \quad X_t \geq a\} \tag{25}$$

$$H^-_{(X_0,a),d}(X) = \inf\{t \geq 0 : t - g_t^{X_0,a} \geq d, X_0 = X_0, \quad X_t \leq a\} \tag{26}$$

Thus $g_t^{X_0,a}(X)$ represents the last time the process $X$ crossed level $a$. As for the Brownian motion case, $g_t^{X_0,a}(X)$ is not a stopping time for the Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$, but for the slow Brownian filtration $(\mathcal{G}_t)_{t\geq 0}$. The random variables $H^+_{(X_0,a),d}(X)$ (resp. $H^-_{(X_0,a),d}(X)$) represent the first instant when the process $X$ spends $d$ units of time above (resp. below) the level $a$. The variables $H^+_{(X_0,a),d}(X)$ and $H^-_{(X_0,a),d}(X)$ are $\mathcal{G}_t$-stopping times, and hence $\mathcal{F}_t$-stopping times. In the notation we use, we indicate the starting point of the process $X$, the level $a$ and the length of time $d$. Although indicating the starting point seems unnecessary, it turns out to be extremely helpful in the context of the Parisian criterion.

Another relevant random variable is the first hitting time of level $a$, which we define below.

$$T_{X_0,a}(X) = \inf\{s : X_0 = X_0, X_s = a\} \tag{27}$$

## 5b Parisian Criterion

According to the notation introduced in Section 2, the investment stopping time $\tau_I$ which satisfies the Parisian criterion corresponds to $H^+_{(V_0,h_I),d_I}(V)$. In order to express in mathematical formulas the disinvestment stopping time $\tau_D$, we need to extend the definition of $H^+_{(V_0,h_I),d_I}(V)$.

Let $\tau$ be any stopping time, $a \in \mathbb{R}$, $X$ a continuous stochastic process, and $g_t^{X_0,a}(X)$ defined in equation (24).

(a) The first instant after $\tau$ when the process $X$ spends $d$ units of time above (resp. below) the level $a$ is given by the stopping time $H^{+,\tau}_{(X_0,a),d}(X)$ (resp. $H^{-,\tau}_{(X_0,a),d}(X)$)

$$H^{+,\tau}_{(X_0,a),d}(X) = \inf\{t \geq \tau : t - g_t^{X_0,a} \geq d, X_0 = X_0, \quad X_t \geq a\} \tag{28}$$

$$H^{-,\tau}_{(X_0,a),d}(X) = \inf\{t \geq \tau : t - g_t^{X_0,a} \geq d, X_0 = X_0, \quad X_t \leq a\} \tag{29}$$

(b) The first hitting time after $\tau$ of level $a$ is the stopping time $T^\tau_{X_0,a}(X)$

$$T^\tau_{X_0,a}(X) = \inf\{s \geq \tau : X_0 = X_0, X_s = a\}. \tag{30}$$

If $X$ has the strong Markov property and $\tau$ is a finite stopping time, we have the following equalities in distribution $H^{+,\tau}_{(X_0,a),d}(X) = H^+_{(X_\tau,a),d}(X)$, $H^{-,\tau}_{(X_0,a),d}(X) = H^-_{(X_\tau,a),d}(X)$, and $T^\tau_{X_0,a}(X) = T_{X_\tau,a}(X)$. Now, we can state the formulas for the stopping times $\tau_I$ and $\tau_D$, which satisfy the Parisian criterion.

**Proposition 5.2** *Let $\tau_I$ and $\tau_D$ be the stopping times corresponding to the Parisian criterion with time windows $d_I$, $d_D$ and levels $h_I$, $h_D$ respectively. Then the following equalities hold*

$$\tau_I = H^+_{(V_0,h_I),d_I}(V), \tag{31}$$

$$\tau_D = H^{-,\tau_I}_{(V_0,h_D),d_D}(V). \tag{32}$$

Otherwise, in terms of the drifted Brownian motion, the Parisian stopping times are

$$\tau_I = H^{+}_{(V_0,h_I),d_I}(V) = H^{+}_{(l_0,l_I),d_I}(X), \quad \text{where } l_0 = 0, \quad \text{and} \quad l_I = \frac{1}{\sigma}\log\left(\frac{h_I}{V_0}\right),$$

and

$$\tau_D = H^{-,\tau_I}_{(V_0,h_D),d_D}(V) = H^{-,\tau_I}_{(l_0,l_D),d_D}(X), \quad \text{where } l_0 = 0, \quad \text{and} \quad l_D = \frac{1}{\sigma}\log\left(\frac{h_D}{V_0}\right).$$

## 5c Parisian Stopping Times and Independence

In this sub-section we prove that the Parisian investment (resp. disinvestment) time and the position of the underlying value process at that time are independent. We rely on the key property of independence of the Brownian meander from the slow Brownian filtration. The independence is a pivotal result that allow us to perform exact calculations of the maximization problem (4) in section 3.

The following proposition helps us to decompose the disinvestment Parisian time, provided that we have proved the independence relationship between the Parisian investment time and the position of the underlying value process.

**Proposition 5.3** *Let $\tau$ be any finite stopping time such that $\tau$ and $V_\tau$ are independent, and assume $h_D \leq V_\tau$ a.s. Then the following equality in distribution holds*

$$H^{-,\tau}_{(V_0,h_D),d_D}(V) = \tau + T_{V_\tau,h_D}(V) + H^{-}_{(h_D,h_D),d_D}(V),$$

*and the terms of the sum are independent. A similar relationship holds for $H^{+,\tau}_{(V_0,h_I),d_I}(V)$ if we assume $V_\tau \leq h_I$ a.s*

**Proof.** The strong Markov property and the continuity of the process $V$ give us the equality. The independence follows from our hypothesis that $\tau$ and $V_\tau$ are independent. ∎

The next theorem shows that the Parisian disinvestment time and the position of the underlying value process at that time are independent.

**Theorem 5.4** *Assume that $h_D \leq h_I$, $V_0 \leq h_I$, let $\tau_D = H^{-,\tau_I}_{(V_0,h_D),d_D}(V)$ and $\mathbb{P}^*$ defined in (6). Then the stopping time $\tau_D$ is finite $\mathbb{P}^*$ a.s. and the random variables $\tau_D$ and $V_{\tau_D}$ are independent under the $\mathbb{P}^*$ measure.*

The proof is broken-down in several steps. The next lemma is one of the main results of this paper. It is the key result we use in our theorem because it proves an independence relationship between two related quantities.

**Lemma 5.5** *Let $X_t = bt + Z_t$, with $b \in \mathbb{R}$, and construct the stopping time $T = H^+_{(0,0),d}(X)$ according to equation (24). The following conclusions hold.*

*(a) The random variables $X_T \cdot 1_{\{T<\infty\}}$ and $T \cdot 1_{\{T<\infty\}}$ are independent under the $\mathbb{P}$ measure if and only if $b \geq 0$,*

*(b) The random variables $X_T \cdot 1_{\{T<\infty\}}$ and $T \cdot 1_{\{T<\infty\}}$ are independent under the $\mathbb{P}^*$ measure for any $b \in \mathbb{R}$.*

*A similar relationship holds when $H^+_{(0,0),d}(X)$ is replaced with $H^-_{(0,0),d}(X)$*

**Proof.** Using Girsanov's theorem, we construct the probability measure $\mathbb{P}^*$ under which $X_t$ becomes a $\mathbb{P}^*$-Brownian motion. Under this probability $X_T = X_{H^+_{(0,0),d}(X)}$ becomes a Brownian meander and thus it is independent of $T = H^+_{(0,0),d}(X)$. Also $1_{\{T<\infty\}} = 1$ a.s. under the measure $\mathbb{P}^*$. Thus we have the required independence under the $\mathbb{P}^*$ measure for any $b \in \mathbb{R}$. We need to show that the independence holds also under the original measure $\mathbb{P}$ if and only if $b \geq 0$.

The independence holds if and only if the Laplace transforms satisfy the equality

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\lambda X_T - \alpha T} \cdot 1_{\{T<\infty\}}\right] = \mathbb{E}_{\mathbb{P}}\left[e^{-\lambda X_T} \cdot 1_{\{T<\infty\}}\right] \cdot \mathbb{E}_{\mathbb{P}}\left[e^{-\alpha T} \cdot 1_{\{T<\infty\}}\right]. \tag{33}$$

Hence, we show that the equality holds if and only if $b \geq 0$.

Let us look at the left hand side of (33) and apply the transformation of the measure. Under the $\mathbb{P}^*$ measure, $1_{\{T<\infty\}} = 1$ a.s., so we do not have to write it.

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\lambda X_T - \alpha T} \cdot 1_{\{T<\infty\}}\right] = \mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda X_T - \alpha T} e^{-\frac{1}{2}b^2 T + bX_T}\right]$$

We know that under the $\mathbb{P}^*$ measure, $X_T$ is independent of $T$, and grouping the factors, the left term of (33) becomes

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\lambda X_T - \alpha T} \cdot 1_{\{T<\infty\}}\right] = \mathbb{E}_{\mathbb{P}^*}\left[e^{(-\lambda+b)X_T}\right] \cdot \mathbb{E}_{\mathbb{P}^*}\left[e^{(-\alpha - \frac{1}{2}b^2)T}\right].$$

Let us now calculate the product from the right hand side of (33),

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\lambda X_T} \cdot 1_{\{T<\infty\}}\right] \cdot \mathbb{E}_{\mathbb{P}}\left[e^{-\alpha T} \cdot 1_{\{T<\infty\}}\right].$$

We change the measure using Girsanov in both terms and get

$$= \mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda X_T} e^{-\frac{1}{2}b^2 T + bX_T}\right] \cdot \mathbb{E}_{\mathbb{P}^*}\left[e^{-\alpha T} e^{-\frac{1}{2}b^2 T + bX_T}\right].$$

We know that under the $\mathbb{P}^*$ measure, $X_T$ is independent of $T$, and grouping the factors we get

$$= \mathbb{E}_{\mathbb{P}^*}\left[e^{(-\lambda+b)X_T}\right] \cdot \mathbb{E}_{\mathbb{P}^*}\left[e^{(-\alpha - \frac{1}{2}b^2)T}\right] \cdot \mathbb{E}_{\mathbb{P}^*}\left[e^{-\frac{1}{2}b^2 T + bX_T}\right].$$

The extra term is

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\frac{1}{2}b^2T+bX_T}\right] = \frac{\phi(b\sqrt{d})}{\phi(|b|\sqrt{d})},$$

where last equality follows from the properties of the Brownian meander. This extra term is equal to 1 if and only if $b \geq 0$, which means that the independence holds if and only if $b \geq 0$. ∎

**Remarks.** It worth noticing that in most of the cited papers, all quantities are transformed and the computations are under the measure $\mathbb{P}^*$, where the independence comes automatically from the properties of the Brownian meander. Those papers that work with $\mathbb{P}$ do not investigate under which condition for $b$ such independence holds.

While studying the independence relationships, we observed an interesting connection with Wald's identity. The following theorem, which relates Wald's identity with the finiteness of the stopping times, is known

**Theorem 5.6** *Let $Z_t$ be a $\mathbb{P}$-Brownian motion and $X_t = bt + Z_t$ be a drifted $\mathbb{P}$-Brownian motion. Let $\mathbb{P}^*$ be the measure under which $X_t$ is a $\mathbb{P}^*$-Brownian motion. Let $T$ be any stopping time and assume $\mathbb{P}^*(T < \infty) = 1$. Then Wald's identity holds*

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\frac{1}{2}b^2T+bX_T}\right] = 1$$

*if and only if $\mathbb{P}(T < \infty) = 1$.*

In our case $T = H^+_{(0,0),d}(X)$. We know from Chesney et al. (1997) that $\mathbb{P}^*(T < \infty) = 1$, and from the properties of the Brownian meander we know that

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\frac{1}{2}b^2T+bX_T}\right] = 1 \quad \text{if and only if } b \geq 0.$$

Thus $\mathbb{P}(T < \infty) < 1$ if and only if $b < 0$. We now ask what $\mathbb{P}(T < \infty)$ equals to. In order to calculate it, we take first a detour and calculate the Laplace transform of $H^+_{(0,0),d}(X)$ under the $\mathbb{P}$ measure.

**Theorem 5.7** *Let $X_t = bt + Z_t$, where $b$ is some fixed real number, and construct the stopping time $H^+_{(0,0),d}(X)$ according to equation (24). Let $\phi$ be the moment generating function defined in equation (18). Then for any $\lambda > 0$, the Laplace transform of $H^+_{(0,0),d}(X)$ is given by*

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\lambda H^+_{(0,0),d}(X)}\right] = \frac{\phi(b\sqrt{d})}{\phi(\sqrt{(2\lambda+b^2)d})}. \tag{34}$$

**Proof.** Using Girsanov's theorem, we construct the probability measure $\mathbb{P}^*$ under which $X_t$ becomes a $\mathbb{P}^*$-Brownian motion. Under this probability $X_{H^+_{(0,0),d}(X)}$ becomes a Brownian meander and thus it is independent of $H^+_{(0,0),d}(X)$. We look at the Laplace transform and apply the transformation of the measure.

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\lambda H^{+}_{(0,0),d}(X)}\right] = \mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda H^{+}_{(0,0),d}(X)} e^{-\frac{1}{2}b^2 H^{+}_{(0,0),d}(X) + bX_{H^{+}_{(0,0),d}(X)}}\right].$$

Grouping terms together and using the independence property just mentioned, we obtain

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\lambda H^{+}_{(0,0),d}(X)}\right] = \mathbb{E}_{\mathbb{P}^*}\left[e^{(-\lambda-\frac{1}{2}b^2)H^{+}_{(0,0),d}(X)}\right]\mathbb{E}_{\mathbb{P}^*}\left[e^{bX_{H^{+}_{(0,0),d}(X)}}\right].$$

We now apply the result for the Laplace transform of the Brownian meander, and use the fact that $\phi$ is the moment generating function of $\frac{1}{\sqrt{d}}H^{+}_{(0,0),d}(X)$ to obtain that

$$\mathbb{E}_{\mathbb{P}}\left[e^{-\lambda H^{+}_{(0,0),d}(X)}\right] = \frac{\phi(b\sqrt{d})}{\phi(\sqrt{(2\lambda+b^2)d})}. \blacksquare$$

We now calculate $\mathbb{P}(T<\infty)$, where we denoted $T = H^{+}_{(0,0),d}(X)$. If $T(\omega)<\infty$, then

$$\lim_{\lambda\searrow 0} e^{-\lambda T(\omega)} = 1;$$

if $T(\omega)=\infty$, then $e^{-\lambda T(\omega)} = 0$ for every $\lambda>0$, so

$$\lim_{\lambda\searrow 0} e^{-\lambda T(\omega)} = 0.$$

Therefore,

$$\lim_{\lambda\searrow 0} e^{-\lambda T(\omega)} = 1_{T<\infty}.$$

Letting $\lambda \searrow 0$ and using the Monotone Convergence theorem in the Laplace transform formula of $T$ we obtain

$$\mathbb{P}(T<\infty) = \frac{\phi(b\sqrt{d})}{\phi(|b|\sqrt{d})}.$$

If $b\geq 0$, then

$$\mathbb{P}(T<\infty) = 1.$$

If $b<0$, then

$$\mathbb{P}(T<\infty) = \frac{\phi(b\sqrt{d})}{\phi(-b\sqrt{d})} < 1.$$

We summarize the results in the following theorem.

**Theorem 5.8** *Let $X_t = bt + Z_t$, where $b$ is some fixed real number, and construct the stopping time $H^{+}_{(0,0),d}(X)$ according to equation (24). Let $\phi$ be the moment generating function defined in* *equation (18). Then*

$$\mathbb{P}(H^{+}_{(0,0),d}(X) < \infty) = \frac{\phi(b\sqrt{d})}{\phi(|b|\sqrt{d})}.$$

**Remarks.** In real option terms, this theorem has the following implications: if $b \geq 0$, then the investment process will take place with probability 1, while there is a positive probability that the disinvestment process will not take place. If $b < 0$, then there is a positive probability that the investment process will not take place. Furthermore, if the investment took place, then the disinvestment process will take place with probability 1.

If we condition on the stopping time to be finite, then we recover the independence under the measure $\mathbb{P}$, for any $b \in \mathbb{R}$.

The next result is a direct consequence of Lemma 5.5, and it proves an independence relationship for the process $V_t$, starting from $h_I$. In order to emphasize that the starting point for our process is $h_I$, we use the notion $V_t^{h_I}$ in the proof of the next lemma.

**Lemma 5.9** *Let $T = H^{+}_{(h_I,h_I),d_I}(V)$ be the stopping time defined in equation (24) and $b, X_t$ be defined in (5). The following conclusions hold.*

*(a) The random variables $T \cdot 1_{\{T<\infty\}}$ and $V_T \cdot 1_{\{T<\infty\}}$ are independent under the $\mathbb{P}$ measure if and only $b \geq 0$. In particular $T \cdot 1_{\{T<\infty\}}$ and $X_T \cdot 1_{\{T<\infty\}}$ are independent under the $\mathbb{P}$ measure if and only if $b \geq 0$,*

*(b) The random variables $T \cdot 1_{\{T<\infty\}}$ and $V_T \cdot 1_{\{T<\infty\}}$ are independent under the $\mathbb{P}^*$ measure for any $b \in \mathbb{R}$. In particular $T \cdot 1_{\{T<\infty\}}$ and $X_T \cdot 1_{\{T<\infty\}}$ are independent under the $\mathbb{P}^*$ measure for any $b \in \mathbb{R}$.*

**Proof.** We know that $V_t$ is the geometric Brownian motion in (3). If it starts from $h_I$, then

$$V_t^{h_I} = h_I e^{\sigma X_t}, \quad \text{where } X_t = bt + Z_t, \ \text{ and } \ b = \frac{\mu - \frac{\sigma^2}{2}}{\sigma}.$$

From the above equality we also get that $H^{+}_{(h_I,h_I),d_I}(V) = H^{+}_{(0,0),d_I}(X)$. Thus we have obtained that

$$V_{H^{+}_{(h_I,h_I),d_I}(V)} = h_I e^{\sigma X_{H^{+}_{(0,0),d_I}(X)}} \ \text{ and } \ H^{+}_{(h_I,h_I),d_I}(V) = H^{+}_{(0,0),d_I}(X).$$

The independence follows now from Lemma 5.5. ∎

We now prove that the Parisian investment time and the position of the underlying value process at that time are independent, this is a remarkable finding of the paper.

**Theorem 5.10** *Assume that $V_0 \leq h_I$, let $b, X_t$ be defined in (5), and recall that the Parisian investment time is $\tau_I = H^{+}_{(V_0,h_I),d_I}(V)$. The the following conclusions hold.*

*(a) If $b \geq 0$, then the stopping time $\tau_I$ is finite $\mathbb{P}$ a.s., and under $\mathbb{P}^*$ $\tau_I$ is finite a.s. for any $b \in \mathbb{R}$.*

*(b) The random variables $\tau_I \cdot 1_{\{\tau_I < \infty\}}$ and $V_{\tau_I} \cdot 1_{\{\tau_I < \infty\}}$ are independent under the $\mathbb{P}$ measure if and only if $b \geq 0$. In particular $\tau_I \cdot 1_{\{\tau_I < \infty\}}$ and $X_{\tau_I} \cdot 1_{\{\tau_I < \infty\}}$ are independent under the $\mathbb{P}$ measure if and only if $b \geq 0$,*

*(c) The random variables $\tau_I \cdot 1_{\{\tau_I < \infty\}}$ and $V_{\tau_I} \cdot 1_{\{\tau_I < \infty\}}$ are independent under the $\mathbb{P}^*$ measure for any $b \in \mathbb{R}$. In particular $\tau_I \cdot 1_{\{\tau_I < \infty\}}$ and $X_{\tau_I} \cdot 1_{\{\tau_I < \infty\}}$ are independent under the $\mathbb{P}^*$ measure for any $b \in \mathbb{R}$,*

**Proof.** Let us prove first the independence under $\mathbb{P}^*$. We apply Proposition 5.3 with $\tau = 0$, and obtain

$$\tau_I = H^+_{(V_0,h_I),d_I}(V) = T_{(V_0,h_I)}(V) + H^+_{(h_I,h_I),d_I}(V),$$

where the terms of sum are independent. If $b \geq 0$, then under $\mathbb{P}$, $\tau_I < \infty$ a.s. because the stopping times $T_{(V_0,h_I)}(V)$ and $H^+_{(h_I,h_I),d_I}(V)$ are finite. Similarly, $\tau_I < \infty$ a.s under $\mathbb{P}^*$ for any $b \in \mathbb{R}$. On the other hand, by the strong Markov property and the continuity of the process $V$, we have the equality in distribution under $\mathbb{P}^*$

$$V_{\tau_I} \cdot 1_{\{\tau_I < \infty\}} = V_{H^+_{(h_I,h_I),d_I}(V)}.$$

We now apply Lemma 5.9, and get that $V_{H^+_{(h_I,h_I),d_I}(V)}$ and $H^+_{(h_I,h_I),d_I}(V)$ are independent. By the strong Markov property, we have that $V_{H^+_{(h_I,h_I),d_I}(V)}$ and $T_{(V_0,h_I)}(V)$ are independent. Putting all together, we obtain that $\tau_I$ and $V_{\tau_I}$ are independent, which is the desired result. The proof of independence under $\mathbb{P}$ is similar to the proof of Lemma 5.5 and we skip it. ∎

We have now all the components to give the proof of Theorem 5.4.

**Proof of Theorem 5.4.** Here we work under the measure $\mathbb{P}^*$. Denote $\tau_D = H^{-,\tau_I}_{(V_0,h_D),d_D}(V)$. We need to prove that $\tau_D$ and $V_{\tau_D}$ are independent. Recall that the Parisian investment time is $\tau_I = H^+_{(V_0,h_I),d_I}(V)$, and by Theorem 5.10, the random variables $\tau_I$ and $V_{\tau_I}$ are independent. We apply Proposition 5.3 with $\tau = \tau_I$, and obtain

$$\tau_D = H^{-,\tau_I}_{(V_0,h_D),d_D}(V) = \tau_I + T_{V_{\tau_I},h_D}(V) + H^-_{(h_D,h_D),d_D}(V),$$

where the terms of sum are independent. Also, under $\mathbb{P}^*$, $\tau_D < \infty$ because the stopping times $\tau_I$, $T_{V_{\tau_I},h_D}(V)$ and $H^-_{(h_D,h_D),d_D}(V)$ are finite. On the other hand, by the strong Markov property and the continuity of the process $V$, we have the equality in distribution under $\mathbb{P}^*$

$$V_{\tau_D} = V_{H^-_{(h_D,h_D),d_D}(V)}.$$

We now apply Lemma 5.9, and get that $V_{H^-_{(h_D,h_D),d_D}(V)}$ and $H^-_{(h_D,h_D),d_D}(V)$ are independent. By the strong Markov property, we have that $V_{H^-_{(h_D,h_D),d_D}(V)}$ is independent of $\tau_I$ and of $T_{V_{\tau_I},h_D}(V)$. Therefore, we obtain that $\tau_I$ and $V_{\tau_I}$ are independent, which is the desired result. ■

## 5d Laplace Transforms and Moment Generating Functions

To obtain an explicit solution for the best disinvestment threshold, and an implicit solution for the best investment threshold we need to calculate all terms that enter into the maximization problem. We first find the Laplace transform of the Parisian investment time under the measure $\mathbb{P}^*$ defined in (6).

**Proposition 5.11** *For any $\lambda > 0$, the following equality holds,*

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda\tau_I}\right] = \left(\frac{V_0}{h_I}\right)^{\frac{\sqrt{2\lambda}}{\sigma}} \frac{1}{\phi(\sqrt{2\lambda d_I})}.$$

**Proof.** From Proposition 5.3 we have

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda\tau_I}\right] = \mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda T_{l_0,l_I}}\right]\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda H^+_{(l_I,l_I),d_I}(X)}\right]$$

using the corresponding Laplace transforms formulas we obtain

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda\tau_I}\right] = e^{-(l_I-l_0)\sqrt{2\lambda}}\frac{1}{\phi(\sqrt{2\lambda d_I})} = \left(\frac{V_0}{h_I}\right)^{\frac{\sqrt{2\lambda}}{\sigma}} \frac{1}{\phi(\sqrt{2\lambda d_I})}. \blacksquare$$

In the next proposition we calculate the moment generating function for the process $X_t$ defined in (5), stopped at the Parisian investment time.

**Proposition 5.12** *For any $\lambda \in \mathbb{R}$, the following equality holds,*

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda X_{\tau_I}}\right] = \left(\frac{h_I}{V_0}\right)^{-\frac{\lambda}{\sigma}} \phi(-\lambda\sqrt{d_I}).$$

**Proof.** Using the definition of $X_{\tau_I}$ we obtain

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda X_{\tau_I}}\right] = \mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda(l_I+m_1\sqrt{d_I})}\right],$$

Now using the definition of $l_I$ and $\phi$ we obtain

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda X_{\tau_I}}\right] = e^{-\lambda l_I}\phi(-\lambda\sqrt{d_I}) = \left(\frac{h_I}{V_0}\right)^{-\frac{\lambda}{\sigma}} \phi(-\lambda\sqrt{d_I}). \blacksquare$$

Following, we calculate the Laplace transform of the first hitting time of $X$, started from the Parisian investment time.

**Proposition 5.13** *For any $\lambda > 0$, the following equality holds,*

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda T_{(X_{\tau_I},l_D)}(X)}\right] = \left(\frac{h_D}{h_I}\right)^{\frac{\sqrt{2\lambda}}{\sigma}} \phi(-\sqrt{2\lambda d_I})$$

**Proof.** Conditioning we write

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda T_{(X_{\tau_I},l_D)}(X)}\right] = \mathbb{E}_{\mathbb{P}^*}\left[\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda T_{(X_{\tau_I},l_D)}(X)}\middle|\mathcal{F}_{\tau_I}\right]\right]$$

since $X_{\tau_I} \geq l_D$ a.s., we can use the Laplace transform of hitting time to obtain

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-(X_{\tau_I}-l_D)\sqrt{2\lambda}}\right].$$

Using the formulas for $X_{\tau_I}$ and $l_d$, we know that $X_{\tau_I} - l_d = \frac{1}{\sigma}\log\frac{V_{\tau_I}}{h_D}$ and hence we obtain

$$\mathbb{E}_{\mathbb{P}^*}\left[\left(\frac{V_{\tau_I}}{h_D}\right)^{-\frac{\sqrt{2\lambda}}{\sigma}}\right] = h_D^{\frac{\sqrt{2\lambda}}{\sigma}}\mathbb{E}_{\mathbb{P}^*}\left[V_{\tau_I}^{-\frac{\sqrt{2\lambda}}{\sigma}}\right] = \left(\frac{h_D}{V_0}\right)^{\frac{\sqrt{2\lambda}}{\sigma}}\mathbb{E}_{\mathbb{P}^*}\left[e^{-\sqrt{2\lambda}X_{\tau_I}}\right]$$

Applying now Proposition 5.12 we arrive to our result. ∎

Then we find the Laplace transform of the Parisian disinvestment time under the measure $\mathbb{P}^*$ defined in (6).

**Proposition 5.14** *For any $\lambda > 0$, the following equality holds,*

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda\tau_D}\right] = \mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda\tau_I}\right]\frac{\phi(-\sqrt{2\lambda d_I})}{\phi(\sqrt{2\lambda d_D})}\left(\frac{h_D}{h_I}\right)^{\frac{\sqrt{2\lambda}}{\sigma}}$$

**Proof.** Using Proposition 5.3, we can write

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda\tau_D}\right] = \mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda\tau_I}\right]\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda T_{(X_{\tau_I},l_D)}(X)}\right]\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda H^-_{(l_D,l_D),d_D}(X)}\right]$$

Now using the corresponding Laplace transforms, we arrive at the desired result. ∎

Again, we calculate the moment generating function for the process $X_t$ defined in (5), stopped at the Parisian disinvestment time.

**Proposition 5.15** *For any $\lambda \in \mathbb{R}$, the following equality holds,*

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda X_{\tau_D}}\right] = \left(\frac{h_D}{V_0}\right)^{-\frac{\lambda}{\sigma}}\phi(\lambda\sqrt{d_D}).$$

**Proof.** Using the definition of $X_{\tau_D}$ we obtain

$$\mathbb{E}_{\mathbb{P}^*}\Big[e^{-\lambda X_{\tau_D}}\Big] = \mathbb{E}_{\mathbb{P}^*}\Big[e^{-\lambda(l_D - m_1\sqrt{d_D})}\Big].$$

Now using the definition of $l_D$ and $\phi$ we obtain the desired result. ■

Finally, we are now able to calculate the first term appearing in the maximization problem (4).

**Proposition 5.16** *The following equality holds,*

$$\mathbb{E}_{\mathbb{P}}\Big[e^{-\rho\tau_I}(V_{\tau_I} - C_I)1_{\{\tau_I<\infty\}}\Big] = \mathbb{E}_{\mathbb{P}^*}\Big[e^{-(\rho+\frac{b^2}{2})\tau_I}\Big]\Big(\frac{h_I}{V_0}\Big)^{\frac{b}{\sigma}}\phi(b\sqrt{d_I})\times$$

$$\times\Big\{h_I\frac{\phi(\sqrt{d_I}(\sigma+b))}{\phi(b\sqrt{d_I})} - C_I\Big\}.$$

**Proof.** Using equation (7) and Proposition 5.12, the left hand side in the above equality becomes

$$\mathbb{E}_{\mathbb{P}^*}\Big[e^{-(\rho+\frac{b^2}{2})\tau_I}\Big]\Big\{V_0\Big(\frac{h_I}{V_0}\Big)^{\frac{\sigma+b}{\sigma}}\phi(\sqrt{d_I}(\sigma+b)) - C_I\Big(\frac{h_I}{V_0}\Big)^{\frac{b}{\sigma}}\phi(b\sqrt{d_I})\Big\}$$

and grouping the terms we arrive at the desired result. ■

Similarly, we calculate the second term appearing in the maximization problem (4).

**Proposition 5.17** *The following equality holds,*

$$\mathbb{E}_{\mathbb{P}}\Big[e^{-\rho\tau_D}(C_D - V_{\tau_D})1_{\{\tau_D<\infty\}}\Big] = \mathbb{E}_{\mathbb{P}^*}\Big[e^{-(\rho+\frac{b^2}{2})\tau_I}\Big]\Big(\frac{h_I}{V_0}\Big)^{\frac{b}{\sigma}}\Big(\frac{h_I}{h_D}\Big)^{\frac{-b-\sqrt{2\rho+b^2}}{\sigma}}\times$$

$$\times\frac{\phi(-\sqrt{(2\rho+b^2)d_I})}{\phi(\sqrt{(2\rho+b^2)d_D})}\phi(-b\sqrt{d_D})\Big\{C_D - h_D\frac{\phi(-(b+\sigma)\sqrt{d_D})}{\phi(-b\sqrt{d_D})}\Big\}$$

**Proof.** Using equation (8), Propositions 5.14, and Proposition 5.15, the left hand side in the above equality becomes

$$\mathbb{E}_{\mathbb{P}^*}\Big[e^{-(\rho+\frac{b^2}{2})\tau_I}\Big]\Big(\frac{h_D}{h_I}\Big)^{\frac{\sqrt{2\rho+b^2}}{\sigma}}\frac{\phi(-\sqrt{(2\rho+b^2)d_I})}{\phi(\sqrt{(2\rho+b^2)d_D})}\Big\{C_D\Big(\frac{h_D}{V_0}\Big)^{\frac{b}{\sigma}}\phi(-b\sqrt{d_D}) -$$

$$-V_0\Big(\frac{h_D}{V_0}\Big)^{\frac{\sigma+b}{\sigma}}\phi(-(b+\sigma)\sqrt{d_D})\Big\}$$

and now factoring out and grouping the terms we arrive at the desired result. ■

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